Weak convergence of random $\mathbf{p}$-mappings and the exploration process of inhomogeneous continuum random trees

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Abstract

We study the asymptotics of the $\mathbf{p}$-mapping model of random mappings on $[n]$ as n gets large, under a large class of asymptotic regimes for the underlying distribution $\mathbf{p}$. We encode these random mappings in random walks which are shown to converge to a functional of the exploration process of inhomogeneous random trees, this exploration process being derived (Aldous-Miermont-Pitman 2003) from a bridge with exchangeable increments. Our setting generalizes previous results by allowing a finite number of "attracting points" to emerge.

Keywords: Random mapping, weak convergence, inhomogeneous continuum random tree

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1 Introduction

We study the asymptotic behavior as $n \to \infty$ of random elements of the set $[n]^{[n]}$ of *mappings* from $[n] = \{1, 2, \dots, n\}$ to $[n]$. Given a probability measure $\mathbf{p} = (p_1, \dots, p_n)$ on $[n]$, define a random mapping M as follows: for each $i \in [n]$, map i to j with probability p_j, independently over different i's, so that

$$P(M = m) = \prod_{i \in [n]} p_{m(i)} , \qquad m \in [n]^{[n]}. \tag{1}$$

The random mapping M is called the $\mathbf{p}$-*mapping*. In what follows, we will not be concerned about keeping track of the labels of the mapping's digraph, so we will suppose that the probability $\mathbf{p}$ is *ranked*, i.e. $p_1 \geq p_2 \geq \dots \geq p_n > 0$.

Now consider a sequence of such probabilities $\mathbf{p}_n = (p_{n1}, \dots, p_{nn})$. Weak convergence of the associated $\mathbf{p}$-mappings M_n as $n \to \infty$ has been studied when $\mathbf{p}_n$ satisfies an asymptotic negligibility condition, namely, letting $\sigma(\mathbf{p}_n) = (\sum_{1 \leq i \leq n} p_{ni}^2)^{1/2}$,

$$\frac{\max_{i \in [n]} p_{ni}}{\sigma(\mathbf{p}_n)} \underset{n \to \infty}{\to} 0. \tag{2}$$

Under this hypothesis, it has been shown [1] that several features of the $\mathbf{p}$-mapping, such as sizes of basins and number of cyclic points, can be described asymptotically in terms of certain functionals of reflected Brownian bridge (this was originally proved in [3] for the uniform case $p_{ni} = 1/n$). The two basic ingredients in the methodology of [1] are:
(i) Code the random mapping into a *mapping-walk* H^{M_n} that contains enough information about the mapping;
(ii) use a random bijection, called the Joyal correspondence [8], that maps $\mathbf{p}$-mappings into random doubly-rooted trees, called $\mathbf{p}$-trees, whose behavior is better understood.
In particular, the limits in law of associated encoding random walks can be shown to converge to twice normalized Brownian excursion under condition (2), and this information lifts back to mappings, implying that the rescaled mapping walks converge weakly to twice standard reflecting Brownian bridge; that is, $\sigma(\mathbf{p}_n) H^{M_n} \to 2B^{|\mathrm{br}|}$ according to a certain topology on càdlàg functions. Results provable via this methodology encompass those proved in [9] by somewhat different methods.

The goal of this paper is to extend this methodology to more general asymptotic regimes for the distribution $\mathbf{p}$, under the natural assumption $\max_{i \in [n]} p_{ni} \to 0$ as $n \to \infty$. In these more general regimes, several $\mathbf{p}$-values are comparable to $\sigma(\mathbf{p}_n)$ instead of being negligible. Precisely, we will assume there exists $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots)$ such that

$$\max_{i \in [n]} p_{ni} \underset{n \to \infty}{\to} 0 \qquad \text{and} \qquad \frac{p_{ni}}{\sigma(\mathbf{p}_n)} \underset{n \to \infty}{\to} \theta_i , \qquad i \geq 1. \tag{3}$$

By Fatou's Lemma, such a limiting $\boldsymbol{\theta}$ must satisfy $\sum_i \theta_i^2 \leq 1$, but $\sum_i \theta_i$ may be finite or infinite. We let $\theta_0 = \sqrt{1 - \sum_i \theta_i^2}$. A vertex $i \geq 1$ with $\theta_i > 0$ then corresponds to a "hub" [4] or "attracting center" [9] for the mapping, because significantly many more integers are likely to be mapped to it as n gets large than to those for which $\theta_i = 0$. Our main result (Theorem 1) roughly states that for $\mathbf{p}_n$ satisfying (3) with $\boldsymbol{\theta} = (\theta_1, \dots, \theta_I, 0, 0, \dots)$ with $\theta_I > 0$ and $\theta_0 > 0$ (the subset of such $\boldsymbol{\theta}$'s is called $\boldsymbol{\Theta}_{\text{finite}}$), we have weak convergence

$$\sigma(\mathbf{p}_n) H^{M_n} \overset{(d)}{\to} Z^{\boldsymbol{\theta}} \tag{4}$$

for a certain continuous process $Z^{\boldsymbol{\theta}}$ to be described in section 2.3, where the topology is in general slightly weaker than the usual Skorokhod topology. We will also provide criteria under which the stronger convergence holds. In turn, we will see how this convergence and related results give information on the size of the basins of M_n, and on the number of cyclic points, which in the limit arise as a kind of local time at 0 for $Z^{\boldsymbol{\theta}}$.

To implement our methodology, the key point is that under (3), the **p**-trees are known to converge in a certain sense (Proposition 1) to an *Inhomogeneous Continuum Random Tree* (ICRT) which we denote by $\mathcal{T}^{\boldsymbol{\theta}}$. This family of trees was first investigated in [4] in the context of the additive coalescent. What is important for this paper is the recent result [2] that a certain class of ICRT's are encoded into random excursion functions $H^{\boldsymbol{\theta}}$, just as the Brownian tree is encoded into twice the normalized Brownian excursion. The definition of $H^{\boldsymbol{\theta}}$ is recalled in section 2.3, where we also give the definition of the process $Z^{\boldsymbol{\theta}}$ as a functional of $H^{\boldsymbol{\theta}}$.

So the contribution of this paper is to show how the ideas from [1] (in particular, the Joyal functional featuring in our Lemma 1) may be combined with the result of [2] to prove the limit result indicated at (4). Once these ingredients are assembled, only a modest amount of new technicalities (e.g part (ii) of Theorem 2 and its use in the proof of Theorem 1) will be required. One reason why "only modest" is our restriction to the case $\boldsymbol{\Theta}_{\text{finite}}$. In [2] it is shown that the construction of $H^{\boldsymbol{\theta}}$ and associated limit results for **p**-trees work in the more general setting where $\sum_i \theta_i < \infty$. It seems very likely that our new result (Theorem 1) also extends to this setting, but the technicalities become more complicated.

While the existence of a limit process $Z^{\boldsymbol{\theta}}$ provides qualitative information about aspects of the **p**-mappings, enabling one to show that various limit distributions exist and equal distributions of certain functionals of $Z^{\boldsymbol{\theta}}$, obtaining explicit formulas for such distributions remains a challenging open problem.

2 Statement of results

2.1 Mappings, trees, walks

We first introduce some notation which is mostly taken from [1]. If m is a mapping on some finite set S, let $\mathcal{D}(m)$ be the directed graph with vertex set S, whose edges are $s \to m(s)$, and let $\mathcal{C}(m)$ be the set of cyclic points, which is further partitioned into disjoint cycles, s and s' belonging to the same cycle if one is mapped to the other by some iterate of m. For $c \in \mathcal{C}(m)$, if we remove the edges $c \to m(c)$ and $c' \to c$ where c' is the unique point of $S \cap \mathcal{C}(m)$ that is mapped to c, the component of $\mathcal{D}(m)$ containing c is a tree $\mathcal{T}_c(m)$ which we root at c. Label the disjoint cycles of m as $\mathcal{C}_1(m), \mathcal{C}_2(m), \ldots$ with some ordering convention, then this in turn induces an order on the *basins* of m:

$$\mathcal{B}_j(m) := \bigcup_{c \in \mathcal{C}_j(m)} \mathcal{T}_c(m).$$

q-biased order. The ordering we will consider in this paper uses a convenient extra randomization, yet we mention that results similar to this paper's could be established for different choices of basins ordering using similar methods. See e.g. [6], where two different choices of ordering lead to two intricate decompositions of Brownian bridge. Given $\mathbf{q}$, a probability distribution on S with $q_s > 0$ for every $s \in S$, consider an i.i.d. $\mathbf{q}$-sample $(X_2, X_3, \ldots)$ indexed by $\{2, 3, \ldots\}$. If m is a random mapping, we choose the $\mathbf{q}$-sample independently of m. Since $q_s > 0$ for every $s \in S$, the following procedure a.s. terminates:

- Let $\tau_1 = 2$ and let $\mathcal{B}_1(m)$ be the basin of m containing X_2.
- If $\cup_{1\le i\le j}\mathcal{B}_i(m) = S$ then end the procedure; else, given τ_j let $\tau_{j+1} = \inf\{k : X_k \notin \cup_{1\le i\le j}\mathcal{B}_i(m)\}$ and let $\mathcal{B}_{j+1}$ be the basin containing $X_{\tau_{j+1}}$.

