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Abstract.

Following our previous papers [1, 2] we complete the construction of the parafermionic theory with the symmetry Z_N based on the second solution of Fateev-Zamolodchikov for the corresponding parafermionic chiral algebra. In the present paper we construct the Z_N parafermionic theory for N even. Primary operators are classified according to their transformation properties under the dihedral group ($Z_N \times Z_2$, where Z_2 stands for the Z_N charge conjugation), as two singlets, doublet $1, 2, \ldots, N/2-1$, and a disorder operator. In an assumed Coulomb gas scenario, the corresponding vertex operators are accommodated by the Kac table based on the weight lattice of the Lie algebra $D_{N/2}$. The unitary theories are representations of the coset $SO_n(N) \times SO_2(N)/SO_{n+2}(N)$, with $n = 1, 2, \ldots$. We suggest that physically they realise the series of multicritical points in statistical systems having a Z_N symmetry.

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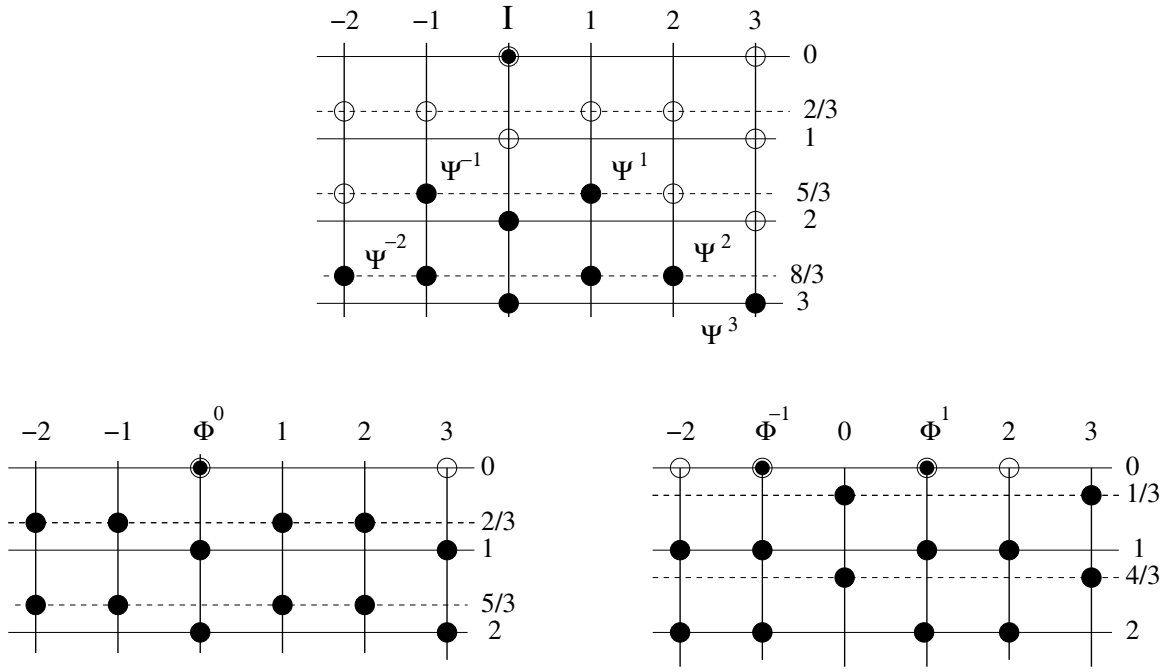

Figure 1: Singlet and doublet modules of the Z_6 theory as derived from the submodule structure of the identity operator module. Empty circles stand for empty states in the module.

In this paper, we complete our previous work [1, 2] on the construction of the Z_N parafermionic conformal field theory based on the second solution[2] of Fateev-Zamolodchikov for the corresponding parafermionic chiral algebra, which has been given in the Appendix of Ref. [3]. The representation theory for these Z_N parafermions has a different structure for N odd and even. The theory with N odd was the subject of Refs. [1, 2]. Its Kac table is based on the weight lattice of the algebra B_r ($N = 2r + 1$). The theory with N even, which will be constructed in this paper, has a Kac table based on the algebra D_r ($N = 2r$). Both cases, N odd and even, correspond to the coset [4]:

$$\frac{SO_n(N) \times SO_2(N)}{SO_{n+2}(N)}. \tag{1}$$

Here $SO_n(N)$ is the orthogonal group, with level n for its affine current algebra.

Physically, we expect that these new parafermionic theories should describe, in particular, the multicritical points of statistical systems having a Z_N symmetry. For the moment this is only an expectation. To our knowledge, the corresponding statistical models which would be realised on a lattice, having Z_N tricritical or multicritical fixed points, are not known at present, except for the cases Z_2 and Z_3.

The local operator algebra of the parafermionic fields $\{\Psi^k(z)\}$ has been given in the Appendix of Ref. [3]. With slightly different notations, which we shall keep in this paper, it is reproduced in Eqs. (2.1)–(2.2) of Ref. [2].

Following Refs. [1, 2], the level structure of the representation modules of singlet and doublet primary fields can be defined from the conformal dimensions of the $\Psi^k(z)$

[2]In the following, we always refer to this *second solution* of the parafermionic chiral algebra.

fields. Each of these dimensions equals a level of the sector with Z_N charge k, within the module of the identity operator. Inside a fixed Z_N sector, the Virasoro subalgebra acts, making the level spacing equal to one. By taking submodules, the level structure of all the modules (singlet, different doublets) can be defined (see Ref. [2] for more details). In Fig. 1 we show as an example the modules of the theory Z_6, which is the first non-trivial theory in the family of Z_N theories, for N even.[3]

We recall the formula for the conformal dimensions of the fields $\Psi^k(z)$ [3]:

$$\Delta_k = \Delta_{-k} = 2\frac{k(N-k)}{N}, \tag{2}$$

where $k = 1, 2, \ldots, N/2$. For Z_6 one has $\Delta_1 = 5/3$, $\Delta_2 = 8/3$, and $\Delta_3 = 3$, cf. Fig. 1.

A particular problem of defining the Z_N theories with N even appears already at this point, when generalising the methods of Ref. [2] for the case of N odd. Considering the Z_6 theory as an example, we observe that if we define the modules of (Φ^{-1}, Φ^1) and (Φ^{-2}, Φ^2) from the corresponding submodules within the module of a singlet Φ^0, as shown in Fig. 1, these two doublet modules will actually be identical. In other words, the doublet modules with Z_N charge $q = \pm 1$ ("doublet 1") and $q = \pm 2$ ("doublet 2") are isomorphic and thus only define one sector of the theory. In the same way, the module of a singlet Φ^3 is identical to that of the singlet Φ^0.

This problem is general for all Z_N with N even. It can be traced back to the fact that $\delta^{N/2-q}_{N/2-k} = \delta^q_k$, where δ^q_k is the level spacing in the module of the doublet q, cf. Eq. (6) below. Thus, the primary fields $\{\Phi^{\pm q}\}$ define distinct doublet modules only for $q = 1, 2, \ldots, q_{\max}$ with $q_{\max} = \lfloor N/4 \rfloor$; here $\lfloor x \rfloor$ denotes the integer part of x. For N odd this problem was absent [2] ($N/2$ in the above formula is not an integer in this case), and accordingly one had $q_{max} = \lfloor N/2 \rfloor$. Summarising, in the Z_N theory with N even, one half of the doublet primary fields appears to be missing.

We shall return to this problem of missing doublets later, to treat it properly. For the moment we observe that in the case of the Z_6 theory one would have a singlet, a doublet 1 and a disorder field.

The disorder fields are defined in a way similar to that described in Refs. [1, 2], with modules having integer and half-integer levels. We refer to the above-mentioned papers for details. The disorder sector does not present major problems when generalising from N odd to N even. A particular difference in the degeneracy structure of the disorder modules will be mentioned later, in Appendix E.

The decompositions of $\{\Psi^k(z)\}$ into mode operators, in the modules of singlets and doublets, have the same form as in Eqs. (2.12)–(2.14) of Ref. [2]:

$$\Psi^k(z)\Phi^q(0) = \sum_n \frac{1}{(z)^{\Delta_k - \delta^q_{k+q}+n}} A^k_{-\delta^q_{k+q}+n}\Phi^q(0), \tag{3}$$

$$A^k_{-\delta^q_{k+q}+n}\Phi^q(0) = 0, \quad \text{for } n > 0, \tag{4}$$

$$A^k_{-\delta^q_{k+q}+n}\Phi^q(0) = \frac{1}{2\pi i}\oint_{C_0} dz (z)^{\Delta_k - \delta^q_{k+q}+n-1}\Psi^k(z)\Phi^q(0), \quad \text{for } n \leq 0. \tag{5}$$

[3]The Z_2 theory is that of a free boson, $c = 1$. And the Z_4 theory factorises into a direct product of two $N = 1$ superconformal theories (each of which having a Z_2 symmetry).

Here δ_k^q is the first descendent level ("gap") in the module of the doublet q, in the Z_N charge sector k. It was established in Ref. [2] that

$$\delta_k^q = 2\frac{(q^2-k^2)}{N} \mod 1. \tag{6}$$

In the same way as in Ref. [2], if δ_{k+q}^q in Eq. (5) happens to vanish, one can define the zero mode eigenvalues $\{h_q\}$:

$$A_0^{\mp 2q}\Phi^{\pm q}(0) = h_q \Phi^{\mp q}(0). \tag{7}$$

Note that the representations Φ^q are characterised by both $\{h_q\}$ and the conformal dimension Δ_q, the latter being just the eigenvalue of the usual Virasoro zero mode L_0.

The commutation relations of the mode operators $\{A^k_{-\delta+n}\}$ have been derived in Ref. [2]. In the present theory, with N even, they have the same general form and they are given by Eqs. (2.17)–(2.24) of Ref. [2].

The fundamental singlets, doublets and disorder operators, one for each sector, are defined as operators whose modules are degenerate at the first possible descendent levels. As already mentioned, each primary operator of the theory is expected to occupy a site of the D_r weight lattice ($N = 2r$). Fundamental operators occupy a portion of the lattice close to the origin, which we shall refer to as the elementary cell. For Z_6, this elementary cell is shown in Fig. 2. In the figure, $\vec{\omega}_1$, $\vec{\omega}_2$, $\vec{\omega}_3$ are the fundamental weights and $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are the simple root vectors of the algebra D_3. We shall comment further on this figure below.

The conformal dimensions of the operators are assumed to take the Coulomb gas form

$$\Delta_{\vec{\beta}} = \Delta_{\vec{\beta}}^{(0)} + B = \left(\vec{\beta} - \vec{\alpha}_0\right)^2 - \vec{\alpha}_0^2 + B, \tag{8}$$

$$\vec{\beta} \equiv \vec{\beta}_{(n_1,n_2,\ldots n_n)(n_1',n_2',\ldots n_n')} = \sum_{a=1}^{r}\left(\frac{1+n_a}{2}\alpha_+ + \frac{1+n_a'}{2}\alpha_-\right)\vec{\omega}_a, \tag{9}$$

$$\vec{\alpha}_0 = \frac{(\alpha_+ + \alpha_-)}{2}\sum_{a=1}^{r}\vec{\omega}_a, \tag{10}$$

where the Coulomb gas parameters α_+, α_- are defined as

$$\alpha_+ = \sqrt{\frac{p+2}{2}}, \quad \alpha_- = -\sqrt{\frac{p}{p+2}} \tag{11}$$

in accordance with the central charge expression for the coset (1) [3, 4, 2]:

$$c = (N-1)\left(1 - \frac{N(N-2)}{p(p+2)}\right), \tag{12}$$

$$p = N - 2 + n. \tag{13}$$

The constant B in Eq. (8) is the *boundary term*, which takes, in general, different values for the different sectors of the theory. To fully determine the Kac table of the theory, one needs to define these sectors, work out the corresponding values of B, and assign the proper sector label to each of the vectors $\vec{\beta}$.

