Collective modes of a trapped Lieb-Liniger gas:a hydrodynamic approach

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We consider a trapped repulsive one-dimensional (1D) Bose gas at very low temperature. In order to study the collective modes of this strongly interacting system, we use a hydrodynamic approach, where the gas is locally described by the Lieb-Liniger model of bosons interacting via a repulsive delta potential. Solving the corresponding linearized hydrodynamic equations, we obtain the collective modes and concentrate more specifically on the lowest compressional mode. This is done by finding models, approaching very closely the exact equation of state of the gas, for which the linearized hydrodynamic equations are exactly solvable. Results are in excellent agreement with those of the sum rule approach of Menotti and Stringari.

I. INTRODUCTION

Cold atomic gases of reduced dimensionality are currently being investigated experimentally [1–3]. A typical situation is that of a dilute 3D Bose-Einstein condensate (BEC) confined in a very anisotropic trap. If the confinement is strong enough in two perpendicular directions, the gas behaves microscopically as a 1D Bose gas in the remaining perpendicular direction. A very interesting model of such an interacting system is that of Lieb and Liniger [4]: it consists of 1D bosons interacting via repulsive zero-range interactions of variable strength. In the *homogeneous* case, this model can be solved exactly, whatever the strength of the interactions, by means of Bethe ansatz [4].

In the present paper, we propose to study the lowest lying collective modes of the *trapped* Lieb-Liniger gas by means of a hydrodynamic approach [5,6]. The linearized hydrodynamic equations are solved using the recently developed method of Combescot and Leyronas [5]. The results, already presented in [7], are compared with those of Menotti and Stringari [6], which were obtained from a sum rule approach.

II. TRAPPED LIEB-LINIGER GAS

The physical system we consider is a 3D dilute BEC at very low temperature ($T \simeq 0$) confined in an elongated cigar-shaped trap. The trap is taken to be harmonic with radial frequency $\omega_\perp = \omega_x = \omega_y$ much greater than the axial frequency ω_z. At very low temperature, repulsive interactions between atoms are fully characterized by the 3D s-wave scattering length $a > 0$.

The temperature T and the chemical potential μ are much smaller than the radial quantum of oscillation $\hbar\omega_\perp$, so that the radial motion is frozen and the gas behaves microscopically as a 1D system along the axial direction of the trap. Olshanii [8] showed that in that case the corresponding effective 1D scattering length a_1 is given by $a_1 = a_\perp^2/a$ [9] when $a_\perp \equiv \sqrt{\hbar/m\omega_\perp} \gg a$, which we assume for simplicity. In order to study such a trapped 1D Bose gas, we assume that it is locally described by the Lieb-Liniger model [4] of hard-core bosons, of mass m, interacting trough a repulsive contact potential $g_1\delta(z)$, where $g_1 = 2\hbar^2/ma_1$ [8].

In the Lieb-Liniger model, the relevant parameter measuring the strength of the interactions is the dimensionless coupling constant $\gamma \equiv mg_1/\hbar^2 n_1 = 2/n_1 a_1$, where n_1 is the 1D density. It is the only dimensionless intensive parameter proportional to the coupling constant g_1 [4]. It shows that the gas is weakly interacting $\gamma \ll 1$ in the dense limit $n_1 a_1 \gg 1$ and strongly interacting $\gamma \gg 1$ in the dilute limit $n_1 a_1 \ll 1$.