This induces an order on basins of m, and then on the corresponding cycles. We add a further order on the cyclic points themselves by letting c_j be the cyclic point of $\mathcal{C}_j(m)$ such that $X_{\tau_j} \in \mathcal{T}_{c_j}$, and by ordering the cyclic points within $\mathcal{C}_j(m)$ as follows:

$$m(c_j) \prec m^2(c_j) \prec \ldots \prec m^{|\mathcal{C}_j(m)|-1}(c_j) \prec c_j.$$

This extends to a linear order on $\mathcal{C}(m)$ by further letting $c_{j-1} \prec m(c_j)$. We call this (random) order on cyclic points and basins the $\mathbf{q}$-*biased random order*. In the special case where $\mathbf{q}$ is the uniform distribution on S, we call it the size-biased order.

Coding trees and mappings with marked walks Let $\mathbf{T}_n^o$ be the set of plane (ordered) rooted trees with n labeled vertices $1, 2, \ldots, n$, so that the children of any vertex v are distinguished as first, second, ... The cardinality of $\mathbf{T}_n^o$ is therefore $n!C_n$ where C_n is the n-th Catalan number. For any $T \in \mathbf{T}_n^o$, we may put its set of vertices in a special linear order $v_1, v_2, \ldots, v_n$ called *depth-first order*: we let $v_1 =$ root, and then v_{j+1} is the first (oldest) child of v_j not in $\{v_1, \ldots, v_j\}$ if any, or the oldest brother of v_j not in $\{v_1, \ldots, v_j\}$ if any, or the oldest brother of the parent of v_j not in $\{v_1, \ldots, v_j\}$, and so on. Write $\mathrm{ht}^T(v)$ for the height of vertex v. For any weight sequence $\mathbf{w} = (w_1, \ldots, w_n)$ with $w_i > 0$ for every i, let

$$H_{\mathbf{w}}^T(s) = \mathrm{ht}^T(v_i) \qquad \text{if } \sum_{j=1}^{i-1} w_{v_j} \le s < \sum_{j=1}^{i} w_{v_j}, \tag{5}$$

and let $H_{\mathbf{w}}^T(\sum_i w_i) = \mathrm{ht}^T(v_n)$. Call $H_{\mathbf{w}}^T$ the height process of T. Notice that any $s \in [0, \sum_i w_i)$ specifies a vertex of T, which is v_i in the case appearing in (5). We say that v_i is *visited at time* s by $H_{\mathbf{w}}^T$. Intuitively, picture a particle touring the vertices in depth-first order during the unit time interval, spending time w_i at vertex i.

Given a mapping m with basins and cyclic points $c_1, \ldots, c_K$ in $\mathbf{q}$-biased order for some $\mathbf{q}$, we may associate to each $\mathcal{T}_{c_i}$ a walk as follows. First, turn these unordered trees into plane trees by putting each set of children of each vertex in random exchangeable order, independently over vertices given $\mathcal{T}_{c_i}$. Then associate to this ordered tree the height process $H_{\mathbf{w}}^{\mathcal{T}_{c_i}}$, with a slight abuse of notation, where we are again given a weight function $\mathbf{w}$ on $[n]$ (though we use only the relevant labels appearing in $\mathcal{T}_{c_i}$). We can now define the walk associated with m to be

$$H_{\mathbf{w}}^m(s) = H_{\mathbf{w}}^{\mathcal{T}_{c_i}}\left(s - \sum_{1\le j<i} w(\mathcal{T}_{c_j})\right) \qquad \text{if } \sum_{j=1}^{i-1} w(\mathcal{T}_{c_j}) \le s < \sum_{j=1}^{i} w(\mathcal{T}_{c_i}),$$

and $H_{\mathbf{w}}^m(\sum_i w_i) = H_{\mathbf{w}}^m(\sum_i w_i -)$, where $w(A) = \sum_{i\in A} w_i$. That is, we concatenate the tree-walks associated to $\mathcal{T}_{c_1}, \ldots, \mathcal{T}_{c_K}$ in this order. Again, there is a natural notion of vertex visited at time $s < \sum_i w_i$.

Further, let $D^m_{\mathbf{w}}(i) = \sum_{j=1}^i w(\mathcal{B}_j(m))$ be the weight of the i first basins, so that $D^m_{\mathbf{w}}(i)$ is the time when the mapping-walk $H^m_{\mathbf{w}}$ has completely visited the vertices of the i-th basin, so $w(\mathcal{B}_i(m)) = D^m_{\mathbf{w}}(i) - D^m_{\mathbf{w}}(i-1)$ for $i \geq 1$ with the convention $D^m_{\mathbf{w}}(0) = 0$. We also let $\ell^m_{\mathbf{w}}(s)$ be the number of cyclic points that have been visited before time s, namely

$$\ell^m_{\mathbf{w}}(s) = \sum_{j=1}^{i} \mathbf{1}_{\{H^m_{\mathbf{w}}(w(\{v_1,\dots,v_j\}))=0\}} \qquad \text{whenever } \sum_{j=1}^{i-1} w_{v_j} \leq s < \sum_{j=1}^{i} w_{v_j},$$

with $\ell^m_{\mathbf{w}}(\sum_i w_i) = \ell^m_{\mathbf{w}}(\sum_i w_i -)$.

2.2 The Joyal functional

We now define a functional $\mathbf{J}^u$ on the Skorokhod space $\mathbb{D}[0,1]$, which translates into the world of encoding paths the Joyal bijection (recalled below) between trees and mappings. Let $u \in [0,1]$. Define the *pre-post infimum* of $f \in \mathbb{D}[0,1]$ before and after u to be the function

$$s \to \underline{f}_s(u) = \begin{cases} \inf_{t\in[s,u]} f_t & \text{for } s < u \\ \inf_{t\in[u,s]} f_t & \text{for } s \geq u. \end{cases}$$

The function $\underline{f}(u)$ is non-decreasing on $[0,u]$ and non-increasing on $[u,1]$. If $[a,b]$ is a maximal flat interval for $\underline{f}(u)$, we call the recentered function $((f - \underline{f}(u))(s+a), 0 \leq s \leq b-a)$ an excursion of f above $\underline{f}(u)$. Such a function may not be an excursion in the usual sense because it might be zero for some $s \in (0, b-a)$. Further, if two distinct such intervals $[a,b]$ and $[c,d]$ satisfy $f(b) = f(c)$, then it must be that $b < u < c$, and in this case we call the function obtained by concatenating the excursion of f above $\underline{f}(u)$ on $[a,b]$ and $[c,d]$ a (generalized) excursion of f above $\underline{f}(u)$. Label as $\varepsilon_1, \varepsilon_2, \dots$ the generalized excursions of f above $\underline{f}(u)$, according to decreasing durations $l_1, l_2, \dots$. Write also h_i for the "height" of the excursion ε_i, i.e. the value taken by $\underline{f}(u)$ on the flat interval of the excursion. We define a function $\mathbf{J}^u(f)$ that arranges these excursions in order of heights:

$$\mathbf{J}^u(f)(s) = \varepsilon_i\left(s - \sum_{j:h_j<h_i} l_j\right) \qquad \text{if } \sum_{j:h_j<h_i} l_j \leq s < \sum_{j:h_j\leq h_i} l_j, \tag{6}$$

with the convention that $\mathbf{J}^u(f)(s) = 0$ on $[\sum_i l_i, 1]$.

To keep track of the structure of the original function, we finally add marks at the points $g^u_i(f) = \sum_{j:h_j<h_i} l_j$ and $d^u_i(f) = \sum_{j:h_j\leq h_i} l_j, i \geq 1$. In particular, if $\mathbf{J}^u(f)$ if non-zero on $(g^u_i(f), d^u_i(f))$, then the ε_i is an "usual" excursion rather than "generalized" excursion.

