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Simple tangential families and perestroikas of their envelopes

Gianmarco Capitanio

Abstract

Tangential families are 1-parameter families of rays emanating tangentially from smooth curves. We classify tangential family germs up to Left-Right equivalence: we prove that there are two infinite series and four sporadic simple singularities of tangential family germs (in addition to two stable singularities). We give their normal forms and miniversal tangential deformations (i.e., deformations among tangential families), and we describe the corresponding envelope perestroikas of small codimension. We also discuss envelope singularities of non simple tangential families.

Keywords : Envelope theory, Left-Right equivalence, Tangential families.
2000 MSC : 14B05, 14H15, 58K25, 58K40, 58K50.

1 Introduction

A tangential family is a system of rays emanating tangentially from a smooth curve. Tangential families naturally arise in the Geometry of Caustics (see e.g. [3]) and in Differential Geometry. For instance, every smooth curve in a Riemannian surface defines the tangential family of its tangent geodesics. Tangential family theory is a generalization of Envelope theory. Indeed, every 1-parameter family of plane curves is tangential, with respect to every generic point of any (geometric) branch of its envelope.

In [10] we classified stable singularities of tangential family germs (under small tangential deformations, i.e. deformations among tangential families), and we proved that their envelopes are smooth or have a second order self-tangency. In [7] and [8] we considered tangential families with singular support (the singularity being a semicubic cusp).

In this paper we classify simple singularities of tangential family germs. We prove that there are two infinite series and four sporadic simple tangential family singularities (in addition to the two stable singularities). We give their normal forms and miniversal tangential deformations, and we describe perestroikas of small codimension occurring to envelopes of simple tangential family germs under small tangential deformations. We also study envelopes of non simple tangential family germs.

Our study of tangential families is related to Envelope Theory and Singularity Theory, namely to the theories of Thom and Arnold on envelopes of families of plane curves (see [1], [2] and [15]), and to several branches of Projection Theory (see [5]), namely the classifications of simple projections of surfaces and of projections of generic surfaces with boundary (due to J.W. Bruce, P.J. Giblin and V.V. Goryunov, see [12], [6] and [13]), in our case the boundary being fixed to be contained in the projection critical set.

2 Simple tangential family germs

Unless otherwise specified, all the objects considered below are supposed to be of class $\mathscr{C}^\infty$; by plane curve we mean an embedded 1-submanifold of $\mathbb{R}^2$.

Let us consider a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane $\mathbb{R}^2$, whose coordinates are denoted by ξ and t. If $\partial_t f$ vanishes nowhere, then $f_\xi := f(\xi, \cdot)$ parameterizes an immersed smooth curve Γ_ξ. Hence, f defines the 1-parameter family of curves $\{\Gamma_\xi : \xi \in \mathbb{R}\}$.

Definition. The family parameterized by f is a *tangential family* whenever (1) the curve parameterized by $f(\cdot, 0)$, called the *support* of the family, is smooth and (2) for every $\xi \in \mathbb{R}$, the family curve Γ_ξ is tangent to the support at $f(\xi, 0)$.

In other terms, a tangential family is a fibration, whose base is the support and whose fibers are the curves Γ_ξ. A *p-parameter tangential deformation* of a tangential family f is a map $F : \mathbb{R}^2 \times \mathbb{R}^p \to \mathbb{R}^2$, such that $F_\lambda := F(\cdot; \lambda)$ is a tangential family for every $\lambda \in \mathbb{R}^p$ and $F_0 = f$. For instance, the tangent geodesics to a curve in a Riemannian surface form a tangential family. A perturbation of the metric induces a tangential deformation on this family.

The *graph* of a tangential family $\{\Gamma_\xi : \xi \in \mathbb{R}\}$ is the surface $\Phi := \cup_{\xi \in \mathbb{R}} \{(\xi, \Gamma_\xi)\} \subset \mathbb{R} \times \mathbb{R}^2$. Let us denote by $\pi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ the projection $(\xi, P) \mapsto P$. The *criminant set* of the tangential family is the critical set of $\pi|_\Phi$. The *envelope* is the apparent contour of the graph in the plane (i.e. the critical value set of $\pi|_\Phi$). By the very definition, the support of the family belongs to its envelope.

Below we will study tangential family germs, so we will consider their local parameterizations as map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. Graphs of tangential family germs are smooth.