In the dense limit, the homogeneous gas is a quasi-condensate, i.e. a condensate with a fluctuating phase. Nevertheless, in a trapped gas, when the temperature is well below the degeneracy temperature [10], the phase is almost constant over the whole sample and the gas behaves as a true condensate: it is, for example, well described by a 1D Gross-Pitaevskii equation [11]. In this limit, we say that the gas is in the "1D mean-field" regime and the chemical potential is given by $\mu = g_1 n_1 = 2\hbar\omega_\perp a n_1$ (with the convention that $\mu = 0$ when $n_1 = 0$). The low density limit

corresponds to the "Tonks-Girardeau" regime, i.e. to a gas of impenetrable bosons. Girardeau [12] showed that in this limit there is an exact mapping between the gas of impenetrable bosons and the 1D free Fermi gas. In the Tonks-Girardeau regime, the chemical potential is given by $\mu = (\hbar\pi n_1)^2/2m$. In the whole range between the 1D mean-field and the Tonks-Girardeau regime, the equation of state $\mu(n_1)$ can be obtained from the exact solution of the homogeneous Lieb-Liniger model [4]. Menotti and Stringari numerically evaluated this equation of state and made their result available [6,13]. As we will see, the equation of state is the essential ingredient needed in the hydrodynamic approach in order to compute the collective modes.

In the following, we use reactive hydrodynamics, where dissipation is negligible and thermal effects can be omitted. This is valid at low enough temperature. The hydrodynamic equations are:

$$\frac{\partial n_1}{\partial t} + \partial_z(n_1 v) = 0 \tag{1}$$

$$m\frac{dv}{dt} = -\partial_z\big[\mu(n_1) + m\omega_z^2 z^2/2\big] \tag{2}$$

where $n_1(z,t)$ is the 1D density at position z and time t and $v(z,t)$ is the velocity along the z axis. The first equation is the continuity equation, the second is the Euler equation at zero temperature. The equation of state for the homogeneous gas $\mu(n_1)$ appears as a needed input in the hydrodynamic equations: it is provided by the Lieb-Liniger microscopic theory.

Here we briefly discuss the validity of the hydrodynamic approach. First, in the 1D mean-field regime, a trapped gas with a finite number of particles behaves as a true condensate at low enough temperature [10]. It is therefore well described by a 1D Gross-Pitaevskii equation. As in the 3D case [14], neglecting the quantum pressure (which is valid when the number of particle is large), the Gross-Pitaevskii equation can be rewritten in the form of reactive hydrodynamics with the equation of state given by the mean-field result $\mu(n_1) = g_1 n_1$. Second, for the whole range of densities (i.e. of coupling constant γ), the low lying excitations of the homogeneous Lieb-Liniger gas are known to be phonon-like with a sound velocity depending on the coupling constant [4]. This property allows the description of the system in term of an effective Luttinger liquid theory [15]. In the case of the trapped gas, when studying collective modes of long wavelength and low energy, the chemical potential is locally given by its value in the homogeneous gas at the same density $\mu[n_1(z)]$. This allows to extend the hydrodynamic approach initially developed for condensates in the mean-field regime [14] to the strongly interacting regime [5,6].

III. COLLECTIVE MODES : SOLVABLE MODELS FOR HYDRODYNAMICS

In this section we use the method recently developed by Combescot and Leyronas [5] in order to solve the linearized hydrodynamic equations. The basic idea is to replace the equation of state $\mu(n_1)$ by a model $\mu_{mod}(n_1)$ for which the linearized hydrodynamic equations can be solved analytically or quasi-analytically. The model has several parameters that allows to fit the actual equation of state as closely as possible. Details of the fitting procedure are given in [7]. We start by briefly reviewing the method of Combescot and Leyronas in the 1D case.

At equilibrium the particle density $n_1^0(z)$ satisfies:

$$\mu[n_1^0(z)] + m\omega_z^2 z^2/2 = \tilde{\mu} \tag{3}$$

where $\tilde{\mu}$ is the constant value over the system of the global chemical potential. For small density fluctuations $\delta n_1(z,t) = n_1(z,t) - n_1^0(z)$ we introduce the departure of the chemical potential from its equilibrium value $w(z,t) = \tilde{\mu}(z,t) - \tilde{\mu} = (\partial\mu/\partial n_1^0)\delta n_1(z,t)$. For a fluctuation occurring at frequency ω, the preceding equation gives $i\omega\delta n_1 = \partial_z(n_1^0 v) = n_1^0\partial_z v + v\partial_z(n_1^0)$ which, together with Euler equation (2) leads to:

$$n_1^0\partial_z^2 w + \partial_z n_1^0\partial_z w + m\omega^2(\partial n_1^0/\partial\mu)w = 0 \tag{4}$$

