Decomposition in bunches of the critical locus of a quasi-ordinary map

Pedro Daniel Gonzalez Perez, Evelia Garcia Barroso

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E.R. García Barroso, P.D. González Pérez

Abstract. [1] A polar hypersurface P of a complex analytic hypersurface germ $f = 0$ can be investigated by analyzing the invariance of certain Newton polyhedra associated to the image of P, with respect to suitable coordinates, by certain morphisms appropriately associated to f. We develop this general principle of Teissier (see [T2] and [T3]) when $f = 0$ is a quasi-ordinary hypersurface germ and P is the polar hypersurface associated to any quasi-ordinary projection of $f = 0$. We build a decomposition of P in bunches of branches which characterizes the embedded topological type of the irreducible components of $f = 0$. This decomposition is characterized also by some properties of the strict transform of P by the toric embedded resolution of $f = 0$ given by the second author in [GP3]. In the plane curve case this result provides a simple algebraic proof of a theorem of Lê, Michel and Weber in [L-M-W].

Introduction

The *polar varieties* or at least their rational equivalence classes play an important role in projective geometry in particular in the study of characteristic classes and numerical invariants of projective algebraic varieties, and also in the study of projective duality (Plücker formulas). In the 1970's local polar varieties began to be used systematically in the study of singularities. Local polar varieties can be used to produce invariants of equisingularity ("topological" invariants of complex analytic singularities) and also to explain why the same invariants appear in apparently unrelated questions. We study here a particular instance of construction and study of such equisingularity invariants.

The *Jacobian polygon*, a plane polygon associated by Teissier to a germ of complex analytical hypersurface defining an isolated singularity at the origin, is an invariant of equisingularity for c-*equinsingularity*, which is equivalent to Whitney condition for a family of isolated hypersurface singularities, implies topological triviality and is equivalent to it for plane curves. The inclinations of the compact edges of this polygon are rational numbers called the *polar invariants* of the germ. If $\phi(X_1, \dots, X_{d+1}) = 0$ is the equation of the hypersurface germ with respect to some suitable coordinates the Jacobian polygon coincides with the Newton polyhedron of image of the critical locus, or *polar variety*, of the morphism $(\mathbf{C}^{d+1}, 0) \to (\mathbf{C}^2, 0)$ defined by $T = \phi(X_1, \dots, X_{d+1})$ and $U = X_1$, with respect to the coordinates (U, T) (see [T2]).

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In the case of a germ of plane irreducible curve, Merle shows in [Me] that the polar invariants determine also the equisingularity class of the curve (or equivalently its embedded topological type). Merle's results has been generalized to the case of *reduced* plane curve germs by Kuo, Lu, Eggers, García Barroso and Wall among others (see [K-L], [Eg], [GB] and [Wa]). They give a decomposition theorem of a generic polar curve in bunches which depends only on the equisingularity class of the curve. To this decomposition is associated a matrix of *partial polar invariants* which determines the equisingularity class of the curve (see [GB])). Lê, Michel and Weber have proven using topological methods that the strict transform of a generic polar curve by the minimal embedded resolution of the curve meets any connected component of a *permitted* subset of the exceptional divisor (see [L-M-W]). Another decomposition in bunches of the generic polar curve, can be defined from this result in a geometrical way, the bunches correspond bijectively to the connected components of the permitted subset. García Barroso has compared these two decompositions and shown that they coincide in [GB].

In this paper we study local polar hypersurfaces of a class of complex analytic hypersurface singularity, called *quasi-ordinary*. This class of singularities, of which the simplest example are the singularities of plane curves, appears naturally in Jung's approach to analyze surface singularities and their parametrizations. A germ of complex analytic variety is *quasi-ordinary* if there exists a finite projection, called quasi-ordinary, to the complex affine space with discriminant locus contained in a normal crossing divisor. By Jung-Abhyankar's theorem any quasi-ordinary projection is provided with a parametrization with fractional power series ([J] and [A]). In the hypersurface case these parametrisations determine a finite set of monomials, called *characteristic* or *distinguished*, which determine quite a lot of the geometry and topology of the singularity. For instance, these monomials constitute a complete invariant of its *embedded topological type* in the analytically irreducible case and conjecturally in the reduced case (see Gau and Lipman works [Gau], [L2] and [L3]), in particular they determine the zeta function of the geometric monodromy as shown in the works of Némethi, McEwan and González Pérez (see [M-N1] and [GPMN]). The characteristic monomials determine also embedded resolutions of the corresponding quasi-ordinary hypersurface singularity, which have been obtained in two different ways by Villamayor [V] and González Pérez (see [GP3] and [GP4]).

We give a decomposition theorem of the polar hypersurfaces $(P,0)$ of a quasi-ordinary hypersurface $(S,0)$ corresponding to a quasi-ordinary projection. If $(S,0)$ is embedded in $(\mathbf{C}^{d+1},0)$ any quasi-ordinary projection can be expressed in suitable coordinates by $(X_1,\dots,X_d,Y)\mapsto(X_1,\dots,X_d)$. Then $(S,0)$ has an equation defined by a Weierstrass polynomial $f\in\mathbf{C}\{X_1,\dots,X_d\}[Y]$ and the associated polar hypersurface $(P,0)$ is defined as the critical space, $f_Y=0$, of the *quasi-ordinary morphism*:

$$\begin{cases} \xi_f \;:\; (\mathbf{C}^{d+1},0) \longrightarrow (\mathbf{C}^{d+1},0) \\ U_1=X_1,\ \dots,\ U_d=X_d,\ T=f(X_1,\dots,X_d,Y) \end{cases}$$

The decomposition is defined in terms of a matrix, generalizing the matrix of partial polar invariants of [GB], which determines and is determined by the partially ordered set of *characteristic monomials* associated to the fixed quasi-ordinary projection. In particular, it defines a complete invariant of the embedded topological type of each irreducible component of $(S,0)$ by using Gau's and Lipman's results. Our decomposition theorem is partially motivated by a result of Popescu-Pampu providing a decomposition of the polar hypersurface $(P,0)$ of the quasi-ordinary hypersurface $(S,0)$, obtained with the additional hypothesis that $(S,0)$ and $(P,0)$ are simultaneously quasi-ordinary with respect to the given quasi-ordinary projection (see [PP1], Chapter 3 or [PP2]). Our decomposition extends also to the *Laurent* quasi-ordinary case studied in [PP1] by analogy to the case of *meromorphic* plane curves of Abhyankar and Assi (see [A-As]).

An important tool which we introduce to prove these results is an *irreducibility criterion* for power series with *polygonal Newton polyhedra* (the maximal dimension of its compact faces is equal to one). Our criterion, which holds for power series with coefficients over any algebraically closed field of arbitrary characteristic, states that if an irreducible series has a polygonal Newton polyhedron then it has only one compact edge. This result generalizes a fundamental property of plane curve germs. Our proof is obtained by using Newton polyhedra in the framework of toric geometry.

The decomposition theorem of the polar hypersurface $(P, 0)$ has a proof inspired by Teissier's works [T2] and [T3]. In the irreducible case we analyze the *discriminant* $\mathcal{D}$ of the quasi-ordinary map ξ_f, i.e., the image of the critical space. We compute the Newton polyhedron $\mathcal{N}_{\mathcal{D}}$ of the discriminant $\mathcal{D}$ in the coordinates $U_1, \dots, U_d, T$ above, and we show that it is a *polygonal polyhedron.* This computation applies the above mentionned results of Popescu-Pampu, after suitable toric base changes already used in [GP1]. We use then the irreducibility criterion to show the existence of a decomposition of $(P, 0)$ in bunches which correspond bijectively to the compact edges of the polyhedron $\mathcal{N}_{\mathcal{D}}$.

We also give a geometrical characterization of the decomposition theorem by analyzing the strict transform of $(P, 0)$ by a modification $p : \mathcal{Z} \to \mathbf{C}^{d+1}$ which is built canonically from the given quasi-ordinary projection by using the characteristic monomials (see [GP4]). Geometrically the bunches of the decomposition of $(P, 0)$ correspond to the union of branches of the polar hypersurface $(P, 0)$ whose strict transforms by p, meet the exceptional fiber $p^{-1}(0)$ at the same irreducible component. A posteriori this analysis can be extended to the the toric *embedded resolutions* of $(S, 0)$ built in [GP3] or [GP4], since they are factored by p. In the plane curve case we apply this result to obtain a simple algebraic proof of the theorem of Lê, Michel and Weber in [L-M-W] which shows the underlying toric structure of the decomposition of the polar curve.

Our results provide answers to some of the questions raised independently by McEwan and Némethi in [M-N2] section III, among some open problems concerning quasi-ordinary singularities. We hope that the results of this paper could apply to the study of polar varieties of hypersurface singularities by using a suitable form of Jung's approach. It is reasonable to expect that this work may have some applications to the metric study of the Milnor fibres of hypersurfaces, at least in the quasi-ordinary case, as suggested by Teissier's and García Barroso's results in the case of plane curve singularities (see [GB-T]); see also Risler's work [Ri] for the real plane curve case.

The proofs are written in the analytic case. The results and proofs of this paper hold also in the algebroid case (over an algebraically closed field of characteristic zero).

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1 Quasi-ordinary polynomials, their characteristic monomials and the Eggers-Wall tree

A germ of complex analytic hypersurface $(S, 0) \subset (\mathbf{C}^{d+1}, 0)$ is *quasi-ordinary* if there exists a finite projection $(S, 0) \to (\mathbf{C}^d, 0)$ which is a local isomorphism outside a normal crossing divisor. The embedding $(S,0) \subset (\mathbf{C}^{d+1},0)$ can be defined by an equation $f=0$ where $f \in \mathbf{C}\{X\}[Y]$ is a *quasi-ordinary polynomial*: a Weierstrass polynomial with discriminant $\Delta_Y f$ of the form $\Delta_Y f = X^\delta \epsilon$ for a unit ϵ in the ring $\mathbf{C}\{X\}$ of convergent (or formal) power series in the variables $X=(X_1,\dots,X_d)$ and $\delta \in \mathbf{Z}^d_{\geq 0}$.

The Jung-Abhyankar theorem guarantees that the roots of the quasi-ordinary polynomial f are fractional power series in the ring $\mathbf{C}\{X^{1/k}\}$ for some suitable integer k, for instance $k = \deg f$ when f is irreducible (see [A]). If the series $\{\zeta^{(l)}\}_{l=1}^{\deg f} \subset \mathbf{C}\{X^{1/k}\}$ are the roots of f, its discriminant is equal to:

$$\Delta_Y f = \prod_{i \neq j} (\zeta^{(i)} - \zeta^{(j)}) \tag{1}$$

hence each factor $\zeta^{(t)} - \zeta^{(r)}$ is of the form $X^{\lambda_{t,r}} \epsilon_{t,r}$ where $\epsilon_{t,r}$ is a unit in $\mathbf{C}\{X^{1/k}\}$. The monomials $X^{\lambda_{t,r}}$ (resp. the exponents $\lambda_{t,r}$) are called *characteristic*.

In the reducible reduced case, if $f = f_1 \dots f_s$ is the factorization in monic irreducible polynomials each factor f_i is a quasi-ordinary polynomial since $\Delta_Y f_i$ divides $\Delta_Y f$ by formula (1).

We define the partial order of $\mathbf{R}^d \cup \{+\infty\}$:

$$u \leq u' \Leftrightarrow u' \in u + \mathbf{R}^d_{\geq 0}. \tag{2}$$

We write $u < u'$ if $u \leq u'$ and $u \neq u'$. If $\alpha \in \mathbf{R}^d$ we set $\alpha < +\infty$. Notice that $u \leq u'$ means that the inequality holds coordinate-wise with respect to the canonical basis. The characteristic exponents have the following property with respect to the order (2), see [L3] and [Z1].

Lemma 1 *Let f_i be an irreducible factor of the reduced quasi-ordinary polynomial f. The set*

$$V_f(f_i) := \left\{ \lambda_{r,t} / \zeta^{(r)} \neq \zeta^{(t)},\ f(\zeta^{(t)}) = 0 \text{ and } f_i(\zeta^{(r)}) = 0 \right\} \tag{3}$$

is totally ordered by $\leq$. ◇

If f_i and f_j are two irreducible factors of the quasi-ordinary polynomial f we define the *order of coincidence* $k(f_i,f_j)$ of their roots by:

$$k(f_i,f_j) = \max\{\lambda_{r,t} / f_i(\zeta^{(r)}) = 0, f_j(\zeta^{(t)}) = 0\}$$

The order of coincidence of f_i with itself is $k(f_i,f_i) := +\infty$. We have the following "valuative" property of the orders of coincidence (see Lemma 3.10 of [GP3]).

$$\min\{k(f_i,f_j), k(f_j,f_r)\} \geq k(f_i,f_r) \text{ with equality if } k(f_i,f_j) \neq k(f_j,f_r) \tag{4}$$

The totally ordered set $V_f(f_i)$ defined by (3) is equal to the union of the non necessarily disjoint sets whose elements are the characteristic exponents $\lambda_1^{(i)} < \dots < \lambda_{g(i)}^{(i)}$ of f_i, if they exist[2], and the orders of coincidence $k(f_i,f_j)$ for $j=1,\dots,s$ and $j \neq i$. We associate to the characteristic exponents of the irreducible factor f_i, for $i=1,\dots,s$, the following sequences of *characteristic lattices and integers*: the lattices are $M_0^{(i)} := \mathbf{Z}^d$ and $M_j^{(i)} := M_{j-1}^{(i)} + \mathbf{Z}\lambda_j^{(i)}$ for $j=1,\dots,g(i)$ with the convention $\lambda_{g(i)+1}^{(i)} = +\infty$; the integers are $n_0^{(i)} := 1$ and $n_j^{(i)}$ is the index of the subgroup $M_{j-1}^{(i)}$ in $M_j^{(i)}$, for $j=1,\dots,g(i)$. We denote the integer $n_j^{(i)} \cdots n_{g(i)}^{(i)}$ by $e_{j-1}^{(i)}$ for $j=1,\dots,g(i)$. We have

[2] The case f_i with no characteristic monomials happens only when $\deg f_i = 1$

that $\deg f_i = e_0^{(i)} = n_1^{(i)} \cdots n_{g(i)}^{(i)}$ (see [L3] and [GP2]) and used When $d = 1$ we have the equality $M_j^{(i)} = (e_j^{(i)})\mathbf{Z}$, and the integer n_j coincides with the first component of the classical characteristic pairs of the plane branch defined by $f = 0$.