Denote by $\mathrm{Diff}(\mathbb{R}^2, 0)$ the group of diffeomorphism germs of the plane keeping fixed the origin and by $\mathfrak{m}_{\xi,t}$ the ring of function germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ in the variables ξ and t. Then the group $\mathscr{A} := \mathrm{Diff}(\mathbb{R}^2, 0) \times \mathrm{Diff}(\mathbb{R}^2, 0)$ acts on the space $(\mathfrak{m}_{\xi,t})^2$ by $(\varphi, \psi) \cdot f := \psi \circ f \circ \varphi^{-1}$, where $(\varphi, \psi) \in \mathscr{A}$ and $f \in (\mathfrak{m}_{\xi,t})^2$. Two elements of $(\mathfrak{m}_{\xi,t})^2$ are said to be *Left-Right equivalent*, or *$\mathscr{A}$-equivalent*, if they belong to the same $\mathscr{A}$-orbit. Thus, envelopes of $\mathscr{A}$-equivalent tangential families are diffeomorphic.

The standard definition of versality of a deformation can be translated in the setting of tangential deformations. A tangential deformation F of a tangential family germ f is said to be *versal* if any tangential deformation of f is $\mathscr{A}$-equivalent to a tangential deformation induced from F. Such a deformation is said to be *miniversal* whenever the base dimension p is minimal. This minimal number p, depending only on the $\mathscr{A}$-orbit of the family, is called the *tangential codimension* of the singularity. A tangential family germ is *stable* if every $\mathscr{A}$-versal tangential deformation is trivial; it is *simple* if by arbitrary sufficiently small tangential deformations of it, we can obtain only a finite number of singularities.

Let us recall that a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ defines, by the formula $f^*g := g \circ f$, a homomorphism from the ring $\mathscr{E}_{x,y}$ of the function germs in the target to the ring $\mathscr{E}_{\xi,t}$ of the function germs in the source. Hence, we can consider every $\mathscr{E}_{\xi,t}$-module as an $\mathscr{E}_{x,y}$-module via this homomorphism. The *tangent space* at f is the $\mathscr{E}_{x,y}$-module defined by

$$Tf := \langle \partial_\xi f, \partial_t f \rangle_{\mathscr{E}_{\xi,t}} + f^*(\mathscr{E}_{x,y}) \cdot \mathbb{R}^2 \ .$$

The *codimension* of f is the dimension of the real vector space $\mathscr{E}^2_{\xi,t}/Tf$.

We state now our main result, proven in Section 6.

Theorem 1. *Every simple tangential family germ is $\mathscr{A}$-equivalent to a tangential family germ $(\xi + t, \varphi)$, of support $y = 0$, where φ is one of the function germs listed in the table below, in which c and τ are the codimension and the tangential codimension of the singularity. A miniversal tangential deformation of the normal form is obtained adding $\sum_i \lambda_i e_i$ to φ according to the table.*

Singularity	φ	c	$\{e_i\}$	τ
I	t^2	0	—	0
II	$t^2\xi$	1	—	0
$S_{1,n}$	$t^2(t+\xi)+t^4+t^{2n+3}$	$n+1$	$t^3, t^5, \ldots, t^{2n+1}$	n
T_n	$t^3+t^2(t+\xi)^{n+1}$	$2n+1$	$t^2, t^2\xi, \ldots, t^2\xi^{n-1}$	n
$S_{2,2}$	$t^2(t+\xi)+t^5+t^6$	3	t^3, t^4	2
$S_{2,3}^{\pm}$	$t^2(t+\xi)+t^5 \pm t^9$	4	t^3, t^4, t^6	3
$S_{2,4}$	$t^2(t+\xi)+t^5$	5	t^3, t^4, t^6, t^9	4

Remarks. (1) This table is a part of the more general classification of the simple projections of surfaces in the plane, due to V. V. Goryunov (see [12]).
(2) In [10] we proved that singularities I and II are all the stable singularities; the corresponding envelopes are respectively smooth and have an order 2 self-tangency.

A curve with a singularity of order b/a is *non symmetric* with respect to a smooth curve tangent to it at the singular point if it is contained into one of the two domains cut off near the singular point by the smooth curve.

Corollary. *The envelope of every simple tangential family germ, having not a singularity* I, *has two tangent branches, one of which is its support. More precisely:*

(i) if the germ has a singularity $S_{1,n}$, the envelope second branch has a $(2n+3)/2$-cusp at the tangency point, this cusp being non symmetric to the support;

(ii) if the germ has a singularity $S_{2,2}$, $S_{2,3}^{\pm}$ or $S_{2,4}$, then the envelope second branch has a singularity of order $5/3$ at its tangency point with the support;

(iii) if the germ has a singularity T_n, the envelope second branch is smooth and it has a tangency of order $3n+2$ with the support.

Envelope singularities of simple tangential family germs are shown in Figure 1.

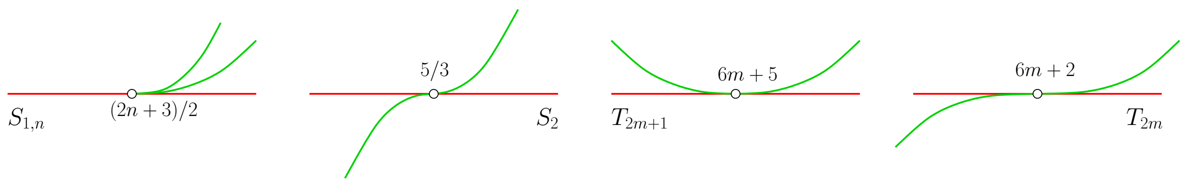

Figure 1: Envelopes of simple tangential family germs.