2.3 The limiting process and main result

Let us recall the construction [2] of the exploration process of the ICRT $\mathcal{T}^{\boldsymbol{\theta}}$ for $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\text{finite}}$. Let $(b_s, 0 \leq s \leq 1)$ be a standard Brownian bridge, $U_1, \dots, U_I$ be independent uniform random variables independent of b, and

$$X^{\mathrm{br},\boldsymbol{\theta}}(s) = \theta_0 b_s + \sum_{i=1}^{I} \theta_i(\mathbf{1}_{\{U_i\leq s\}} - s), \qquad 0 \leq s \leq 1.$$

Such a process has a.s. a unique time where it attains its overall minimum, and this time is a continuity time, call it $s_{\min}$. Define the *Vervaat transform* of $X^{\mathrm{br},\boldsymbol{\theta}}$ by

$$X^{\boldsymbol{\theta}}(s) = X^{\mathrm{br},\boldsymbol{\theta}}(s + s_{\min}[\mathrm{mod}\ 1]) - X^{\mathrm{br},\boldsymbol{\theta}}(s_{\min}), \qquad 0 \le s \le 1,$$

and let $t_i = U_i - s_{\min}[\mathrm{mod}\ 1], 1 \le i \le I$ be the jump times of $X^{\boldsymbol{\theta}}$. Let $T_i = \inf\{s \ge t_i : X^{\boldsymbol{\theta}}_s = X^{\boldsymbol{\theta}}_{t_i-}\}$ and write

$$R^{\boldsymbol{\theta}}_i(s) = \begin{cases} \inf_{t_i \le u \le s} X^{\boldsymbol{\theta}}_u - X^{\boldsymbol{\theta}}_{t_i-} & \text{if } s \in [t_i, T_i] \\ 0 & \text{else.} \end{cases}$$

Last, let $Y^{\boldsymbol{\theta}} = X^{\boldsymbol{\theta}} - \sum_{i=1}^I R^{\boldsymbol{\theta}}_i$. This process $Y^{\boldsymbol{\theta}}$ is continuous, and it is intuitively described by: "take away all the jumps of $X^{\boldsymbol{\theta}}$ and reflect the process above its infimum after these jumps until $X^{\boldsymbol{\theta}}$ gets back to the level it started at before jumping". The exploration process of $\mathcal{T}^{\boldsymbol{\theta}}$ is then defined as $H^{\boldsymbol{\theta}} = \frac{2}{\theta_0^2} Y^{\boldsymbol{\theta}}$.

The open set $\{s \in (t_i, T_i) : H^{\boldsymbol{\theta}}(s) > H^{\boldsymbol{\theta}}(t_i)\}$ associated with jump i can be decomposed into disjoint open intervals $(t_{ij}, T_{ij}), j \ge 1$, ranked by decreasing order of lengths.

Now take a uniform$(0,1)$ random variable U independent of $Y^{\boldsymbol{\theta}}$, and consider the process $Z^{\boldsymbol{\theta}} = \mathbf{J}^U(H^{\boldsymbol{\theta}})$. Recall that this process has marks $g^U_i(H^{\boldsymbol{\theta}}), d^U_i(H^{\boldsymbol{\theta}})$, which we more simply call g_i, d_i. The following facts are simple consequences of usual properties of Brownian motion and Brownian bridge:

- The sum of durations of generalized excursions of $H^{\boldsymbol{\theta}}$ above $\underline{H}^{\boldsymbol{\theta}}(U)$ is 1, meaning $\sum_{i \ge 1}(d_i - g_i) = 1$.
- The corresponding excursion heights $h_i, i \ge 1$ are a.s. everywhere dense in $[0, H^{\boldsymbol{\theta}}_U]$.

Now let $V_1, V_2, \ldots$ be independent uniform$(0,1)$ variables, independent of U and $H^{\boldsymbol{\theta}}$. Define recursively a sequence $D_0 = 0 < D_1 < D_2 < \ldots < 1$ by

$$D_n = \inf\{s : s > D_{n-1} + V_n(1 - D_{n-1}) \text{ and } \exists i \ge 1, s = d_i\} \qquad n \ge 1.$$

Last, we define the local time function of $Z^{\boldsymbol{\theta}}$ as follows: for s in an excursion interval of $Z^{\boldsymbol{\theta}}$ above 0, let $L^{\boldsymbol{\theta}}_s$ be the "height" of the corresponding generalized excursion of $H^{\boldsymbol{\theta}}$ above $\underline{H}^{\boldsymbol{\theta}}$. This defines $L^{\boldsymbol{\theta}}$ on a dense subset of $[0,1]$ as an increasing function, which can be extended to the whole interval $[0,1]$ uniquely as a continuous function, because $H^{\boldsymbol{\theta}}$ is itself continuous, and the excursion heights are dense in $[0, H^{\boldsymbol{\theta}}_U]$. Notice that this "local" time has the unusual property that its increase times do not exactly match with the zero set of $Z^{\boldsymbol{\theta}}$; rather, the set of increase times is the closure of $\{g_i, d_i, i \ge 1\}$.

Now let $\mathbf{p}_n, \mathbf{q}_n, \mathbf{w}_n$ be three sequences of probabilities on $[n]$ charging every point. Consider a $\mathbf{p}_n$-mapping M_n with basins in $\mathbf{q}_n$-biased order, and let $H^{M_n}_{\mathbf{w}_n}$ be the associated walk. We let $H^{M_n} := H^{M_n}_{\mathbf{p}_n}$. Our main result is

Theorem 1 *Suppose* $\max_i q_{ni} \to 0$ *as* $n \to \infty$.

(i) *Under the asymptotic regime (3) for* $\mathbf{p}_n$*, with limiting* $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\text{finite}}$*, and if (8,9) below are satisfied, then for any weight function* $\mathbf{w}$ *satisfying* $\max_i w_{ni} \to 0$*, we have the convergence in law in the usual Skorokhod topology on* $\mathbb{D}[0,1]$

$$\sigma(\mathbf{p}_n) H^{M_n}_{\mathbf{w}_n} \overset{(d)}{\to} Z^{\boldsymbol{\theta}}.$$

(ii) *Moreover, jointly with the above convergence, the marks* $D^{M_n}_{\mathbf{w}_n}(1), D^{M_n}_{\mathbf{w}}(2), \ldots$ *converge in law to* $D_1, D_2, \ldots$.

(iii) *Jointly with the above convergences,* $\sigma(\mathbf{p}_n)\ell^{M_n}_{\mathbf{w}_n} \overset{(d)}{\to} L^{\boldsymbol{\theta}}$ *for the uniform topology.*

(iv) *In general, under the asymptotic regime (3) for* $\mathbf{p}_n$*, with limiting* $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\text{finite}}$*, one has convergence in law for the* $*$*-topology defined in [5]*

$$\sigma(\mathbf{p}_n)H^{M_n} \overset{(d)}{\to} Z^{\boldsymbol{\theta}},$$

and the convergences of (ii),(iii) *hold jointly for* $\mathbf{w}_n = \mathbf{p}_n$.

We echo [1, Corollary 1] by stating

Corollary 1 *Under (3,8,9) with finite-length limiting* $\boldsymbol{\theta}$*, and for any weight function* $\mathbf{w}_n, \mathbf{q}_n$ *with* $\max_i \max(w_{ni}, q_{ni}) \to 0$ *as* $n \to \infty$*, we have*

$$(\mathbf{w}_n(\mathcal{B}_j(M_n)), \sigma(\mathbf{p}_n)\text{Card}\, \mathcal{C}_j(M_n), j \geq 1) \overset{(d)}{\to} (D_j - D_{j-1}, L^{\boldsymbol{\theta}}_{D_j} - L^{\boldsymbol{\theta}}_{D_{j-1}}, j \geq 1).$$

Notice that for uniform $\mathbf{w}_n$, the first component equals $n^{-1}\text{Card}\,(\mathcal{B}_j(M_n))$.