The information provided by the characteristic monomials is structured in a tree which encodes the embedded topological type of the irreducible components of $f = 0$ (characterized by the work of Gau and Lipman in terms of the characteristic exponents, see [Gau] and [L3]). This tree is introduced by Popescu-Pampu (see [PP1] and [PP2]) following a construction of Wall [Wa] and Eggers [Eg] used to study the polar curves of a plane curve germ (see also [GB]).

The *elementary branch* $\theta_f(f_i)$ associated to f_i is the abstract simplicial complex of dimension one with vertices running through the elements of the totally ordered subset $V_f(f_i) \cup \{0, +\infty\}$ of $\mathbf{Q}^d \cup \{\infty\}$, and edges running through the segments joining consecutive vertices. The underlying topological space is homeomorphic to the segment $[0, +\infty]$. We denote the vertex of $\theta_f(f_i)$ corresponding to $\lambda \in V_f(f_i) \cup \{0, +\infty\}$ by $P_\lambda^{(i)}$. The simplicial complex $\theta_f(f)$ obtained from the disjoint union $\bigsqcup_{i=1}^s \theta_f(f_i)$ by identifying in $\theta_f(f_i)$ and $\theta_f(f_j)$ the sub-simplicial complexes corresponding to $\overline{P_0^{(i)} P_{k(f_i,f_j)}^{(i)}}$ and $\overline{P_0^{(j)} P_{k(f_i,f_j)}^{(j)}}$ for $1 \leq i < j \leq s$ is a tree.

We give to a vertex $P_\lambda^{(i)}$ of $\theta_f(f)$ the valuation $v(P_\lambda^{(i)}) = \lambda$. The restriction of the valuation v to the set of non extremal vertices of $\theta_f(f)$ is a 0-chain with coefficients in $\frac{1}{k}\mathbf{Z}^d$. The set of vertices of $\theta_f(f)$ is partially ordered by $P \leq P'$ if P, P' are vertices of the same elementary branch of the tree and $v(P) \leq v(P')$. The valuation v defines an orientation on the tree $\theta_f(f)$. The boundary operator ∂ is the linear map of integral 1-chains defined on the segments by $\partial(\overline{PP'}) = P' - P$ if $v(P) < v(P')$.

If $v(P) \neq +\infty$ the value $v(P)$ is the d-uple of coordinates of an element of the lattice $\frac{1}{k}\mathbf{Z}^d$, which we denote by $\tilde{v}(P)$, with respect to the canonical basis of $\mathbf{Z}^d$. If $v(P) = +\infty$ we set $\tilde{v}(P) = +\infty$. This defines a *lattice valuation* $\tilde{v}$ the vertices of $\theta_f(f)$ which is preserved by certain modifications (see section 4). We recover the valuation v from the lattice valuation $\tilde{v}$ and the *reference lattice cone:* $(\mathbf{R}^d_{\geq 0}, \mathbf{Z}^d)$ associated to the exponents of the monomials of $\mathbf{C}\{X\}$.

For $i = 1, \ldots, s$ we define an integral 1-chain whose segments are obtained by by subdividing the segments of the chain

$$\overline{P_0^{(i)} P_{\lambda_1^{(i)}}^{(i)}} + n_1^{(i)} \overline{P_{\lambda_1^{(i)}}^{(i)} P_{\lambda_2^{(i)}}^{(i)}} + \cdots + n_1^{(i)} \cdots n_{g(i)}^{(i)} \overline{P_{\lambda_{g(i)}^{(i)}}^{(i)} P_{+\infty}^{(i)}} \tag{5}$$

with the points corresponding to the orders of coincidence of f_i, the coefficient of an oriented segment in the subdivision is the same as the coefficient of the oriented segment of (5) containing it. It follows that these 1-chains paste on $\theta_f(f)$ defining a 1-chain which we denote by γ_f. The vertex $P_\lambda^{(i)}$, if $\lambda \neq 0, +\infty$ does not correspond to a characteristic exponent of f_i if and only if the vertex $P_\lambda^{(i)}$ appears in two segments of $\theta_f(f_i)$ with the same coefficient.

Definition 2 *The Eggers-Wall tree of the quasi-ordinary polynomial f is the simplicial complex $\theta_f(f)$ with the chains γ_f and v. We denote by $\tilde{\theta}_f(f)$ the simplicial complex $\theta_f(f)$ with the chains γ_f and $\tilde{v}$.*

As shown by the work of Wall [Wa] and Popescu-Pampu [PP2], the Eggers-Wall tree is useful to represent the information provided by the orders of coincidence of the roots of f with the roots of h in the set $\mathcal{RC}_f$ of polynomials *radically comparable* with the polynomial f:

$$\mathcal{RC}_f := \{h \in \mathbf{C}\{X\}[Y] \, / h \text{ monic and the product } fh \text{ is quasi-ordinary } \}.$$

Any $h \in \mathcal{RC}_f$ is a quasi-ordinary polynomial and the difference of its roots with those of f has a dominant monomial (viewed in $\mathbf{C}\{X^{1/k}\}$ for some suitable k). If $h \in \mathcal{RC}_f$ we consider the sub-tree $\theta_{fh}(f) = \bigcup_{i=1}^{s} \theta_{fh}(f_i)$ of $\theta_{fh}(fh)$ as a subdivision of $\theta_f(f)$ induced by h. If h is irreducible the point $P^{(i)}_{k(f_i,h)}$ is the point of bifurcation of the elementary branches $\theta_{fh}(f_i)$ and $\theta_{fh}(h)$ in $\theta_{fh}(fh)$; we denote by $P^h_{k(h,f)}$ the point of bifurcation of the elementary branch $\theta_{fh}(h)$ from the tree $\theta_{fh}(f)$. If $h \in \mathcal{RC}_f$ and if $h = h_1 \dots h_t$ is the factorization of h as a product of monic irreducible polynomials, the *contact chain* $[h]^{(f)}$ is the integral 0-chain on $\theta_{fh}(f)$ defined by:

$$[h]^{(f)} = \sum_{j=1}^{t} \deg h_j P^{h_j}_{k(h_j,f)}.$$

The contact chain $[h]^{(f)}$ is associated with the decomposition $h = b_1 \dots b_{s(f,h)}$ in the ring $\mathbf{C}\{X\}[Y]$, where the factors b_j are the products of those irreducible factors of h having the same order of coincidence with each irreducible factor of f.

2 Decomposition in bunches and Newton polyhedra of images

We show that a quasi-ordinary polynomial $f \in \mathbf{C}\{X\}[Y]$ defines in a natural way a decomposition in bunches for certain class of polynomials which contains the derivative f_Y of f. We state the characterization of the decomposition in bunches of the polar hypersurface $f_Y = 0$ and of the Newton polyhedron of the equation defining the image of $f_Y = 0$ by the quasi-ordinary morphism ξ_f, in terms of the tree $\theta_f(f)$.

If $f \in \mathbf{C}\{X\}[Y]$ is a quasi-ordinary polynomial the derivative $f_Y := \frac{1}{n}\frac{\partial f}{\partial Y}$ belongs to the set $\mathcal{C}_f$ of polynomials *comparable*[3] with f:

$$\mathcal{C}_f := \left\{ h \in \mathbf{C}\{X\}[Y] \, / h \text{ monic}, \quad \text{Res}_Y(f,h) = X^{\rho(f,h)} \epsilon_{f,h} \text{ with } \epsilon_{f,h}(0) \neq 0 \text{ and } \rho(f,h) \in \mathbf{Z}^d \right\},$$

where $\text{Res}_Y(f,h)$ denotes the resultant of the polynomials f and h.

We have an inclusion $\mathcal{RC}_f \subset \mathcal{C}_f$: it is sufficient to notice that $\text{Res}_Y(f,h)$ divides $\Delta_Y(fh)$, a statement which follows from the classical properties of resultants and discriminants (see [G-K-Z]):

$$\text{Res}_Y(f, h_1 \cdots h_t) = \prod_{i=1}^{t} \text{Res}_Y(f, h_i), \quad \Delta_Y(fh) = \Delta_Y(f)\Delta_Y(h)(\text{Res}_Y(f,h))^2 \tag{6}$$

Remark 3 *The inclusion $\mathcal{RC}_f \subset \mathcal{C}_f$ is strict in general. In particular the derivative f_Y of a quasi-ordinary polynomial $f \in \mathbf{C}\{X_1, \dots, X_d\}[Y]$ is not radically comparable to f in general.*

- For instance, if $f = Y$ we have that $\mathcal{C}_f$ is the set of monic polynomials $h \in \mathbf{C}\{X\}[Y]$ such that $h(0)$ is of the form $h(0) = X^{\rho(f,h)}\epsilon_{f,h}$ with $\epsilon_{f,h}(0) \neq 0$. On the other hand $\mathcal{RC}_f$ is the set of monic polynomials $h \in \mathbf{C}\{X\}[Y]$ such that the product Yh is a quasi-ordinary polynomial. We have that $h = Y^2 + (X_1 + X_2)Y + X_1 X_2^2 \in \mathcal{C}_f \backslash \mathcal{RC}_f$.

- This exemple is already given by Popescu-Pampu. The polynomial $f = Y^3 + X_1X_2Y^2 + X_1^3X_2Y + X_1X_2$ is quasi-ordinary thus $f_Y \in \mathcal{C}_f$ but $f_Y \notin \mathcal{RC}_f$ (see [PP1], page 127).

[3] Our notion of *comparable* polynomials generalizes that of *radically comparable*, (which corresponds to the notion of *comparable* polynomials of Popescu-Pampu in [PP1] or [PP2]).

If $f = f_1 \cdots f_s$ is the factorization of f in monic irreducible polynomials we deduce from (6) and the definitions that $\mathcal{C}_f = \bigcap_{i=1}^{s} \mathcal{C}_{f_i}$. We define an equivalence relation in the set $\mathcal{C}_f$:

$$h, h' \in \mathcal{C}_f,\ h \sim h' \Leftrightarrow \frac{\rho(f_i, h)}{\deg h} = \frac{\rho(f_i, h')}{\deg h'} \text{ for } i = 1, \dots, s. \tag{7}$$

By (6) if $h \in \mathcal{C}_f$ the irreducible factors of h are also in $\mathcal{C}_f$. We denote by $s(f,h)$ the number of classes of the restriction of the equivalence relation (7) to the set of irreducible factors of h.

Definition 4 *The f-bunch decomposition of a polynomial $h \in \mathbf{C}\{X_1, \dots, X_d\}[Y]$ comparable to the quasi-ordinary polynomial $f \in \mathbf{C}\{X_1, \dots, X_d\}[Y]$ is $h = b_1 \cdots b_{s(f,h)}$, where the b_i, for $i = 1, \dots, s(f,h)$, are the products of the irreducible factors of h which are in the same class. The type of the f-bunch decomposition of h is the collection of vectors:*

$$\left\{ \left(\frac{\rho(f_1, h_j)}{\deg h_j}, \dots, \frac{\rho(f_s, h_j)}{\deg h_j}; \deg b_j \right) \right\}_{j=1}^{s(f,h)}, \tag{8}$$

where h_j is any irreducible factor of b_j for $j = 1, , \dots, s(f,h)$.

If f is clear from the context, we write bunch decomposition instead of f-bunch decomposition, in particular we will do this for the polynomial f_Y.

If $h \in \mathcal{RC}_f$ the type of the f-bunch decomposition of h is studied by using the orders of coincidence of $h \in \mathcal{RC}_f$ with the irreducible factors of f. The following proposition extends the classical relation between the intersection multiplicity and the order of coincidence in the plane branch case (see Proposition 3.7.15 of [PP1]). We introduce some notations: if $\lambda \in V_{hf}(f_i)$ we denote by $c_\lambda^{(i)}$ the integer $c_\lambda^{(i)} = \max(\{j / \lambda_j^{(i)} < \lambda\} \cup \{0\})$, if $\lambda = k(h, f_i)$ we denote $c_\lambda^{(i)}$ also by $c_{(h,f_i)}$.

Proposition 5 *Let $h \in \mathcal{RC}_f$ irreducible. If τ is any root of h in $\mathbf{C}\{X^{1/k}\}$ (for some integer $k > 0$) we have that $f_i(\tau)$ is of the form:*

$$f_i(\tau) = X^{\frac{\rho(f_i,h)}{\deg h}} \epsilon_{i,\tau} \textit{ where } \epsilon_{i,\tau} \textit{ is a unit in } \mathbf{C}\{X^{1/k}\} \textit{ and} \tag{9}$$

$$\frac{\rho(f_i, h)}{\deg h} = e^{(i)}_{c_{(h,f_i)}} k(h, f_i) + \sum_{k=1}^{c_{(h,f_i)}} (e^{(i)}_{k-1} - e^{(i)}_k) \lambda^{(i)}_k \tag{10}$$

◇

Definition 6 *We associate to the factor f_i of f the valuation ν_i of the vertices of $\theta_{fh}(f)$:*

$$\nu_i(P_\lambda^{(j)}) := \begin{cases} e^{(i)}_{c^{(i)}_\lambda} \lambda + \sum_{k=1}^{c^{(i)}_\lambda} (e^{(i)}_{k-1} - e^{(i)}_k) \lambda^{(i)}_k & \textit{if} \quad P_\lambda^{(j)} \in \theta_{fh}(f_i), \lambda \neq 0, +\infty, \\ \nu_i(P^{(i)}_{k(f_i,f_j)}) & \textit{if} \quad P_\lambda^{(j)} \notin \theta_{fh}(f_i), \\ 0 & \textit{if} \quad \lambda = 0, \\ +\infty & \textit{if} \quad j = i \textit{ and } \lambda = +\infty. \end{cases} \tag{11}$$

Remark 7 *If $h \in \mathcal{RC}_f$ is irreducible then $\nu_i(P^h_{k(h,f)}) = \frac{\rho(f_i,h)}{\deg h}$.*

If $h \in \mathcal{C}_f$ and in particular when $h = f_Y$, we study the f-bunch decomposition of h by analysing the Newton polyhedra with respect to suitable coordinates, of the polynomials defining the images of $h = 0$ under *quasi-ordinary morphisms* associated to the irreducible factors of f.