We end the classification of the simple singularities describing their adjacencies. A singularity L is said to be *adjacent* to a singularity K, and we write $L \to K$, if every map germ in L can be deformed into a map germ in K by an arbitrary small tangential deformation. If $L \to K \to K'$, then L is also adjacent to K'; in this case we omit the arrow $L \to K'$. It follows from Theorem 1 that the adjacencies of the simple singularities of tangential family germs are as follows.

$$
\begin{tikzcd}
\mathrm{I} & \mathrm{II} \arrow[l] & T_1 \arrow[l] & T_2 \arrow[l] & T_3 \arrow[l] & T_4 \arrow[l] & T_5 \arrow[l] & \cdots \arrow[l] \\
& & S_{1,1} \arrow[ul] & S_{1,2} \arrow[l] & S_{1,3} \arrow[l] & S_{1,4} \arrow[l] & S_{1,5} \arrow[l] & \cdots \arrow[l] \\
& & & S_{2,2} \arrow[ul] & S^{\pm}_{2,3} \arrow[l] & S_{2,4} \arrow[l] & &
\end{tikzcd}
$$

3 Bifurcation diagrams of simple singularities

In this section we discuss the bifurcation diagrams of small codimension simple singularities of tangential family germs, and the perestroikas of the corresponding envelopes.

Let L be a singularity, f a map germ in L and F a miniversal tangential deformation of f. The *discriminant* of L is, up to diffeomorphisms, the germ at the origin of the set formed by the deformation parameter values λ for which the envelope of F_λ has more complicated singularities than for some arbitrary close value of λ. The *bifurcation diagram* of a singularity is its discriminant, together with the envelopes of the deformed tangential family germs. In order to study the small codimension bifurctions diagrams of simple singularities, we consider the miniversal tangential deformations provided by Theorem 1.

We start with singularity $S_{1,n}$. By the corollary of Theorem 1, the envelope has two branches: the first is the family support γ, the second has a $(2n+3)/2$-cusp tangent to γ.

Remark. The discriminant of $S_{1,n}$ contains the flag $V_{n-1} \supset \cdots \supset V_0$, where V_i is defined by $\lambda_1 = ... = \lambda_{n-i} = 0$. A tangential family germ belonging to the stratum $V_i \setminus V_{i-1}$ has a singularity $S_{1,i}$, according to adjacency $S_{1,n} \to S_{1,n-1}$ (where $S_{1,0} := \mathrm{II}$).

For singularity $S_{1,1}$, the envelope perestroika is shown in Figure 2.

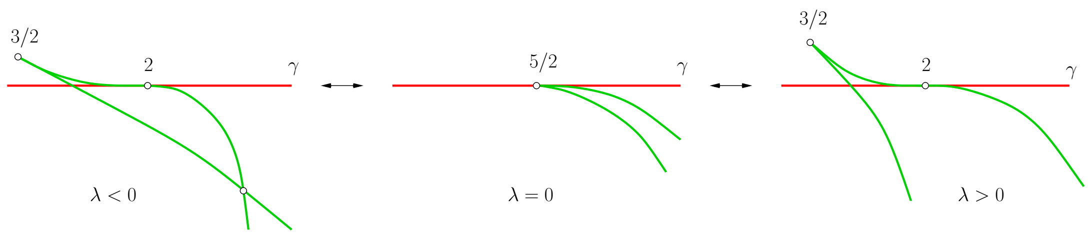

Figure 2: Envelope perestroika of singularity $S_{1,1}$.

Remark. The envelopes of the perturbed families form a singular surface germ in $\mathbb{R}^3 = \{x, y, \lambda\}$, diffeomorphic to a folded umbrella with a cubically tangent smooth surface; the half line of the umbrella self-intersections is tangent to the smooth surface.

The discriminant of singularity $S_{1,2}$ is represented in Figure 3. It has been found for me experimentally by Francesca Aicardi. Note that it contains the flag $V_1 \supset V_0$ and a curve, corresponding to a self-tangency of the envelope second branch.