The essential point of the $*$-topology is the following property [1]. One has $f^n \to f$ for the $*$-topology, where $f_n \in \mathbb{D}[0,1]$ and $f \in C[0,1]$ is continuous, if and only of there exist $g_n, h_n \in \mathbb{D}[0,1]$ with $f_n = g_n + h_n$ such that $g_n \to f$ uniformly, $h_n \geq 0$ and $\text{Leb}\{x : h_n(x) > 0\} \to 0$. Thus the $*$-convergence asserted in (iv) is compatible with the possible presence of upward "spikes" on the mapping-walk, which have arbitrary large height but vanishing weight. In particular, Theorem 1 (iv) allows us to deduce the asymptotic "height" (distance to the set $\mathcal{C}(M_n)$) of a randomly $\mathbf{p}_n$-chosen vertex, but not the behavior of the asymptotic maximum height over all vertices, which is however handled under the hypotheses in (i). Under the same hypotheses, we can handle quantities such as the diameter of the random mapping (the maximal k such that there exists v with $v, m(v), \ldots, m^{k-1}(v)$ pairwise distinct).

Although this result leaves a large degree of freedom for choosing the order of basins, we stress that other orderings are possible, such as ordering the basins according to increasing order of the least vertices they contain, or ordering cycles by order of least vertex they contain. The first order is in fact equivalent to the size-biased order described above, up to relabeling, and the second order could be also handled by our methods, although the marks D_i would have to be defined in a different way, see [6].

Last, we stress that the hypotheses (8,9) below are by no means necessary, we believe that they are in fact quite crude (see [2] for further discussion). Also, as discussed below, we believe that Theorem 1 (iv) remains true for much more general $\mathbf{w}_n$.

3 Proofs

3.1 p-trees and associated walks

p-trees and their walks. We now define the random trees whose asymptotics are related to the process $H^{\boldsymbol{\theta}}$, namely $\mathbf{p}$-trees. Let $\mathbf{T}_n$ be the set constituted of the n^{n-1}

(unordered) rooted labeled trees on $[n]$. For $\mathbf{p}$ a probability measure charging every point of $[n]$, let $\mathcal{T}^{\mathbf{p}}$ be the random variable in $\mathbf{T}_n$ with law

$$P(\mathcal{T}^{\mathbf{p}} = t) = \prod_{i\in[n]} p_i^{c_i(t)}, \qquad t \in \mathbf{T}_n, \tag{7}$$

where $c_i(t)$ is the number of children of i in t. The fact that (7) indeed defines a *probability* measure amounts to the Cayley multinomial expansion for trees [10]. For $t \in \mathbf{T}_n$, we can associate a random $\mathbf{T}_n^o$-valued tree t^o by putting each set of children of a given vertex in uniform random order, independently over distinct vertices, so given a weight function $\mathbf{w}$ on $[n]$ we may associate to t the random walk $H^t := H^{t^o}$ as defined in section 2.1. We will now apply this to the random trees $\mathcal{T}^{\mathbf{p}}$ and their associated height processes $H^{\mathbf{p}}_{\mathbf{w}} := H^{\mathcal{T}^{\mathbf{p}}}_{\mathbf{w}}$. When $\mathbf{w} = \mathbf{p}$ we let $H^{\mathbf{p}} := H^{\mathbf{p}}_{\mathbf{p}}$.

Asymptotics. We introduce two extra hypothesis on the sequence $\mathbf{p}_n$ besides (3). The first one prevents exponentially small (in the scale $\sigma(\mathbf{p})$) $\mathbf{p}$-values from appearing:

$$(\min_i p_{ni})^{-1} = o(\exp(\alpha/\sigma(\mathbf{p}_n))), \qquad \forall\, \alpha > 0. \tag{8}$$

The second states that "small" $\mathbf{p}$-values are of rough order $\sigma(\mathbf{p})^2$. Suppose there exists some non-negative finite r.v. Q such that, letting $\bar{\mathbf{p}} = (0,\ldots,0,p_{I+1},p_{I+2},\ldots)$,

$$\lim_{n\to\infty} E\left[\exp\left(\frac{\lambda \bar{p}_{n\xi}}{\sigma(\mathbf{p}_n)^2}\right)\right] = E[\exp(\lambda Q)] < \infty \tag{9}$$

for every λ in a neighborhood of 0. Here, ξ denotes a random variable with law $\mathbf{p}$, so $\bar{p}_{n\xi}$ is its $\bar{\mathbf{p}}_n$-value.

The key results on $\mathbf{p}$-trees are the following variations of [2, Theorems 1,3]. For $k \geq 1$, let $X_2,\ldots,X_k$ be independent $\mathbf{p}$-sampled vertices of $\mathcal{T}^{\mathbf{p}}$, independent of $\mathcal{T}^{\mathbf{p}}$. Let $r_k(\mathcal{T}^{\mathbf{p}})$ be the subtree of $\mathcal{T}^{\mathbf{p}}$ spanned by the root and $X_2,\ldots,X_k$, re-interpreted as a *tree with edge-lengths*, in the sense that two vertices separated by a single edge are at distance 1, and we delete all the nodes that have degree 2, so the distance between two vertices on the final tree is equal to the number of deleted nodes plus 1. The tree $r_k(\mathcal{T}^{\mathbf{p}})$ is thus a discrete rooted tree with at most k leaves, which has no degree 2 vertices, and with lengths attached to each of its edges. The notion of convergence on the space of trees with edge-length is the usual convergence for the product topology, so $\mathbf{t}_n \to \mathbf{t}$ if both trees have the same discrete structures for all sufficiently large n, and the vector of edge-lengths of $\mathbf{t}_n$ converges to that of $\mathbf{t}$. Last, for $a > 0$ we let $a \otimes \mathbf{t}$ be the tree with edge-length with same discrete structure as $\mathbf{t}$, and where all distances have been multiplied by a.

Proposition 1 ([7]) *Suppose that $\mathbf{p}_n$ satisfies (3). For every k, the tree $\sigma(\mathbf{p}_n)\otimes r_k(\mathcal{T}^{\mathbf{p}_n})$ converges in distribution to $\mathcal{T}^{\boldsymbol{\theta}}_k$, the k-th marginal of the ICRT described in section 4.*

Theorem 2 (i) *Suppose that $\mathbf{p}_n$ satisfies (3,8,9), with $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\text{finite}}$, and that $\mathbf{w}_n$ satisfies $\max_i w_{ni} \to 0$. Then*

$$\sigma(\mathbf{p}_n) H^{\mathbf{p}_n}_{\mathbf{w}_n} \xrightarrow{(d)} H^{\boldsymbol{\theta}}$$

for the usual Skorokhod topology (and hence for the uniform topology since the limit is continuous).

(ii) *Under the assumptions of (i), for each $1 \leq i \leq I$, there exist random sequences $t_i^{\mathbf{p}_n}, T_i^{\mathbf{p}_n}$ and $t_{ij}^{\mathbf{p}_n}, T_{ij}^{\mathbf{p}_n}, j \geq 1$ with*

$$H_{\mathbf{w}_n}^{\mathbf{p}_n}(t_i^{\mathbf{p}_n}) = H_{\mathbf{w}_n}^{\mathbf{p}_n}(t_{ij}^{\mathbf{p}_n}) = H_{\mathbf{w}_n}^{\mathbf{p}_n}(T_{ij}^{\mathbf{p}_n}) \text{ for every } j \geq 1,$$

and $H_{\mathbf{w}_n}^{\mathbf{p}_n}(s) \geq H_{\mathbf{w}_n}^{\mathbf{p}_n}(t_i^{\mathbf{p}_n})$ for $s \in [t_i^{\mathbf{p}_n}, T_i^{\mathbf{p}_n}]$, such that jointly with the convergence of (i), *one has convergence in law*

$$(t_i^{\mathbf{p}_n}, T_i^{\mathbf{p}_n}, t_{ij}^{\mathbf{p}_n}, T_{ij}^{\mathbf{p}_n}, 1 \leq i \leq I, j \geq 1) \xrightarrow{(d)} (t_i, T_i, t_{ij}, T_{ij}, 1 \leq i \leq I, j \geq 1),$$

with the notations of section 2.3.

(iii) *Suppose only that $\mathbf{p}_n$ satisfies (3), then*

$$\sigma(\mathbf{p}_n) H^{\mathbf{p}_n} \xrightarrow{(d)} H^{\boldsymbol{\theta}}$$

in the $*$-topology. Moreover, the statement of (ii) *still holds for $\mathbf{w}_n = \mathbf{p}_n$.*

Proof. Except for the last sentence, point (iii) is a consequence of [5, Proposition 7] which states that the convergence of marginals of $\mathbf{p}$-trees to that of the limiting ICRT [2, Proposition 1 and (23)] is equivalent to the $*$-convergence of the rescaled walk $\sigma(\mathbf{p}_n)H^{\mathbf{p}_n}$ with weights $\mathbf{w}_n = \mathbf{p}_n$ to $H^{\boldsymbol{\theta}}$.