The equilibrium relation (3) implies $(\partial\mu/\partial n_1^0)(\partial n_1^0/\partial z) = -m\omega_z^2 z$ which gives:

$$zw'' + zL'(z)w' - \nu^2 L'(z)w = 0 \tag{5}$$

where $\nu^2 \equiv \omega^2/\omega_z^2$ is the squared reduced mode frequency and we defined $L(z) \equiv \ln[n_1^0(z)]$ with $L'(z) \equiv dL/dz$. Let us introduce the function $v(z)$ by $w(z) = z^l v(z)$, with $l = 0$ for an even mode with respect to z and $l = 1$ for an odd mode, and restrict to $z > 0$. With this change of function, equation (5) becomes:

$$zv'' + [2l + zL'(z)]v' - (\nu^2 - l)L'(z)v = 0 \tag{6}$$

This equation is invariant under the replacement $z \to z/R$. We therefore take the cloud radius R as unity in the following. The cloud radius is given by $\mu[n_1^0(0)] = m\omega_z^2 R^2/2$. Next we make the change of variable $y = z^\alpha$ to obtain:

$$y\frac{d^2v}{dy^2} + (\Delta + y\frac{dL}{dy})\frac{dv}{dy} - \frac{\nu^2 - l}{\alpha}\frac{dL}{dy}v = 0 \tag{7}$$

where $\Delta = 1 + (2l-1)/\alpha$. Finally, we define the reduced density $\bar{n}_1(z) \equiv n_1(z)/n_1(0)$ and the normalized local chemical potential $\bar{\mu}(z) \equiv \mu[n_1(z)]/\mu[n_1(0)] = 1 - z^2$. We note that in the mode equation (7), the equation of state only enters trough the quantity dL/dy.

The simplest model introduced by Combescot and Leyronas is the two parameter $\alpha - p$ model [5]. It consists in replacing the quantity dL/dy by the model $-p/(1-y)$ in equation (7), which then reduces to the hypergeometric differential equation:

$$y(1-y)\frac{d^2v}{dy^2} + [\Delta - y(p+\Delta)]\frac{dv}{dy} + p\frac{\nu^2 - l}{\alpha}v = 0 \tag{8}$$

The solution to this equation (with proper boundary conditions) is a polynomial [5]. The corresponding mode frequencies are:

$$\nu^2 = l + \frac{\alpha}{p}\, n\, (n + p + \frac{2l-1}{\alpha}) \tag{9}$$

where n is a positive integer and $l = 0$ or 1. The $\alpha - p$ model corresponds explicitely to the equilibrium density $\bar{n}_1(z) = (1-z^\alpha)^p$ and to the equation of state $\bar{\mu} = 1 - (1 - \bar{n}_1^{1/p})^{2/\alpha}$. We can check on equation (9) that the "dipole mode" ($n = 0$ and $l = 1$) occurs at frequency $\omega = \omega_z$ in agreement with Kohn's theorem. The particular case $\alpha = 2$ corresponds to the power law equation of state $\bar{\mu} = \bar{n}_1^{1/p}$. This is precisely the functional dependence of the equation of state in the two limiting regimes that we discussed above: the Tonks-Girardeau regime corresponds to $p = 1/2$ and the 1D mean-field regime to $p = 1$. In the Tonks-Girardeau case, the mode frequencies are given by $\omega = k\omega_z$, where $k \equiv 2n + l$ is any positive integer (it is convenient to notice that $l^2 = l$). This result was already obtained in [16,6]. In the 1D mean field regime, the mode frequencies are given by $\nu^2 = k(1+k)/2$, where $k \equiv 2n + l$ is any positive integer, in agreement with [6].