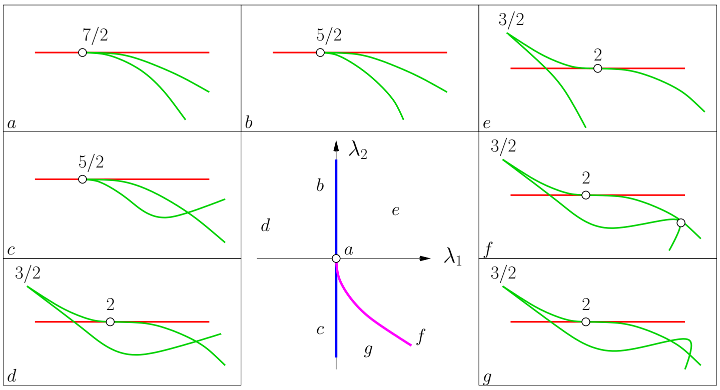

Figure 3: Bifurcation diagram of singularity $S_{1,2}$.

The $S_{2,2}$-discriminant in the plane $\{\lambda_1, \lambda_2\}$, shown in Figure 4, is the union of the λ_2-axis and four curve germs, tangent to it at the origin, the tangency order of all these curves being 1. To my knowledge, this bifurcation diagram did not appear earlier in Projection Theory.

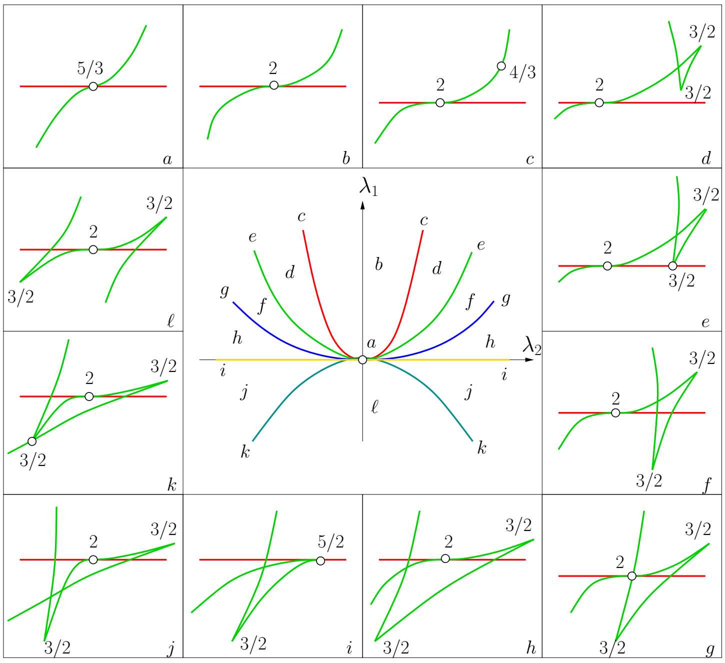

Figure 4: Bifurcation diagram of the singularity $S_{2,2}$.

Finally, we consider singularities T_n.

Proposition 1. *The discriminant of the singularity T_n is an n-dimensional swallowtail.*

Proof. The second branch of the envelope is the graph $y = \frac{4}{27} Q_\lambda^3(x)$, where $Q_\lambda(x) := x^{n+1} + \lambda_n x^{n-1} + \cdots + \lambda_1$, is a miniversal deformation of x^{n+1} for the Left equivalence in the space of function germs $(\mathbb{R},0) \to \mathbb{R}$. The T_n-discriminant is formed by the values λ for which P_λ has roots of multiplicity greater than 3, i.e., for which Q_λ has multiple roots. Thus, it is a swallowtail. $\square$

Bifurcation diagrams of singularities T_n are depicted in Figure 5 for $n = 1, 2, 3$.