Point (i) was proved in [2, Theorem 3, Corollary 3] in the two special cases where $\mathbf{w}_n = \mathbf{p}_n$ and where $\mathbf{w}_n = (1/n, \ldots, 1/n)$ (n times). The general case uses the same proof as Corollary 3 in the stated paper. By the weak law of large numbers for sampling without replacement applied to $\mathbf{w}_n$, we have $\sup_{0 \leq t \leq 1} |S_{\mathbf{w}_n,0}(t) - t| \to 0$ in probability, where $S_{\mathbf{w}_n,0}$ is the linear interpolation between the points $((\sum_{1 \leq k \leq i} w_{n\pi(k)}, i/n), 1 \leq i \leq n)$, and where π is a uniformly distributed random permutation on $[n]$. This implies the result because, as shown in [2], the depth-first order on vertices $v_1, v_2, \ldots, v_n$ of a $\mathbf{p}_n$-tree is a (random) shift of a uniform permutation of $[n]$. Therefore, the linear interpolation $S_{\mathbf{w}_n}$ between points $((\sum_{1 \leq k \leq i} w_{nv_k}, i/n), 1 \leq i \leq n)$ also uniformly converges to the identity, and the conclusion follows from the fact that $H_{\mathbf{w}_n}^{\mathbf{p}_n} = H^{\mathbf{p}_n} \circ (S_{\mathbf{p}_n})^{-1} \circ S_{\mathbf{w}_n}$.

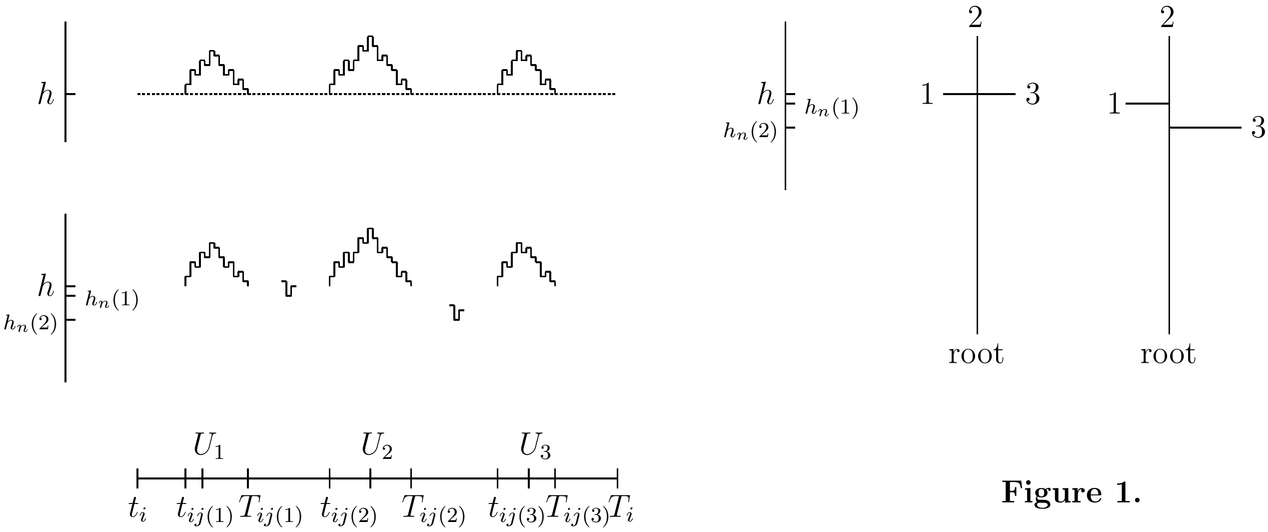

Figure 1.

Point (ii) refines one aspect of (i). First consider the case $\mathbf{w}_n = \mathbf{p}_n$. By Skorokhod's representation theorem, suppose that the convergence in law of (i) holds almost-surely. Fix i. Figure 1 shows schematically (top left) three of the excursions of $H^{\boldsymbol{\theta}}$ associated with jump i. All have the same height, h say. The lower left diagram in Figure 1 shows corresponding parts of $H^{\mathbf{p}_n}$. Consider the minimum value $h_n(1)$ of $H^{\mathbf{p}_n}$ between $T_{ij(1)}$ and $t_{ij(2)}$, and the minimum value $h_n(2)$ of $H^{\mathbf{p}_n}$ between $T_{ij(2)}$ and $t_{ij(3)}$. The key claim is

$$h_n(1) = h_n(2) \text{ for all large } n. \tag{10}$$

To verify (10), take three independent uniform random variables U_1, U_2, U_3 on $[0,1]$ independent of $H^{\boldsymbol{\theta}}, H^{\mathbf{p}_n}, n \geq 1$. These random variables specify three $\mathbf{p}_n$-chosen vertices on $\mathcal{T}^{\mathbf{p}_n}$, namely those which are visited by $H^{\mathbf{p}_n}$ at these times. On an event of positive probability we have $U_k \in (t_{ij(k)}, T_{ij(k)}),\ k = 1,2,3$. Consider the subtree of $\mathcal{T}^{\mathbf{p}_n}$ spanned by the root and the three vertices encoded by U_1, U_2, U_3. If $h_n(1) \neq h_n(2)$ then, on the above event, the subtree has an edge of length $|h_n(2) - h_n(1)|$ (as shown in rightmost tree in Figure 1), but this is not converging to the correct limit asserted in Proposition 1 (in the sense of convergence of discrete structures mentioned above Proposition 1) because the limit tree (the second-right tree in Figure 1) has different tree shape. Thus we can deduce (10) using Proposition 1. It is then straightforward to deduce the full assertion of (ii) from the case (10) of three excursions.

Treating the case of general weights $\mathbf{w}_n$ is done by asking (again by the Skorokhod representation theorem) that the uniform convergence of $S_{\mathbf{w}_n}^{-1} \circ S_{\mathbf{p}_n}$ to identity is also almost-sure. Then replace $U_k, k = 1,2,3$ by $U_k^{\mathbf{w}_n} = S_{\mathbf{w}_n}^{-1} \circ S_{\mathbf{p}_n}(U_k), k = 1,2,3$, so the new variables encode again $\mathbf{p}_n$-chosen vertices. The case of $*$-convergence (for $\mathbf{w}_n = \mathbf{p}_n$) is similar (see also the proof of [1, Lemma 2]). □

Remark. To prove (iii) for more general weights $\mathbf{w}_n$, we could try to use the same method as above (first treating the case of uniform weights). But if $f_n \to f$ for the $*$-topology with f continuous, and if S_n is a strictly increasing piecewise linear continuous function that converges uniformly to the identity on $[0,1]$, then $f_n \circ S_n$ need not converge to f for the $*$-topology. Indeed, with the above notation, this convergence is equivalent to $\mathrm{Leb}\{x \in [0,1] : h_n \circ S_n(x) > 0\} \to 0$. But this last quantity is $\int_0^1 \mathbf{1}_{\{h_n > 0\}} (S_n^{-1})'(x) \mathrm{d}x$. So we would need a sharper result than the weak law of large numbers for sampling without replacemement to estimate the values of the derivative at points where $h_n > 0$. However, it was proved in [5, Theorem 25] by different methods that in the asymptotically negligible regime (2), Theorem 1 (i) is still valid for general weights $\mathbf{w}_n$ satisfying $\max_i w_{ni} \to 0$. It would therefore be surprising if the same result did not hold here.

3.2 The Joyal correspondence.

Let us now describe the Joyal correspondence between trees and mappings, designed to push the distribution of $\mathbf{p}$-trees onto the distribution of $\mathbf{p}$-mappings. Let $\mathbf{q}$ be a probability distribution charging every point. Let X_0 be the root of the $\mathbf{p}$-tree $\mathcal{T}^{\mathbf{p}}$ and X_1 be random with law $\mathbf{p}$ independent of $\mathcal{T}^{\mathbf{p}}$. We consider X_1 as a second root, and call the path $X_0 = c_1, c_2, \ldots, c_K = X_1$ from X_0 to X_1 the *spine*. Deleting the edges $\{c_1, c_2\}, \{c_2, c_3\}, \ldots$ splits $\mathcal{T}^{\mathbf{p}}$ into subtrees rooted at $c_1, c_2, \ldots, c_K$, which we call $\mathcal{T}_{c_1}, \ldots, \mathcal{T}_{c_K}$. Orient the edges of these trees by making them point towards the root. Now let $X_2, X_3, \ldots$ be an i.i.d. $\mathbf{q}$-sample independent of $\mathcal{T}^{\mathbf{p}}$. Consider the following procedure.