Combescot and Leyronas introduced other models which lead to a generalized hypergeometric differential equation with quasi-polynomial solutions [5]. Here, we only describe the "3 parameters quasi-polynomial" model. The quantity dL/dy is modeled by $-(p_0 + p_1 y)/(1-y)$ depending on the three parameters α, p_0 and p_1. It is solution of:

$$y(1-y)\frac{d^2v}{dy^2} + (-p_1 y^2 - (\Delta + p_0)y + \Delta)\frac{dv}{dy} + \frac{\nu^2 - l}{\alpha}(p_1 y + p_0)v = 0 \tag{10}$$

This differential equation admits solutions that are very rapidly converging series which can safely be truncated above some order and that we call quasi-polynomials [5]. The mode frequencies are easily obtained numerically. Details are given in [7].

From now on, we will concentrate on the lowest compressional mode ($n = 1$ and $l = 0$). In order to obtain the mode frequency, we will use the following procedure: (i) fit the actual equation of state by a model (either $\alpha - p$ or 3 parameters quasi-polynomial) to obtain the best set of parameters; (ii) calculate the mode frequency using this set of parameters (with equation (9) in the case of the $\alpha - p$ model, for example). In the next section, we show how to improve on this zero-order result for the mode frequency by using perturbation theory.

IV. PERTURBATION THEORY

In order to account for the difference between the actual equation of state and the model equation of state we can rewrite the mode equation (7) in the form of a Schrödinger-like equation and use standard perturbation theory. This allows to correct the results obtained for the mode frequency calculated using the model equation of state and to come closer to the mode frequency corresponding to the actual equation of state.

The following change of function $\psi(y) = v(y)\big[y^\Delta n_1^0(y)\big]$ is made in equation (7) leading to:

$$-\frac{d^2\psi}{dy^2} + \Big[\frac{1}{2}L'' + \frac{1}{4}L'^2 + (\frac{\nu^2 - l}{\alpha} + \frac{\Delta}{2})\frac{L'}{y} + \frac{\Delta(\Delta-2)}{4y^2}\Big]\psi = 0 \tag{11}$$

which is a Schrödinger equation with an effective potential (term between brackets [ ]) and corresponding to zero energy (and $\hbar^2/2m = 1$). The effective potential can be decomposed into a zero-order term corresponding to the model equation of state and a perturbation term corresponding to the difference of effective potential between the actual and the model equation of state. The perturbation is treated using standard first order perturbation theory. This leads to a first order change of the energy in the Schrödinger equation. In order to keep the energy equal to zero, we have to give a compensating variation $\delta\nu^2$ for the frequency:

$$\delta\nu^2 = \alpha \frac{\int_0^1 dy\, \delta L'\, n_1^0\, y^{\Delta-1} v[\frac{\nu^2-l}{\alpha} v - yv']}{\int_0^1 dy\, (-n_1^{0\prime})\, y^{\Delta-1} v^2} \tag{12}$$

where $\delta L' \equiv L' - L'_{mod}$. This result is proven in [7].

V. LOWEST COMPRESSIONAL MODE

We apply now the previously developed method to study the lowest compressional mode ($n = 1$ and $l = 0$), which physically corresponds to a breathing motion of the gas along the axis. The needed input is the equation of state $\mu(n_1)$ of the Lieb-Liniger model. Using the numerical evaluation of this equation of state by Menotti and Stringari, we compute the mode frequency as follows: for each value of the dimensionless density at the center of the trap $n_1(0)a_1$, (i) we fit the equation of state (for $n_1(z)$ varying between 0 and $n_1(0)$) with either the $\alpha - p$ or the 3 parameters quasi-polynomial models; (ii) the zero order mode frequency is then obtained by inserting the value of the best set of parameters in the formula giving the mode frequency for the model; (iii) we then compute the first order perturbation correction to the mode frequency with equation (12).