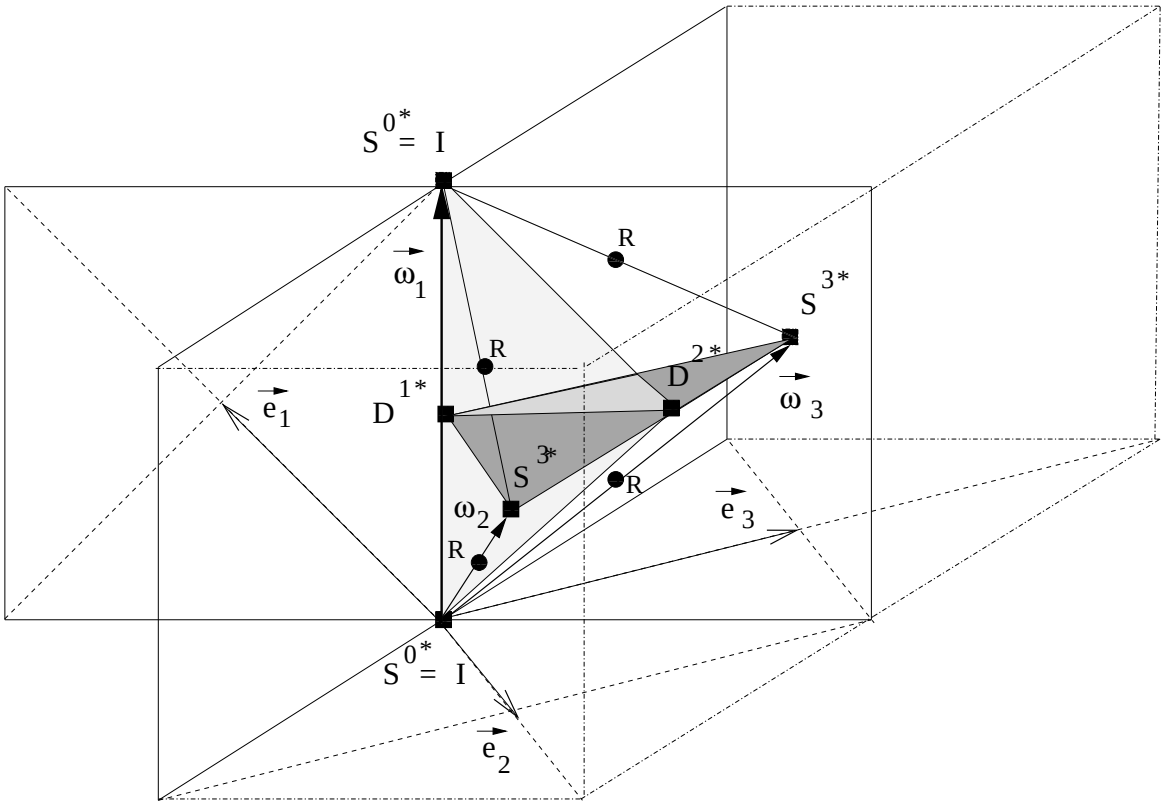

Figure 2: Elementary cell of the theory Z_6. The fundamental operators, the independent ones after applying the symmetry $Z_2 \times Z_2$, are: the trivial singlet $S^{0^*} = I = \Phi_{(1,1,1)}$, the doublets $D^{1^*} = \Phi_{(1)} \equiv \Phi_{(2,1,1)}$, $D^{2^*} = \Phi_{(2)} \equiv \Phi_{(1,2,2)}$, the non-trivial singlet $S^{3^*} = \Phi_{(3)} \equiv \Phi_{(1,3,1)}$, and the disorder operator $R = \Phi_{(1,2,1)}$.
The vectors $\vec{\beta}$ in Eq. (9), which constitute the lattice in this figure, can be represented as $\vec{\beta} = 2\vec{\alpha_0} - \tilde{\vec{\beta}}$, with $\tilde{\vec{\beta}} = \sum_{a=1}^{r} (\frac{1-n_a}{2}\alpha_+ + \frac{1-n'_a}{2}\alpha_-)\vec{\omega}_a$. Accordingly, the vectors corresponding to the sites of this lattice, with the origin at $S^{0^*} = I$, represent the part $-\tilde{\vec{\beta}}_{(1,1,1)(n'_1,n'_2,n'_3)} = \sum_{a=1}^{3} \frac{n'_a-1}{2}\alpha_-\vec{\omega}_a$ of $\vec{\beta}_{(1,1,1)(n'_1,n'_2,n'_3)}$, while $2\vec{\alpha}_0$ is being attributed to the origin.

Concerning the sector label assignment, we shall see that it is, in fact, sufficient to properly place the fundamental operators in the elementary cell of the Kac table. The assignment of sector labels to the rest of the lattice will then be realised by Weyl-type reflections of the elementary cell. We shall make this comment more precise in what follows.

First of all, the elementary cell of the Z_N theory is defined as the finite part of the Kac table which corresponds, in particular, to the physical domain of the trivial theory, i.e., the theory with level $n = 0$ in Eq. (1) and central charge $c = 0$ in Eq. (12).

The unitary theories correspond to n taking positive integer values in Eqs. (12)–(13). For a particular unitary theory, corresponding to a given value of the parameter p, the physical domain of the Kac table is delimited as follows:

$$n'_1 + 2\sum_{a=2}^{r-2} n'_a + n'_{r-1} + n'_r \;\leq\; p+1, \tag{14}$$

$$n_1 + 2\sum_{a=2}^{r-2} n_a + n_{r-1} + n_r \;\leq\; p-1, \tag{15}$$

where n'_a, n_a are positive integers. The reason is that on the outer side of the hyperplanes

$$n'_1 + 2\sum_{a=2}^{r-2} n'_a + n'_{r-1} + n'_r \;=\; p+2, \tag{16}$$

$$n_1 + 2\sum_{a=2}^{r-2} n_a + n_{r-1} + n_r \;=\; p \tag{17}$$

the sites of the Kac table are occupied by "ghosts" (reflections of primary submodule operators) which decouple, in correlation functions, from the operators of the physical domain. For more comments, see Refs. [1, 2]. Note that the delimitations (14)–(15) are different from those of the WD_r conformal theory [5].

For the trivial Z_6 theory ($c = 0$, $n = 0$) the inequalities (14)–(15) take the form

$$n'_1 + n'_2 + n'_3 \;\leq\; 5, \tag{18}$$

$$n_1 + n_2 + n_3 \;\leq\; 3. \tag{19}$$

The α_+ side of the Kac table is reduced to one site, $n_1 = n_2 = n_3 = 1$; a non-trivial domain exists only on the α_- side, cf. Eq. (18). This domain is shown in Fig. 2.

Also in the general case, the physical domain of the trivial $c = 0$ theory is confined by Eqs. (14)–(15) to trivial values $n_a = 1$ of the α_+ indices. The α_- side is non-trivial and defines the elementary cell whose operators we must identify. The identification of the other operators in the Kac table will follow from this analysis.

We consider as being natural the following assumptions:

1. The elementary cell, which, in particular, is the physical domain of the $c = 0$ theory, should contain all possible types of operators: the singlet(s), all different doublets, and disorder. This implies, in particular, that the fundamental operators (the ones with degeneracies at the first available levels, one operator for each sector) should all be contained in the elementary cell of a given Z_N theory.

2. The theory $c = 0$ being trivial, we shall assume that the conformal dimensions of the operators in the elementary cell all become equal to zero when $c = 0$.

The second assumption allows us to define the values of all the boundary terms, i.e., the constant B appearing in Eq. (8) for the dimensions of primary operators. For a given $N = 2r$, the $c = 0$ theory corresponds to $p = N - 2$, cf. Eqs. (12)–(13). From Eq. (14), the elementary cell will be delimited by the inequality:

$$n'_1 + 2\sum_{a=2}^{r-2} n'_a + n'_{r-1} + n'_r \leq N - 1 = 2r - 1. \tag{20}$$

The independent operators in this cell could be ordered in the following way:

$$\begin{aligned}
\Phi_{(0)} &\equiv \Phi_{(1,1,1,\ldots,1,1)} = I \\
\Phi_{(1)} &\equiv \Phi_{(2,1,1,\ldots,1,1)} \\
\Phi_{(2)} &\equiv \Phi_{(1,2,1,\ldots,1,1)} \\
\ldots &\ldots \ldots \\
\Phi_{(r-2)} &\equiv \Phi_{(1,1,1,\ldots,2,1,1)} \\
\Phi_{(r-1)} &\equiv \Phi_{(1,1,1,\ldots,1,2,2)} \\
\Phi_{(r)} &\equiv \Phi_{(1,1,1,\ldots,1,3,1)}
\end{aligned} \tag{21}$$

plus one more operator

$$R = \Phi_{(1,1,1\ldots,1,2,1)} \tag{22}$$

which will be shown to be a disorder operator. The operators in Eq. (21) will be shown to be all possible singlets and doublets admitted by the symmetry Z_N. We remind that, by the definition of the elementary cell, all α_+ indices have trivial values $n_a = 1$, and so we have suppressed them in Eqs. (21)–(22), showing only the set of α_- indices $\{n'_a\}$.

All other operators inside the domain (20) can be shown to be related to the ones in Eqs. (21)–(22) by the symmetries of the elementary cell. As an example, we show in Fig. 2 the elementary cell of the theory Z_6, its operators, and its symmetries.

In general, the symmetry of the elementary cell of the theory Z_N, with N even, is of the type $Z_2 \times Z_2$. It corresponds to reflections into two mutually orthogonal hyperplanes, and will be made explicit below. Eventually, one quarter of the elementary cell contains all the independent operators. They are all the fundamental ones, as we shall see shortly.

The geometry of the lattice generated by the vectors $\{\frac{\vec{\omega}_a}{2},\ a = 1, 2, \ldots, r\}$ is contained in the matrix of scalar products $\omega_{a,b} = (\vec{\omega}_a, \vec{\omega}_b)$. For the D_r lattice these scalar products take the following values:

$$\begin{aligned}
\omega_{a,b} &= a, \quad a \leq b \leq r-2; \\
\omega_{a,r-1} = \omega_{a,r} &= \frac{a}{2}, \quad a \leq r-2; \\
\omega_{r,r} = \omega_{r-1,r-1} &= \frac{r}{4}; \\
\omega_{r-1,r} &= \frac{r-2}{4}.
\end{aligned} \tag{23}$$

It is appropriate at this point to make precise the normalisation conventions which we assume in our formulae. We fix the overall normalisation of the vectors $\{\vec{\omega}_a\}$ from the values of the scalar products $\omega_{a,b}$ given in Eq. (23). The normalisation of the simple roots $\{\vec{e}_b\}$ is then fixed by defining their scalar products with $\{\vec{\omega}_a\}$ to be

$$(\vec{\omega}_a, \vec{e}_b) = |\vec{e}_b|^2 \, \delta_{ab}. \tag{24}$$

In the case of D_r this implies that

$$|\vec{e}_a|^2 = \frac{1}{2}, \text{ for all } a. \tag{25}$$

From the relation (24) between $\{\vec{\omega}_a\}$ and $\{\vec{e}_a\}$, one then finds that the decomposition of $\vec{e}_a$ in the basis of $\{\vec{\omega}_b\}$ takes the form

$$\vec{e}_a = \sum_{b=1}^{r} A_{ab} \frac{\vec{\omega}_b}{2}, \tag{26}$$

where A_{ab} is the Cartan matrix

$$A_{ab} = \frac{2(\vec{e}_a, \vec{e}_b)}{|\vec{e}_b|^2}. \tag{27}$$

We shall return now to the problem of determining the boundary terms for all the operators (21)–(22) occupying the elementary cell. They can be obtained from Eqs. (8)–(11) by demanding that the conformal dimension of the corresponding operator vanish upon setting p equal to $N-2 = 2r-2$, the value which corresponds to the $c = 0$ theory, cf. Eqs. (12)–(13). Performing the calculations by using the scalar products in Eq. (23), one finds that the constant B should take the following values for the operators $\Phi_{(a)}$ and R defined in Eqs. (21)–(22):

$$B_{(a)} = \frac{a(r-a)}{4r}, \quad a = 0, 1, 2, \ldots, r \tag{28}$$

$$B_R = \frac{r-1}{16}. \tag{29}$$

One could object that in Eqs. (8)–(11) for the conformal dimensions there is, in principle, a freedom in the overall normalisation of the vectors $\{\vec{\omega}_a\}$, on which the values of the boundary terms would depend, following the logic given above. In fact, this freedom is removed by using the method of reflections which we shall describe now. The normalisation will then get fixed exactly as in Eq. (23).

In a way analogous to the BRST structure of the (Virasoro algebra based) minimal models [6], the reflections in the hyperplanes which border the physical domain (14)–(15) put in correspondence the operators outside the physical domain with the degenerate combinations of descendent fields inside the modules of physical operators (i.e., operators positioned within the physical domain).

The simple reflections $s_{\vec{e}_a} \equiv s_a$, with $a = 1, 2, \ldots, r$, act on the weight lattice as the generators of the Weyl group associated with D_r. They are defined by[4]

$$s_a \vec{\beta}_{(1,1,\ldots,1)(n'_1,n'_2,\ldots,n'_r)} = \vec{\beta}_{(1,1,\ldots,1)(n'_1,n'_2,\ldots,n'_r)} - n'_a \alpha_- \vec{e}_a. \tag{30}$$

[4]Eq. (30) is taken from Eq. (3.6) of Ref. [2], but with the sign corrected.