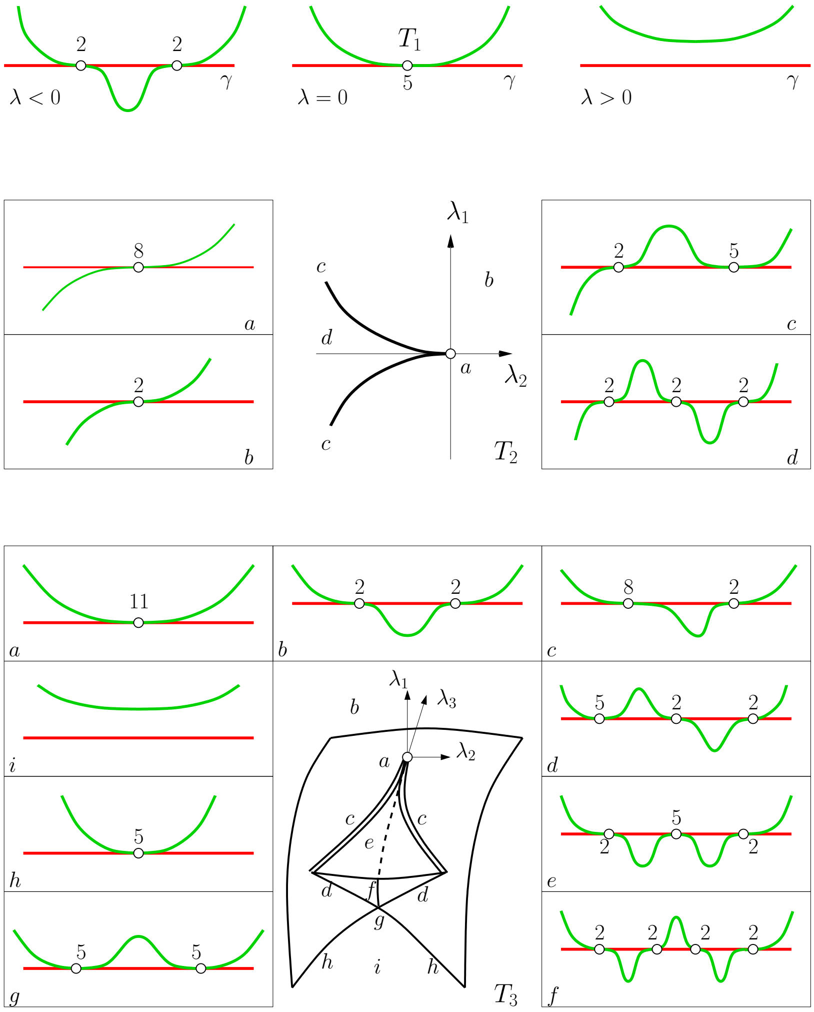

Figure 5: Bifurcation diagrams of singularities T_n for $n = 1, 2, 3$.

4 Classification of tangential family germs

The set of tangential family germs is naturally decomposed according to the configurtions of their criminant sets. We define now this decomposition.

Given a tangential family germ, the fiber $\pi^{-1}(0,0)$ defines a *vertical direction* in the tangent plane to its graph at the origin. The germ is said to be of *first type* if its criminant set has only one branch, of *second type* if it has exactly two branches and these branches are smooth, non vertical and transversal each other. In [10] we proved that a tangential family germ has a singularity I (resp., II) if and only if it is of first type (resp., of second type).

Definition. Let $n \in \mathbb{N} \cup \{\infty\}$. A tangential family germ is said to be (1) *of type* S_n if its criminant set has two branches, both smooth, one of which has an order n tangency with the vertical direction; (2) *of type* T_n if its criminant set has two branches, both smooth, which have an order n tangency; (3) *of type* U if its criminant set has more than two branches or at least a singular branch.

These singularity classes, in addition to stable singularities I and II, cover all the tangential family singularities.

Definition. A tangential family germ of type S_1 is said to be *of type* $S_{1,n}$ if its envelope has a cusp of order $(2n+3)/2$ at the origin.

Theorem 2. *For every $n \in \mathbb{N}$, a tangential family germ has a singularity $S_{1,n}$ (resp., T_n) if and only if it is of type $S_{1,n}$ (resp., T_n). A tangential family germ has a singularity $S_{2,2}$, $S_{2,3}^{\pm}$ or $S_{2,4}$ if and only if it is of type S_2.*

The next result describes envelopes of non simple tangential family germs of type $S_{n\geq 3}$.

Theorem 3. *For $n \in \mathbb{N}$, $n \geq 3$, the envelope of a S_n-type tangential family germ has, in addition to the support, a branch with a singularity of order $(n+3)/(n+1)$, tangent to the support. If n is even, this singularity is a cusp, non symmetric to the support.*

Theorems 2 and 3 are proven in Sections 5 and 6.

We end this section with the description of the hierarchy of non simple singularities. We denote by (L) and (K) some classes of non simple singularities. The arrow $(L) \to (K)$ means that there exists two singularities $L' \subset (L)$ and $K' \subset (K)$ such that $L' \to K'$. The main adjacencies of non simple singularities are as follows.

$$\begin{tikzcd}
 & (S_\infty) \arrow[r] & (S_n)_{n\geq 3} \arrow[r] \arrow[d] & S_{2,4} \\
(U) \arrow[ur] \arrow[r] \arrow[dr] & (S_{1,\infty}) \arrow[r] & S_{1,m} & \\
 & (T_\infty) \arrow[r] & T_r &
\end{tikzcd}$$

5 Preliminary results

Here we introduce some technical results we will use in next section. We start describing standard parameterizations of tangential family germs.

Let us fix a tangential family germ. One easily verifies (see [10]) that in a coordinate system in which the family support is the germ of $y = 0$, the tangential family is parameterized by a map germ $(\xi + t, \varphi)$, where φ is of the form $\alpha t^3 + t^2 \sum_{i=0}^{\infty} k_i \xi^i + t^3 \cdot \delta(0) + R(\xi)$, where $\alpha, k_i \in \mathbb{R}$, $\delta(n)$ denotes any function of two variables with vanishing n-jet at the origin and R is flat (i.e., its Taylor expansion is zero). Such a parameterization is called *prenormal form*. The prenormal form of a tangential family germ is not unique.