If $f \in \mathbf{C}\{X_1, \dots, X_d\}[Y]$ is a quasi-ordinary polynomial we say that the morphism

$$\begin{cases} \xi_f \;:\; (\mathbf{C}^{d+1}, 0) \longrightarrow (\mathbf{C}^{d+1}, 0) \\ U_1 = X_1, \;\dots,\; U_d = X_d, \;\; T = f(X_1, \dots, X_d, Y) \end{cases}$$

is *quasi-ordinary.* By definition, *the critical space* of the morphism ξ_f is the *polar hypersurface*, $f_Y = 0$, associated to the given quasi-ordinary projection $(X_1, \dots, X_d, Y) \mapsto (X_1, \dots, X_d)$. The *discriminant space* is the image of the the critical space by ξ_f, see [T1].

More generally, if $h \in \mathbf{C}\{X\}[Y]$ the equation defining the image of the hypersurface $h = 0$ by ξ_f is obtained by eliminating $X_1, \dots, X_d, Y$ from the equations: $h = 0$, $T - f = 0$, $U_1 = X_1, \dots, U_d = X_d$, i.e., by the vanishing of:

$$\psi_f(h) := \mathrm{Res}_Y(T - f, h). \tag{12}$$

The degree of the polynomial $\psi_f(h) \in \mathbf{C}\{U\}[T]$ is equal to $\deg h$. If $h = 0$ is analytically irreducible at the origin the same holds for its image $\psi_f(h) = 0$ thus $\psi_f(h)$ is an irreducible polynomial. If $h = h_1 \cdots h_t$ then it follows from (6) that $\psi_f(h) = \prod_{r=1}^{t} \psi_f(h_r)$.

We analyse the Newton polyhedron of $\psi_{f_i}(h)$ for f_i any irreducible factor of f. Recall that the Newton polyhedron $\mathcal{N}(\phi) \subset \mathbf{R}^d$ of a non zero series $\phi = \sum c_\alpha X^\alpha \in \mathbf{C}\{X\}$ with $X = (X_1, \dots, X_d)$ is the convex hull of the set $\bigcup_{c_\alpha \neq 0} \alpha + \mathbf{R}^d_{\geq 0}$. The Newton polyhedron of a polynomial $F \in \mathbf{C}\{X\}[Y]$ is the polyhedron $\mathcal{N}(F) \subset \mathbf{R}^d \times \mathbf{R}$ of F viewed as a series in $X_1, \dots, X_d, Y$. We need the following notation:

Notation 8 *We denote the Newton polyhedron of $Y^p - X^a \in \mathbf{C}\{X_1, \dots, X_d\}[Y]$ by the symbol $\frac{p}{q}$ where $q := \frac{a}{p} \in \mathbf{Q}^d$ is the inclination of the edge of $\mathcal{N}(Y^p - X^a)$. Our notation $\frac{p}{q}$ is inspired by the one $\{\frac{b}{a}\}$ used by Teissier with a different meaning ($\{\frac{b}{a}\} := \frac{b}{\frac{a}{b}}$) to describe elementary Newton polygons in [T1] and [T3]. We have the following property of the Minkoski sum: $\frac{p}{q} + \frac{p'}{q} = \frac{p+p'}{q}$.*

We prove that the tree $\theta_f(f)$ determines the Newton polyhedra $\mathcal{N}(\psi_{f_i}(f_Y))$ and $\mathcal{N}(\psi_f(f_Y))$.

Theorem 1 *Let $f \in \mathbf{C}\{X_1, \dots, X_d\}[Y]$ be a quasi-ordinary polynomial with irreducible factors $f_1, \dots, f_s$. The Newton polyhedron of $\psi_{f_i}(f_Y)$ (resp. of $\psi_f(f_Y)$) is the Minkowski sum:*

$$\mathcal{N}(\psi_{f_i}(f_Y)) = \sum_j \frac{c_j}{\nu_i(P_j)} \qquad \left(\text{resp. } \mathcal{N}(\psi_f(f_Y)) = \sum_j \frac{c_j}{\nu_1(P_j) + \dots + \nu_s(P_j)} \right), \tag{13}$$

where in both cases P_j runs through the set of non extremal vertices of $\theta_f(f)$.

We use this result to characterize the type of the bunch decomposition of the polar hypersurface $f_Y = 0$ in terms of the tree $\theta_f(f)$ and conversely, generalizing a theorem of García Barroso's for a generic polar curve of a plane curve germ (see Théorème 6.1 of [GB]).

Theorem 2 *Let $f \in \mathbf{C}\{X_1, \dots, X_d\}[Y]$ be a quasi-ordinary polynomial with irreducible factors $f_1, \dots, f_s$.*

1. *The type of the bunch decomposition of the partial derivative* f_Y *is:*

$$\{(\nu_1(P_j), \ldots, \nu_s(P_j);\ c_j)\}_j \tag{14}$$

where P_j *runs through the set of non extremal vertices of* $\theta_f(f)$ *and* c_j *is the coefficient of* P_j *in the chain* $-\partial\gamma_f$ *. In particular, when* f *is irreducible with characteristic exponents* $\lambda_1, \ldots, \lambda_g$ *the type of* f_Y *is*

$$\{(\nu(P_{\lambda_j});\ n_0 n_1 \cdots n_{j-1}(n_j - 1))\}_{j=1}^g.$$

2. *The type of the bunch decomposition of* f_Y *and the degrees of the irreducible factors of* f *determine the Eggers-Wall tree of* f *.*

Remark 9 *Assertion 1 of Theorem 2 generalizes Popescu-Pampu's Theorem 3.8.5 of [PP1] (or Theorem 6.3 of [PP2]) obtained in the case of a quasi-ordinary derivative (when* $f_Y \in \mathcal{RC}_f$ *).*

Popescu Pampu's theorem is based on a generalization of a result of Kuo and Lu [K-L] Lemma 3.3, which is also essential to prove the properties of the bunch decomposition of the polar curve given by Eggers, García Barroso and Wall (see [Eg], [GB] and [Wa]). Kuo-Lu's lemma compares in the case of a plane curve germ, $F(X,Y) = 0$, the dominant terms of the differences of any fixed root $Y = \zeta(X)$ of F with the other roots of F and of $\zeta(X)$ with the roots of F_Y. The additional hypotesis needed to generalize Kuo-Lu's lemma to the case of a quasi-ordinary polynomial $f \in \mathbf{C}\{X_1, \ldots, X_d\}[Y]$, i.e., to compare the *roots* of the derivative f_Y with the roots of f, is that the polynomial f_Y should be radically comparable to f. With this hypothesis Popescu-Pampu's decomposition follows by extending in a natural way the approach of Wall in the plane curve case (see [Wa]). If $f_Y \notin \mathcal{RC}_f$ the quasi-ordinary projection $(X_1, \ldots, X_d, Y) \mapsto (X_1, \ldots, X_d)$ may be replaced by a base change defined by an embedded resolution of the discriminant $\Delta_Y(f_Y \cdot f) = 0$, in such a way that the transforms of f and f_Y become simultaneously quasi-ordinary with respect to the same quasi-ordinary projection over any point of the exceptional divisor. However, it is not clear that the decompositions obtained in this way come from a decomposition of $f_Y = 0$ since base changes do not preserve irreducible components in general.

3 Newton polyhedra and toric geometry

We introduce in the following subsections the tools needed to prove the main results.

3.1 Polygonal Newton polyhedra and their dual Newton diagrams

If $\phi \in \mathbf{C}\{X\}$ is a non zero series in the variables $X = (X_1, \ldots, X_d)$ we have that any linear form $w \in (\mathbf{R}^d)^*$ in the cone $\Delta_d := (\mathbf{R}^d)^*_{\geq 0}$ defines a *face* $\mathcal{F}_w$ of the polyhedron $\mathcal{N}(\phi)$:

$$\mathcal{F}_w = \{v \in \mathcal{N}(\phi) \,/\, \langle w, v\rangle = \inf_{v' \in \mathcal{N}(\phi)} \langle w, v'\rangle\}.$$

All faces of the polyhedron $\mathcal{N}(\phi)$ can be recovered in this way. The face of $\mathcal{N}(\phi)$ defined by w is compact if and only if w belongs to the interior $\overset{\circ}{\Delta}_d$ of the cone Δ_d. The *cone* $\sigma(\mathcal{F}) \subset \Delta_d$ *associated to the face* $\mathcal{F}$ of the polyhedron $\mathcal{N}(\phi)$ is $\sigma(\mathcal{F}) := \{u \in \Delta_d \,/\, \forall v \in \mathcal{F}, \langle u, v\rangle = \inf_{v' \in \mathcal{N}(\phi)} \langle u, v'\rangle\}$. The *dual Newton diagram* $\Sigma(\mathcal{N}(\phi))$ is the set of cones $\sigma(\mathcal{F})$, for $\mathcal{F}$ running through the set of faces of the polyhedron $\mathcal{N}(\phi)$ (see [Kho]).

Remark 10 *If* $\phi = \phi_1 \dots \phi_r$ *, the elements of the dual Newton diagram of* ϕ *are the intersections* $\cap_{i=1}^r \sigma_i$ *for* σ_i *running through* $\Sigma(\mathcal{N}(\phi_i))$ *for* $i = 1, \dots, r$ *.*

We deduce this property by duality from:

$$\mathcal{N}(\phi) = \mathcal{N}(\phi_1) + \cdots + \mathcal{N}(\phi_r). \tag{15}$$

The set of compact faces of a polygonal Newton polyhedron $\mathcal{N}(\phi)$ is combinatorially isomorphic to a finite subdivision of a compact segment: since $\mathcal{N}(\phi)$ is polygonal the cones of the dual Newton diagram which intersect $\overset{\circ}{\Delta}_d$ are of dimensions d and $d-1$ by duality.

Lemma 11 *If* $\phi \in \mathbf{C}\{X\}$ *has a polygonal Newton polyhedron any irreducible factor of* ϕ *which is not associated to* X_i *, for* $i = 1, \dots, d$ *, has a polygonal Newton polyhedron.*

Proof. It follows from (15) that the compact face of $\mathcal{N}(\phi)$ determined by $w \in \overset{\circ}{\Delta}_d$ is the Minkowski sum of the compact faces, determined by w, on the Newton polyhedra of the factors. Since the polyhedron $\mathcal{N}(\phi)$ is polygonal the dimension of these compact faces is zero or one. It follows that the Newton polyhedron of an irreducible factor of ϕ is polygonal or a translation of $\mathbf{R}^d_{\geq 0}$, and in the latter case this irreducible factor is associated to one variable. $\diamond$

3.2 Coherent polygonal paths

The Newton polyhedron of a polynomial $0 \neq F \in \mathbf{C}\{X\}[Y]$ is contained in $\mathbf{R}^d \times \mathbf{R}$. Any *irrational* vector $w \in \Delta_d$, i.e., with linearly independent coordinates over $\mathbf{Q}$, defines a *coherent polygonal path* on the compact edges of $\mathcal{N}(H)$ (the terminology comes from the combinatorial convexity theory, see [Bi-S]). This path is defined by $c_w(t) = (u_w(t), t)$ where $u_w(t)$ is the unique point of the hyperplane section $v = t$ of $\mathcal{N}(H)$ where the minimal value of the linear function w is reached for $t \in [\mathrm{ord}_Y H, \deg H]$ (for ord_Y the order of F as a series in Y). The point $u_w(t)$ is unique because the vertices of the polyhedron $\mathcal{N}(H)$ are rational, i.e., they belong to the lattice $\mathbf{Z}^d \times \mathbf{Z}$. Any compact edge which is not parallel to the hyperplane $v = 0$ belongs to some path $c_w(t)$ for some irrational vector $w \in \Delta_d$. The maximal segments of the polygonal path $c_w(t)$ are of the form $\varepsilon_i = [p_i, p_{i+1}]$ where $p_j = (u_j, v_j)$ for $j = i, i+1$ and $v_i < v_{i+1}$. We call the vector $q_i = \frac{u_i - u_{i+1}}{v_{i+1} - v_i}$ the *inclination* and the integer $l_i = v_{i+1} - v_i$ the *height* of the edge ε_i (see [GP1] where this construction is related to generalizations of Newton Puiseux Theorem).