Note again the trivial set of indices $\{n_a\}$ on the α_+ side. As we shall continue to be interested in the operators of the elementary cell, we shall suppress these indices again in the following. The vector $\vec{\beta}_{(\ldots)(n'_1,n'_2,\ldots,n'_r)}$ in Eq. (30) is to be taken outside the physical domain, in one of the adjacent regions, while the result of the reflection, i.e., the vector appearing on the right-hand side, should belong to the physical domain.

In the case of unitary theories, the set of simple reflections (30) has to be completed by a further reflection in the hyperplane (16). We denote this reflection by $s_{\vec{e}_{r+1}} \equiv s_{r+1}$. Expressed in terms of the simple roots $\{\vec{e}_a\}$ (which, in a standard way, correspond to screening operators) s_{r+1} can be cast in a form similar to that of Eq. (30):[5]

$$s_{r+1}\vec{\beta}_{(1,1,\ldots,1)(n'_1,n'_2,\ldots,n'_r)} = \vec{\beta}_{(1,1,\ldots,1)(n'_1,n'_2,\ldots,n'_r)} + n'_{r+1}\alpha_-\vec{e}_{r+1}, \tag{31}$$

with

$$\begin{aligned} n'_{r+1} &= N - n'_1 - 2\sum_{a=2}^{r-2} n'_a - n'_{r-1} - n'_r \\ \vec{e}_{r+1} &= e_1 + 2\sum_{a=2}^{r-2}\vec{e}_a + \vec{e}_{r-1} + \vec{e}_r, \end{aligned} \tag{32}$$

where $\vec{e}_{r+1}$ is the affine simple root.

Since a given simple reflection connects a ghost operator (outside the physical domain) and a degenerate (or singular) state inside the module of a physical operator (inside the physical domain), the difference of conformal dimensions of the ghost operator and the corresponding physical operator should be compatible with the levels available in the module. These levels are given by Eq. (6).

For the difference of dimensions one obtains, from Eq. (8),

$$\Delta_{\vec{\beta}} - \Delta_{s_a\vec{\beta}} = \Delta^{(0)}_{\vec{\beta}} - \Delta^{(0)}_{s_a\vec{\beta}} + B_{\vec{\beta}} - B_{s_a\vec{\beta}} \tag{33}$$

Simple calculations using Eqs. (8)–(11), (24) and (25)—as well as the definition of $\vec{\beta}$ and $s_a\vec{\beta}$ in Eq. (30)—lead to the result:

$$\Delta^{(0)}_{\vec{\beta}} - \Delta^{(0)}_{s_a\vec{\beta}} = -\frac{n'_a}{2} = \frac{|n'_a|^2}{2} \tag{34}$$

for $a = 1, 2, \ldots, r$.

It should be observed that the vector

$$\vec{\beta}_{(\tilde{n}'_1,\tilde{n}'_2,\ldots,\tilde{n}'_r)} = \vec{\beta}_{(n'_1,n'_2,\ldots,n'_r)} - n'_a\alpha_-\vec{e}_a \tag{35}$$

which must correspond to an operator inside the physical domain, satisfies $\tilde{n}'_a = -n'_a$. This is due to the decomposition (26) of $\vec{e}_a$ in the basis of $\{\frac{\vec{\omega}_b}{2}\}$ and to the fact that the diagonal elements of the matrix A_{ab} are always equal to 2. Now, by definition of the physical domain, the indices $\tilde{n}'_1, \tilde{n}'_2, \ldots, \tilde{n}'_r$ should all be positive. And so, the index n'_a

[5]Note the different signs of the second term on the right-hand sides of Eqs. (30)–(31). This means that the simple reflections (30) map ghosts *into* the physical domain, while the reflection (31) is a mapping *out of* the physical domain.

of the ghost operator $\vec{\beta}_{(n'_1,n'_2,\ldots,n'_r)}$ has to be negative. This explains the last equality in (34). This implies also, in the case of the simple reflection s_a, that $|n'_a|$ is the index of the corresponding physical operator, $\tilde{n}'_a = |n'_a|$.

Combining Eqs. (33)–(34) one finally obtains:

$$\Delta_{\vec{\beta}} - \Delta_{s_a\vec{\beta}} = \frac{|n'_a|}{2} + B_{\vec{\beta}} - B_{s_a\vec{\beta}} \tag{36}$$

The boundary terms here carry indices $\vec{\beta}$ and $s_a\vec{\beta}$ in order to recall that these constants (which are independent of c, and of the parameters α_+, α_-) take, in general, different values for different sites of the $\vec{\beta}$ lattice, according to the positioning of the various singlet, doublet and disorder sector operators.

Our purpose is to classify the operators (21)–(22) of the elementary cell as singlet, doublet and disorder operators, in accordance with the Z_N symmetry of the theory. For each of these operators, the position on the weight lattice is known, so we know the first term on the right-hand side of Eq. (36). The boundary terms of the operators (21)–(22) are known from Eqs. (28)–(29); this provides the third term on the right-hand side of Eq. (36), $B_{s_a\vec{\beta}}$. The ghost boundary term $B_{\vec{\beta}}$ is not given in advance. However, its value must be found among those of Eqs. (28)–(29), since by Assumption 1 [made just after Eq. (19)] the operators of the elementary cell cover all possible sectors of the theory.

On the other hand, as has been said before, the left-hand side of Eq. (36) should match with the levels available in the modules. These are given by the numbers δ_k^q in Eq. (6). This constraint is actually even stronger, because a particular level in the module corresponds only to a limited number of Z_N charge sectors. Thus, when computing the right-hand side of Eq. (36)—trying successively all possible values (28)–(29) of the unknown term $B_{\vec{\beta}}$—one needs to stay compatible with the available levels, and at the same time the charge sector of the available state in the module should match with the sector of the ghost. These constraints are sufficiently strong to allow to identify almost uniquely the nature of all the operators in the elementary cell.

This method has already been used in Ref. [2] for the case of odd N. It leads to a unique identification of operators for N sufficiently small, namely $N \leq 13$. For larger N extra solutions appear, which are however non-regular in N. In the following, when applying this method to the case of N even, we shall ignore these sporadic possibilities and assume that we are looking for a theory which is regular in N.

In the case of the theory with N even there is an extra problem to resolve, that of missing doublets. We have already mentioned that by inspecting the structure of the modules, with the levels and Z_N charge sectors defined by Eq. (6), one finds that the number of different modules is only one half the number of expected representations for the group Z_N. For instance, for Z_6 the modules of $q = 0$ (singlet) and $q = \pm 1$ (doublet 1) are found to be distinct, while the module of $q = \pm 2$ is identical to that of $q = \pm 1$ and $q = 3$ is identical to that of $q = 0$.

Another mismatch with the number of representations of Z_N is in the number of different values of boundary terms. The values of the boundary terms for the operators (21) are given by Eq. (28), which possesses a symmetry with respect to $a \to r - a \equiv N/2 - a$. As a result, the number of different values is again reduced by a factor of two.

On the other hand, the number of operators in Eq. (21) remains identical to the number of different representations of Z_N.

To judge from these observations, one could be tempted to identify the sectors of those operators in Eq. (21) which have identical boundary terms, reducing in this way the number of sectors in the theory by a factor of two. This might appear to be consistent with the number of different modules, which is also twice smaller.

This reduction could be obtained by assuming a different symmetry of the elementary cell, reducing in this way the number of independent operators in it. This new symmetry would be that of Z_4, replacing the $Z_2 \times Z_2$ symmetry which was assumed earlier. In the elementary cell of the Z_6 theory, shown in Fig. 2, this Z_4 symmetry would be generated by a rotation of the cell through an angle $\frac{\pi}{2}$ around the axis joining the sites D^{1^*} and D^{2^*}, followed by a reflection in a plane which is perpendicular to this axis and passes through its mid-point.

One could compare the symmetries of the elementary cell to those appearing in the W-type theories based on the same Lie algebra. In Ref. [2], the symmetries of the elementary cells (and, more generally, of the physical domains for the $c > 0$ unitary theories) of the Z_N theory with N odd was found to coincide with those of the WB_r theory [5]. For even N, one could compare to the WD_r theory [5], whose elementary cell has a $Z_2 \times Z_2$ symmetry when r is even, and a Z_4 symmetry when r is odd. In the particular case of the Z_6 theory, the symmetry of the elementary cell would be Z_4 if the correspondence with the WD_3 theory were to be maintained.

The only mismatch of the promising scenario suggested above is that the number of different sectors of the theory would not correspond to the number of representations of Z_N; it would be twice smaller. As a consequence, we shall abandon the scenario described above and look for another one which would respect the number of representations of Z_N.[6]

So far our definition of the modules has been based on the submodule structure of the identity operator module, which is filled with the parafermionic currents and their derivatives (and normal ordered products thereof) [1, 2]. In particular, in this construction the Z_N charge sectors of the representation fields Φ^q were the same as those of the parafermionic operators $\Psi^k(z)$, i.e., q and k were taking the same set of values

$$q, k = \pm 1, \pm 2, \ldots, \pm \lfloor N/2 \rfloor. \tag{37}$$

The problem that has occurred in the case of the theory with N even is that this method only produces distinct modules for $|q| \leq \lfloor N/4 \rfloor$. Namely, the modules corresponding to $\Phi^{\pm q}$ with $\lfloor N/4 \rfloor < q \leq \lfloor N/2 \rfloor$ are in fact identical to those of $0 \leq q \leq \lfloor N/4 \rfloor$.

The attempt of recovering a number of sectors equal to the number of representations of Z_N has led us to suggest that one must also consider half-integer values of q. Thus, we have assumed that the allowed values of q for the *primary*[7] representation fields should be

$$q = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots, \pm \lfloor N/4 \rfloor. \tag{38}$$

[6] At this point the knowledge of the actual symmetry of the theory is crucial. The reference to the corresponding coset would not be useful in any way.

[7] I.e., primary with respect to the parafermionic algebra.

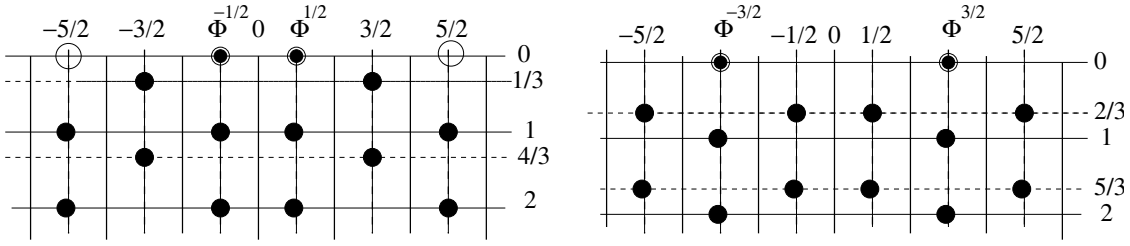

Figure 3: $q = \pm 1/2$ and $q = \pm 3/2$ modules of the Z_6 theory.

The levels in the modules corresponding to half-integer values of q can still be calculated from Eq. (6). In fact, this formula could alternatively be re-derived by considering the associativity requirements for the four-point function

$$\left\langle \Phi^{-q}(\infty)\Psi^{-k}(1)\Psi^{k}(z)\Phi^{q}(0)\right\rangle = \frac{P_l(z)}{(1-z)^{2\Delta_k}(z)^{\Delta_k-\delta^q_{k+q}}}. \tag{39}$$

Here $P_l(z)$ is a polynomial of degree l; the integer l is determined by the condition that $\Phi^q(0)$ be a primary (highest weight) field with respect to the currents $\Psi^k(z)$, i.e., that the gaps δ^q_k have to take the smallest possible non-negative values. It is sufficient to analyse the limit $z \to \infty$ of the function above, by comparing the analytic form on the right-hand side of Eq. (39) with the expression obtained by using the operator algebra for the fields on the left-hand side.

Within this approach one could perfectly well consider q as being half-integer valued, while the charges k of the parafermions remain integer valued.

As an example, we show in Fig. 3 the modules of $\Phi^{\pm 1/2}$ and of $\Phi^{\pm 3/2}$ for the Z_6 theory. Figs. 1 and 3 taken together give the complete list of modules for this theory.

In Appendix B we explicitly construct a $(\Phi^{-1/2}, \Phi^{+1/2})$ module (cf. Fig. 3) which is degenerate on the first available descendent levels. The result for the conformal dimension $\Delta_{\Phi^{1/2}}$ of the operators $\Phi^{\pm 1/2}$ is in perfect agreement with the dimension of the operator $\Phi_{(1)} \equiv \Phi_{(2,1,1)}$ of the elementary cell (21), the latter being computed with the appropriate boundary term (28). A simple analysis using the method of Weyl reflections, based on Eq. (36), predicts that the module of the operator $\Phi_{(2,1,1)}$ has to be degenerate twice at the level 1/3 and once at the level 1. This prediction is also confirmed by a direct calculation in Appendix B.

