PROOF IN CLASSROOM SOCIAL PRACTICE

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How does the construction of proof relate to the social practice developed in the mathematics classroom? This report addresses the role of diagrams in order to focus the complementarity of participation and reification in the process of constructing a proof and negotiating its meaning. The discussion is based on the analysis of the mathematical practice developed by a group of four 9th grade students and is inspired by the social theory of learning.

INTRODUCTION

The present study dealt with the problem of proof in school mathematics (Rodrigues, 2008). Its main goal was to identify the ways in which students validate their mathematical results, relating them to the social practice developed in the classroom. The questions posed were: 1) what is the nature of proof in a school context?, 2) what is the role of proof in students' mathematical activity?, and 3) how does the construction of proof relate to the social practice developed in the mathematics classroom? We will present in this paper just some results related with the third question.

The study's framework is rooted in the theoretical frame of social practice in the line of Wenger (1998). Mathematics learning is seen as a situated and a social phenomenon (Lave, 1997; Matos, 2010). As a social participation, it is the process of being an active participant in the practice of social communities and constructing identities in relation to those communities (Wenger, 1998). "Such participation shapes not only what we do, but also who we are and how we interpret what we do" (Wenger, 1998, p. 4). The social theory of learning includes components that are interrelated and characterize the social participation as a process of learning and of knowing: (a) community, (b) identity, (c) practice, and (d) meaning.

The construct of *community of practice* is a central one in this theory. The basic structure of a community of practice is composed of three elements: (1) the *domain* of knowledge that defines the area or the set of shared topics; (2) the *community* of people, concerned with the domain, creating relationships and a sense of belonging; and (3) the shared *practice* developed by people to deal with the domain, consisting of the body of shared knowledge and resources that enables the community to

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proceed efficiently (Wenger, McDermott, & Snyder, 2002). The relation, by which practice is the source of coherence of a community, has three dimensions: (1) the mutual engagement, (2) the joint enterprise, and (3) the shared repertoire (Wenger, 1998). The communities of practice may not be spontaneous and the introduction of a member may not be voluntary. Nevertheless, the maintenance of a community depends on the energy produced by the proper community and not by an external mandate.

In this paper we also analyse the role of a diagram in proof construction through the concept of *reification* — "the process of giving form to our experience by producing objects that congeal this experience into 'thingness'" (Wenger, 1998, p. 58). This concept refers both to a process and its product. In fact, they always imply each other.

PROOF: THEORETICAL ISSUES

Proof is inherent to the nature of mathematics as a science (Jahnke, 2010). The notion of proof has evolved throughout the history of mathematics and it is nowadays the subject of debate among mathematicians. Yet, proof maintains a central role in mathematics (Hanna & Jahnke, 1996; Thurston, 1995). Discussing the epistemological status of proof, we have to examine issues, in the philosophy of mathematics, such as (a) the nature of mathematical objects; (b) the relationship between the experimental reality, the natural and human world and mathematics; and (c) the issue of truth. We assume mathematics as a human and social construction, but non-arbitral. It is this non-arbitrary nature that explains the parallelism between the physical reality and the mathematical one (Hersh, 1997). Mathematical knowledge develops through conjectures and refutations (Lakatos, 1991) and relies on linguistic knowledge, conventions and rules.

We also need to look at the curriculum in general terms and specifically the mathematics curricula, regarding how proof should be integrated. Many mathematics educators attach great importance to proof in the curriculum. Two essential reasons justify the relevance of the teaching of proof: (a) a more comprehensive vision of the nature of mathematics (de Villiers, 2004; Hanna, 2000; Jahnke, 2010), and (b) the promotion of mathematical understanding through the primordial function of proof in mathematics education, the explaining function (Hanna, 2000; Hersh, 1997). The more recent curricular documents, in Portugal and in other countries, have attached major importance to proof, advocating that from elementary to upper level there should be a gradual and continuous transition from justification and explanation activities to the proof itself. This curricular perspective regards proof as a process that evolves along all the school years. Counterexample proof is a particular method that can be introduced very early as a way of proving the falsehood of a statement or conjecture. According to Harel and Sowder (2007), upper elementary school children can deal with proof if they are taught appropriately.

However, internationally, studies in mathematics education provide empirical evidence that students reveal a great difficulty in understanding the need for proof (Rodrigues, 2008), understanding the functions of proof (Harel & Sowder, 2007) and constructing proofs (Healy & Hoyles, 2000). The majority of students of various levels (from the more basic to the first years of university level) use specific cases to establish the truth of conjectures they make (Chazan, 1993; Harel & Sowder, 2007; Healy & Hoyles, 2000; Rodrigues, 2000; 2008).

The discussion of mathematical ideas, developed within a small group of students orchestrated by the teacher within the class, plays a decisive role (a) in the emergence of proof meaning, (b) in the motivation to prove mathematical statements, and (c) in changing the spontaneous attitude of students towards construction of proof.