- Let $\tau_1 = 2$ and k_1 be such that $\mathcal{T}_{c_{k_1}}$ contains X_2. Bind the trees $\mathcal{T}_{c_1}, \ldots, \mathcal{T}_{c_{k_1}}$ by adding oriented edges $c_1 \to c_2 \to \ldots \to c_{k_1} \to c_1$. Let $\mathcal{C}_1 = \{c_1, \ldots, c_{k_1}\}$ and $\mathcal{B}_1 = \cup_{1 \leq i \leq k_1} \mathcal{T}_{c_i}$.

- Given $\tau_i, k_i, \mathcal{C}_i, \mathcal{B}_i, 1 \leq i \leq j$ as long as $\cup_{1 \leq i \leq j} \mathcal{B}_i \neq [n]$, let $\tau_{j+1} = \inf\{k : X_k \notin \cup_{1 \leq i \leq j} \mathcal{B}_i\}$ and k_{j+1} be such that $\mathcal{T}_{c_{k_{j+1}}}$ contains $X_{\tau_{j+1}}$. Then add edges $c_{k_j+1} \to c_{k_j+2} \to \ldots \to c_{k_{j+1}} \to c_{k_j+1}$, let $\mathcal{C}_{j+1} = \{c_{k_j+1}, \ldots, c_{k_{j+1}}\}$, $\mathcal{B}_{j+1} = \cup_{k_j+1 \leq i \leq k_{j+1}} \mathcal{T}_{c_i}$.

When it terminates, say at stage r, the procedure yields a digraph with r connected components $\mathcal{B}_1, \ldots, \mathcal{B}_r$, and each component contains exactly one cycle of the form $c_{k_j+1} \to \ldots \to c_{k_{j+1}} \to c_{k_j+1}$. Let $J(\mathcal{T}^{\mathbf{p}}, X_i, i \geq 1)$ be the mapping whose digraph equals the one given by the procedure. Then, as an easy variation of [1, Proposition 1],

Proposition 2 *The random mapping $J(\mathcal{T}^{\mathbf{p}}, X_i, i \geq 1)$ is a $\mathbf{p}$-mapping, and the order on its basins $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r$ induced by the algorithm is $\mathbf{q}$-biased order.*

3.3 Consequences for associated walks

From now on, let $\mathcal{T}^{\mathbf{p}}$ be a $\mathbf{p}$-tree, and $H^{\mathbf{p}}_{\mathbf{w}}$ be its associated height process. Let $v_1, v_2, \ldots, v_n$ be the vertices of $\mathcal{T}^{\mathbf{p}}$ in depth-first order, and let $S_{\mathbf{w}}$ be the linear interpolation between points $((\sum_{1 \leq j \leq i} w_j, i/n), 0 \leq i \leq n)$. Given a random variable U uniform on $[0,1]$ and independent of $H^{\mathbf{p}}$, let $X_1 = X_1(U)$ be the vertex that is visited by the walk at time $U^{\mathbf{w}} = S^{-1}_{\mathbf{w}} \circ S_{\mathbf{p}}(U)$, so this vertex is a $\mathbf{p}$-distributed random variable independent of $\mathcal{T}^{\mathbf{p}}$. We also let $X_2, X_3, \ldots$ be an independent $\mathbf{q}$-sample, independent of $\mathcal{T}^{\mathbf{p}}, U$. Let $M = J(\mathcal{T}^{\mathbf{p}}, X_i, i \geq 1)$ be the $\mathbf{p}$-mapping associated to $\mathcal{T}^{\mathbf{p}}$ by the Joyal correspondence. We will prove Theorem 1(i) by showing that the mapping-walk associated to M converges in law to $Z^{\boldsymbol{\theta}}$.

Consider the slight variation of the process $\underline{H}^{\mathbf{p}}_{\mathbf{w}}(u)$:

$$K^{\mathbf{p}}_{\mathbf{w}}(u)(s) = \begin{cases} \underline{H}^{\mathbf{p}}_{\mathbf{w}}(u)(s) & \text{if } s \text{ is not a time when a vertex of the spine is visited} \\ \underline{H}^{\mathbf{p}}_{\mathbf{w}}(u)(s) + 1 & \text{else.} \end{cases}$$

This process thus "lifts" the heights of the vertices of the spine by 1. Recall from the proof of [1, Lemma 3] (with a slightly more general context that incorporates the weights $\mathbf{w}$) that these vertices are visited precisely at the times for which the reversed pre-minimum process $s \mapsto \underline{H}^{\mathbf{p}}_{(u-s)-}(u)$ jumps downward, so in $K^{\mathbf{p}}_{\mathbf{w}}(u)$ we just delay these jumps by the corresponding $\mathbf{w}$-mass of the vertex. What we now call "excursion" or generalized excursion of $H^{\mathbf{p}}_{\mathbf{w}}$ above $K^{\mathbf{p}}_{\mathbf{w}}(u)$ is just the same as before, that is a recentered portion of the path of $H^{\mathbf{p}}_{\mathbf{w}}$ on a flat interval of $K^{\mathbf{p}}_{\mathbf{w}}(u)$, with the convention that two excursions on two flat intervals with same heights (here and below the term "height" refers to the flat intervals) are merged together as a single generalized excursion. By contrast with the above, these excursions may take negative values, but only at times when cyclic vertices are visited, where the excursions' value is -1. As above, let $\widetilde{\mathbf{J}}^u(H^{\mathbf{p}}_{\mathbf{w}})$ be the process obtained by merging the excursions of $H^{\mathbf{p}}_{\mathbf{w}}$ above $K^{\mathbf{p}}_{\mathbf{w}}(u)$ in increasing order of height. A slight variation of [1, Lemma 3] gives

Lemma 1

$$\widetilde{\mathbf{J}}^{U^{\mathbf{w}}}(H^{\mathbf{p}}_{\mathbf{w}}) = H^M_{\mathbf{w}} - 1.$$

Notice in particular that $H^M_{\mathbf{w}}$ is a functional of $\mathcal{T}^{\mathbf{p}}$ and $X_1(U)$ alone, and does not depend on $X_2, X_3, \ldots$.

Proof of Theorem 1. Let $\mathbf{p}_n$ satisfy (3) with finite-length limit $\boldsymbol{\theta}$. We use Theorem 2 and Skorokhod's representation theorem, so we suppose that the convergence of $\sigma(\mathbf{p}_n)H^{\mathbf{p}_n} \to H^{\boldsymbol{\theta}}$ (either in $*$-topology or Skorokhod topology according to the hypotheses) is almost-sure, as well as the convergence of $S_{\mathbf{p}_n}, S_{\mathbf{w}_n}, S_{\mathbf{q}_n}$ to the identity. We also suppose that the convergence of Theorem 2 (ii) is almost-sure.

Fix $\epsilon > 0$. For (Lebesgue) almost-every $u \in [0,1]$, u is not a local minimum of $H^{\boldsymbol{\theta}}$ on the right or on the left. Fix such a u. Since $u^{\mathbf{w}_n} := S^{-1}_{\mathbf{w}_n} \circ S_{\mathbf{p}_n}(u) \to u$ as $n \to \infty$, it is easily checked that for any $\eta > 0$ and $n > N_1$ large enough, the processes $\underline{H}^{\boldsymbol{\theta}}(u^{\mathbf{w}_n})$ and $\underline{H}^{\boldsymbol{\theta}}(u)$ (resp. $K^{\mathbf{p}_n}_{\mathbf{w}_n}(u^{\mathbf{w}_n})$ and $K^{\mathbf{p}_n}_{\mathbf{w}_n}(u)$) coïncide outside the interval $(u-\eta, u+\eta)$. Let $\varepsilon_1, \varepsilon_2, \ldots$ be the generalized excursions of $H^{\boldsymbol{\theta}}$ above $\underline{H}^{\boldsymbol{\theta}}(u)$, ranked by decreasing order of their durations $l_1, l_2, \ldots$, call $h_1, h_2, \ldots$ the corresponding (pairwise distinct) heights. Let $\alpha > 0$ be such that $\omega(h) := \sup_{h \in [-\alpha,\alpha]} ||H^{\boldsymbol{\theta}}_{\cdot+h} - H^{\boldsymbol{\theta}}_{\cdot}||_\infty < \epsilon/3$. Notice that for $n > N_2$ large enough, we also have $\omega_n(h) := \sigma(\mathbf{p}_n) \sup_{h\in[-\alpha,\alpha]} ||H^{\mathbf{p}_n}_{\mathbf{w}_n}(\cdot + h) - H^{\mathbf{p}_n}_{\mathbf{w}_n}(\cdot)||_\infty \leq \varepsilon/2$. Next, take k such that $\sum_{i=1}^k l_i \geq 1 - \alpha/2$, and choose $\eta < \alpha/4$ such that none of the intervals of constancy of $\underline{H}^{\boldsymbol{\theta}}(u)$ corresponding to these k excursions intersect $(u-\eta, u+\eta)$.