Lemma 12 *Let* $\{u_i\}_{i=1}^r$ *be* r *different non zero vectors in* $\mathbf{Q}^d$ *such that* $0 < u_r \leq \cdots \leq u_1$ *(with respect to the order (2)) and integers* $l_1, \dots, l_r \in \mathbf{Z}_{>0}$ *. The Minkowski sum:*

$$\mathcal{N} = \sum_{i=1}^r \frac{l_i}{u_i} \tag{16}$$

is a polygonal polyhedron in $\mathbf{R}^{d+1}$ *. It has* r *compact edges* $\mathcal{E}_i$ *of inclinations* u_i *and heights* l_i *for* $i = 1, \dots, r$ *. The polyhedron* $\mathcal{N}$ *determines the terms of the Minkowski sum (16).*

Proof. The vector hyperplane h_i orthogonal to the compact edge of $\frac{l_i}{u_i}$ defines two half-spaces which subdivide the interior of the cone Δ_{d+1} since $0 < u_i$. The condition $u_i < u_j$ implies that the hyperplanes h_i and h_j do not intersect in the interior $\overset{\circ}{\Delta}_{d+1}$ of the cone Δ_{d+1}. It follows that the possible codimensions of the cones of the dual diagram of $\mathcal{N}$, intersecting $\overset{\circ}{\Delta}_{d+1}$, are 0 and 1. The codimension one case corresponds to the cones defined by the hyperplane sections h_i. By duality the polyhedron $\mathcal{N}$ is polygonal. The edge defined by $u \in h_i \cap \overset{\circ}{\Delta}_{d+1}$ is the Minkowski sum of the faces defined by u on each of the terms of (16), i.e, it is a translation of the polyhedron $\frac{l_i}{u_i}$. ◇

Lemma 13 *Let $F \in \mathbf{C}\{X\}[Y]$ be a monic polynomial of degree > 0 with $0 \neq F(0)$ a non unit. If the polygonal path $c_w(t)$ does not depend on the irrational vector of $\omega \in \Delta_d$ then the inclinations of the edges of $c_w(t)$ are totally ordered with respect to the order (2) and the polyhedron $\mathcal{N}(F)$ is polygonal.*

Proof. We label the edges of $c_w(t)$ by $\varepsilon_0, \dots, \varepsilon_r$ in such a way that $v_0 = 0 < v_1 < \cdots < v_r < v_{r+1} = \deg F$ with the previous notations. The irrational vector w defines the total order of $\mathbf{Q}^d$ defined by $u \leq_w u' \Leftrightarrow \langle u, w\rangle \leq \langle u', w\rangle$ and we have that

$$q_r <_w q_{r-1} <_w \cdots <_w q_0 \tag{17}$$

(see Lemme 5 of [GP1]). By hypothesis the path $c_w(t)$ does not depend on the irrational $w \in \Delta_d$. It follows that the inequality (17) holds for all irrational vector $w \in \Delta_d$ therefore $q_r \leq q_{r-1} \leq \cdots \leq q_0$ with respect to the order (2). It follows that the polyhedron $\mathcal{N}(F)$ is of the form (16) therefore it is polygonal by lemma 12. ◇

3.3 A reminder of toric geometry

We give some definitions and notations (see [Ew], [Od] or [KKMS] for proofs). If $N \cong \mathbf{Z}^d$ is a lattice we denote by M the dual lattice, by $N_{\mathbf{R}}$ the real vector space spanned by N. In what follows a *cone* mean a *rational convex polyhedral cone*: the set of non negative linear combinations of vectors $a^1, \dots, a^s \in N$. The cone σ is *strictly convex* if σ contains no linear subspace of dimension > 0; the cone σ is *regular* if the primitive integral vectors defining the 1-dimensional faces belong to a basis of the lattice N. The *dual* cone $\sigma^\vee$ (resp. *orthogonal* cone $\sigma^\perp$) of σ is the set $\{w \in M_{\mathbf{R}} / \langle w, u\rangle \geq 0$, (resp. $\langle w, u\rangle = 0$) $\forall u \in \sigma\}$. A *fan* Σ is a family of strictly convex cones in $N_{\mathbf{R}}$ such that any face of such a cone is in the family and the intersection of any two of them is a face of each. The *support* of the fan Σ is the set $\bigcup_{\sigma\in\Sigma} \sigma \subset N_{\mathbf{R}}$. The fan Σ is *regular* if all its cones are regular. If σ is a cone in the fan Σ the semigroup $\sigma^\vee \cap M$ is of finite type, it spans the lattice M and defines the affine variety $Z^{\sigma^\vee \cap M} = \operatorname{Spec} \mathbf{C}[\sigma^\vee \cap M]$, which we denote also by $Z_{\sigma,N}$ or by Z_σ when the lattice is clear from the context. If $\sigma \subset \sigma'$ are cones in the fan Σ then we have an open immersion $Z_\sigma \subset Z_{\sigma'}$; the affine varieties Z_σ corresponding to cones in a fan Σ glue up to define the *toric variety* Z_Σ. The toric variety Z_Σ is non singular if and only if the fan Σ is regular. The torus, $(\mathbf{C}^*)^{d+1}$, is embedded as an open dense subset $Z_{\{0\}}$ of Z_Σ, which acts on each chart Z_σ; these actions paste to an action on Z_Σ which extends the action of the torus on itself by multiplication. The correspondence which associates to a cone $\sigma \in \Sigma$ the Zariski closed subset $\mathbf{O}_\sigma$ of Z_σ, defined by the ideal $(X^w / w \in (\sigma^\vee - \sigma^\perp) \cap M)$ of $\mathbf{C}[\sigma^\vee \cap M]$, is a bijection between Σ and the set of orbits of the torus action in Z_Σ. For example the set of faces of a cone σ defines a fan such that the associated toric variety coincides with Z_σ.

We say that a fan Σ' is a *subdivision* of the fan Σ if both fans have the same support and if every cone of Σ' is contained in a cone of Σ. If $\Sigma' \ni \sigma' \subset \sigma \in \Sigma$ we have the morphism $Z_{\sigma'} \to Z_\sigma$ defined by the inclusion of semigroups $\sigma^\vee \cap M \to \sigma'^\vee \cap M$. These morphisms glue up and define the *toric modification* $\pi_{\Sigma'} : Z_{\Sigma'} \to Z_\Sigma$. Given any fan Σ there exists a regular fan Σ' subdividing Σ (see [Od]). The associated toric modification $\pi_{\Sigma'}$ is a desingularization.

For instance, if we denote the lattice $\mathbf{Z}^d$ by N we have that $\Delta_d \subset N_{\mathbf{R}}$, the toric variety Z_Δ is the affine space $\mathbf{C}^d$ and the orbits correspond to the strata of the stratification defined by the coordinate hyperplanes $X_i = X^{e_i} = 0$ (where e_i runs through the elements of the basis of M defined by the edges of $\Delta_d^\vee$). Any fan Σ supported on Δ_d defines the toric modification $\pi_\Sigma : Z_\Sigma \to \mathbf{C}^d$. Taking away the cone σ from the fan of the cone σ means geometrically to take away the orbit $\mathbf{O}_\sigma$ from the variety Z_σ. This implies that the exceptional fiber $\pi_\Sigma^{-1}(0)$ of the toric modification π_Σ is

$$\pi_\Sigma^{-1}(0) = \pi_\Sigma^{-1}(\mathbf{O}_{\Delta_d}) = \bigcup_{\tau \in \Sigma, \overset{\circ}{\tau} \subset \overset{\circ}{\Delta}_d} \mathbf{O}_\tau \tag{18}$$

Let $\mathcal{V}$ be a subvariety of $\mathbf{C}^d$ such that the intersection with the torus is a non singular dense open subset of $\mathcal{V}$. The *strict transform* $\mathcal{V}_\Sigma \subset Z_\Sigma$ is the subvariety of $\pi_\Sigma^{-1}(\mathcal{V})$ such that the restriction $\mathcal{V}_\Sigma \to \mathcal{V}$ is a modification. If $0 \neq \phi = \sum c_a X^a \in \mathbf{C}\{X\}$ is a non zero series in $X = (X_1, \dots, X_d)$ the dual Newton diagram $\Sigma(\mathcal{N}(\phi))$ is a subdivision of Δ_d. The *symbolic restriction* of ϕ to a set $\mathcal{F} \subset M_{\mathbf{R}}$ is $\phi_{|\mathcal{F}} := \sum_{a \in \mathcal{F}} c_a X^a$.

Lemma 14 *Let $\phi = \sum c_a X^a \in \mathbf{C}\{X_1, \dots, X_d\}$ be an irreducible series, not associated to any of the variables $X_1, \dots, X_d$, defining the germ $(\mathcal{V}, 0) \subset (\mathbf{C}^d, 0)$. Let Σ be any subdivision of the dual Newton diagram $\Sigma(\mathcal{N}(\phi))$. If $\sigma \in \Sigma$ and if $\overset{\circ}{\sigma} \subset \overset{\circ}{\Delta}_d$, the intersection $\mathbf{O}_\sigma \cap \mathcal{V}_\Sigma$ is defined by the vanishing of $X^{-u}\phi_{\mathcal{F}_\sigma} \in \mathbf{C}[\sigma^\perp \cap M]$ where u is any vertex of the compact face $\mathcal{F}_\sigma$ of $\mathcal{N}(\phi)$ defined by σ.*

Proof. Let $v \in M$ such that $-v + \mathcal{N}(\phi) \subset \sigma^\vee$. Then all the terms in $X^{-v}\phi$ vanish on the orbit $\mathbf{O}_\sigma$ (since their exponents belong to $\sigma^\vee - \sigma^\perp$) unless the vector v belongs to the affine hull $\mathrm{Aff}(\mathcal{F}_\sigma)$ of the compact face $\mathcal{F}_\sigma$. In this case we have that $(X^{-v}\phi)|\mathbf{O}_\sigma = X^{-v}\phi_{|\mathcal{F}_\sigma}$. If $v, v' \in \mathrm{Aff}(\mathcal{F}_\sigma) \cap M$ we have that $v - v'$ belongs to $\sigma^\perp \cap M$ therefore the polynomials $X^{-v}\phi_{|\mathcal{F}_\sigma}$ and $X^{-v'}\phi_{|\mathcal{F}_\sigma}$ are related by the invertible function $X^{v-v'}$ on the torus $\mathbf{O}_\sigma$. It follows from this that $X^{-v}\phi_{|\mathcal{F}_\sigma}$ defines the ideal of the intersection $\mathcal{V}_\Sigma \cap \mathbf{O}_\sigma$. ⋄

3.4 An irreducibility criterion for series with polygonal Newton polyhedra

We use Theorem 1 to prove Theorem 2 by translating the existence of the bunch decomposition of f_Y in geometrical terms by means of an *irreducibility criterion* for power series with *polygonal* Newton polyhedron:

Definition 15 *A polyhedron is polygonal if the maximal dimension of its compact faces is one.*

Polygonal polyhedra share some properties with classical Newton polygons of plane curve germs. For instance, any Newton polygon is the Minkowski sum of *elementary Newton polygons* up to translations (see [T3]). The criterion holds when the field $\mathbf{C}$ of complex numbers is replaced by an algebraically closed field of arbitrary characteristic. The criterion generalizes a fundamental property of plane curves. We introduce some definitions and notations:

Let $\mathcal{E} = [\alpha, \alpha']$ be a compact segment joining two elements α, α' of the lattice $\mathbf{Z}^d$, and denote by u the primitive integral vector parallel to $\mathcal{E}$, i.e., we have an equality of the form $\alpha' - \alpha = lu$ for some maximal integer $l \in \mathbf{Z}_{\geq 1}$. If $\phi = \sum c_a X^a$ is a series in $\mathbf{C}\{X_1, \dots, X_d\}$ we have that

$$X^{-\alpha}(\sum_{a \in \mathcal{E} \cap \mathbf{Z}^d} c_a X^a) = \sum_{i=0}^{l} c_{iq} X^{iq} = p(\mathcal{E}, \phi)(X^u) \tag{19}$$

where $p(\phi,\mathcal{E})$ is the polynomial $p(\phi,\mathcal{E}) = \sum_{i=0}^{l} c_{iq} t^i$. Obviously, this definition depends on the order of the vertices: the polynomial obtained by interchanging the vertices α and α' of $\mathcal{E}$ is equal to $t^l p(\phi,\mathcal{E})(t^{-1})$. In both cases these two polynomials define isomorphic subschemas of $\mathbf{C}^*$. If in addition ϕ is a monic polynomial in $\mathbf{C}\{X_1,\dots,X_{d-1}\}[X_d]$ we fix the order of the vertices $\alpha=(a,t)$ and (a',t') of $\mathcal{E}$ by the convention $t<t'$, and we obtain that the polynomial $p(\phi,\mathcal{E})$ is defined in a unique way by (19).

Theorem 3 *If $\phi \in \mathbf{C}\{X_1,\dots,X_d\}$ is irreducible and has a polygonal Newton polyhedron $\mathcal{N}(\phi)$, then the polyhedron $\mathcal{N}(\phi)$ has only one compact edge $\mathcal{E}$ and the polynomial $p(\phi,\mathcal{E})$ has only one root in $\mathbf{C}^*$.*

Proof. Since $\mathcal{N}(\phi)$ is polygonal the cones σ in the dual Newton diagram Σ of the polyhedron $\mathcal{N}(\phi))$, such that $\overset{\circ}{\sigma} \subset \overset{\circ}{\Delta}_d$ are of codimensions 0 or 1 (the possible dimensions of the compact faces of $\mathcal{N}(\phi)$). We keep notations of lemma 14. If $\dim\sigma = d$ it follows from lemma 14 that the intersection $\mathcal{V}_\Sigma \cap \mathbf{O}_\sigma$ is empty. If $\dim\sigma = d-1$ the cone σ corresponds to the compact edge $\mathcal{E}$ of the polyhedron $\mathcal{N}(\phi)$. By lemma 14 the intersection $\mathcal{V}_\Sigma \cap \mathbf{O}_\sigma$ is defined by the vanishing of $X^{-u}\phi_{\mathcal{E}}$ on the torus $\mathbf{O}_\sigma$ (the vector u being one of the vertices of the edge $\mathcal{E}$). We have that the coordinate ring $\mathbf{C}[\sigma^\perp \cap M]$ of the orbit $\mathbf{O}_\sigma \cong \mathbf{C}^*$ is isomorphic to $\mathbf{C}[X^{\pm u}]$, where u the primitive integral vector parallel to the edge $\mathcal{E}$. By formula (19) the polynomial $X^{-u}\phi_{\mathcal{E}}$ corresponds to the polynomial $p(\mathcal{E},\phi)(X^u)$. It follows that the intersection $\mathcal{V}_\Sigma \cap \mathbf{O}_\sigma$ is a finite set of points counted with multiplicities which correspond to the zeroes of the polynomial $p(\mathcal{E},\phi)$. By (18) the fiber of the modification $\pi_\Sigma|\mathcal{V}_\Sigma : \mathcal{V}_\Sigma \to \mathcal{V}$ is equal to the discrete set $\bigcup(\mathbf{O}_\sigma \cap \mathcal{V}_\Sigma)$, where σ runs through the cones $\sigma \in \Sigma$ such that $\overset{\circ}{\sigma} \subset \overset{\circ}{\Delta}_d$. Since by hypothesis the germ $\mathcal{V}$ is analytically irreducible at the origin this fiber is a connected set by Zariski's Main Theorem (see [Mu] and [Z2]) thus it is reduced to one point. This implies that the Newton polyhedron $\mathcal{N}(\phi)$ has only one compact edge $\mathcal{E}$ and that the polynomial $p(\mathcal{E},\phi)$ has only one root in $\mathbf{C}^*$. ◇

We will need the following lemma in section 5. We denote by M (resp. by M') the lattice spanned by the exponents of monomials in $\mathbf{C}\{X_1,\dots,X_d\}$ (resp. in $\mathbf{C}\{X_1,\dots,X_d,Y\}$).