The fact that the levels of degeneracy predicted by the reflection technique agree with the outcome of direct calculations is important, even when the result for the conformal dimension is trivial. Therefore, we expose briefly in Appendix A the degeneracy calculation for the module of the identity operator $\Phi_{(0)} \equiv \Phi_{(1,1,1)} = I$.

Similarly, the direct calculation in Appendix C of degeneracies for the module (Φ^{-1}, Φ^{1}), cf. Fig. 1, provides an agreement with the operator $\Phi_{(2)} \equiv \Phi_{(1,2,2)}$ of the elementary cell (21). (A technical point regarding this calculation is deferred to Appendix D.) It should be noticed that the calculations for the operators $\Phi_{(1)}$ and $\Phi_{(2)}$ are actually generalised to the Z_N theory with arbitrary even $N > 6$. This is done towards the end of Appendices B–C.

Finally, the calculation in Appendix E classifies the operator $\Phi_{(1,2,1)}$ in Eq. (22) as the fundamental disorder operator.

The direct degeneracy calculations of Appendices A–E clearly support the classification of sectors that we have proposed above. In particular, Appendix B corroborates the argument in favour of half-integer q.

On the other hand, it can be checked that once the modules with half-integer q have been admitted, the analysis using the reflection method described after Eq. (36) makes the assignment of the operators given above the only possible.

This is but with one correction. Application of the reflection method alone cannot determine which one of the operators $\Phi_{(1)} = \Phi_{(2,1,1)}$ and $\Phi_{(2)} = \Phi_{(1,2,2)}$ should be related, respectively, to the doublets $(\Phi^{-1/2}, \Phi^{1/2})$ and (Φ^{-1}, Φ^{1}). This is because the boundary terms and the level spacings are the same in the above two sectors, as witnessed by Eq. (28), and by Figs. 1 and 3.

To resolve this ambiguity we examine more closely the degeneracy structure of the corresponding modules. According to the reflections, the module of the operator $\Phi_{(2,1,1)}$ should be degenerate twice at the level $1/3$ and once at the level 1. Similarly, the module of the operator $\Phi_{(1,2,2)}$ should be degenerate once at the level $1/3$ and twice at the level 1. Using this difference in the degeneracy patterns we could set up a simple test: we could assume the assignment of $\Phi_{(2,1,1)}$ to the doublet (Φ^{-1}, Φ^{1}) and demand, accordingly, that the module of this doublet be degenerate twice at the level $1/3$. Calculation of the degeneracy under this assumption, and for the level $1/3$ alone, is very simple. It leads immediately to a non-acceptable value of Δ for this doublet. We must therefore conclude that $\Phi_{(2,1,1)}$ should be assigned to the doublet $(\Phi^{-1/2}, \Phi^{1/2})$ and the other operator, $\Phi_{(1,2,2)}$, to the doublet (Φ^{-1}, Φ^{1}).

Once the above ambiguity is resolved, with a slight intervention of the degeneracy calculation, the assignment of the elementary cell operators and of all their surrounding ghosts is uniquely defined.

We emphasise again that even though the boundary terms of the operators $\Phi_{(1)} \equiv \Phi_{(2,1,1)}$ and $\Phi_{(2)} = \Phi_{(1,2,2)}$ happen to be equal, their respective modules and the degeneracy patterns are different. They therefore represent different sectors of the theory. In particular, this difference is a strong argument against the would-be Z_4 symmetry of the elementary cell that we tentatively discussed above.

So far we have not discussed the operator $\Phi_{(3)} \equiv \Phi_{(1,3,1)}$, which is the last fundamental operator in the list (21), in the case of the Z_6 theory. Its position in the elementary cell is shown in Fig. 2. By the analysis of reflections, cf. Eq. (36), the module of this operator should be degenerate twice at the level $2/3$ and once at the level $5/3$. The direct calculation of the degeneracy at the level $5/3$ is complicated. In general, the deeper the level, the more involved is the calculation. This is because the number of states to be analysed grows rapidly with the level. On the other hand, the assignment of this operator to the sector $q = 3/2$, the second module shown in Fig. 3, is the unique possibility allowed by the combined argument of reflections [cf. Eq. (36)], the available modules [cf. Figs. 1 and 3], and the content of the elementary cell [cf. Fig. 2]. There is however one important difference with respect to the other operators having non-zero q charge. Namely, we shall argue below that despite the appearance of Fig. 3, $\Phi_{(1,3,1)}$ is *not* a doublet operator $(\Phi^{-3/2}, \Phi^{3/2})$, but rather another non-trivial singlet operator, whose

module is built on the state $\Phi^{3/2}$ alone at the summit.

As has been shown above, the Z_N charges of the primary fields Φ^q are required to take half-integer values. This is because they have been defined initially with respect to the Z_N charges of the parafermionic fields Ψ^k. For Ψ^k we have admitted, initially, the natural set of values $k = \pm 1, \pm 2, ..., \pm(N/2-1), N/2$.

To obtain a more natural notation, and to bring out the structural similarities of the final results for even and odd N, it is convenient to redefine the initial Z_N charges by multiplying them by two:

$$Q = 2q; \quad \Phi^{Q^\star} = \Phi^q, \tag{40}$$

in order that the new charges of the primary fields take the natural set of values for the Z_N group:

$$Q = 0, \pm 1, \pm 2, \ldots, \pm(N/2-1), N/2. \tag{41}$$

Both definitions, q and Q, have already been used in Ref. [2] for the Z_N theory with N odd. For N odd the difference between the two notations is less pronounced and does not lead to the appearance of half-integer values. As in Ref. [2], we shall adopt a notation in which the charges in the Q notation are marked with an asterisk (we have already tacitly done so in Eq. (40)). For instance, when N is even, $\Phi^{1^\star} = \Phi^{\frac{1}{2}}$ will correspond to Φ with $Q = 1$, or $q = \frac{1}{2}$. When N is odd [2], this same value, $Q = 1$, corresponds to the maximal possible q charge, $q = \lfloor N/2 \rfloor$.

We can now complete the argument that $\Phi_{(1,3,1)}$ is in fact a singlet. As the Z_N charges of the primary fields should be defined modulo N (once again, in order to be consistent with the number of representations of the group Z_N), the charges $-N/2$ and $N/2$ of a primary field Φ^q should be taken as equal. This implies that the corresponding operator is not a doublet $(\Phi^{-N/2^*}, \Phi^{N/2^*})$, but rather a singlet state $\Phi^{N/2^*}$. In the special case of Z_6, this maximal Q state will be the singlet $\Phi^{3^\star} \equiv S^{3^\star} = \Phi^{3/2}$, shown in Fig. 2.

Within the convention of doubled charges, we shall denote by K the Z_N charges of the parafermions:

$$K = 2k = \pm 2, \pm 4, ..., \pm(N-2), N; \quad \Psi^{K^*} = \Psi^k. \tag{42}$$

The K charges then add up modulo $2N$. As a consequence, the Z_N charges of the descendent fields will also be defined modulo $2N$. At the same time, in every given module only N distinct Z_N charge sectors will be occupied, as shown in Figs. 1 and 3. This is consistent with the fact that the modules correspond to representations of Z_N, whose number should be N, and not $2N$.

The above feature that the Z_N charges of the parafermions take a set of doubled values (as compared to charges of the primary fields) is characteristic of a self-dual Z_N theory. This is explained in detail in Ref. [3], within the context of the parafermionic theory based on the *first solution* for the Z_N chiral algebra. Note that in our construction self-duality has not been assumed from the beginning, neither has it been used anywhere. Rather, it has emerged by itself, in the process of constructing a consistent theory.[8]

[8] In principle, one does not need the spin operator $\sigma^k \equiv \Phi^k$ and its dual, the Z_N disorder operator $\mu^k \equiv \tilde{\Phi}^k$, to produce parafermions by taking products: $\sigma^k \times \mu^k \sim \Psi^{2k}$. The parafermionic fields are produced equally well in the products of different Z_2 disorder operators. In Ref. [1] we have established the operator product expansion $\Psi^1(z)R_a(0) \sim R_{a-2}(0)$, and used it to derive characteristic equations. This relation implies, obviously, that Ψ^1 is produced in the product $R_{a-2}(z)R_a(0)$.

Before leaving the q notation completely, we should remark that only in the Z_N theories with $N = 2r$ and *even* r is the notion of half-integer charges q mandatory. Namely, for any r, the doublet modules (Φ^{-q}, Φ^{q}) are isomorphic to doublet modules $(\Phi^{N/4-q}, \Phi^{N/4+q})$ having their summits centered in $N/4$. For even r, the charges of these two equivalent positionings of the module are either both integer or both half-integer. For odd r, the charges of one of them is integer and the other half-integer. In this latter case, one can therefore avoid half-integer q by taking one half of the doublet modules centered in 0 and one half centered in $N/4$.

The identification of the elementary cell operators (and of their ghosts), which we have described above in some detail for the example of the Z_6 theory, are very similar in the Z_8 case. Like in the case of Z_6, the application of the method of reflections leaves one ambiguity unresolved, that of assigning the elementary cell operators $\Phi_{(2,1,1,1)}$ and $\Phi_{(1,1,2,2)}$ to the sectors, respectively, of the doublets $(\Phi^{-1/2}, \Phi^{1/2})$ and $(\Phi^{-3/2}, \Phi^{3/2})$, or in the opposite order. This ambiguity is resolved in a way similar to the Z_6 case, by demanding that the module of the doublet $(\Phi^{-3/2}, \Phi^{3/2})$ be triply degenerate on the first available level, which is $1/4$ in this case. Similarly, this leads to a wrong value for Δ.

For the Z_{10} case additional ambiguities appear in the reflection method. They could still all be resolved, this time with a more important intervention of the direct degeneracy calculations. In particular, one would need to use the results obtained in the Appendices B and C for the modules $q = 1/2$ and $q = 1$, for general (even) N.

After these first cases of Z_6, Z_8 and Z_{10} have been analysed explicitly, the generalisation to higher values of N is straightforward. It is additionally checked by the results of Appendices B and C for the doublets $q = 1/2$ and 1, and by the results of Appendix E for the disorder sector.

We could now summerise. The elementary cell of the Z_N theory, with $N = 2r$ even, is delimited by the inequality (20). This is also the physical domain of the $c = 0$ theory. The symmetry of this cell, with respect to the operator content of the $c = 0$ theory, is $Z_2 \times Z_2$. This is realised by reflections in two orthogonal hyperplanes, defined as:

$$\begin{array}{lll} \underline{\text{First } Z_2}: & n'_a \to n'_a, & a = 1, 2, \ldots, r-2 \\ & n'_{r-1} \to n'_r & \\ & n'_r \to n'_{r-1} & \\ \underline{\text{Second } Z_2}: & n'_1 \to 2r - n'_1 - 2\sum_{b=2}^{r-2} n'_b - n'_{r-1} - n'_r & \\ & n'_a \to n'_a, & a = 2, 3, \ldots, r \end{array} \tag{43}$$

The independent operators, after applying the above symmetry, are listed in Eqs. (21)–(22). In this list there is precisely one operator for each sector of the theory. Their identification is the following:

$$\begin{aligned} \Phi_{(0)} &\equiv \Phi_{(1,1,1,\ldots,1,1,1)} = I = S^{0^*} \\ \Phi_{(1)} &\equiv \Phi_{(2,1,1,\ldots,1,1,1)} = D^{1^*} \\ \Phi_{(2)} &\equiv \Phi_{(1,2,1,\ldots,1,1,1)} = D^{2^*} \\ \ldots &\quad \ldots \quad \ldots \\ \Phi_{(r-2)} &\equiv \Phi_{(1,1,1,\ldots,2,1,1)} = D^{r-2^*} \end{aligned}$$

$$\begin{aligned} \Phi_{(r-1)} &\equiv \Phi_{(1,1,1,\ldots,1,2,2)} = D^{r-1^*} \\ \Phi_{(r)} &\equiv \Phi_{(1,1,1,\ldots,1,3,1)} = S^{r^*} \\ &\quad \Phi_{(1,1,1,\ldots,1,2,1)} = R \end{aligned} \tag{44}$$

Here S^{0^*} (charge $Q = 0$) is a trivial singlet; D^{Q^*} (charge $Q = 1, 2, \ldots, r-1$) are $r-1$ different doublets; note that D^{Q^*} represents both components of the doublet $\left(\Phi^{-Q^*}, \Phi^{Q^*}\right)$; and S^{r^*} (charge $Q = r$) is a nontrivial singlet. Finally, R is the Z_2 (charge conjugation) disorder operator.

The boundary terms for all these different sectors (i.e., the constant B in the Kac formula (8)) have been given in Eqs. (28)–(29).