There is empirical evidence that a classroom environment rich in social interactions among students and between the teacher and the students can foster the development of the actual *proof schemes* of students. "A person's (or a community's) proof scheme consists of what constitutes ascertaining and persuading for that person (or community)" (Harel & Sowder, 2007, p. 809). However, according to Balacheff (1991), there are situations of social interaction that do not guarantee effective involvement in mathematical discussion and a construction of a proof at the end.

> Our point is that in some circumstances social interaction might become an obstacle, when students are eager to succeed, or when they are not able to coordinate their different points of view, or when they are not able to overcome their conflict on a scientific basis. In particular these situations can favour naïve empiricism, or they can justify the use of crucial experiments in order to obtain an agreement instead of proofs at a higher level. (Balacheff, 1991, p. 188)

All the efforts of children in elaborating their arguments should be valued but the teacher should insist on the need to improve them to become successively more general. The teacher must also give back to the students the responsibility of validating their statements (Balacheff, 1991; Harel & Sowder, 2007).

METHODOLOGY OF THE STUDY

The methodology adopted has an interpretative nature because it is suited to the aims of the study. It focused on the participants' meanings. The unit of analysis was proof scheme of students. Through the analysis of school mathematics practice, we tried to understand how students reason within this practice, how the meaning of the proof is negotiated, and how the process of proving evolves over time, studying the phenomenon in its natural setting — the mathematics classroom. For that reason, we paid attention to all aspects concerned with students practice: their utterances, their acting, their facial expressions and the mediating resources.

Data was collected in a state school in a class of the 9th grade, over one school year. A group of four students was selected to be videotaped and audiotaped. The researcher played the role of participant observer, having observed and participated in all the mathematical activities of the class during the 16 lessons in which inquiry tasks were carried out. To collect data we used (a) video recording of mathematical activities of students, (b) audio recording of students' dialogues, (c) field notes made by the researcher, (d) video recording of students and teacher semi-structured interviews, and (e) documental analysis of the work done by students and of video and audio recordings.

DISCUSSION OF SOME RESULTS

In this section we present results related to the carrying out of a single task: "*What is the relationship between the bisectors of supplementary adjacent angles?*" The task included a small note suggesting the drawing up of a diagram with the angles and their bisectors.

Proof by insight

Within the group, this was Ricardo, who suddenly, by insight, discovered the problem solution. He read the question of the task attentively and then he exclaimed: "Yeah!! I know! Bisectors of supplementary angles because added together they give an angle of 90º." This is the transcript of the following moment:

Ricardo: Well, I´ll explain. These are two supplementary angles. It gives 180. (…). When we divide, it's this and this. (*drawing the bisectors)*
Sara: It is the bisector.
Ricardo: Hang on!... We can add these two, it is half of 180 outside. So, this is a right angle. So, this is 90º.
Sara: I don't understand. It's what? We've got a mathematician here! Seriously, I don't understand.
Ricardo: I know you don't… Neither do I.
Sara: Ah! You don't understand! Good!
Ricardo: I'm sure that it is correct. But now I don't understand…

The discovery of the solution problem was made by a narrative and informal proof, constructed individually by Ricardo, without relation to the social interactions in the group. His fast process of solving the problem includes conceiving a proof: a general and a deductive argument. The Figure 1 illustrates the structure of Ricardo's argument, using Toulmin (1969) model.

All Ricardo's efforts in sharing the proof with his classmates came up against the communication difficulties presented by the prematurity of the moment: his classmates hadn't yet assimilated the task sense yet. When Ricardo says that he didn't understand (lines 8 and 10) this is because he had difficulty in communicating his thinking. Therefore it is his understanding that gives him the certainty "that it is correct" (line 10). But his thinking is regarded by him as a tourbillion, in a syncretic phase, as something that needs to be dissected so that all his classmates can understand clearly what he saw and knew to be correct.

Data
2 supplementary adjacent angles and their bisectors

Claim
So the bisectors form a right angle.

Warrant
Because the amplitude of the angle formed by the bisectors is the sum of the two supplementary semi-angles amplitudes.

Figure 1: Structure of Ricardo's argument

Despite the fact that Ricardo was the single owner of the original meaning of the proof, he never concealed it from his classmates. On the contrary, he used various resources, including the specialization and the diagram, to make transparent to them his individual process of reasoning:

Ricardo: (*looking at Sara and pointing to the diagram)* For example, for example, I'll use crazy numbers, well… Here it is 60, here it is 120. The bisector making this, it gives here 30 and here 60. 30+60, 90. Right angle. Have you got it now?