Figure 1 shows the results of four different calculations of the lowest compressional mode using respectively the $\alpha - p$ model and the 3 parameters quasi-polynomial model without perturbative correction, and the same models corrected by first order perturbation theory. Actually, at this scale, it is hard to see any difference between these calculations, except for the $\alpha - p$ model which gives a slightly lower mode frequency.

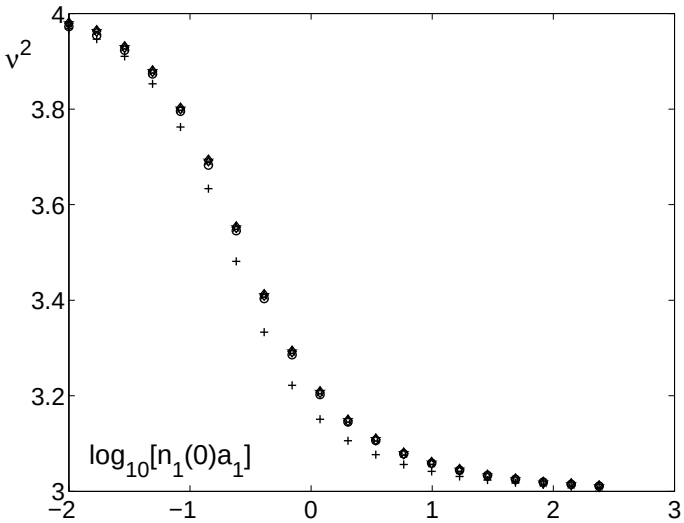

FIG. 1. Lowest compressional mode of the trapped Lieb-Liniger gas. The squared mode frequency $\nu^2 = \omega^2/\omega_z^2$ is plotted as a function of $\log_{10}[n_1(0)a_1]$. The crosses (+) correspond to the $\alpha - p$ model; the stars ($\star$) to the corrected $\alpha - p$ model; the circles ($\circ$) to the 3 parameters quasi-polynomial model; and the diamonds ($\diamond$) to the corrected 3 parameters quasi-polynomial model.

In order to compare the various calculations, we plot the same data in a different manner, see figure 2. As discussed in [7] we expect the corrected 3 parameters model to be the most precise of our results. We therefore take this corrected model as a reference and plot the differences between the mode frequencies calculated within the three other approaches ($\alpha - p$, corrected $\alpha - p$ and 3 parameters quasi-polynomial models) and this reference, as shown in figure 2. We can see on this figure that, once corrected, the results of the $\alpha - p$ and the 3 parameters quasi-polynomial models agree remarkably well, at the absolute 10^{-3} level for ν^2. As already mentioned, the $\alpha - p$ model is exact in the 1D mean-field and in the Tonks-Girardeau limits where it predicts $\nu^2 = 3$ and $\nu^2 = 4$, respectively. In between

these limits, the uncorrected $\alpha - p$ model has some difficulties in predicting precisely the correct value of the mode frequency. For a discussion of this issue see [7].