When $c > 0$, the unitary theories with integer $p > N-2$ have their physical domains delimited by the inequalities (14)–(15), which replace the inequality (20) of the $c = 0$ theory. The $Z_2 \times Z_2$ symmetry of the physical domain is realised as:

$$\begin{array}{ll} & n'_a \to n'_a, \quad n_a \to n_a, \quad a = 1, 2, \ldots, r-2 \\ \underline{\text{First } Z_2}\text{:} & n'_{r-1} \to n'_r, \quad n_{r-1} \to n_r \\ & n'_r \to n'_{r-1}, \quad n_r \to n_{r-1} \\ & \\ \underline{\text{Second } Z_2}\text{:} & n'_1 \to p + 2 - n'_1 - 2\sum_{b=2}^{r-2} n'_b - n'_{r-1} - n'_r \\ & n_1 \to p - n_1 - 2\sum_{b=2}^{r-2} n_b - n_{r-1} - n_r \\ & n'_a \to n'_a, \quad n_a \to n_a, \quad a = 2, 3, \ldots, r \end{array} \tag{45}$$

which replaces (43).

The physical domain of a given $c > 0$ unitary theory will contain more (independent) operators than those listed in Eq. (44). Still, the notion of the elementary cell remains significant. Firstly, because the list (44) presents all the fundamental operators for a given theory. Secondly, the full set of operators delimited by the inequality (20) (with trivial indices on the α_+ side) still preserves the symmetry (43) *as far as the sector assignment is concerned*. And thirdly, because the assignment of sector labels to all the operators in the Kac table can be obtained by reflecting repeatedly the elementary cell as we shall describe below.

The problem of filling the Kac table (i.e., assigning sector labels to all the sites of the weight lattice) has already been dealt with in Refs. [1, 2], for the theory with N odd. The method was based on Weyl reflections, evoking the assignment of sector labels to the ghost operators, and on fusion of singlets ($q = 0$) with other operators. This method can be simplified considerably by turning once again to the trivial theory, $c = 0$, which has the advantage that its physical domain coincides with the elementary cell. Thus, all physical operators lie in the "basic layer" [2], defined by the absence of excitations on the α_+ side. The ghost environment of the fundamental operators, the only physical ones of the $c = 0$ theory, is known from the preceding analysis. The rest of the operators in the elementary cell are obtained by applying the $Z_2 \times Z_2$ symmetry to the fundamental ones, cf. Fig. 2 for the theory Z_6. Their ghost environment is also obtained by this symmetry. To explain the resulting ghosts one has to use all the simple reflections, including the reflection (31)–(32) with respect to the direction $\vec{e}_{r+1}$, for all the operators in the elementary cell. The two dimensional basic layer of the Z_5 theory [1] is particularly adapted to show how the method of reflections is applied. The elementary cell of this theory and the

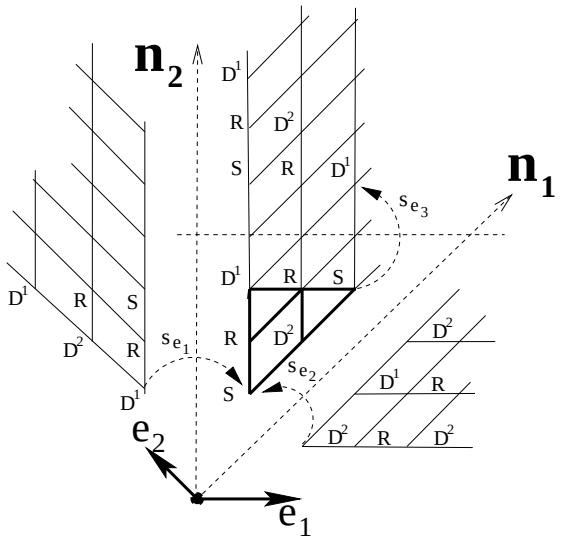

Figure 4: The ghost environment of the elementary cell of the Z_5 theory is obtained by considering the reflections $s_{\vec{e}_1}$, $s_{\vec{e}_2}$ and $s_{\vec{e}_3}$ (dashed arrows). Then, by using the fusions with the singlets S in the figure, the whole basic layer can be filled.

resulting ghost environment is shown in Fig. 4. The resulting ghost environment of the elementary cell is sufficient to fill the whole basic layer, by using fusions with the singlet ($q = 0$) operators. As in Ref. [2] we assume that the principal channel amplitudes are non-vanishing in all fusions of singlets with other operators.

The final result of this analysis can be expressed in a much simpler way. Once sector labels have been assigned to the operators of the elementary cell, the assignment of the rest of the operators in the basic layer is obtained by repeatedly reflecting the elementary cell in all its faces, filling progressively in this way the whole lattice. In Fig. 5 we show this procedure in the case of the Z_5 theory. The reflections employed in this process (henceforth referred to as Weyl reflections of the second kind) are on a different footing than the reflections evoking ghosts, which we have been discussing all the way before (henceforth we refer to these as Weyl reflections of the first kind). In particular, the reflections of the second kind are with respect to a given face of the elementary cell, and not with respect to an intermediate plane as in the case of reflections of the first kind, cf. Figs. 4–5. Also, the reflections of the second kind are always applied to the elementary cell, even in the case of $c \neq 0$ theories, while the reflections of the first kind are applied, in general, to the physical domain of a given unitary theory. This can be seen from the definitions (30)–(32).

The second kind of reflections has no direct relation to the structure of modules of primary operators. Its only significance is with respect to the sector assignment.

We remark that the second kind of Weyl reflections appears also in a general analysis of the coset-based conformal theories, with respect to the distribution over the lattice of the boundary terms [7, 8]. The methods we are using in this paper are quite different from those of Refs. [7, 8]. Also, the results that we have obtained are more complete: we distinguish the different sectors not only by the actual values of the boundary term (which in the present case of the Z_N theory with N even exhibits degeneracies), but also

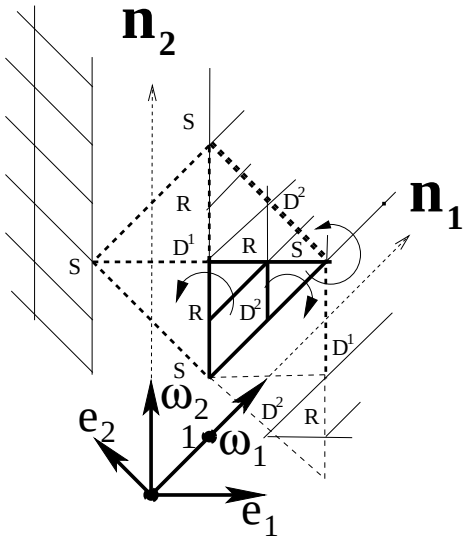

Figure 5: The method of filling the lattice, by repeatedly reflecting the elementary cell, is sketched for the case of the Z_5 theory.

by the actual symmetry content (i.e., the transformation properties under the group Z_N).

On the other hand, the demonstration of the coset-based results of Refs. [7, 8] is made using more general techniques, and applies in a more general context. Consequently, the convergence of the two approaches in the particular case of the Z_N theory can only be considered as being satisfactory and positive for both methods.

Saying it differently, the (second kind of) Weyl reflections symmetry of the Kac table, established in Refs. [7, 8], in relation to the values of the boundary terms, extends to the complete sector labeling that we have found. One may hope that a complementary use of both methods might prove to be useful in the future.

To complete our analysis of the sector labeling, it should be observed that, as usual, having assigned the sector labels to the operators of the basic layer, the labeling of the sites of the whole Kac table is obtained by applying translations, as in Refs. [1, 2]. Evidently, the assignment of sector labels, which we have been analysing above in the context of the $c = 0$ theory, holds for $c \neq 0$ theories in general.

The final result on the labeling of the Kac table can also be stated algebraically, in a way similar to that in Ref. [2]. We use the notation $\tilde{n}_a = |n_a - n'_a|$, and define the following quantities:

$$\begin{aligned} x_a &= \tilde{n}_a, \qquad a = 1, 2, \ldots, r-2 \\ x_{r-1} &= \tilde{n}_r, \\ x_r &= (\tilde{n}_{r-1} - \tilde{n}_r)/2. \end{aligned} \tag{46}$$

The operator $\Phi_{(n_1,\ldots,n_r)(n'_1,\ldots,n'_r)}$ is then a disorder operator if x_r is non-integer. Otherwise, if x_r is integer, we have a singlet or a doublet operator with an associated charge Q.

This charge Q can be written in several different ways. It can be noted that Q only

depends on $\{x_a\}$ through $\tilde{x}_a = x_a \bmod 2$. We then have the recursive formula

$$Q(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{k-1}, 1, 0, \ldots, 0) = k - Q(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{k-1}, 0, 0, \ldots, 0), \tag{47}$$

with the initial condition $Q(0, \ldots, 0) = 0$. This formula for Q is actually not the simplest possible, but we have given it here because it is identical to the one reported in Ref. [2] for the case of N odd. (Note however that in that case $\tilde{x}_{r-1}$ and $\tilde{x}_r$ were defined in a different way.) A more appealing direct formula is the following:

$$Q(x_1, x_2, \ldots, x_r) = \sum_{a=1}^{r} \left[\left(\sum_{b=a}^{r} x_b \right) \bmod 2 \right]. \tag{48}$$

The sector labeling can alternatively be stated by giving the assignment corresponding to the vector

$$\sum_{a=1}^{r} \frac{\tilde{n}_a}{2} \vec{\omega}_a \tag{49}$$

with respect to an orthonormal (hypercubic) basis, rather than with respect to the Dynkin labels, cf. Fig. 2 for the $r = 3$ case. This basis can be chosen such that

$$\begin{aligned} \vec{\omega}_a &= (1, \ldots, 1, 0, \ldots, 0), \qquad a = 1, 2, \ldots, r-2 \\ \vec{\omega}_{r-1} &= (1/2, \ldots, 1/2, -1/2), \\ \vec{\omega}_r &= (1/2, \ldots, 1/2). \end{aligned} \tag{50}$$

In the first line, the first a entries of $\vec{\omega}_a$ are 1. Note that this choice satisfies the scalar products (23). Now let y_a be the coordinates of the vector (49) with respect to this basis. The periodicity of the assignment is such that the sector label only depends on the values of $\tilde{y}_a = y_a \bmod 1$. The result is the following:

- If Q of the coordinates $\{\tilde{y}_a\}$ are equal to $1/2$ and the remaining $r-Q$ coordinates are zero, we have a singlet/doublet operator of charge Q. In other words, $Q = 2\sum_{a=1}^{r} \tilde{y}_a$ in this case. Note in particular that there are $\binom{r}{Q}$ charge Q operators within each cubic unit cell.
- If each of the $\{\tilde{y}_a\}$ is either $1/4$ or $3/4$, we have a disorder operator R. In particular, there are 2^r disorder operators within each cubic unit cell.

Let us finally remark that this same result holds true in the B_r algebra case (with $N = 2r + 1$ odd), provided that one chooses the basis

$$\begin{aligned} \vec{\omega}_a &= (1, \ldots, 1, 0, \ldots, 0), \qquad a = 1, 2, \ldots, r-1 \\ \vec{\omega}_r &= (1/2, \ldots, 1/2). \end{aligned} \tag{51}$$

A Singlet $q = 0$

We here expose the degeneracy calculation of the fundamental singlet S^0 in the Z_6 theory. According to the technique of Weyl reflections, the corresponding module should be three times degenerate at level $2/3$.

Extensive use of commutation relations shows that the unreduced module has three states at level 2/3:

$$A^1_{-\frac{2}{3}}\Phi^0, \qquad A^2_0 A^{-1}_{-\frac{2}{3}}\Phi^0, \qquad A^2_{-\frac{2}{3}}\Phi^0. \tag{A.1}$$

Note that the second of these states exploits the zero mode A^2_0 at level 2/3. We have checked that repeated actions of the various zero modes at level 2/3 do not produce additional independent states.

In order to have the required three degeneracies, we impose the complete degeneracy of the states (A.1). This results in the following three independent conditions:

$$A^{-1}_{\frac{2}{3}}\left(A^1_{-\frac{2}{3}}\Phi^0\right) = 0, \tag{A.2}$$

$$A^{-1}_{\frac{2}{3}}\left(A^2_0 A^{-1}_{-\frac{2}{3}}\Phi^0\right) = 0, \tag{A.3}$$

$$A^{-1}_{\frac{2}{3}}A^{-1}_0\left(A^2_{-\frac{2}{3}}\Phi^0\right) = 0. \tag{A.4}$$

The first condition, Eq. (A.2), can be rewritten through

$$A^{-1}_{\frac{2}{3}}A^1_{-\frac{2}{3}}\Phi^0 = \frac{10\Delta_{\Phi^0}}{3c}\Phi^0, \tag{A.5}$$

fixing $\Delta_{\Phi^0} = 0$; this was expected since the most relevant singlet must be the identity operator $\Phi_{(1,1,1)}$. The second condition, Eq. (A.3), further implies the vanishing of some matrix elements at level 1:

$$A^{-1}_{\frac{2}{3}}A^1_{\frac{1}{3}}A^1_{-\frac{1}{3}}A^{-1}_{-\frac{2}{3}} = A^1_{\frac{2}{3}}A^{-1}_{\frac{1}{3}}A^1_{-\frac{1}{3}}A^{-1}_{-\frac{2}{3}} = 0. \tag{A.6}$$

Finally, the third condition, Eq. (A.4), can be written as

$$A^{-2}_{\frac{2}{3}}A^2_{-\frac{2}{3}}\Phi^0 = 0. \tag{A.7}$$

Exploiting the commutation relation

$$\left(A^{-2}_{\frac{5}{3}}A^2_{-\frac{5}{3}} - \frac{7}{3}A^{-2}_{\frac{2}{3}}A^2_{-\frac{2}{3}} + A^2_{\frac{2}{3}}A^{-2}_{-\frac{2}{3}}\right)\Phi^0 = \frac{16\Delta_{\Phi^0}}{3c}\Phi^0, \tag{A.8}$$

this means that we have $A^{-2}_{\frac{5}{3}}A^2_{-\frac{5}{3}}\Phi^0 = 0$ as well.