In [10] we proved that *a tangential family germ is of first type if and only if $k_0 \neq 0$; it is of second type if and only if $k_0 = 0$ and $k_1 \neq 0, \alpha$.* Therefore, for any tangential family germ neither of first nor of second type, we have $k_0 = 0$ and $k_1(k_1 - \alpha) = 0$.

Lemma 1. *Let $(\xi + t, \varphi)$ be a prenormal form of a tangential family germ, where φ is as above. The germ is of type $(S_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$ or U if and only $\alpha = k_1 \neq 0$, $\alpha \neq k_1 = 0$ or $\alpha = k_1 = 0$ respectively. More precisely,*

(i) the family is of type S_n if and only if $\varphi(0,t) = \alpha t^3 + O(t^{2n+3})$;

(ii) the family is of type T_n if and only if $k_0 = \cdots = k_n = 0$ and $\alpha, k_{n+1} \neq 0$,

(iii) the family is of type U if and only if $\varphi(\xi,t) = t^2 \cdot \delta(1)$.

Theorem 3 follows from Lemma 1 by explicit computation of the envelope.

Remark 1. Let $(\xi + t, \varphi)$ the prenormal form of a tangential family germ. Then every p-parameter tangential deformation of it is equivalent to a tangential deformation of the form $(\xi + t, \varphi + t^2\psi)$, where $\psi = \psi(\xi, t; \lambda)$ is a function germ $(\mathbb{R}^2 \times \mathbb{R}^p, 0) \to (\mathbb{R}^2, 0)$.

Let us consider now the space $\mathbb{R}[[\xi, t]]$, endowed with a quasihomogeneous filtration defined by weights $\deg(\xi) = a$ and $\deg(t) = b$, where a and b are coprime natural numbers. We denote by $\tilde{\mathfrak{m}}^p_{\xi,t}$ the $\mathscr{E}_{\xi,t}$-ideal generated by the monomials of weighted degree p, and by $\tilde{\delta}(p)$ any function germ with zero weighted p-jet.

The *reduced tangent space* at f is the $\mathscr{E}_{x,y}$-module defined by $T_r f := \mathfrak{g}_+ + \mathcal{M}$, where $\mathfrak{g}_+$ is the space of the vector field germs having positive weighted order, $\mathcal{M}$ is the $\mathscr{E}_{x,y}$-module $f^*\left(\mathfrak{m}^2_{x,y} \oplus \mathbb{R} \cdot y\right) \times f^*\left(\mathfrak{m}^2_{x,y} \oplus \mathbb{R} \cdot x\right)$, and $\mathfrak{m}^2_{x,y}$ is the second power of the maximal ideal $\mathfrak{m}_{x,y}$.

Remark 2. Let R be an element of $\tilde{\mathfrak{m}}^p \times \tilde{\mathfrak{m}}^q$, such that $R \in T_r f/(\tilde{\mathfrak{m}}^{p+1} \times \tilde{\mathfrak{m}}^{q+1})$. Then the weighted (p,q)-jet of f is $\mathscr{A}$-equivalent to that of $f + R$.

6 Proofs

Here we prove Theorems 1 and 2. For the computations we skip, we refer to [9]. A *vector monomial* is by definition an element of $(\mathfrak{m}_{\xi,t})^2$ having a component which is a monomial, the other component being zero.

S_1-type tangential families. Set $\deg(t) = 1$, $\deg(\xi) = 2$. Consider a tangential family germ of type S_1; by Lemma 1, its prenormal form is $\mathscr{A}$-equivalent to $(\xi, t^4 + t^2\xi + t^2 \cdot \tilde{\delta}(2))$. It follows from the geometry of the Newton Diagram of $t^4 + t^2\xi$ that the preceding map germ is formally $\mathscr{A}$-equivalent to $(\xi, t^4 + t^2\xi) + \sum_{i=1}^{\infty} b_i(0, t^{2i+3})$, for some $b_i \in \mathbb{R}$. Such a germ is of type $S_{1,n}$ if and only if $b_i = 0$ for $i < n$ and $b_n \neq 0$. In this case we may assume $b_n = 1$ after rescaling.

We set $f_n(\xi, t) := (\xi, t^4 + t^2\xi + t^{2n+3})$. Since $T_r f_n$ contains $\tilde{\mathfrak{m}}^{2n+2}_{\xi,t} \times \tilde{\mathfrak{m}}^{2n+4}_{\xi,t}$ and the vector monomials $(0, t^{2\ell+4})$ for $\ell \in \mathbb{N}$, Remark 2 implies that every $S_{1,n}$ type tangential family germ is $\mathscr{A}$-equivalent to f_n $(n \in \mathbb{N})$, and hence to the $S_{1,n}$-normal form f'_n listed in Theorem 1.