Next, consider hypothesis (i) of Theorem 1. If $[a,b]$ is an interval of constancy of $\underline{H}^{\boldsymbol{\theta}}(u^{\mathbf{w}_n})$ (or $\underline{H}^{\boldsymbol{\theta}}(u)$) not intersecting $(u-\eta, u+\eta)$, then there exists for n large enough a constancy interval of $K^{\mathbf{p}_n}_{\mathbf{w}_n}(u)$, which we denote by $[a^n, b^n]$, such that $(a^n, b^n) \to (a,b)$, implying by Theorem 2(i) that

$$(\sigma(\mathbf{p}_n)(H^{\mathbf{p}_n}_{\mathbf{w}_n}(a^n + s) - H^{\mathbf{p}_n}_{\mathbf{w}_n}(a^n)), 0 \leq s \leq b^n - a^n) \to (H^{\boldsymbol{\theta}}(a+s) - H^{\boldsymbol{\theta}}(a), 0 \leq s \leq b-a)$$

uniformly. Moreover, for u as chosen above, if $u \in (t_i, T_i)$ (notice $u = T_i$ or $u = t_i$ is not possible) then there exists some t_{ij}, T_{ij} with $t_{ij} < u < T_{ij}$. Thus, for such u and as a consequence of Theorem 2 (ii), if there exists a second such flat interval $[c,d]$ with same height as the initial one (with say $b < c$), then there also exists a constancy interval $[c^n, d^n]$ of $K^{\mathbf{p}_n}_{\mathbf{w}_n}(u)$ with $(c^n, d^n) \to (c,d)$, with the **same** height as the first one. Therefore, these two intervals do merge to form the interval of a generalized excursion of $\sigma(\mathbf{p}_n)H^{\mathbf{p}_n}_{\mathbf{w}_n}$ above $\sigma(\mathbf{p}_n)K^{\mathbf{p}_n}_{\mathbf{w}_n}(u)$ with length $(b^n - a^n) + (d^n - c^n)$, that converges uniformly to the generalized excursion of $H^{\boldsymbol{\theta}}$ above $\underline{H}^{\boldsymbol{\theta}}(u)$ with height $H^{\boldsymbol{\theta}}_a$ and duration $(b-a)+(d-c)$. As a conclusion, one has $\varepsilon^n_i \to \varepsilon_i$ uniformly for every $1 \leq i \leq k$, where ε^n_i is the generalized excursion of $\sigma(\mathbf{p}_n)H^{\mathbf{p}_n}_{\mathbf{w}_n}$ above $\sigma(\mathbf{p}_n)K^{\mathbf{p}_n}_{\mathbf{w}_n}(u^{\mathbf{w}_n})$ with i-th largest duration l^n_i. Call h^n_i its height.

Now $(h^n_1, \ldots, h^n_k) \to (h_1, \ldots, h_k)$, and $\sum_{1\leq i\leq k} |l^n_i - l_i| \to 0$ as $n \to \infty$. Thus, if $n > \max(N_1, N_2)$ is also chosen so that

- $\sum_{1\leq i\leq k} |l^n_i - l_i| \leq \alpha/2$,
- $h^n_1, \ldots, h^n_k$ are in the same order as $h_1, \ldots, h_k$ (recall these are pairwise distinct),
- $\sup_{1\leq i\leq k} ||\varepsilon^n_i - \varepsilon_i||_\infty < \epsilon/2$,

then necessarily, the uniform distance between $\sigma(\mathbf{p}_n)\widetilde{\mathbf{J}}^{u^{\mathbf{w}_n}}(H^{\mathbf{p}_n}_{\mathbf{w}_n})$ and $\mathbf{J}^{u^{\mathbf{w}_n}}(H^{\boldsymbol{\theta}})$ is at most ϵ. Indeed, for $x \in [0,1]$, if $x \in (g^n_i, d^n_i) \cap (g_i, d_i)$ for some $i \leq k$, then

$$|\sigma(\mathbf{p}_n)\widetilde{\mathbf{J}}^{u^{\mathbf{w}_n}}(H^{\mathbf{p}_n}_{\mathbf{w}_n})(x) - \mathbf{J}^{u^{\mathbf{w}_n}}(H^{\boldsymbol{\theta}})(x)| \leq ||\varepsilon_i - \varepsilon^n_i||_\infty + \sup_{|h|<\alpha} ||\varepsilon_i(\cdot) - \varepsilon_i(\cdot + h)||_\infty \leq \epsilon,$$

and else the value taken by this difference does not exceed $\omega(h)+\omega_n(h) \leq \epsilon$ because there must be a zero of both processes at distance $< \alpha$ from x. Apply this to $u = U$, which a.s. does not belong to the set of local minima (on the left or on the right) of $H^{\boldsymbol{\theta}}$. Using Lemma 1 establishes the assertion of (i).

The case (iv) of $*$-convergence follows the same lines as in [1, Lemma 2]. We suppose up to extracting subsequences that $\sigma(\mathbf{p}_n)H^{\mathbf{p}_n}$ can be written as g_n+h_n with g_n converging uniformly to $H^{\boldsymbol{\theta}}$ and $h_n(u) = 0$ ultimately for almost-every u. Then, up to modifying slightly the constancy intervals of $K^{\mathbf{p}_n}$, the same result as above holds for g_n, so this proves that $\sigma(\mathbf{p}_n)\mathbf{J}^U(H^{\mathbf{p}_n})$ converges to $Z^{\boldsymbol{\theta}}$ in probability for the $*$-metric.

Points (ii,iii) in Theorem 1 then follow the same lines as in the proof of [1, Theorem 1]. We give some details for (ii). Let $U_2, U_3, \ldots$ be uniform independent random variables, independent of $H^{\boldsymbol{\theta}}, U, (H^{\mathbf{p}_n}, n \geq 1)$. Let $U_i^{\mathbf{q}_n} = S_{\mathbf{w}_n}^{-1} \circ S_{\mathbf{q}_n}(U_i)$ for $i \geq 2$. Recall that the walk $H_{\mathbf{w}_n}^{M_n}$ can be defined using only $H^{\mathbf{p}_n}, U$, so we are allowed to make the following choice for $X_2, X_3, \ldots$: we let X_i be the vertex visited by $H_{\mathbf{w}_n}^{M_n}$ at time $U_i^{\mathbf{q}}$. Therefore, the marks $D_{\mathbf{w}_n}^{M_n}(i)$ are obtained recursively as follows: let v be the vertex visited by the first $U_j^{\mathbf{q}} > D_{\mathbf{w}_n}^{M_n}(i)$, then $D_{\mathbf{w}_n}^{M_n}(i+1)$ is the first time when a cyclic point is visited strictly after v, i.e. at the right end of the generalized excursion of $H_{\mathbf{w}_n}^{M_n}$ straddling this $U_j^{\mathbf{q}}$. Passing to the limit, we find that $(D_{\mathbf{w}_n}^{M_n}(i), 1 \leq i \leq j)$ converges a.s. to $(D'_i, 1 \leq i \leq j)$ defined recursively by: D'_{i+1} is the first point of $\{d_1, d_2, d_3, \ldots\}$ that occurs after the first $U_j > D'_i$. It is easy to see that this defines a sequence with the same law as $D_i, i \geq 1$. □