B Doublet $q = \pm 1/2$

According to the reflection method, the doublet $D^{\pm 1/2} = \Phi_{(2,1,1)}$ of the Z_6 theory is doubly degenerate at level 1/3 and degenerate once at level 1.

We begin by investigating the $q = 3/2$ sector at level 1/3. There is just a single state:

$$A^1_{-\frac{1}{3}}\Phi^{1/2} \propto A^2_{-\frac{1}{3}}\Phi^{-1/2}, \tag{B.1}$$

and we require it to be completely degenerate. This amounts to imposing the condition $A^{-1}_{\frac{1}{3}}A^1_{-\frac{1}{3}}\Phi^{1/2} = 0$, which, by the use of commutation relations, can be rewritten as

$$\left(-\frac{1}{9} + \frac{10}{3c}\Delta_{\Phi^{1/2}} - A^1_0 A^{-1}_0\right)\Phi^{1/2} = 0. \tag{B.2}$$

This fixes the zero-mode eigenvalue, defined by $A^1_0\Phi^{-1/2} \equiv h\Phi^{1/2}$, as

$$h^2 = -\frac{1}{9} + \frac{10}{3c}\Delta_{\Phi^{1/2}}. \tag{B.3}$$

Imposing also the conjugate degeneracy in the $q = -3/2$ sector at level $\frac{1}{3}$, we can now reduce the module by setting $A^{\pm 1}_{-\frac{1}{3}}\Phi^{\pm 1/2} = 0$. The reduced module is then completely empty at level 1/3.

The next available level is level 1, and we here examine the sector $q = 1/2$ (the case $q = -1/2$ is equivalent, by charge conjugation symmetry). In the reduced module, there are only two independent ways of descending to this position, and we consider a linear combination of the corresponding two states:

$$\chi^{1/2}_{-1} = aL_{-1}\Phi^{1/2} + bA^1_{-1}\Phi^{-1/2}. \tag{B.4}$$

We impose degeneracy of $\chi^{1/2}_{-1}$ by demanding that $L^1\chi^{1/2}_{-1} = A^{-1}_1\chi^{1/2}_{-1} = 0$. In terms of the matrix elements defined by

$$\mu_{11}\Phi^{1/2} \equiv L_1L_{-1}\Phi^{1/2}, \tag{B.5}$$

$$\mu_{12}\Phi^{1/2} \equiv L_1A^1_{-1}\Phi^{-1/2}, \tag{B.6}$$

$$\mu_{21}\Phi^{-1/2} \equiv A^{-1}_1L_{-1}\Phi^{1/2}, \tag{B.7}$$

$$\mu_{22}\Phi^{-1/2} \equiv A^{-1}_1A^1_{-1}\Phi^{-1/2}, \tag{B.8}$$

the degeneracy constraint can be written

$$\mu_{11}\mu_{22} - \mu_{12}\mu_{21} = 0. \tag{B.9}$$

The four matrix elements are easily evaluated:

$$\mu_{11} = 2\Delta_{\Phi^{1/2}}, \tag{B.10}$$

$$\mu_{12} = \mu_{21} = \Delta_1 h, \tag{B.11}$$

$$\mu_{22} = \frac{1}{3}h^2 + \frac{2}{9} + \frac{10}{3c}\Delta_{\Phi^{1/2}}. \tag{B.12}$$

Inserting this in Eq. (B.9)—and using also Eq. (B.3) for h^2—we obtain the physical solutions

$$\Delta_{\Phi^{1/2}} = \frac{1}{48}\left(25 - c \pm \sqrt{(c-5)(c-125)}\right). \tag{B.13}$$

The calculation can readily be generalised from the Z_6 theory to the case of Z_N with $N \geq 6$ even. The complete degeneracy in the $q = \pm 3/2$ sector takes the form $A^{\pm 1}_{-1+4/N}\Phi^{\pm 1/2} = 0$ and leads to

$$h^2 = -\frac{N-2}{N^2} + \frac{4(N-1)}{Nc}\Delta_{\Phi^{1/2}}. \tag{B.14}$$

Following Ref. [2] we now assume that $N/2 - 3$ further degeneracies can be imposed at the levels predicted by the Weyl reflections, without fixing $\Delta_{\Phi^{1/2}}$, and in such a way

that the reduced module will be completely empty at all levels strictly between 0 and 1. The correctness of this assumption can be explicitly checked for the first few values of N.

Turning to level 1, we once again impose the degeneracy of the state (B.4). The matrix elements (B.10)–(B.11) are unchanged, whereas

$$\mu_{22} = \frac{N-4}{N}h^2 + \frac{N+2}{N^2} + \frac{4(N-1)}{Nc}\Delta_{\Phi^{1/2}}. \tag{B.15}$$

Solving Eq. (B.9) we obtain the solutions

$$\Delta_{\Phi^{1/2}} = \frac{1}{2N(N-2)}\left((N-1)^2 - c \pm \sqrt{(c-(N-1))\,(c-(N-1)^3)}\right). \tag{B.16}$$

generalising Eq. (B.13). This corresponds to the dimension of the operator $\Phi_{(2,1,1,\ldots,1,1)}$.

C Doublet $q = \pm 1$

In the Z_6 case, which we consider first, the method of Weyl reflections indicates the existence of a doublet $\Phi_{(1,2,2)}$ with one degeneracy at level 1/3 and two degeneracies at level 1. We here show that this is the doublet D^1.

First, to obtain a consistent doublet module, we need to make sure that $A_0^1\Phi^1 = 0$. In particular, this implies $A_0^{-1}A_0^1\Phi^1 = 0$, and using the commutation relation $\{\Psi^1, \Psi^{-1}\}\Phi^1$ we obtain

$$A^1_{\frac{1}{3}}A^{-1}_{-\frac{1}{3}}\Phi^1 = \left(-\frac{1}{9} + \frac{10}{3c}\Delta_{\Phi^1}\right)\Phi^1. \tag{C.1}$$

Note that this consistency requirement does not count as a degeneracy within the reflection method.

At level 1/3, the $q = 3$ sector is now empty. This follows from $\{\Psi^1, \Psi^1\}\Phi^1$:

$$A^1_{-\frac{1}{3}}A^1_0\Phi^1 = \frac{1}{2}\lambda_2^{1,1}A^2_{-\frac{1}{3}}\Phi^1 = 0. \tag{C.2}$$

On the other hand, the two states at level 1/3 in the $q = 0$ sector can be used to form a singular state:

$$\chi^0_{-\frac{1}{3}} = aA^1_{-\frac{1}{3}}\Phi^{-1} + bA^{-1}_{-\frac{1}{3}}\Phi^1, \tag{C.3}$$

for some constants a and b. Imposing the constraints $A^{\pm 1}_{\frac{1}{3}}\chi^0_{-\frac{1}{3}}$, and defining the matrix elements

$$\mu_{1,1}\Phi^1 \equiv A^1_{\frac{1}{3}}A^1_{-\frac{1}{3}}\Phi^{-1}, \tag{C.4}$$

$$\mu_{1,-1}\Phi^1 \equiv A^1_{\frac{1}{3}}A^{-1}_{-\frac{1}{3}}\Phi^1, \tag{C.5}$$

the degeneracy condition takes the form

$$\mu_{1,1} = \pm\mu_{1,-1}. \tag{C.6}$$

The matrix element $\mu_{1,-1}$ is fixed by Eq. (C.1), and $\mu_{1,1}$ follows from $\{\Psi^1,\Psi^1\}\Phi^{-1}$:

$$A^1_{\frac{1}{3}}A^1_{-\frac{1}{3}}\Phi^{-1} = \lambda_2^{1,1}A_0^2\Phi^{-1} \equiv \lambda_2^{1,1}h\Phi^1, \tag{C.7}$$

where h is the zero-mode eigenvalue. The degeneracy criterion then reads

$$\lambda_2^{1,1}h = \pm\left(-\frac{1}{9}+\frac{10}{3c}\Delta_{\Phi^1}\right). \tag{C.8}$$

In the corresponding reduced module there is now only one state at level 1/3:

$$A^1_{-\frac{1}{3}}\Phi^{-1} = A^{-1}_{-\frac{1}{3}}\Phi^1. \tag{C.9}$$

At level 1, we can now choose to impose a further degeneracy either in the $q=1$ sector or in the $q=2$ sector (the cases $q=-1$ and $q=-2$ are equivalent by Z_N charge conjugation). Extensive use of the commutation relations reveals that among the many possible ways of descending to level 1 (including the use of zero modes at level 1), only a few are independent. More precisely, in the $q=1$ sector only the two states

$$L_{-1}\Phi^1 \text{ and } A^1_{-\frac{2}{3}}A^{-1}_{-\frac{1}{3}}\Phi^1 \tag{C.10}$$

are linearly independent. Likewise, in the $q=2$ sector we have the two independent states

$$A^1_{-1}\Phi^1 \text{ and } A^2_{-\frac{2}{3}}A^{-1}_{-\frac{1}{3}}\Phi^1. \tag{C.11}$$

We first consider the possibility of degenerating a linear combination of the states (C.10) in the $q=1$ sector. Setting

$$\chi^1_{-1} = aL_{-1}\Phi^1 + bA^1_{-\frac{2}{3}}A^{-1}_{-\frac{1}{3}}\Phi^1, \tag{C.12}$$

we require $L_1\chi^1_{-1}=0$ and $A^{-1}_{\frac{2}{3}}\chi^1_{-1}=0$. (The constants a and b are not related to those in Eq. (C.3).) In terms of the matrix elements

$$\mu_0\Phi^1 \equiv L_1L_{-1}\Phi^1, \tag{C.13}$$

$$\mu_1\Phi^1 \equiv L_1A^1_{-\frac{2}{3}}A^{-1}_{-\frac{1}{3}}\Phi^1, \tag{C.14}$$

$$\mu_2\left(A^{-1}_{-\frac{1}{3}}\Phi^1\right) \equiv A^{-1}_{\frac{2}{3}}L_{-1}\Phi^1, \tag{C.15}$$

$$\mu_3\left(A^{-1}_{-\frac{1}{3}}\Phi^1\right) \equiv A^{-1}_{\frac{2}{3}}A^1_{-\frac{2}{3}}A^{-1}_{-\frac{1}{3}}\Phi^1, \tag{C.16}$$

the degeneracy criterion reads

$$\mu_0\mu_3 - \mu_1\mu_2 = 0. \tag{C.17}$$

The evaluation of the first three matrix elements is straightforward:

$$\mu_0 = 2\Delta_{\Phi^1}, \tag{C.18}$$

$$\mu_1 = \frac{4}{3}\left(-\frac{1}{9}+\frac{10}{3c}\Delta_{\Phi^1}\right), \tag{C.19}$$

$$\mu_2 = \frac{4}{3}. \tag{C.20}$$

The last one follows from the commutation relation

$$\left(A^{-1}_{\frac{2}{3}}A^{1}_{-\frac{2}{3}}-\frac{1}{3}A^{-1}_{-\frac{1}{3}}A^{1}_{\frac{1}{3}}+A^{1}_{-\frac{1}{3}}A^{-1}_{\frac{1}{3}}\right)A^{-1}_{-\frac{1}{3}}\Phi^1=\frac{10}{3c}\left(\Delta_{\Phi^1}+\frac{1}{3}\right)A^{-1}_{-\frac{1}{3}}\Phi^1, \tag{C.21}$$

which gives

$$\mu_3=\frac{1}{3}\left(-\frac{1}{9}+\frac{10}{3c}\Delta_{\Phi^1}\right)-\lambda_2^{1,1}h+\frac{10}{3c}\left(\Delta_{\Phi^1}+\frac{1}{3}\right). \tag{C.22}$$

Inserting this in Eq. (C.17) one obtains a quadratic equation for Δ_{Φ^1}, whose solutions depend on the sign chosen in Eq. (C.8). With the plus sign, we obtain the physically acceptable solutions

$$\Delta_{\Phi^1}=\frac{1}{30}\left(25-c\pm\sqrt{(c-5)(c-125)}\right), \tag{C.23}$$

whilst the minus sign leads to the unacceptable solutions

$$\Delta_{\Phi^1}=\frac{1}{210}\left(25+2c\pm\sqrt{4c^2-460c+625}\right). \tag{C.24}$$

Instead of imposing a degeneracy at level 1 in the $q=1$ sector, one could have chosen to degenerate a linear combination of the states (C.11) in the $q=2$ sector. Note that the symmetry of the unreduced module is such that the $q=1$ and $q=2$ sectors are equivalent. However, in the construction of the doublet module we have broken that symmetry by placing the summits in the $q=1$ sector. There is thus a priori no reason to expect that these two sectors should be equivalent as far as the submodules are concerned. And indeed we have seen that at level 1/3 there are zero states in the $q=3$ sector and two states in the $q=0$ sector (of which one has been degenerated).