The $\mathscr{E}_{\xi,t}$-ideal $\langle f_n \rangle$ is generated by ξ and t^4. By the Preparation Theorem of Malgrange and Mather (see e.g. [14]), $\{1, t, t^2, t^3\}$ is a generator system of the $\mathscr{E}_{x,y}$-module $\mathscr{E}_{\xi,t}$. Now, Tf_n contains $\tilde{\mathfrak{m}}^{2n}_{\xi,t} \times \tilde{\mathfrak{m}}^{2n+2}_{\xi,t}$ and the vector monomials $(0, t^{2\ell})$ for $\ell \in \mathbb{N}$. Hence, the vectors $(0, t^{2i+1})$, $i = 0, \ldots, n$, form a basis of the real vector space $\mathscr{E}^2_{\xi,t}/Tf_n$. Therefore, the codimension of f_n is $n + 1$ and $f_n + \sum_{i=0}^{n}(0, \lambda_i t^{2i+1})$ is an $\mathscr{A}$-miniversal deformation of f_n. As a consequence,

$f'_n + \sum_{i=0}^n (0, \lambda_i t^{2i+1})$ is an $\mathscr{A}$-miniversal deformation of the normal form f'_n; moreover, the singularities of class $S_{1,\infty}$ have infinite codimension.

Let us consider a tangential deformation $F = F(\xi, t; \alpha)$ of the tangential family germ f'_n. By the very definition of versality, there exist function germs $\lambda_i(\alpha)$ such that F is equivalent to $f'_n + \sum_{i=0}^n (0, \lambda_i(\alpha) t^{2i+1})$. By Remark 1, such a deformation is tangential if and only if $\lambda_0 \equiv 0$. Thus, $f'_n + \sum_{i=1}^n (0, \lambda_i t^{2i+1})$ is a miniversal tangential deformation of f'_n.

S_2-type tangential families. Set $\deg(\xi) = 3$, $\deg(t) = 1$. By Lemma 1, every tangential family germ of type S_2 is $\mathscr{A}$-equivalent to a map germ $g + (0, \tilde{\delta}(5))$, where $g(\xi, t) := (\xi, t^5 + t^2\xi)$. It follows from the inclusion $\tilde{\mathfrak{m}}_{\xi,t}^8 \times \tilde{\mathfrak{m}}_{\xi,t}^{10} \subset T_r g$ and a well known Theorem of Gaffney (see [11], [4]) that g is 9-$\mathscr{A}$-determined. One checks that there are exactly four $\mathscr{A}$-orbits in $(\mathfrak{m}_{\xi,t})^2$ over the weighted 5-jet g, represented by the map germs $g_2 := g + (0, t^6)$, $g_3^{\pm} := g \pm (0, t^9)$ and $g_4 := g$ (the forthcoming computations being identical for g_3^+ and for g_3^-, we omit the sign $\pm$). These germs are $\mathscr{A}$-equivalent to the normal forms g'_i given in Theorem 1.

In all the four cases we deal with, the quotient space $\mathscr{E}_{\xi,t}/\langle g_i \rangle$, where $i = 2, 3, 4$, is generated by $\{t, t^2, t^3, t^4\}$ (Preparation Theorem). Set $e_0 := t$, $e_1 := t^3$, $e_2 := t^4$, $e_3 := t^6$, $e_4 := t^9$. Using the inclusions $\tilde{\mathfrak{m}}_{\xi,t}^3 \times \tilde{\mathfrak{m}}_{\xi,t}^5 \subset T_r g_2$, $\tilde{\mathfrak{m}}_{\xi,t}^5 \times \tilde{\mathfrak{m}}_{\xi,t}^7 \subset T_r g_3$ and the above inclusion concerning g_4, one proves that $\{(0, e_j) : j = 1, \ldots i\}$ is a basis of the real vector space $\mathscr{E}_{\xi,t}^2/T g_i$, so the map germs $g_i + \sum_{j=0,\ldots,i} \lambda_j (0, e_j)$ are $\mathscr{A}$-miniversal deformations of the map germs g_i; in particular, the codimensions of g_2, g_3 and g_4 are 3, 4 and 5. Thus, the map germs $g'_i + \sum_{j=0,\ldots,i} \lambda_j (0, e_j)$ are $\mathscr{A}$-miniversal deformations of the normal forms g'_i As for $S_{1,n}$-type tangential families, we see that these deformations are miniversal tangential deformations if and only if $\lambda_0 \equiv 0$.

T-type tangential families. Set $\deg(t) = n + 1$ and $\deg(\xi) = 1$ and fix $n \in \mathbb{N}$. By Lemma 1, any T_n-type tangential family germ is $\mathscr{A}$-equivalent to $h_n + (0, \tilde{\delta}(3n+3))$, where $h_n(\xi, t) := (\xi, t^3 + t^2\xi^{n+1})$. Since $\tilde{\mathfrak{m}}_{\xi,t}^2 \times \tilde{\mathfrak{m}}_{\xi,t}^{3n+4} \subset T_r h_n$, every such a germ is $\mathscr{A}$-equivalent to h_n, due to Remark 2, and then to the normal form h'_n listed in Theorem 1.