Lemma 16 *Let $h,h' \in \mathbf{C}\{X_1,\dots,X_d\}[Y]$ be monic polynomials. We suppose that the polyhedron $\mathcal{N}(h')$ is polygonal with compact edge of the inclination $\lambda \in M_{\mathbf{Q}}$. We denote by $\mathcal{E}$ the compact edge of the polyhedron $\frac{\deg h}{\lambda}$. If $\mathcal{N}(h) \subset \frac{\deg h}{\lambda}$ and if $p(\mathcal{E},h) = t^{\deg h}$, the strict transform of $h=0$ by π_Σ, for $\Sigma = \Sigma(\frac{\deg h}{\lambda})$, intersects $\pi_\Sigma^{-1}(0)$ only at the zero dimensional orbit $\mathbf{O}_\tau$ where τ is the cone associated to the vertex $((\deg h)\lambda, 0)$ of the polyhedron $\frac{\deg h}{\lambda}$.*

Proof. Denote by v the vertex $(0,\deg h)$ of $\frac{\deg h}{\lambda}$. We deduce from the hypothesis $\mathcal{N}(h) \subset \frac{\deg h}{\lambda}$ and $p(\mathcal{E},h) = t^{\deg h}$ that the series $Y^{-\deg h}h$ has terms in $\mathbf{C}[\sigma^\vee \cap M']$, constant term equal to one and the exponents of non constant terms do not belong to $\sigma^\perp$, for $\sigma = \sigma(\{v\})$ or $\sigma = \sigma(\mathcal{E})$. It follows that the strict transform of $h=0$ does not meet $\mathbf{O}_\sigma$ since the terms $\neq 1$ vanish on $\mathbf{O}_\sigma$. We deduce from (18) that the strict transform of h intersects the exceptional fiber $\pi_\Sigma^{-1}(0)$ at the closed orbit $\mathbf{O}_\tau$ (which is reduced to a point since τ is of maximal dimension $d+1$). ◇

4 The proofs of the results on the type of f_Y

We apply the irreducibility criterion to clarify the relation between the type of the f-bunch decomposition of $h \in \mathcal{C}_f$ and the Newton polyhedra $\mathcal{N}(\psi_{f_i}(h))$.

Lemma 17 *The restriction of the valuation* ν_i *to* $\theta_{fh}(f_i)$ *is an order-preserving bijection. The characteristic exponents of* f_i *and the valuation* $\nu_i(P^{(i)}_\lambda)$ *determine* λ *.*

Proof. We denote the characteristic exponents of f_i by $\lambda_1, \dots, \lambda_g$. Let P_λ be a vertex of $\theta_{fh}(f_i)$. For simplicity we drop the index i. If $P_\lambda < P_{\lambda_1}$ then we have that $\nu(P_\lambda) = n\lambda$. Otherwise there exists a unique $1 \leq j \leq g$ such that $P_{\lambda_j} \leq P_\lambda < P_{\lambda_{j+1}}$ since $\theta_{fh}(f_i)$ is totally ordered. Then the first assertion follows from the inequality:

$$\begin{aligned} \nu(P_{\lambda_j}) &= e_{j-1}\lambda_j + \textstyle\sum_{k=1}^{j-1}(e_{k-1} - e_k)\lambda_k = e_j\lambda_j + \sum_{k=1}^{j}(e_{k-1} - e_k)\lambda_k \leq \\ &\leq e_j\lambda + \textstyle\sum_{k=1}^{j}(e_{k-1} - e_k)\lambda_k = \nu(P_\lambda) < e_j\lambda_{j+1} + \sum_{k=1}^{j}(e_{k-1} - e_k)\lambda_k = \nu(P_{\lambda_{j+1}}). \end{aligned}$$

If we know the characteristic exponents of f_i then there is a unique j such that $\nu(P_{\lambda_j}) \leq \nu(P_\lambda) < \nu(P_{\lambda_{j+1}})$ where we convey that $\lambda_0 = 0$. Then we recover λ from equation (11). ◇

If $h \in \mathcal{RC}_f$ we show below that the Newton polyhedron of $\psi_{f_i}(h)$ is determined by the f-type of h.

Proposition 18 *If* $\{(q_{1,r}, \dots, q_{s,r}; c_r)\}_{r=1}^{s(f,h)}$ *is the type of the* f*-bunch decomposition of a polynomial* $h \in \mathcal{RC}_f$ *then the Newton polyhedron of* $\psi_{f_i}(h)$ *is the Minkowski sum:*

$$\mathcal{N}(\psi_{f_i}(h)) = \sum_{r=1}^{s(f,h)} \frac{c_r}{q_{i,r}}. \tag{20}$$

Proof. If $\{\tau_r^{(j)}\}_{j=1,\dots,\deg b_r}$ are the roots of the factor b_r of the f-bunch decomposition of h for $r = 1, \dots, s(f,h)$, viewed in some suitable ring extension of the form $\mathbf{C}\{X^{1/k}\}$, it follows from proposition 5 and the definition of the bunches that:

$$f_i(\tau_r^{(j)}) = X^{q_{i,r}}\epsilon_{i,r,j}, \text{ where } \epsilon_{i,r,j} \text{ is a unit in } \mathbf{C}\{X^{1/k}\}. \tag{21}$$

By general properties of the resultant we have that: $\psi_f(b_r) = \prod_{j=1}^{\deg h}(T - f(\tau_r^{(j)}))$. We deduce from this and (21) that the polyhedron $\mathcal{N}(\psi_{f_i}(b_r)) = \sum_{j=1}^{c_r} \mathcal{N}(T - f_i(\tau_r^{(j)}))$ is equal to $\frac{c_r}{q_{i,r}}$ and equality (20) follows from the property (15). ◇

Remark 19 *If* $h \in \mathcal{C}_f$ *the assertion of proposition 18 is not true in general (see remark 3).*

The following proposition generalizes Proposition 3.4.8 [PP1].

Proposition 20 *Given* $\theta_f(f)$*, if* $h \in \mathcal{RC}_f$ *the following informations determine each other:*

1. *The contact chain* $[h]^{(f)}$.
2. *The type of the bunch decomposition of* h *induced by* f.
3. *The collection of Newton polyhedra of* $\psi_{f_i}(h)$ *for* $i = 1, \dots, s$.

Proof. 1. ⇒ 2. : If the contact chain is $[h]^{(f)} = \sum c_i P_i$ the type of the f-bunch decomposition of h is $\{(\nu_1(P_i), \dots, \nu_s(P_i); c_i)\}_i$ by remark 7. The implication 2. ⇒ 3. is a direct consequence of proposition 18. 3. ⇒ 1. : The Newton polyhedron of $\psi_{f_i}(h)$ is polygonal by proposition 18. We recover the set of vertices $\{P_j^{(i)}\}_j$ of $\theta_{fh}(f_i)$ corresponding to the orders of coincidence of f_i with the irreducible factors of h from the inclinations of the compact edges by lemma 17 (since $\theta_f(f_i)$ is given). The maximal point $P^{(i)}_{j_0}$ of the set $\{P^{(i)}_j\}_j$ corresponds to a factor b_1 of the f-bunch decomposition of degree c_1 equal to the height of the edge of $\mathcal{N}(\psi_{f_i}(h))$ of maximal inclination $\nu_i(P^{(i)}_{j_0})$. Then we can replace h by $h' = h/b_1$ and continue in the same way. The Newton polyhedra of $\psi_{f_i}(h')$ is obtained from $\mathcal{N}(\psi_{f_i}(h'))$ by subtracting the elementary polyhedra $\frac{c_1}{\nu_i(P^{(i)}_{j_0})}$ (the subtraction makes sense by lemma 12). ⋄

Proposition 21 *Given $\theta_f(f)$ and $h \in \mathcal{C}_f$, if the Newton polyhedra $\psi_{f_i}(h)$ for $i = 1, \dots, s$, are polygonal the following informations determine each other:*

1. *The type of the bunch decomposition of h induced by f.*
2. *The collection of Newton polyhedra of $\psi_{f_i}(h)$ for $i = 1, \dots, s$.*

Proof. If $\deg h = m > 0$ the type of h determines two vertices of $\mathcal{N}(\psi_{f_i}(h))$: the vertex $(0, m)$ which corresponds to the monomial T^m and the vertex $(\rho(f_i, h), 0)$ which corresponds to the dominant term of $(\psi_{f_i}(h))_{|T=0} = \mathrm{Res}_Y(-f_i, h)$. In particular if h is irreducible and if $\psi_{f_i}(h)$ has a polygonal Newton polyhedron then by the irreducibility criterion $\mathcal{N}(\psi_{f_i}(h))$ has only one compact edge equal to $[(0, m), (\rho(f_i, h), 0)]$, hence we obtain that $\mathcal{N}(\psi_{f_i}(h)) = \frac{\deg h}{\frac{\rho(f_i,h)}{\deg h}}$. In the general case, if the monic polynomial $\psi_{f_i}(h)$ has a polygonal Newton polyhedron the same holds for its irreducible factors by lemma 11. It follows then that the type of h determines the Newton polyhedron of $\psi_{f_i}(h)$ (see the proof of proposition 18). Conversely, if we are given the tree $\theta_f(f)$ and polygonal polyhedron $\mathcal{N}(\psi_{f_i}(h))$ for $i = 1, \dots, s$ we recover the type from the inclinations and heights of the compact edges of $\mathcal{N}(\psi_{f_i}(h))$ following the method of proposition 20. ⋄

We determine the structure of the Newton polyhedron of $\psi_{f_i}(f_Y)$ by using adequate toric base changes, which reduce to the case $f_Y \in \mathcal{RC}_f$, in such a way that we can recover the coherent polygonal paths on the Newton polyhedron of $\psi_{f_i}(f_Y)$.

Lemma 22 *If $f_Y \in \mathcal{RC}_f$ then the Newton polyhedra of $\psi_{f_i}(f_Y)$ are polygonal, for $i = 1, \dots, s$.*

Proof. If $f_Y \in \mathcal{RC}_f$ the type of f_Y is given in terms of the Eggers-Wall tree by formula (14) by applying Popescu-Pampu decomposition (see remark 9). Then we obtain the polyhedra $\psi_{f_i}(f_Y)$ for $i = 1, \dots, s$, from the type of f_Y by formula (20). These polyhedra are polygonal by lemmas 17 and 12. ⋄

We describe first the toric base changes, already used in [GP1] and [GP2], we use to reduce to the quasi-ordinary derivative case. The ring of convergent (or formal) complex power series in $X = (X_1, \dots, X_d)$ can be denoted by $\mathbf{C}\{\Delta_d^\vee \cap M\}$ where M denotes the lattice $\mathbf{Z}^d$ and Δ_d denotes the cone $(\mathbf{R}^d)^*_{\geq 0}$. The advantage of this notation is that we can define ring homomorphisms by changing the lattice M or the cone Δ_d.

Let $\tau \subset \Delta_d$ be a regular[4] cone of dimension d (see [Ew] for its existence). The dual cone $\tau^\vee$ is generated by a basis $a_1, \dots, a_d$ of the lattice M and contains $\Delta_d^\vee$. We define from the semigroup inclusion $\Delta_d^\vee \cap M \hookrightarrow \tau^\vee \cap M$ the local ring extension $\mathbf{C}\{\Delta_d^\vee \cap M\} \hookrightarrow \mathbf{C}\{\tau^\vee \cap M\}$. The local ring $\mathbf{C}\{\tau^\vee \cap M\}$ is equal to $\mathbf{C}\{V_1, \dots, V_d\}$ where $V_i = X^{a_i}$ for $i = 1, \dots, d$. We denote by $H^{(\tau)}$ the image of a polynomial $H \in \mathbf{C}\{\Delta_d^\vee \cap M\}[Y]$ in the ring $\mathbf{C}\{\tau^\vee \cap M\}[Y]$.

By definition of Newton polyhedron we deduce that:

[4]The regularity of the cone τ is not essential in what follows (see [GP1] and [GP3]).

Remark 23 *Let* $\phi \in \{\Delta^\vee \cap M\}$ *and* $H \in \mathbf{C}\{\Delta_d^\vee \cap M\}[Y]$ *be non zero. We have that:*

$$\mathcal{N}(\phi^{(\tau)}) = \mathcal{N}(\phi) + \tau^\vee, \quad \textit{and} \quad \mathcal{N}(H^{(\tau)}) = \mathcal{N}(H) + (\tau^\vee \times \{0\}). \tag{22}$$

We deduce from (22) the following lemma:

Lemma 24 *If* $\tau \subset \Delta_d$ *is a regular cone the coherent polygonal paths defined by an irrational vector* $w \in \tau$ *on the edges of the polyhedra* $\mathcal{N}(F)$ *and* $\mathcal{N}(F^{(\tau)})$ *coincide.* ◇

The main argument of the following proposition is already used in Proposition 2.14 of [GP2].

Lemma 25 *If* $\tau \subset \Delta_d$ *is a regular cone and if* $f \in \mathbf{C}\{X\}[Y]$ *is a quasi-ordinary polynomial then* $f^{(\tau)}$ *is a quasi-ordinary polynomial and* $\tilde{\theta}_f(f) = \tilde{\theta}_{f^{(\tau)}}(f^{(\tau)})$.