We therefore consider the singular vector

$$\chi^2_{-1}=aA^1_{-1}\Phi^1+bA^2_{-\frac{2}{3}}A^{-1}_{-\frac{1}{3}}\Phi^1, \tag{C.25}$$

subject to the constraints $A_1^{-1}\chi^2_{-1}=0$ and $A^{-2}_{\frac{2}{3}}\chi^2_{-1}=0$. Defining the matrix elements

$$\tilde{\mu}_0\Phi^1 \equiv A_1^{-1}A^1_{-1}\Phi^1, \tag{C.26}$$

$$\tilde{\mu}_1\Phi^1 \equiv A_1^{-1}A^2_{-\frac{2}{3}}A^{-1}_{-\frac{1}{3}}\Phi^1, \tag{C.27}$$

$$\tilde{\mu}_2\left(A^{-1}_{-\frac{1}{3}}\Phi^1\right) \equiv A^{-2}_{\frac{2}{3}}A^1_{-1}\Phi^1, \tag{C.28}$$

$$\tilde{\mu}_3\left(A^{-1}_{-\frac{1}{3}}\Phi^1\right) \equiv A^{-2}_{\frac{2}{3}}A^2_{-\frac{2}{3}}A^{-1}_{-\frac{1}{3}}\Phi^1, \tag{C.29}$$

the degeneracy criterion is

$$\tilde{\mu}_0\tilde{\mu}_3-\tilde{\mu}_1\tilde{\mu}_2=0. \tag{C.30}$$

Once again, the evaluation of the first three matrix elements is straightforward:

$$\tilde{\mu}_0 = \frac{2}{9}+\frac{10}{3c}\Delta_{\Phi^1}, \tag{C.31}$$

$$\tilde{\mu}_1 = \lambda_2^{1,1}h^2+\frac{3}{5}\lambda_1^{-1,2}\left(-\frac{1}{9}+\frac{10}{3c}\Delta_{\Phi^1}\right), \tag{C.32}$$

$$\tilde{\mu}_2 = h+\frac{3}{5}\lambda_1^{-1,2}. \tag{C.33}$$

Evaluating $\tilde{\mu}_3$ is considerably more involved, calling for the use of numerous commutation relations (see Appendix D). The result is

$$\tilde{\mu}_3 = h^2 + \frac{\lambda_1^{-1,2}}{\lambda_2^{1,1}}\left(-\frac{1}{45} + \frac{8\Delta_{\Phi^1}}{3c}\right) + \frac{\lambda_3^{1,2}\lambda_{-2}^{1,3}}{2\lambda_2^{1,1}}\left(h + \frac{\lambda_1^{-1,2}}{5}\right) - \frac{\lambda_1^{-1,2}}{5}\left(h + \frac{3\lambda_1^{-1,2}}{5}\right). \quad \text{(C.34)}$$

Inserting these matrix elements—and the values of the various structure constants—in Eq. (C.30), one finds that the physically acceptable solutions for Δ_{Φ^1} are once again given by Eq. (C.23).

According to the reflection technique, there should be a second degeneracy at level 1, apart from the one that we have imposed either in the $q = 1$ or in the $q = 2$ sector (with identical results). This second degeneracy will not have any consequence for the determination of Δ_{Φ^1} or of h, but will ensure that the ghost state at level 1 has a valid doublet structure. In other words, it imposes a constraint on the module of the ghost operator at level 0; this is analoguous to the constraint $A_0^1\Phi^1 = 0$, discussed above Eq. (C.1), which we have imposed on the module of the physical operator $\Phi_{(1,2,2)}$.

The result (C.23) can be generalised to the case of Z_N with $N > 6$ even, as follows. We begin by imposing the full degeneracy of the first descendent $q = 2$ state, which is $A^1_{-1+6/N}\Phi^1$. The constraint $A^{-1}_{1-6/N}A^1_{-1+6/N}\Phi^1$ then leads to the fixation of the matrix element

$$A^1_{2/N}A^{-1}_{-2/N}\Phi^1 = -\frac{2(N-4)}{N^2} + \frac{4(N-1)}{Nc}\Delta_{\Phi^1}. \quad \text{(C.35)}$$

This generalises Eq. (C.1), but note that here it counts as a genuine degeneracy, rather than just a consistency requirement, as was the case for $N = 6$.

Imposing next a degeneracy in the $q = 0$ sector at level $2/N$ will fix the zero mode eigenvalue:

$$\lambda_2^{1,1}h = \pm\left(-\frac{2(N-4)}{N^2} + \frac{4(N-1)}{Nc}\Delta_{\Phi^1}\right). \quad \text{(C.36)}$$

This is completely analogous to Eq. (C.8).

Taking our cue from Appendix B (and from Ref. [2]) we now assume that $N/2 - 4$ further degeneracies can be imposed at the levels predicted by the Weyl reflections, without fixing Δ_{Φ^1}, and in such a way that the reduced module will be completely empty at all levels strictly between 0 and 1 in the sectors $q \neq 0$. However, note that the criterion (C.36) still leaves one state at level $2/N$ in the $q = 0$ sector.

Finally we demand the degeneracy of

$$\chi^1_{-1} = aL_{-1}\Phi^1 + bA^1_{-1+2/N}A^{-1}_{-2/N}\Phi^1 \quad \text{(C.37)}$$

cf. Eq. (C.12). Defining μ_1, μ_2 and μ_3 as the obvious generalisations of Eqs. (C.14)–(C.16), the commutation relations lead to

$$\mu_1 = \frac{2(N-2)}{N}\left(-\frac{2(N-4)}{N^2} + \frac{4(N-1)}{Nc}\Delta_{\Phi^1}\right), \quad \text{(C.38)}$$

$$\mu_2 = \frac{2(N-2)}{N}, \quad \text{(C.39)}$$

$$\mu_3 = \frac{N-4}{N}\left(-\frac{2(N-4)}{N^2} + \frac{4(N-1)}{Nc}\Delta_{\Phi^1}\right) - \lambda_2^{1,1}h + \frac{4(N-1)}{Nc}\left(\Delta_{\Phi^1} + \frac{2}{N}\right) \quad \text{(C.40)}$$

The condition (C.17) then yields the physically acceptable solution

$$\Delta_{\Phi^1} = \frac{1}{N(N-1)}\left((N-1)^2 - c \pm \sqrt{(c-(N-1))\,(c-(N-1)^3)}\right), \tag{C.41}$$

generalising Eq. (C.23). This corresponds to the dimension of the operator $\Phi_{(1,2,1,\ldots,1,1)}$ for $N > 6$, and to $\Phi_{(1,2,2)}$ for $N = 6$.

D Calculation of $\tilde{\mu}_3$

We wish to evaluate the matrix element $\tilde{\mu}_3$ defined by Eq. (C.29). In the following we shall make extensive use of various commutation relations; which relations are being used can be inferred from the presence of the corresponding structure constants. We first have:

$$\begin{aligned}\tilde{\mu}_3\left(A^{-1}_{-\frac{1}{3}}\Phi^1\right) &\equiv A^{-2}_{\frac{2}{3}}A^{2}_{-\frac{2}{3}}A^{-1}_{-\frac{1}{3}}\Phi^1 \\ &= A^{-2}_{\frac{2}{3}}\left(A^{-1}_{\frac{1}{3}}A^{2}_{-\frac{4}{3}} - \frac{1}{5}\lambda_1^{-1,2}A^1_{-1}\right)\Phi^1.\end{aligned} \tag{D.1}$$

The first term of the right-hand side of Eq. (D.1) also enters in:

$$\left(A^{-2}_{\frac{2}{3}}A^{-1}_{\frac{1}{3}} - A^{-1}_{-\frac{1}{3}}A^{-2}_{\frac{4}{3}}\right)A^2_{-\frac{4}{3}}\Phi^1 = \frac{1}{3}\lambda_3^{1,2}A^3_1A^2_{-\frac{4}{3}}\Phi^1, \tag{D.2}$$

where we have used that $A^{-1}_{\frac{4}{3}}A^2_{-\frac{4}{3}}\Phi^1 = 0$. To simplify Eq. (D.2) we shall use that

$$\lambda_2^{1,1}A^2_{-\frac{4}{3}}\Phi^1 = A^1_{-\frac{1}{3}}A^1_{-1}\Phi^1. \tag{D.3}$$

The second term on the left-hand side of Eq. (D.2) contains the piece $A^{-2}_{\frac{4}{3}}A^2_{-\frac{4}{3}}\Phi^1$, which also enters in:

$$\left(A^{-2}_{\frac{4}{3}}A^1_{-\frac{1}{3}} - A^1_{\frac{1}{3}}A^{-2}_{\frac{2}{3}}\right)A^1_{-1}\Phi^1 = \frac{1}{5}\lambda_1^{-1,2}A^{-1}_1A^1_{-1}\Phi^1, \tag{D.4}$$

where we have used that $A^1_{\frac{2}{3}}A^1_{-1}\Phi^1 = 0$.

Eq. (D.4) calls for the evaluation of $A^{-2}_{\frac{2}{3}}A^1_{-1}\Phi^1$, which is also required to get the second term on the right-hand side of Eq. (D.1). We have

$$\left(A^{-2}_{\frac{2}{3}}A^1_{-1} - A^1_{-\frac{1}{3}}A^{-2}_0\right)\Phi^1 = \frac{3}{5}\lambda_1^{-1,2}A^{-1}_{-\frac{1}{3}}\Phi^1, \tag{D.5}$$

where we have used that $A^1_0\Phi^1 = 0$. Thus

$$A^{-2}_{\frac{2}{3}}A^1_{-1}\Phi^1 = hA^1_{-\frac{1}{3}}\Phi^{-1} + \frac{3}{5}\lambda_1^{-1,2}A^{-1}_{-\frac{1}{3}}\Phi^1. \tag{D.6}$$

Acting on this with $A^1_{\frac{1}{3}}$ we find

$$A^1_{\frac{1}{3}}A^{-2}_{\frac{2}{3}}A^1_{-1}\Phi^1 = h^2\lambda_2^{1,1}\Phi^1 + \frac{3}{5}\lambda_1^{-1,2}\left(-\frac{1}{9} + \frac{10}{3c}\Delta_{\Phi^1}\right)\Phi^1, \tag{D.7}$$

which is the second term on the left-hand side of Eq. (D.4).

To obtain the right-hand side of Eq. (D.4) we use that

$$A_1^{-1}A_{-1}^1\Phi^1 = \left(\frac{2}{9}+\frac{10}{3c}\Delta_{\Phi^1}\right)\Phi^1. \tag{D.8}$$

Rearranging Eq. (D.4), and inserting Eqs. (D.7)–(D.8), one finds that

$$A_{\frac{4}{3}}^{-2}A_{-\frac{1}{3}}^1A_{-1}^1\Phi^1 = \left(h^2\lambda_2^{1,1}+\frac{3}{5}\left(-\frac{1}{9}+\frac{10}{3c}\Delta_{\Phi^1}\right)\lambda_1^{-1,2}+\frac{1}{5}\left(\frac{2}{9}+\frac{10}{3c}\Delta_{\Phi^1}\right)\lambda_1^{-1,2}\right)\Phi^1. \tag{D.9}$$

Using Eq. (D.3), and doing some simplification, we arrive at

$$A_{\frac{4}{3}}^{-2}A_{-\frac{4}{3}}^2\Phi^1 = \left(h^2+\frac{\lambda_1^{-1,2}}{\lambda_2^{1,1}}\left(-\frac{1}{45}+\frac{8}{3c}\Delta_{\Phi^1}\right)\right)\Phi^1, \tag{D.10}$$

which is essentially the second term on the left-hand side of Eq. (D.2).