Now, $\langle h_n \rangle$ is generated by ξ and t^3. Since $\mathscr{E}_{\xi,t} \times \tilde{\mathfrak{m}}_{\xi,t}^{3n+2} \subset T h_n$, the Preparation Theorem implies that the vectors $(0, t\xi^j), j = 0, \ldots, n$ and $(0, t^2\xi^i), i = 0, \ldots, n-1$ form a basis of the real vector space $\mathscr{E}_{\xi,t}^2/T f_n$. Therefore, the codimension of h_n is $2n+1$ (so, the codimension of any T_∞ singularity is infinite). Actually, we get still a basis if we replce the vectors $(0, t\xi^j)$ and $(0, t^2\xi^i)$ by $v_j := (0, t(\xi - t)^j)$ and $w_i := j(0, t^2(\xi - t)^i)$, so $h_n + \sum_{i=1}^n \lambda_i w_i + \sum_{j=0}^n \mu_j v_j$ is an $\mathscr{A}$-miniversal deformation of h_n. Thus, $h'_n + \sum_{i=1}^n \lambda_i (0, t^2\xi^{i-1}) + \sum_{j=0}^n \mu_j (0, t\xi^j)$ is an $\mathscr{A}$-miniversal deformation of the normal form h'_n. Any tangential deformation of the T_n-normal form is equivalent to the above map germ, in which λ_j and μ_i are function germs depending on the parameter of the tangential deformation. Due to Remark 1, all the function germs μ_i have to be identically zero.

The above computations end the proof of Theorem 2. To complete the proof of Theorem 1, it remains to show that the singularity classes $S_{1,\infty}$, T_∞, $S_{n\geq 3}$ and U are non simple. For singularities of classes $S_{1,\infty}$ and T_∞, this follows from the adjacency of $S_{1,\infty}$ (resp., T_∞) to $S_{1,n}$ (resp., T_n) for every $n \in \mathbb{N}$.

We prove now that the singularities of classes $S_{n\geq 3}$ are non simple. Set $\deg(\xi) = 4$, $\deg(t) = 1$. The germ $f_a(\xi, t) := (\xi, t^6 + t^2\xi + t^7 + at^9)$, $a \in \mathbb{R}$, has a singularity of class S_3, due to Lemma 1. By a direct computation, one verifies that the weighted 9-jets f_a and f_b are not $\mathscr{A}$-equivalent if $a \neq b$. Thus, in the neighboring of each such a map germ, there are infinitely many $\mathscr{A}$-orbits; that is, the germ is non simple. Let us consider a tangential family f of class S_3. By Lemma 1, its prenormal form is $f(\xi, t) = (\xi, t^6 + t^2\xi + \tilde{\delta}(6))$. It is easy to see that the weighted 9-jet of f is $\mathscr{A}$-equivalent to $(\xi, t^6 + t^2\xi + At^7 + Bt^9)$ for some $A, B \in \mathbb{R}$. If $A \neq 0$, we normalize it to 1. Now the weighted 9-jet of f is of the form f_a for some a, so f is non simple. If $A = 0$, we consider the deformation $F_\varepsilon(\xi, t) := f(\xi, t) + (0, \varepsilon t^7)$. For every $\varepsilon \neq 0$, F_ε is non simple, so f is also non simple. Finally, the singularities belonging to $S_{n>3}$ and S_∞ are adjacent to S_3, so they are not simple.

It remains to prove that the singularities of class U are non simple. Let us set $f_a(\xi, t) := (\xi, t^4/4 + 2at^3\xi/3 + t^2\xi^2/2)$. Due to Lemma 1, f_a is of type U for every a. The vertical direction at the origin and the three branches of the complex criminant set define four points on the projective complex line, whose complex cross ratio is $2a^2 - 1 + 2\sqrt{a^2 - 1}$. This number is an invariant of tangential family germs, i.e. $\mathscr{A}$-equivalent tangential families have the same cross ratio. Therefore, two germs whose 4-jets are f_a and f_b, are not $\mathscr{A}$-equivalent whenever their cross ratios are different, i.e. $|a| \neq |b|$. Consider now the prenormal form $f(\xi, t) = (\xi, t^3 \cdot \delta(1))$ of any U-type tangential family. Set $F_\varepsilon(\xi, t) := f(\xi, t) + (0, \varepsilon(t^4 + t^2\xi^2))$. For every ε small enough, F_ε is $\mathscr{A}$-equivalent to a non simple map germ, so f is non simple. This ends the proof of Theorem 1.

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