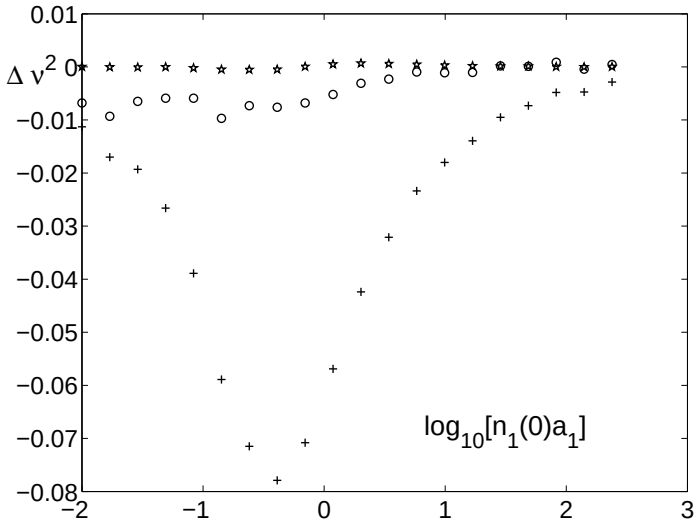

FIG. 2. Lowest compressional mode of the trapped Lieb-Liniger gas. The squared mode frequency $\nu^2 = \omega^2/\omega_z^2$ calculated using the corrected 3 parameters quasi-polynomial model is taken as a reference. The difference between this reference and three different calculations ($\alpha - p$, corrected $\alpha - p$ and 3 parameters quasi-polynomial models) are plotted as a function of $\log_{10}[n_1(0)a_1]$. The crosses (+) correspond to the $\alpha - p$ model; the stars ($\star$) to the corrected $\alpha - p$ model; and the circles ($\circ$) to the 3 parameters quasi-polynomial model.

Next we want to compare our most precise result (the corrected 3 parameters quasi-polynomial model) with the results of Menotti and Stringari [6], who calculated the lowest compressional mode frequency using a sum rule technique [14], which is basically a variational approach. This method is known to provide an exact upper bound to the lowest mode frequency. Both results are shown in figure 3. It is seen that they agree very well over the whole range of $n_1(0)a_1$.

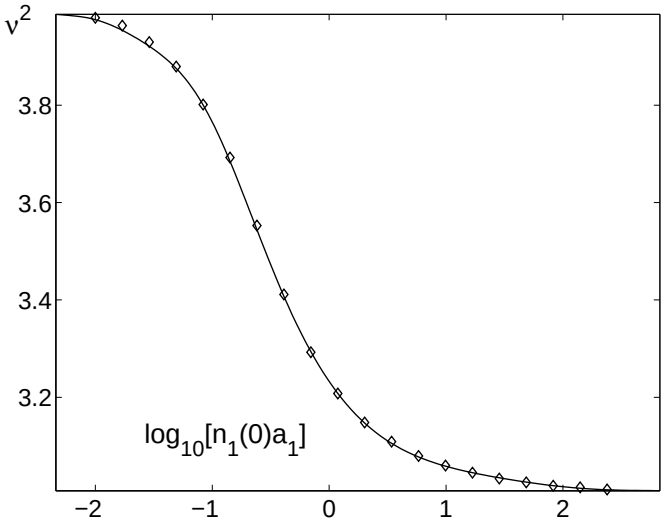

FIG. 3. Lowest compressional mode of the trapped Lieb-Liniger gas. The squared mode frequency $\nu^2 = \omega^2/\omega_z^2$ is plotted as a function of $\log_{10}[n_1(0)a_1]$. The diamonds ($\diamond$) correspond to the corrected 3 parameters quasi-polynomial model, and the full line is the result of Menotti and Stringari.

VI. DISCUSSION

In this paper, we used a recently developed method to solve the linearized hydrodynamic equations of the trapped Lieb-Liniger gas at $T \simeq 0$. We obtained the collective modes with high precision and especially studied the lowest compressional mode which is relevant to current experiments. Moritz et al. [3] recently measured this mode in a trapped 1D Bose gas in the mean-field regime. They found $\nu^2 \simeq 3$ when the gas is degenerate, in agreement with our calculations. As first emphasized in [6], measuring the lowest compressional mode should allow to reveal the transition from the 1D mean-field to the Tonks-Girardeau regime, where fermionization is expected.

More generally, we showed, in the particular case of the trapped Lieb-Liniger gas, how the hydrodynamic approach allows to obtain collective modes even in a strongly interacting system. We also checked that the sum rule approach used by Menotti and Stringari [6] indeed gives very good results for the trapped 1D Bose gas. For a discussion of a system where this is not necessarily true see [7].

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