Proof. If $\zeta^{(i)} \in \mathbf{C}\{\Delta_d^\vee \cap \frac{1}{k}M\}$ is a root of f then $(\zeta^{(i)})^{(\tau)} \in \mathbf{C}\{\tau^\vee \cap \frac{1}{k}M\}$ is a root of $f^{(\tau)}$. Extending the cone does not modify the support of the series nor the lattices spanned by the exponents. It follows then from Lipman's characterization of roots of quasi-ordinary polynomials that $\zeta^{(i)}$ and $(\zeta^{(i)})^{(\tau)}$ have characteristic exponents defined by the same elements of the lattice $\frac{1}{k}M$, and that if f is irreducible the same holds for $f^{(\tau)}$ (see Proposition 1.5 of [L2] or Proposition 1.3 [Gau]). Then the equality $\tilde{\theta}_f(f) = \tilde{\theta}_{f^{(\tau)}}(f^{(\tau)})$ follows from the fact that the ring extension $\mathbf{C}\{\Delta_d^\vee \cap \frac{1}{k}M\} \hookrightarrow \mathbf{C}\{\tau^\vee \cap \frac{1}{k}M\}$ sends monomials to monomials and units to units. ◇

Remark 26 *If* $\tau \neq \Delta_d$ *then* $\theta_f(f) \neq \theta_{f^{(\tau)}}(f^{(\tau)})$ *since the valuations of the vertices, denoted* v *and* $v^{(\tau)}$, *take on any non extremal vertex* P *of* $\tilde{\theta}_f(f) = \tilde{\theta}_{f^{(\tau)}}(f^{(\tau)})$ *values equal to the coordinates of* $\tilde{v}(P) = \tilde{v}^{(\tau)}(P)$ *with respect to two fixed different basis of the lattice* $\frac{1}{k}M$.

Proof of Theorem 1. We discuss first the case of the Newton polyhedron of $\psi_{f_i}(f_Y)$. If $f_Y \notin \mathcal{RC}_f$, let Σ a regular subdivision of the dual Newton diagram of $\Delta_Y(f_Y)$. Let $\tau \in \Sigma$ be a cone of dimension d. It follows that $(f^{(\tau)})_Y = (f_Y)^{(\tau)}$ is polynomial in $\mathcal{RC}_{f^{(\tau)}}$ by lemma 25 and the definitions. By remark 24, the coherent polygonal path $c_w(t)$ determined by an irrational vector $w \in \tau$ on $\mathcal{N}(\psi_{f_i}(f_Y))$ and on $\mathcal{N}(\psi_{f_i^{(\tau)}}(f_Y^{(\tau)}))$ coincide. By lemma 22 the polygonal path c_w is completely determined by the tree $\tilde{\theta_f}(f)$. It follows that $c_w(t)$ gives the same value for irrational w and τ therefore the polyhedron $\mathcal{N}(\psi_{f_i}(f_Y))$ is defined by the first formula of (13) by lemma 13.

The case of the Newton polyhedron of $\psi_f(f_Y)$ is discussed analogously using that if $f_Y \in \mathcal{RC}_f$ and $f = b_1 \dots b_{s(f,f_Y)}$ is its bunch decomposition the result follows: the Minkowski sum (13) corresponds to the decomposition $\psi_f(f_Y) = \prod_{i=1}^{s(f,f_Y)} \mathrm{Res}_Y(T - f_1 \cdots f_s, b_i)$. ◇

Remark 27 *The Newton polyhedron of* $\psi_f(f_Y)$ *is not necessarily polygonal since the set* $\{v(P)\}$, *for* P *running through the non extremal vertices of* $\theta_f(f)$ *is not totally ordered in general.*

Proof of Theorem 2. By Theorem 1 the Newton polyhedra $\psi_{f_i}(f_Y)$ for $i = 1, \dots, s$, are polygonal and coincide with the those obtained assuming the hypothesis of $f_Y \in \mathcal{RC}_f$. Then assertion 1 follows by proposition 21.

To prove assertion 2 we consider the matrix $\mathcal{M} = (m_{i,j})$ whose columns are the t-uples of vectors defining the type of f_Y:

$$\left(\frac{\rho(f_1, h_j)}{\deg h_j}, \dots, \frac{\rho(f_s, h_j)}{\deg h_j}; \deg b_j \right) \text{ for } j = 1, \dots, s(f, f_Y).$$

By definition, the columns of the matrix $\mathcal{M}$ correspond bijectively to the bunches of the decomposition of f_Y induced by f. By theorem 2 these bunches correspond bijectively with the non extremal vertices of the tree $\theta_f(f)$, in such a way that if the column j corresponds to the vertex P_j then $m_{i,j} = \nu_i(P_j)$ for $i = 1, \ldots, s$. We build the tree $\theta_f(f)$ from the matrix $\mathcal{M}$ by identifying the columns of $\mathcal{M}$ with those non extremal vertices of $\theta_f(f_r)$ for $r = 1, \ldots, s$ separately:

We begin by analysing the row r. If $a \in \{m_{r,j}\}_{j=1}^{s(f,f_Y)}$ the set of columns $\mathcal{K}_a^r := \{j / m_{r,j} = a\}$ of $\mathcal{M}$ is non empty and is clearly in bijection with the set $\mathcal{P}_a^r = \{P \text{ vertex of } \theta_f(f) / \nu_r(P) = a\}$. Since the set $\mathcal{P}_a^r$ has a minimum for the valuation v, namely the vertex Q of $\theta_f(f_r)$ such that $\nu_r(P) = a$, it follows by lemma 17 that there is a unique column $l \in \mathcal{K}_a^r$, corresponding to Q, such that $m_{t,l} \leq m_{t,k}$ for $t = 1, \ldots, s$ and $k \in \mathcal{K}_a^r$. This procedure defines a partial order in the columns. We recover the skeleton of the tree $\theta_f(f)$ by repeating this procedure for the rows $r = 1, \ldots, s$ of $\mathcal{M}$. The vertex of bifurcation of $\theta_f(f_r)$ and $\theta_f(f_k)$ is the greatest common column defined by the rows r and k.

To determine the chain γ_f we use the row $s(f, f_Y) + 1$ of $\mathcal{M}$ and the degrees of the irreducible factors of f. By theorem 2 we know that the integer $m_{s(f,f_Y)+1,j}$ is the coefficient of the vertex P_j, corresponding to the column j, in the chain $-\partial\gamma_f$. The coefficient of the extremal edge containing the the vertex $P_{+\infty}^{(i)}$ in the chain γ_f is equal to $\deg f_i$ for $i = 1, \ldots, s$.

Since $\theta_f(f)$ is a tree, we recover recursively the coefficients appearing in the segments of the chain γ_f from the chain $-\partial\gamma_f$ and the coefficients $\deg f_i$. The chain γ_f defines the vertices of $\theta_{f_i}(f_i)$ and the associated characteristic integers. We recover from them, by using (10) and the ν_i valuation, the valuations $v(P)$ for those non extremal vertices P of $\theta_f(f_i)$. ◇

5 A geometrical characterization of the bunch decomposition

We give a geometrical characterization of the bunch decomposition of f_Y in terms of the *partial embedded resolution* $p : \mathcal{Z} \to \mathbf{C}^{d+1}$ of $f = 0$ built in [GP3] and [GP4], by González Pérez. The morphism p is a a composition of toric modifications which are canonically determined by the given quasi-ordinary projection, by using the tree $\theta_f(f)$. An embedded resolution of $f = 0$ is obtained by composing the modification p with any toroidal modification defining resolution of singularities of $\mathcal{Z}$, which always exists (see [KKMS]).

The exceptional fiber $p^{-1}(0)$ of the modification p is a curve, its irreducible components are complex projective lines. The definition of the modification p induces a bijection $P \mapsto C(P)$ between the non extremal vertices of the tree $\theta_f(f)$ and the irreducible components of the exceptional fiber of p.

Theorem 2 establishes a canonical bijection $P \to b_P$ between the non extremal vertices of the tree $\theta_f(f)$ and the bunches of the decomposition of f_Y induced by f. An irreducible factor h of f_Y is a factor of of b_P if and only if the following equality holds:

$$\left(\frac{\rho(f_1, h)}{\deg h}, \ldots, \frac{\rho(f_s, h)}{\deg h}\right) = (\nu_1(P), \ldots, \nu_s(P)). \tag{23}$$

Let $h \in \mathcal{C}_f$ be irreducible, we say that h is *associated* to a non extremal vertex P of the tree $\theta_f(f)$ if the equality (23) holds, or equivalently $\mathcal{N}(\psi_{f_i}(h)) = \frac{\deg h}{\nu_i(P)}$ for $i = 1, \ldots, s$; this equivalence is deduced easily by arguing as in the proof of theorem 1. The following theorem implies that the f-bunch decomposition of f_Y is compatible with the bijections above.

Theorem 4 *If $h \in \mathcal{C}_f$ is associated to a non extremal vertex P of the tree $\theta_f(f)$ then the strict transform of the hypersurface $h=0$ only intersects the irreducible component $C(P)$ of $p^{-1}(0)$. The strict transform of $h=0$ does not intersect the strict transform of $f=0$.*

We describe the procedure used to build the modification p (for details see [GP3]). The modification p is a composition $p = \pi_1 \dots \pi_l$ of toric modifications. The irreducible components of the exceptional fiber $p^{-1}(0)$ are complex projective lines which can be ordered by $C' < C$ iff there exists $t > 1$ such that the image of C' by the modification $p' := \pi_t \circ \cdots \circ \pi_l$ is a point of $p'(C)$ and $p'(C)$ is not reduced to a point. By definition the minimal components of $p^{-1}(0)$ with respect to this relation are the irreducible components of the exceptional fiber $\pi_1^{-1}(0)$ of the first toric modification.

The morphism π_1 is the toric modification defined by the dual Newton diagram of $\Sigma(f)$ of the quasi-ordinary polynomial $f \in \mathbf{C}\{X\}[Y]$, when Y is a *good coordinate*. Such a good coordinate is built by a $\mathbf{C}\{X\}$-automorphism of the polynomial ring $\mathbf{C}\{X\}[Y]$ of the form $Y \mapsto Y + r(X)$. These automorphisms are compatible with the sets $\mathcal{C}_f$ and $\mathcal{RC}_f$ since they preserve resultants and discriminants of polynomials. We suppose from now on that Y is a good coordinate for f, this means that the Newton polyhedron $\mathcal{N}(f)$ is polygonal and it is completely determined from the Eggers-Wall tree $\theta_f(f)$. In order describe this polyhedron we define

$$\mathcal{A}_i^{(f)} := (M \cap \{k(f_i, f_j)\}_j) \cup \{\lambda_1^{(i)}\} \text{ for } 1 \leq i \leq s.$$

By lemma 1, if the set $\mathcal{A}_i^{(f)}$ is non empty it is totally ordered and we define then

$$\lambda_{\kappa(i)}^{(f)} := \left\{ \begin{array}{l} \min \mathcal{A}_i^{(f)} \text{ if } \mathcal{A}_i^{(f)} \neq \emptyset \\ +\infty \text{ otherwise} \end{array} \right\} \text{ for } i = 1, \dots, s. \tag{24}$$

The polyhedron $\mathcal{N}(f_i)$ has only one compact edge (since is a Minkowski term of a polygonal polyhedron) and since f_i is irreducible it has only one compact edge of inclination equal to $\lambda_{\kappa(i)}^{(f)}$ if $\lambda_{\kappa(i)}^{(f)} \neq +\infty$ (the case $\lambda_{\kappa(i)}^{(f)} = +\infty$ may happen only for one index i and in that case f_i is a good coordinate for f, i.e., we will suppose that $f_i = Y$).

Lemma 28 *If the term X^λ appears in the expansions of the roots of f_j and if $\lambda_{\kappa(i)}^{(f)} \not\leq \lambda \notin M$ then we have that $\lambda \geq k(f_i, f_j)$ and the equality $\lambda = k(f_i, f_j)$ implies that $k(f_i, f_j) = \lambda_1^{(j)}$. The set $\{\lambda_{\kappa(1)}^{(f)}, \dots, \lambda_{\kappa(s)}^{(f)}\}$ is totally ordered by (2) and its intersection with the reference lattice M is defined by its maximal element or empty. We have that*

$$k(f_i, f_j) < \lambda_{\kappa(i)}^{(f)} \Leftrightarrow \lambda_{\kappa(j)}^{(f)} < \lambda_{\kappa(i)}^{(f)}.$$

Proof. For the assertions 1 and 2 see lemma 3.15 of [GP3]. Suppose that $k(f_i, f_j) < \lambda_{\kappa(i)}^{(f)}$, if $k(f_i, f_j) \notin M$ it follows from assertion 1 that $k(f_i, f_j) = \lambda_1^{(j)}$. In any case it follows from the definition that $\lambda_{\kappa(j)}^{(f)} < \lambda_{\kappa(i)}^{(f)}$. Conversely, if $\lambda_{\kappa(j)}^{(f)} < \lambda_{\kappa(i)}^{(f)}$ then $\lambda_{\kappa(j)}^{(f)} \notin M$ by assertion 2 thus $\lambda_{\kappa(j)}^{(f)} = \lambda_1^{(j)}$ by definition, therefore $k(f_i, f_j) \leq \lambda_1^{(j)} < \lambda_{\kappa(i)}^{(f)}$ by assertion 1. ⋄

The exceptional fiber of the toric modification π_1 is described by (18). We denote by $C(P^{(i)}_{\lambda_{\kappa(i)}^{(f)}})$ the irreducible component of $\pi_1^{-1}(0)$ which is the closure of the orbit associated to the cone of $\Sigma(f)$ orthogonal to the compact edge of $\mathcal{N}(f)$ with inclination $\lambda_{\kappa(i)}^{(f)} \neq +\infty$.

The following lemma describes some properties of the strict transform of f by π_1.