The diagram drawn by Ricardo is a *structuring resource* (Lave, 1997) since it shaped the process of constructing and communicating the proof. A structuring resource is something — concept, object, people, activity — that supports a situation giving it structural form. Probably Ricardo visualized the diagram in his insight and then he used it for the purpose of communication. First, drawing the diagram had a social motive of explaining, that is to say, of making Ricardo's thinking intelligible to others, without any relationship with the task suggestion of a diagram. Later, the student group drew up a diagram, in reply to what was asked in the task:

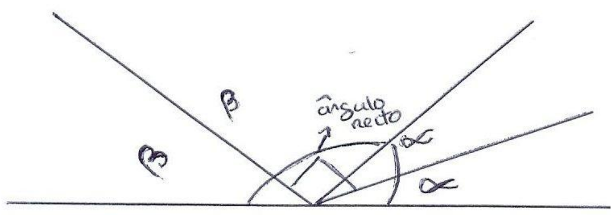

Figure 2: Diagram drawn up by student group

The bisectors were drawn with a ruler reifying a product in such a way the students consider appropriate to give to their teacher. The angles were seen in their generality and the diagram highlights theoretical properties such as the notion that a bisector creates two congruent angles. So the students did not care if it was drawn exactly in half and they did not measure angles. It was a conceptual congruency and the diagram was a support to thinking. Even when Ricardo refers to specific cases of angles (line 12), they were used to illustrate the general properties. So the specialization functioned as a communicative resource.

There is an equilibrium of power between Ricardo and Sara and communication benefits from this equilibrium. Ricardo speaks almost exclusively to Sara. It is her understanding that concerns him. Both Bernardo and Maria withhold their incomprehension, thus seeming to build up *identities of non-participation* (Wenger, 1998) within the group, maintaining a marginal position since their participation is restricted by non-participation.

The algebraic proof

When the teacher came up to the group, Ricardo said:

> Teacher, it gives an angle of 90º. The relationship is that it forms an angle of 90º. Now, I can't explain it to the others. The others…

The teacher did not validate Ricardo's statement and Ricardo did not want any validation either. He was certain of his deductive conclusion and his single worry was the difficulty of communication. Then the teacher negotiated the use of Greek letters to label the angles in the diagram and the construction of a formal and algebraic proof:

$$\begin{cases} 2\alpha + 2\beta = 180^{\circ} \\ \dfrac{2\alpha}{2} + \dfrac{2\beta}{2} = \dfrac{180^{\circ}}{2} \end{cases}$$

$$\begin{cases} 2\alpha + 2\beta = 180^{\circ} \\ \alpha + \beta = 90^{\circ} \end{cases}$$

> R: The relationship between the bisectors is that they form an angle of 90º. (Rodrigues, 2008, p. 678)

The written group report does not include any mark of the narrative and informal proof in the terms used by Ricardo to communicate it. The algebraic proof is the result of teacher negotiation leading to its reification.

This proof is strongly linked to the diagram since it translates the relations observed in the diagram. The symbolic notation by assigning equal letters to congruent angles led the students to concentrate on the essential, distinguishing the angles of interest (the angles inside the bisectors) and ignoring those which were not of interest (the angles outside the bisectors). The students, implicitly, treated a geometric situation as algebraic, assuming angles as quantities. The succinctness of the diagram focuses the negotiation of meaning produced in the process of constructing the proof. And in this sense, the communicative ability of this artifact depended on how negotiating meaning of the proof was distributed between reification of a diagram and participation in the carrying out of the task by each student.

Ricardo had a leading role in the entire construction of the algebraic proof. It was Ricardo that dictated the final answer. The other members' group drew up their own diagrams but they needed to look at Ricardo's and they copied the algebraic proof which Ricardo wrote. Because Ricardo's writing was untidy, it was always his classmates that wrote on the final sheet to be given to the teacher. The mutual engagement of the team members is characterized by complementary contributions. The whole process of constructing the proof, anchored by the drawing up of the diagram, increased ownership of meaning for all the members of the team in different degrees.

CONCLUDING REMARKS

The group of students can be seen as a community of practice since all members share a concern with their mathematical tasks, in the classroom, create relationships and a sense of belonging to the team, and develop a shared practice to deal with the tasks set. In this paper it was illustrated that this community of practice does not entail homogeneity. The mutual engagement is characterized by diversity and it is inherently partial. The members assume different roles depending on their competence. It is a community of practice where people help each other. Ricardo shared his original proof meaning and through the process of sharing, all the members of the team increased their ownership of meaning in different degrees depending on the degree of participation. Participation is both a kind of action and a mode of belonging. The degree of ownership of meaning refers to the degree to which anyone can make use of or assert as his or her own the meanings that negotiate. The process of Ricardo communicating his deductive reasoning gives him the opportunity of clarifying his mathematical thinking. As stated by Wenger (1998), the resulting relations of a shared practice are diverse, reflecting the complexity of doing things together, and they are not reducible to a single principle such as power or collaboration.

Regarding the roles played by different elements in the social practice, we can focus on the teacher and the diagram. The intervention of the teacher led students to express the written proof in a formal and algebraic format. The diagram played an important role in the process of sharing and increasing the ownership of meaning of proof by highlighting the relevant properties.

Finally, we must pay attention if there is a split between production and adoption of meaning within a group of students because this split compromises learning, reflecting an enduring patterns of engagement among members that can result in non-participation and marginality.

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