It remains to get the right-hand side of Eq. (D.2). Using again (D.3), we have

$$A_1^3A_{-\frac{4}{3}}^2\Phi^1 = \frac{1}{\lambda_2^{1,1}}A_1^3A_{-\frac{1}{3}}^1A_{-1}^1\Phi^1, \tag{D.11}$$

which can be simplified through

$$\left(A_1^3A_{-\frac{1}{3}}^1-A_{-\frac{1}{3}}^1A_1^3\right)A_{-1}^1\Phi^1 = \frac{1}{2}\lambda_{-2}^{1,3}A_{\frac{2}{3}}^{-2}A_{-1}^1\Phi^1 \tag{D.12}$$

To make this explicit, we need $A_1^3A_{-1}^1\Phi^1$ and $A_{\frac{2}{3}}^{-2}A_{-1}^1\Phi^1$. The first of these reads

$$A_1^3A_{-1}^1\Phi^1 = \lambda_{-2}^{1,3}A_0^{-2}\Phi^1 = h\lambda_{-2}^{1,3}\Phi^{-1}, \tag{D.13}$$

whilst the second is obtained from

$$\left(A_{\frac{2}{3}}^{-2}A_{-1}^1-A_{-\frac{1}{3}}^1A_0^{-2}\right)\Phi^1 = \frac{3}{5}\lambda_1^{-1,2}A_{-\frac{1}{3}}^{-1}\Phi^1, \tag{D.14}$$

yielding

$$A_{\frac{2}{3}}^{-2}A_{-1}^1\Phi^1 = hA_{-\frac{1}{3}}^1\Phi^{-1}+\frac{3}{5}\lambda_1^{-1,2}A_{-\frac{1}{3}}^{-1}\Phi^1. \tag{D.15}$$

Inserting these bits and pieces in Eqs. (D.11)–(D.12) we get

$$A_1^3A_{-\frac{4}{3}}^2\Phi^1 = \frac{1}{\lambda_2^{1,1}}\left(\frac{3}{2}h\lambda_{-2}^{1,3}A_{-\frac{1}{3}}^1\Phi^{-1}+\frac{3}{10}\lambda_{-2}^{1,3}\lambda_1^{-1,2}A_{-\frac{1}{3}}^{-1}\Phi^1\right). \tag{D.16}$$

Finally, we inject Eqs. (D.10) and (D.16) in Eq. (D.2) in order to isolate $A_{\frac{2}{3}}^{-2}A_{\frac{1}{3}}^{-1}A_{-\frac{4}{3}}^2\Phi^1$. Inserting this, and also Eq. (D.6), in Eq. (D.1) we infer the value of $\tilde{\mu}_3$ given in Eq. (C.34).

E Disorder operators

The general properties of the disorder operators (fusion rules, analytic continuation, etc.) are discussed in detail in Appendix B of Ref. [1].

The decompositions of $\{\Psi^k\}$ into mode operators have the same form as in Eqs. (2.102)–(2.104) of Ref. [2]:

$$\Psi^k(z)R_a(0) = \sum_n \frac{1}{(z)^{\Delta_k+\frac{n}{2}}} A^k_{\frac{n}{2}} R_a(0), \qquad k=1,2,\ldots,\frac{N}{2}. \tag{E.1}$$

$$\Psi^{-k}(z)R_a(0) = \sum_n \frac{(-1)^n}{(z)^{\Delta_k+\frac{n}{2}}} A^1_{\frac{n}{2}} \mathsf{U}^k R_a(0), \qquad k=1,2,\ldots,\frac{N}{2} \tag{E.2}$$

$$A^k_{\frac{n}{2}} R_a(0) = \frac{1}{4\pi i}\oint_{C_0} \mathrm{d}z\,(z)^{\Delta_k+\frac{n}{2}-1}\Psi^k(z)R_a(0), \tag{E.3}$$

where $a=1,2,\ldots,N$ is the index of the disorder operator, and U is a $N\times N$ matrix which rotates this index, $\mathsf{U}R_a(0)=R_{a-1}(0)$. In Eq. (E.3), the action of the parafermionic modes $A^k_{\frac{n}{2}}$ has been expressed in terms of contour integrals which are defined by letting z turn twice around the operator $R_a(0)$ at the origin. The representation (E.3) has been used to determine the commutation relations between the modes, given in Eqs. (2.105)–(2.107) of Ref. [2].

The decomposition of the local products $\Psi^k(z)R_a(0)$ into half-integer powers of z is due to the non-abelian monodromy of the disorder operator $R_a(z,\bar z)$ with respect to the chiral fields $\Psi^{\pm k}(z)$, with $k=1,2,\ldots,N/2-1$.

On the other hand, the monodromy of the disorder operator with respect to the parafermionic field with maximal charge $\Psi^{N/2}$ turns out to be abelian. Thus, in the expansion of $\Psi^{N/2}(z)R_a(0)$ only integer powers of z are present, as can be seen by comparing the expansions (E.1) and (E.2), and taking into account that $\Psi^{-N/2}\equiv\Psi^{N/2}$. In other words, we have $A^{N/2}_{\frac{n}{2}}R_a(0)=0$ for n odd. This peculiarity marks a difference in the degeneracy structure of the fundamental disorder operators for the cases of N even and odd.

It is seen from the expansion (E.1) that there are $N/2$ zero modes A^k_0 (with $k=1,2,\ldots,N/2$) associated with the parafermion Ψ^k which acts between the N summits of the module:

$$A^k_0 R_a = h_k\,\mathsf{U}^{2k}R_a. \tag{E.4}$$

This defines $N/2$ eigenvalues $\{h_k\}$ which characterise, in addition to the conformal dimension Δ_R, each representation R_a. Actually, as described in Ref. [2], there is a relation between the eigenvalues h_1 and h_2 which holds irrespectively of the details of the representation of a particular operator R_a:

$$2h_1^2 = \lambda_2^{1,1}2^{\Delta_2-3}h_2 + 2^{-\Delta_2-2}\left[\kappa(0)+\frac{16\Delta_1}{c}\Delta_R\right], \tag{E.5}$$

where

$$\kappa(n) = (2\Delta_1+n-1)(2\Delta_1+n-2) - (2\Delta_1+n-1)(\Delta_2+1) + \frac{(\Delta_2+1)(\Delta_2+2)}{4}. \tag{E.6}$$

As witnessed by the expansions (E.1)–(E.2), each disorder module has N summits, labeled by the components R_a, and has only integer and half-integer levels. When N is odd, for a given primary operator R_a there are $(N-1)/2$ states at level $1/2$:

$$\left(\chi_a^{(k)}\right)_{-\frac{1}{2}} = A^k_{-\frac{1}{2}}\mathsf{U}^{-2k}R_a = \mathsf{U}^{-2k}A^k_{-\frac{1}{2}}R_a, \qquad k = 1, 2, \ldots, \frac{N-1}{2}. \tag{E.7}$$

Imposing that all the states (E.7) be primary, and using Eq. (E.5), gives all the $(N+1)/2$ conditions required to fix the values of all the eigenvalues $\{h_k\}$ and of the conformal dimension Δ_R. We showed in Ref. [2] that the solutions of the resulting system of equations give the conformal dimensions of the fundamental disorder operators.

When N is even, we have only $(N-2)/2$ states at level $1/2$:

$$\left(\chi_a^{(k)}\right)_{-\frac{1}{2}} = A^k_{-\frac{1}{2}}\mathsf{U}^{-2k}R_a = \mathsf{U}^{-2k}A^k_{-\frac{1}{2}}R_a, \qquad k = 1, 2, \ldots, \frac{N-2}{2}, \tag{E.8}$$

the "missing" state being $A^{N/2}_{-\frac{1}{2}}R_a = 0$, as we have seen above. Following the exemple of the case N odd, we shall demand that all the states (E.8) be singular. We therefore require that

$$A^{k'}_{+\frac{1}{2}}(\chi_a^{(k)})_{-\frac{1}{2}} = 0 \tag{E.9}$$

for each $k' = 1, 2, \ldots, (N-2)/2$, and for each $k = 1, 2, \ldots, (N-2)/2$. The degeneracy condition (E.9) and the relation (E.5) result in a system of $N/2$ independent equations. The number of equations is not sufficient to determine the $N/2+1$ unknown variables, i.e., the $N/2$ zero mode eigenvalues and the conformal dimension. Therefore—in contradistinction to case of N odd—we shall have to require the module be degenerate also at the next available level, i.e., at level 1.

After imposing the condition (E.9), all the states (E.8) can be put equal to zero. After this reduction, the level $1/2$ will be completely empty. With this in mind, it is not difficult to verify that at level 1 we have to consider the state

$$\left(\chi_a^{(1)}\right)_{-1} = aA^1_{-1}\mathsf{U}^{-2}R_a + bL_{-1}R_a \tag{E.10}$$

and require the following conditions to be satisfied:

$$L_{+1}\left(\chi_a^{(1)}\right)_{-1} = 0, \quad A^1_{+1}\left(\chi_a^{(1)}\right)_{-1} = 0. \tag{E.11}$$

In terms of the matrix elements μ_{ij} defined by

$$L_{+1}L_{-1}R_a = \mu_{11}R_a, \tag{E.12}$$

$$L_{+1}A^1_{-1}\mathsf{U}^{-2}R_a = \mu_{12}R_a, \tag{E.13}$$

$$A^{-1}_{+1}L_{-1}R_a = \mu_{21}\mathsf{U}^2R_a, \tag{E.14}$$

$$A^{-1}_{+1}A^1_{-1}\mathsf{U}^{-2}R_a = \mu_{22}\mathsf{U}^2R_a, \tag{E.15}$$

the degeneracy criterion reads

$$\mu_{11}\mu_{22} - \mu_{12}\mu_{21} = 0. \tag{E.16}$$

Using the commutation relations given in Eqs. (2.105)-(2.107) of Ref. [2], the matrix elements are readily computed:

$$\mu_{11} = 2\Delta_R, \tag{E.17}$$

$$\mu_{12} = \mu_{21} = \Delta_1 h_1, \tag{E.18}$$

$$\mu_{22} = \lambda_2^{1,1} 2^{\Delta_2 - 3} h_2 + 2^{-\Delta_2 - 2}\kappa(2) + 2^{-\Delta_2+1}\frac{2\Delta_1}{c}\Delta_R - \gamma h_1^2, \tag{E.19}$$

where we used the abbreviation:

$$\gamma = 2\Delta_1 + 4 - \frac{3}{2}\Delta_2 + \frac{1}{2}\Delta_2^2. \tag{E.20}$$

By making extensive use of the commutation relations given in Ref. [2], the equations (E.5), (E.9) and (E.16) can be solved. They admit two solutions Δ_R for the conformal dimension the fundamental operator R_a:

$$\Delta_R^{(1)} = \frac{1}{16}\frac{(N-1)(p+N)}{p}, \qquad \Delta_R^{(2)} = \frac{1}{16}\frac{(N-1)(p+2-N)}{p+2}. \tag{E.21}$$

The above solutions have the same form as the corresponding solutions found in Ref. [2] for N odd. They correspond respectively to:

$$\Delta_{(1,1,\ldots,1,2,1)(1,1,\ldots,1,1,1)} \text{ and } \Delta_{(1,1,\ldots,1,1,1)(1,1,\ldots,1,2,1)} \tag{E.22}$$

with the boundary term

$$B_R = \frac{r-1}{16}. \tag{E.23}$$

This explicit calculation of the degeneracies of the fundamental disorder operators confirms the assignment made in Eq. (22), and the value of the boundary term given in Eq. (29). Furthermore, the degeneracy structure is in perfect agreement with the one predicted by the method of reflections. Following this method it is easy to see that the module of the operator R possesses $(N-2)/2$ singular states at level $1/2$ (using the $r-1$ simple reflections s_a with $a = 1, 2, \ldots, r-2, r$) and one singular state at level 1 (obtained from the simple reflection s_{r-1}).

References

[1] Vl. S. Dotsenko, J. L. Jacobsen and R. Santachiara, Nucl. Phys. B **656**, 259 (2003).

[2] Vl. S. Dotsenko, J. L. Jacobsen and R. Santachiara, Nucl. Phys. B **664**, 477 (2003).

[3] V. A. Fateev and A. B. Zamolodchikov, Sov. Phys. JETP **62**, 215 (1985).

[4] P. Goddard and A. Schwimmer, Phys. Lett. B **206**, 62 (1988).

[5] V. A. Fateev and S. I. Luk'yanov, Sov. Sci. Rev. A Phys. **15**, 1–117 (1990).

[6] G. Felder, Nucl. Phys. B **317**, 215 (1989); Nucl. Phys. B **324**, 548 (1989).

[7] J. Soda and H. Yoshii, Prog. Theor. Phys. **80**, 941 (1988).

[8] P. Christe and F. Ravanini, Int. J. Mod. Phys. A **4**, 897 (1989).