Lemma 29 *The strict transform of* $f_i = 0$ *by* π_1 *is a germ at the point of intersection* $o_1^{(i)}$ *with* $\pi_1^{-1}(0)$. *This point belongs to only one irreducible component of* $\pi_1^{-1}(0)$ *which is equal to* $C(P^{(i)}_{\lambda^{(f)}_{\kappa(i)}})$ *if* $\lambda^{(f)}_{\kappa(i)} \neq +\infty$ *or to* $C(P^{(*)}_{\max\{\lambda^{(f)}_{\kappa(j)}\}_{j=1}^{s}})$ *otherwise. If* $\lambda^{(f)}_{\kappa(i)} \neq +\infty$ *then we have that:*

$$C(P^{(i)}_{\lambda^{(f)}_{\kappa(i)}}) = C(P^{(j)}_{\lambda^{(f)}_{\kappa(j)}}) \Leftrightarrow k(f_i, f_j) \geq \lambda^{(f)}_{\kappa(i)} \text{ and in this case } o_1^{(i)} = o_1^{(j)} \Leftrightarrow\ k(f_i, f_j) > \lambda^{(f)}_{\kappa(i)}$$

Proof. See Proposition 3.32 of [GP3]. The main assertion is also a direct consequence of the proof of theorem 3 and lemma 28. ◇

Remark 30

1. *The point* $o_1^{(i)}$ *is parametrized by the only root* c_{f_i} *of the polynomial in one variable* $p(f_i, \mathcal{E}_i)$ *defined from the symbolic restriction of* f *to the compact edge* $\mathcal{E}_i$ *of* $\mathcal{N}(f_i)$ *by (19).*
2. *If* $\lambda^{(f)}_{\kappa(i)} \neq +\infty$ *and if* $h \in \mathbf{C}\{X\}[Y]$ *is such that* $\mathcal{N}(h) = \frac{\deg h}{\lambda^{(f)}_{\kappa(i)}}$ *and the polynomial* $p(h, \mathcal{E})$ *defined from the symbolic restriction of* h *to the compact edge* $\mathcal{E}$ *of the polyhedron* $\frac{\deg h}{\lambda^{(f)}_{\kappa(i)}}$ *by (19) has only one root* $c_h = c_{f_i}$, *then the strict transform of* $h = 0$ *by* π_1 *is a germ at the point* $o_1^{(i)}$ *of* $p_1^{-1}(0)$.
3. *Suppose that* $\lambda^{(f)}_{\kappa(i)} = \max\{\lambda^{(f)}_{\kappa(l)}\} < +\infty$, *if* $h \in \mathbf{C}\{X\}[Y]$ *is any polynomial such that* $\mathcal{N}(h) \subset \frac{\deg h}{\lambda^{(f)}_{\kappa(i)}}$ *and that the polynomial* $p(h, \mathcal{E})$ *has only one root* $c_h = 0$, *the strict transform of* $h = 0$ *by* π_1 *only intersects the component* $C(P^{(i)}_{\lambda^{(f)}_{\kappa(i)}})$.

Proof. For the first and the second assertion of the following remark see the proof of theorem 3. The third assertion is obtained by the same argument used in the proof of lemma 16. ◇

The key inductive step in the embedded resolution procedure is that the strict transform of $f_i = 0$ at the point $o_1^{(i)}$ is a *toric quasi-ordinary singularity*[5], with a canonical "quasi-ordinary" projection. These singularities are hypersurfaces of affine toric varieties, for instance the strict transform of $f = 0$ at the point $o_1^{(i)}$ are defined by the vanishing of a monic polynomial f' in one variable with coefficients in the ring $\mathbf{C}\{\Delta_d^\vee \cap (M + \lambda^{(f)}_{\kappa(i)}\mathbf{Z})\}$. The polynomial f' is *quasi-ordinary*: its discriminant is the product of a monomial by a unit of this ring. The definition of characteristic monomials and the Eggers-Wall tree introduced in the first section generalize to this setting by replacing the valuation v by the lattice valuation $\tilde{v}$ and the reference lattice cone introduced in the first section (see [GP3]). In particular the Eggers-Wall tree of f' is determined from that of f by the following proposition (see Proposition 3.22 of [GP3]):

Proposition 31 *If* $\lambda^{(f)}_{\kappa(i)} \neq +\infty$, *then we have that the Eggers-Wall tree* $\theta_{f'}(f')$ *associated to the strict transform of* f *at the point* $o_1^{(i)}$ *is obtained from* $\theta_f(f)$ *by removing the segment* $[P_0^{(j)}, P^{(j)}_{\lambda^{(f)}_{\kappa(i)}}[$ *from the sub-tree of* $\theta_f(f)$ *given by* $\bigcup \theta_f(f_j)$, *for those irreducible factors* f_j *with order of coincidence* $> \lambda^{(f)}_{\kappa(i)}$ *with* f_i. *The new lattice valuation is* $\tilde{v}'(P) = \tilde{v}(P) - \lambda^{(f)}_{\kappa(i)}$. *The coefficients of the* 1*-chain* $\gamma_{f'}$ *are obtained from those of* γ_f *by division by the index of* $\lambda^{(f)}_{\kappa(i)}$ *over the old reference lattice* M. *The new reference lattice cone is* $(\Delta_d^\vee, M + \lambda^{(f)}_{\kappa(i)}\mathbf{Z})$. ◇

[5]This means that there is a finite projection onto a germ of affine toric variety which is unramified outside its torus (see [GP3] and [GP1])

Proposition 31 allows us to extend the natural bijection between the components of the exceptional fiber of π_1 and the subset $\{P^{(1)}_{\lambda^{(f)}_{\kappa(1)}}, \dots, P^{(s)}_{\lambda^{(f)}_{\kappa(s)}}\}$ of vertices of $\theta_f(f)$, inductively between the the components of $p^{-1}(0)$ and the set of non extremal vertices of $\theta_f(f)$. The following lemma which translates some of the properties stated in terms of the notion of order of coincidence in terms of Newton polyhedra.

Lemma 32 *Let $h \in \mathcal{C}_f$ be an irreducible polynomial associated to the non extremal vertex P of $\theta_f(f)$. Let $C(P^{(i)}_{\lambda^{(f)}_{\kappa(i)}})$ be the unique component of $p^{-1}(0)$ which is $\leq C(P)$. Then we have that $\mathcal{N}(h) \subset \frac{\deg h}{\lambda^{(f)}_{\kappa(i)}}$ with equality if $\lambda^{(f)}_{\kappa(i)} \neq \max\{\lambda^{(f)}_{\kappa(j)}\}_{j=1}^{s}$. The polynomial $p(h, \mathcal{E}) \in \mathbf{C}[t]$ obtained from the symbolic restriction of h to the compact edge $\mathcal{E}$ of the polyhedron $\frac{\deg h}{\lambda^{(f)}_{\kappa(i)}}$ by (19) has only one complex root c_h and we have that $c_h = c_{f_i} \Leftrightarrow v(P) > \lambda^{(f)}_{\kappa(i)}$ (resp. $c_h \neq c_{f_i} \Leftrightarrow v(P) = \lambda^{(f)}_{\kappa(i)}$). The case $c_h = 0$ may happen only if $v(P) = \max\{\lambda^{(f)}_{\kappa(j)}\}_{j=1}^{s}$.*

Proof. If $h \in \mathcal{RC}_f$, we consider it as an irreducible factor of the quasi-ordinary polynomial fh. The hypothesis means that the vertex $P^h_{k(f,h)}$ of $\theta_{fh}(f)$ belongs to $\theta_f(f)$, therefore $\theta_{fh}(f) = \theta_f(f)$ and $P = P^h_{k(f,h)}$ hence we have:

$$v(P) = k(f,h) = \max_{l=1,\dots,s}\{k(f_l,h)\} = k(f_i,h) = \lambda^{(f)}_{\kappa(i)}, \text{ and} \tag{25}$$

$$\lambda^{(fh)}_{\kappa(l)} = \lambda^{(f)}_{\kappa(l)} \text{ for } l = 1, \dots, s. \tag{26}$$

The exponents appearing on the parametrizations of f_i are $\geq \lambda^{(f)}_{\kappa(i)}$, since Y is a good coordinate for f. The equality $k(f_i, h) = \lambda^{(f)}_{\kappa(i)}$ implies that the same property holds for the parametrizations of h and therefore $\mathcal{N}(h) \subset \frac{\deg h}{\lambda^{(f)}_{\kappa(i)}}$. If these polyhedra are not equal the intersection of the compact face $\mathcal{E}$ with $\mathcal{N}(h)$ is reduced to the point $(0, \deg h)$ therefore $c_h = 0$.

These polyhedra are equal if and only if $c_h \neq 0$. We show that this is always the case when $\lambda^{(f)}_{\kappa(i)} < \lambda^{(f)}_{\kappa(j)}$ for some index j: We deduce from lemma 28 and (25) that $k(f_i, h) = \lambda^{(i)}_1$. We have that $k(h, f_j) \leq k(h, f_i) < \lambda^{(fh)}_{\kappa(j)} = \lambda^{(f)}_{\kappa(j)}$ by (25) and (26), therefore $k(h, f_j) \notin M$ by definition of $\lambda^{(fh)}_{\kappa(j)}$. By lemma 28 we deduce that $k(h, f_j) = \lambda^{(h)}_1 \leq k(h, f_i) = \lambda^{(i)}_1$ hence $\lambda^{(h)}_1 = k(h, f_i) = \lambda^{(f)}_{\kappa(i)}$ by definition of order of coincidence. Therefore, we obtain that $\mathcal{N}(h) = \frac{\deg h}{\lambda^{(f)}_{\kappa(i)}}$. The rest of the assertion in this case follows from lemma 29.

We prove that this result holds also for $h \in \mathcal{C}_f$ by using toric base changes of section 4. Let $\tau \subset \Delta_d$ be any regular cone of dimension d such that $h^{(\tau)} \in \mathcal{RC}_{f^{(\tau)}}$. The polynomial $p(h, \mathcal{E})$ coincides with $p(h^{(\tau)}, \mathcal{E})$ by lemmas 25 and 24. The same argument provides the assertion about the inequality $\mathcal{N}(h) \subset \frac{\deg h}{\lambda^{(f)}_{\kappa(i)}}$. ⋄

Proof of Theorem 4. If $P = P^{(i)}_{\lambda^{(f)}_{\kappa(i)}}$ for some i, we deduce from lemma 32 and remark 30 that the strict transform $\mathcal{H}'$ of $h = 0$ by π_1 only meets the component $C(P)$ of $\pi_1^{-1}(0)$ and it does not intersect the strict transforms of $f = 0$. This implies that a neighborhood of $\mathcal{H}'$ wont be modified by the toric modifications $\pi_2, \dots, \pi_l$ and proves the theorem in this case.

If $C(P^{(i)}_{\lambda^{(f)}_{\kappa(i)}}) < C(P)$ we consider the polynomial h' defining the strict transform of h (which is a germ at the point $o^{(i)}_1$). It follows from proposition 31 and another application of lemmas 25 and 24 that h' is associated to P viewed on the tree $\theta_{f'}(f')$. If $C(P)$ is minimal between those components of $p^{-1}(0)$ corresponding to the non extremal vertices of $\theta_{f'}(f')$ we can apply the arguments of the previous case, otherwise we obtain the result by iterating the procedure. $\diamond$

5.1 A theorem of Lê, Michel and Weber revisited

If $d = 1$, then $f \in \mathbf{C}\{X\}[Y]$ defines the germ of a complex analytic plane curve in $(\mathbf{C}^2, 0)$. The *minimal embedded resolution* of $(S, 0)$ is the modification $\Pi : \mathcal{X} \to \mathbf{C}^2$ defined by the composition of the minimal sequence of points blow ups, such that the total transform of $f = 0$ is a normal crossing divisor. The *dual graph* $\mathcal{G}(\Pi, 0)$ (resp. the *total dual graph* $\mathcal{G}(\Pi, f)$) is the graph obtained from the exceptional divisor $\Pi^{-1}(0)$ (resp. from the total transform $\Pi^{-1}(\{f = 0\})$) by associating a vertex to any irreducible component and joining with a segment those vertices whose associated components have non empty intersection. The *valency* of a vertex P of a finite graph is the number of edges of the graph which contain the vertex P. We denote by $\#1$ the component of $\mathcal{G}(\Pi, f)$ which corresponds to the first blow up. We define for a vertex P of $\mathcal{G}(\Pi, f)$ the integer

$$\omega(P) := \begin{cases} \text{valency of } P \text{ in } \mathcal{G}(\Pi, f) & \text{if} \quad P \neq \#1 \\ 1 + \text{ valency of } P \text{ in } \mathcal{G}(\Pi, f) & \text{if} \quad P = \#1 \end{cases} \tag{27}$$

A vertex P of $\mathcal{G}(\Pi, f)$ with $\omega(P) = 1$ and $P \neq \#1$ (resp. $\omega(P) \geq 3$) is called an *extremal vertex* (resp. a *rupture vertex*). A *dead arc* of the graph $\mathcal{G}(\Pi, f)$ is a closed polygonal in $\mathcal{G}(\Pi, 0)$ joining a extremal vertex to any rupture vertex of $\mathcal{G}(\Pi, f)$, and which does not contain any other rupture vertex. The dual graph $\mathcal{G}(\Pi, f)$ defines a natural stratification of $\Pi^{-1}(\{f = 0\})$, the 0-dimensional strata are in bijection with the segments of $\mathcal{G}(\Pi, f)$, each segment corresponds to the intersection of the irreducible components associated to the vertices. The 1-dimensional strata are in bijection with the vertices, the stratum corresponding to a vertex of $\mathcal{G}(\Pi, f)$ is the set of points of the corresponding component which do not belong to any other component of $\Pi^{-1}(\{f = 0\})$. We can associate by this correspondence a subset of $\Pi^{-1}(\{f = 0\})$ to any subgraph of $\mathcal{G}(\Pi, f)$.