4 Inhomogeneous continuum random tree interpretation

Let us briefly introduce the details of the limiting ICRT's *stick-breaking* construction [7, 4]. Let $\boldsymbol{\theta} = (\theta_0, \theta_1, \theta_2, \ldots)$ satisfy $\sum_{i\geq 0} \theta_i^2 = 1$. Consider a Poisson process $(U_j, V_j), j \geq 1$ on the first octant $\mathbb{O} = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x\}$, with intensity θ_0^2 per unit area. For each $i \geq 1$ consider also homogeneous Poisson processes $(\xi_{i,j}, j \geq 1)$ with intensity θ_i per unit length, and suppose these processes are independent, and independent of the first Poisson process. The points of $\mathbb{R}_+$ that are either equal to some $U_i, i \geq 1$ or some $\xi_{i,j}, j \geq 2$ will be called *cutpoints*. To a cutpoint η we associate a *joinpoint* η^*: if η is of the form U_i, let $\eta^* = V_i$, while if $\eta = \xi_{i,j}$ for some $i \geq 1, j \geq 2$, we let $\eta^* = \xi_{i,1}$. Under the hypothesis $\sum_i \theta_i^2$, one shows that we may order the cutpoints as $0 < \eta_1 < \eta_2, \ldots$ almost-surely. We build recursively a consistent family of trees whose edges are line-segments by first letting $\mathcal{T}_1^{\boldsymbol{\theta}}$ be the segment $[0, \eta_1]$ rooted at 0, and then, given $\mathcal{T}_J^{\boldsymbol{\theta}}$, by attaching the left-end of the segment $(\eta_J, \eta_{J+1}]$ at the corresponding joinpoint η_J^*, which has been already placed somewhere on $\mathcal{T}_J^{\boldsymbol{\theta}}$. Further, we relabel the joinpoints of the form $\xi_{i,1}$ as i, and we relabel the leaves $\eta_1, \eta_2, \ldots$ as $1+, 2+, \ldots$. When all the branches are attached, we obtain a random metric space whose completion we call $\mathcal{T}^{\boldsymbol{\theta}}$ (it can therefore be interpreted as the completion of a special metrization of $[0, \infty)$). We let $[[v, w]]$ be the only injective path from v to w, and $]]v, w]] = [[v, w]] \setminus \{v\}$.

Together with the ICRT comes one natural measure, which is the length measure inherited from Lebesgue measure on $[0, \infty)$. When $\boldsymbol{\theta}$ satisfies the further hypothesis $\theta_0 > 0$ or $\sum_i \theta_i = \infty$, the tree can be endowed with another measure μ, which is a probability measure obtained as the weak limit of the empirical distribution μ_J on the leaves $1+, 2+, \ldots, J+$ as $J \to \infty$. We call μ the *mass measure*.

If $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\text{finite}}$, it has been shown in [2] that $H^{\boldsymbol{\theta}}$ is the *exploration process* of $\mathcal{T}^{\boldsymbol{\theta}}$. To explain what this means, note first that $H^{\boldsymbol{\theta}}$ induces a special pseudo-metric on $[0,1]$ by letting

$$d(u,v) = H^{\boldsymbol{\theta}}_u + H^{\boldsymbol{\theta}}_v - 2 \inf_{w \in [u,v]} H^{\boldsymbol{\theta}}_w.$$

It turns out that the quotient space $\mathcal{T}$ obtained by identifying points of $[0,1]$ at distance 0 has the same "law" as $\mathcal{T}^{\boldsymbol{\theta}}$, where the mass measure is the measure on the quotient induced by Lebesgue measure on $[0,1]$. Precisely,

Theorem 3 ([2]) *If $U_1, \ldots, U_J$ are independent uniform variables on $[0,1]$, independent of $H^{\boldsymbol{\theta}}$, then the subtree of $\mathcal{T}^{\boldsymbol{\theta}}$ spanned by the (equivalence classes of the) U_i's has the same law as $\mathcal{T}^{\boldsymbol{\theta}}_J$.*

Conceptually, the stick-breaking construction provides an "algorithmic construction" of the ICRT, whereas the process $H^{\boldsymbol{\theta}}$ plays a rôle similar to that of Brownian excursion in our methodology described in point (ii) in the introduction.

We now show how some consequences of our main theorem can be formulated in terms of the stick-breaking construction of the ICRT. For $v \in \mathcal{T}^{\boldsymbol{\theta}}$, let $\texttt{junc}(v)$ be the branchpoint between v and $1+$. Define recursively a sequence $0 = c_0, c_1, \ldots$ of vertices of the spine $[[\text{root}, 1+]]$ with increasing heights recursively using the rule

> Given c_j let $k_{j+1}+$ be the first leaf of $\{2+, 3+, 4+, \ldots\}$ with $\texttt{junc}(k_{j+1}+) \notin [[\text{root}, c_j]]$ and let $c_{j+1} = \texttt{junc}(k_{j+1}+)$.

Corollary 2 *Under regime (3) with limiting $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\text{finite}}$,*

$$\begin{aligned}&(\mathbf{p}_n(\mathcal{B}_j(M_n)), \sigma(\mathbf{p}_n)\text{Card}\,(\mathcal{C}_j(M_n)), j \geq 1)\\ &\quad\to \left(\lim_{k\to\infty} \frac{1}{k}\text{Card}\,\{1 \leq i \leq k : \texttt{junc}(i+) \in]]c_{j-1}, c_j]]\}, \text{ht}(c_j) - \text{ht}(c_{j-1}), j \geq 1\right)\end{aligned}$$

Proof. The $n \to \infty$ limit of the left side is (by Corollary 1) the law of

$$(D_j - D_{j-1}, L^{\boldsymbol{\theta}}_{D_j} - L^{\boldsymbol{\theta}}_{D_{j-1}}, j \geq 1). \tag{11}$$

By the description of μ as the $k \to \infty$ limit of the empirical distribution on leaves $\{1+, 2+, \ldots, k+\}$, the $k \to \infty$ limit of the right side of Corollary 2 becomes

$$(\mu\{v \in \mathcal{T}^{\boldsymbol{\theta}} : \texttt{junc}(v) \in]]c_{j-1}, c_j]]\}, \text{ht}(c_j) - \text{ht}(c_{j-1}), j \geq 1). \tag{12}$$

So the issue is to show equality in law of (11) and (12). But Theorem 3 identifies the law (12) with the law

$$(\text{Leb}\{v \in (0,1) : \texttt{junc}(v) \in]]c_{j-1}, c_j]]\}, H^{\boldsymbol{\theta}}_{c_j} - H^{\boldsymbol{\theta}}_{c_{j-1}}, j \geq 1) \tag{13}$$

where the quantities involved can be redefined as follows. Take $U_1, U_2, U_3, \ldots$ uniform on $(0,1)$, independent of $H^{\boldsymbol{\theta}}$. Let $\texttt{junc}(v)$ be the point at which $\inf_{[v,U_1]} H^{\boldsymbol{\theta}}_\cdot$ or $\inf_{[U_1,v]} H^{\boldsymbol{\theta}}_\cdot$ is attained. Given c_j, let $c_{j+1} = \texttt{junc}(U')$ where U' is the first of $\{U_2, U_3, U_4, \ldots\}$ such that $H^{\boldsymbol{\theta}}_{\texttt{junc}(U')} > H^{\boldsymbol{\theta}}_{c_j}$.

On the other hand, D_1 is by definition equal in law to the sum of the lengths of the generalized excursions of $H^{\boldsymbol{\theta}}$ above $\underline{H}^{\boldsymbol{\theta}}(U_1)$ whose heights are less than or equal to that of the excursion containing an independent uniform U_2, while $L^{\boldsymbol{\theta}}_{D_1}$ is the height of the corresponding excursion. Recursively, $D_{j+1} - D_j$ is equal in law to the sum of the durations of the excursions with heights between the height of the previously explored excursions (strictly) and the height of the excursion straddling the first U_i that falls in an excursion interval with height larger than the previous ones; $L^{\boldsymbol{\theta}}_{D_j} - L^{\boldsymbol{\theta}}_{D_{j-1}}$ is then the difference of these heights. This identifies the law (11) with the law (13). $\square$

Remark. Corollary 2 could alternatively be proved, for more general limit regimes, by an argument based directly on the Joyal correspondence, without using the detour through exploration processes.

5 Final remarks

The regimes (3) are basically the only possible ones, if we require a limit distribution for the number $|\mathcal{C}(M_n)|$ of cyclic vertices.

Lemma 2 *If $c_n(|\mathcal{C}(M_n)| - d_n)$ converges in law to some non-trivial distribution on $\mathbb{R}_+$ for some renormalizing sequences c, d, then there exists $\boldsymbol{\theta}$ such that $\mathbf{p}$ satisfies (3) up to elementary rescaling, that is, there exists $\alpha \in (0, \infty)$ and $\beta \in \mathbb{R}$ such that $c_n/\sigma(\mathbf{p}_n) \to \alpha$ and $c_n d_n \to \beta$.*

This lemma is a direct consequence of [7, Theorem 4] and of Proposition 2, which implies that the number of cyclic points of a $\mathbf{p}$-mapping has same distribution as one plus the distance from the root to a $\mathbf{p}$-sampled vertex of a $\mathbf{p}$-tree.

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