With these notations the main result of Lê, Michel and Weber in [L-M-W] is:

Theorem 5 *Denote by $\mathcal{Q}(\Pi, f)$ the set associated to the subgraph of $\mathcal{G}(\Pi, f)$ defined by the rupture vertices and dead arcs. If $X = 0$ is not contained in the tangent cone of the plane curve germ $f = 0$ the intersection of the exceptional divisor $\Pi^{-1}(0)$ with the strict transform of the polar curve $f_Y = 0$ is contained $\mathcal{Q}(\Pi, f)$ and meets any connected component of $\mathcal{Q}(\Pi, f)$.*

Proof. If $X = 0$ is not contained in the tangent cone of the plane curve germ $f = 0$ the minimal embedded resolution Π is the composition of the partial embedded resolution $p : \mathcal{Z} \to \mathbf{C}^2$ used in the previous section, with a finite number of local toric modifications at the isolated singular points of the normal variety $\mathcal{Z}$. See [GP3], section 3.3.4 for details. The notions of *dual graph* $\mathcal{G}(p, 0)$ and *total dual graph* $\mathcal{G}(p, f)$ can be defined in an analogous way for p. In particular, we have that $\mathcal{G}(p, 0)$ is combinatorially isomorphic to the tree $\theta_f(f)$ minus its extremal segments and that there is a natural inclusion of the vertices of $\mathcal{G}(p, 0)$ in the vertices of $\mathcal{G}(\Pi, 0)$ whose image is the subset of rupture vertices of $\mathcal{G}(\Pi, f)$. The dual graph $\mathcal{G}(p, f)$ is associated in an analogous manner to a natural stratification of $p^{-1}(\{f = 0\})$, in such a way that we can associate by this correspondence a subset of $p^{-1}(\{f = 0\})$ to any subgraph of $\mathcal{G}(p, f)$. We denote by $\mathcal{Q}(p, f)$ the subset of $p^{-1}(\{f = 0\})$ corresponding to the set of vertices of $\mathcal{G}(p, 0)$.

An irreducible factor h of f_Y is associated to a non extremal vertex P of $\theta_f(f)$ by Theorem 2. The strict transform of h by p intersects $p^{-1}(0)$ at a smooth point o_h, which belongs to the component $C(P)$ associated to P and does not intersect the strict transform of f, by theorem 4. We deduce that the point o_h belongs to $\mathcal{Q}(p, f)$ and that if $\Pi = p' \circ p$ then $\mathcal{Q}(\Pi, f) = (p')^{-1}(\mathcal{Q}(p, f))$ and the result follows. $\diamond$

Remark 33 *If $C(P)$ is minimal for $p^{-1}(0)$ the component corresponding to $C(P)$ in $\Pi^{-1}(0)$ belongs to a dead arc if and only if $v(P) = \max\{\lambda^{(f)}_{\kappa(j)}\}$, which is necessarily $< +\infty$. If $C(P)$ is not minimal, after some toric modifications, $C(P)$ is minimal for the strict transform of f and an analogous result holds.*

By definition there is no dead arc corresponding to $\min\{\lambda^{(f)}_{\kappa(j)}\}$ since the extremal point #1 is not considered for the definition of dead arcs.

6 The case of Laurent quasi-ordinary polynomials

The class of *Laurent quasi-ordinary polynomials* is introduced by Popescu-Pampu in [PP1], see also [PP2], by analogy with the case of *meromorphic plane curves* studied by Abhyankar and Assi (see [A-As]). He proves a decomposition theorem for the derivative of any polynomial in the class such that the derivative itself is also Laurent quasi-ordinary. In this section we generalize this result to any Laurent quasi-ordinary polynomial by translating in an equivalent manner, the properties of the bunch decomposition on the derivative of a quasi-ordinary polynomial from the Laurent case to holomorphic case (characterized in Theorem 2). This reduction is partially inspired by an argument of Kuo and Parusinski comparing the plane curve meromorphic case to the holomorphic case (see [K-P]).

We denote by $\mathbf{C}\langle X\rangle$ the ring of *Laurent power series* in $X = (X_1, \dots, X_d)$, that is the ring of fractions $\mathbf{C}\{X\}[X_1^{-1}, \dots, X_d^{-1}]$. A Laurent polynomial $F \in \mathbf{C}\langle X\rangle[Y]$ admits a Newton polyhedron which is defined, as usual in terms of the power series expansion $F = \sum c_{\alpha,i} X^\alpha Y^i$, as the convex hull of the set $\bigcup_{c_{\alpha,i}\neq 0} (\alpha, i) + (\mathbf{R}^d_{\geq 0} \times \{0\})$. A Laurent monic polynomial $F \in \mathbf{C}\langle X\rangle[Y]$ is *quasi-ordinary* if the discriminant $\Delta_Y(f)$ is of the form $\Delta_Y(f) = X^\delta \epsilon$ where $\delta \in \mathbf{Z}^d$ and ϵ is a unit in the ring of power series $\mathbf{C}\{X\}$. We can extend the definition of polynomials comparable to a quasi-ordinary polynomial to the Laurent case. In particular if F is a Laurent quasi-ordinary polynomial we define the type of F_Y as in the holomorphic case (see Definition 4). We relate Laurent monic polynomials with monic holomorphic polynomials by:

Lemma 34 *Let $F \in \mathbf{C}\langle X\rangle[Y]$ be a monic polynomial. Then there exists a vector $q \in \mathbf{Z}^d$ such that the monic polynomial f defined by:*

$$f := X^{-\deg(F)\, q} F(X^q Y) \tag{28}$$

belongs to $\mathbf{C}\{X\}[Y]$. In this case we have:

1. *If F is quasi-ordinary the same holds for f.*
2. *If $F = F_1 \cdots F_s$ is the factorization in irreducible monic polynomials the same holds for $f = f_1 \cdots f_s$ where the polynomials f_i are defined from F_i by (28).*
3. *The polynomial r defined from $R = F_Y$ by (28) is equal to $r = f_Y$.*

Proof. The polyhedron $\mathcal{N}(F) \subset \mathbf{R}^d \times \mathbf{R}$ is contained in an affine cone W of the form:

$$W = [(0, \deg F), (a, 0)] + (\mathbf{R}^d \times \{0\})$$

for some integral vector $a \in (\deg F)\mathbf{Z}^d$. If $q := \frac{1}{\deg F} a$ then we obtain that the polynomial f defined by (28) belongs to $\mathbf{C}\{X\}$. The idea is that the 1-dimensional face $[(0, \deg F), (a, 0)]$ of the cone W corresponds to the segment $[(0, \deg f), (0, 0)]$ in the Newton polyhedron of f. If $F = \sum_{i=0}^{n} a_i Y^i$ then $f = X^{-\deg(F) q}(\sum_{i=0}^{n} a_i X^{iq} Y^i)$ and if $R = \frac{\partial F}{\partial Y}$ then we obtain that the polynomial

$$\frac{\partial f}{\partial Y} = X^{(-\deg(F)+1)q} \left(\sum_{i=1}^{n} i a_i X^{(i-1)q} Y^{i-1} \right)$$

is equal to r. It follows from quasi-homogeneity and homogeneity properties of the generic discriminant and resultant (see [G-K-Z], pag. 398-399) that f is quasi-ordinary if F is. See also the proof of theorem 3 of [GP1] for details.

It remains to prove assertion 2. Recall that if a domain A is integrally closed with fraction field K then the factorization of a monic polynomial in $A[Y]$ as product of monic irreducible factors coincides over $A[Y]$ and $K[Y]$. It is easy to see that the rings $\mathbf{C}\{X\} \subset \mathbf{C}\langle X \rangle$ are integrally closed and have the same fraction field L. For any fixed $q \in \mathbf{Z}^d$ the multiplicative endomorphism of $\mathbf{C}\langle X \rangle[Y] \backslash \{0\}$ defined by (28) is an automorphism which preserves degrees and monic polynomials, and extends to a multiplicative autormorphism of $L[Y] \backslash \{0\}$. Therefore the factorization in monic irreducible factors of a monic polynomial $F \in \mathbf{C}\langle X \rangle[Y]$ corresponds by this mapping to the factorization in monic irreducible factors of f in $L[Y]$, if in addition f belongs to $\mathbf{C}\{X\}[Y]$ then this factorization holds over $\mathbf{C}\{X\}[Y]$ since $\mathbf{C}\{X\}$ is integrally closed. ◇

Let F be a Laurent quasi-ordinary polynomial. The vector $q \in \mathbf{Z}^d$ will be that built in the proof of lemma 34. If $F_Y = H_1 \cdots H_r$ is the factorization in monic irreducible polynomials the same holds for $f_Y = h_1 \dots h_r$ by Lemma 34 (the polynomials h_i being defined from H_i by (28)). We obtain, by using the quasi-homogeneity and homogeneity properties of the generic resultant (see [G-K-Z] pag. 398-399), that for any i, j:

$$\mathrm{Res}_Y(f_j, h_i) = X^{-\deg(H_i)\, \deg(F_j)\, q}\, \mathrm{Res}_Y(F_j, H_i)$$

and since $\deg H_i = \deg h_i$ and $\deg F_i = \deg f_i$ for all i, we obtain that:

$$\frac{\rho(F_j, H_i)}{\deg H_i} = \frac{\rho(f_j, h_i)}{\deg h_i} + \deg(f_j)\, q. \qquad (29)$$

We have proved the following result:

Proposition 35 *The bunches of the F-decomposition of F_Y correspond to the bunches of the f-decomposition of f_Y by the transformation (28). The type of F_Y is obtained from the type of f_Y by (29).* ◇

7 An example

The polynomial $f_{i,j} := (Y^2 - iX_1^3 X_2^2)^2 - jX_1^5 X_2^4 Y$ is quasi-ordinary with characteristic exponents $\lambda_1 = (\frac{3}{2}, 1)$ and $\lambda_2 = (\frac{7}{4}, \frac{3}{2})$, and integers $n_1 = n_2 = 2$ for any $i, j \in \mathbf{C}^*$. The equation is obtained by defining a deformation of the monomial variety associated to a quasi-ordinary hypersurface (see [GP3]) in an analogous manner as the deformation of the monomial curve associated to a plane branch studied in [T4].

The Eggers-Wall tree associated to the polynomial $f = f_{1,1}f_{1,2}f_{2,1}f_{2,2}$ is below. We have that $v(P_1) = \lambda_1$ and $v(P_2) = v(P_3) = \lambda_2$. The edges are labeled with the coefficients of the chain γ_f thus we have that $-\partial\gamma_f = 4P_1 + 6P_2 + 6P_3$.

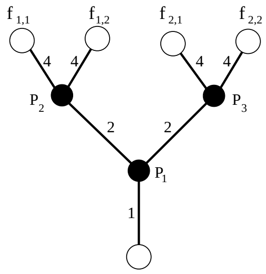

Figure 1: The Eggers-Wall tree associated to f

We determine the type of f_Y by using Proposition 5 and theorem 2:

$f_{1,1}$	$(6,4)$	$(\frac{13}{2},5)$	$(6,4)$
$f_{1,2}$	$(6,4)$	$(\frac{13}{2},5)$	$(6,4)$
$f_{2,1}$	$(6,4)$	$(6,4)$	$(\frac{13}{2},5)$
$f_{2,2}$	$(6,4)$	$(6,4)$	$(\frac{13}{2},5)$
	3	6	6
	–	–	–
	P_1	P_2	P_3

We compute the Newton polyhedra of the polynomials $\psi_{f_{1,1}}(f_Y)$ and $\psi_f(f_Y)$ by using Theorem 1. We obtain that: $\mathcal{N}(\psi_{f_{i,j}}(f_Y)) = \frac{3}{(6,4)} + \frac{6}{(6,4)} + \frac{6}{(\frac{13}{2},5)} = \frac{9}{(6,4)} + \frac{6}{(\frac{13}{2},5)}$ for $i,j \in \{1,2\}$ and $\mathcal{N}(\psi_f(f_Y)) = \frac{3}{(24,16)} + \frac{6}{(25,18)} + \frac{6}{(25,18)} = \frac{3}{(24,16)} + \frac{12}{(25,18)}$. It follows that the polyhedra $\mathcal{N}(\psi_{f_{i,j}}(f_Y))$ coincide in this example for $i,j \in \{1,2\}$. In particular, the example shows that the only datum of these polyhedra does not allow us to distinguish between the different irreducible factors of f. The polyhedron obtained for $\mathcal{N}(\psi_{f_{i,j}}(f_Y))$ (resp. for $\mathcal{N}(\psi_f(f_Y))$) is of the form of given in Figure 2 below where the vertices are $A = ((0,0),15)$, $B = ((54,36),6)$ and $C = ((93,66),0)$; (resp. $A = ((0,0),15)$, $B = ((72,48),12)$ and $C = ((372,264),0)$).

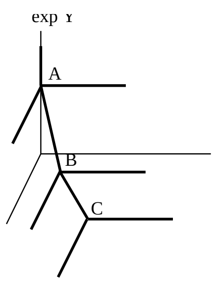

Figure 2: A polygonal polyhedron

If we are given the type of f we recover the skeleton of the tree by noticing that $\theta_f(f_{i,j})$ has two non extremal vertices, corresponding to the different values appearing on the associated row. The corresponding minimal columns for $i = j = 1$ (and for $i = 1$, $j = 2$ are $P_1 < P_2$. Thus the column P_2 corresponds to the point of bifurcation of $\theta_f(f_{1,1})$ and $\theta_f(f_{1,2})$. The columns corresponding to the non extremal vertices of $\theta_f(f_{i,j})$ for $i = 2, j = 1, 2$ are $P_1 < P_3$, the bigger column corresponds with the point of bifurcation of $\theta_f(f_{2,1})$ and $\theta_f(f_{2,2})$ and the first is the point of bifurcation of $\theta_f(f_{1,1})$ and $\theta_f(f_{2,1})$. The coefficient of the edge $\overline{P_1 P_2}$ on γ_f is equal to $\deg f_{1,1} + \deg f_{1,2} - c_{P_2} = 4 + 4 - 6 = 2$ and we obtain the same value for $\overline{P_1 P_3}$. Then we recover the characteristic exponents $v(P_i)$ by using proposition 5.

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Evelia R. García Barroso
Dpto. Matemática Fundamental, Fac. Matemáticas, Universidad de La Laguna
38271, La Laguna, Tenerife, Spain

ergarcia@ull.es

Pedro Daniel González Pérez
Université de Paris 7, Institut de Mathématiques, Equipe Géométrie et Dynamique
Case 7012; 2, Place Jussieu, 75005 Paris, France.

gonzalez@math.jussieu.fr