Legendrian graphs generated by Tangential Families

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Abstract

We construct a Legendrian version of Envelope theory. A tangential family is a 1-parameter family of rays emanating tangentially from a smooth plane curve. The Legendrian graph of the family is the union of the Legendrian lifts of the family curves in the projectivized cotangent bundle $PT^*\mathbb{R}^2$. We study the singularities of Legendrian graphs and their stability under small tangential deformations. We also find normal forms of their projections into the plane. This allows to interprete the beaks perestroika as the apparent contour of a deformation of the Double Whitney Umbrella singularity $A_1^{\pm}$.

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1 Introduction

A tangential family is a 1-parameter family of "rays" emanating tangentially from a smooth plane curve. Tangential families and their envelopes (or caustics) are natural objects in Differential Geometry: for instance, every curve in a Riemannian surface defines the tangential family of its tangent geodesics. The theory of tangential families is related to the study developed by Thom and Arnold for plane envelopes (see [1], [2] and [14]). In [7] and [8] we studied stable and simple singularities of tangential family germs (with respect to deformations among tangential families).

In this paper we construct a Legendrian version of tangential family theory. The envelope of a tangential family is viewed as the apparent contour of the surface, called Legendrian graph, formed by the union of the Legendrian lifts of the family curves in the projectivized cotangent bundle of the plane.

We classify the Legendrian graph singularities that are stable under small tangential deformations of the generating tangential families. We prove that, in addition to a regular Legendrian graph, there exists just one more local stable singularity, the Double Whitney Umbrella $A_1^{\pm}$. Furthermore, we find normal forms of typical projections of Legendrian graphs into the plane. This allows to interprete the beaks perestroika as the apparent contour of a non-tangential deformation of the Double Whitney Umbrella singularity in the projectivization of the cotangent bundle $T^*\mathbb{R}^2$.

Our results are related to several theories, concerning Maps from the Space to the Plane (Mond [13]), Projections of Manifolds with Boundaries (Goryunov [11], Bruce and Giblin [5]), Singular Lagrangian Varieties and their Lagrangian mappings (Givental [12]).
Acknowledgments. This paper contains a part of the results of my PhD Thesis ([6]). I wish to express my deep gratitude to my advisor V.I. Arnold.

2 Legendrian graphs and their singularities

Unless otherwise specified, all the objects considered below are supposed to be of class $\mathscr{C}^\infty$; by plane curve we mean an embedded 1-submanifold of the plane.

In this section we recall basic facts about tangential families and we define their Legendrian graphs in the projectivized cotangent bundle $PT^*\mathbb{R}^2$. We study the typical singularities of these graphs up to Left-Right equivalence. This classification considers neither the fiber nor the contact structure of $PT^*\mathbb{R}^2$. A classification of Legendrian graphs taking into account the fiber bundle structure is the object of Section 3.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping of the source plane, equipped with the coordinates ξ and t, to another plane. If $\partial_t f$ vanishes nowhere, then $f_\xi := f(\xi, \cdot)$ parameterizes an immersed curve Γ_ξ. Hence, f parameterizes the 1-parameter family of curves $\{\Gamma_\xi : \xi \in \mathbb{R}\}$.

Definition. The family parameterized by f is a *tangential family* if $f(\cdot, 0)$ parameterizes an embedded smooth curve, called the *support*, and Γ_ξ is tangent to γ at $f(\xi, 0)$ for every $\xi \in \mathbb{R}$.

The *graph* of the family is the surface $\Phi := \cup_{\xi\in\mathbb{R}}\{(\xi, \Gamma_\xi)\} \subset \mathbb{R}^3$. The *envelope* is the apparent contour of Φ under the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $\pi(\xi, P) := P$ (i.e. the critical value set of $\pi|_\Phi$); the *criminant set* is the critical set of $\pi|_\Phi$. By the very definition, the support of a tangential family belongs to its envelope.

A p-parameter deformation $F : \mathbb{R}^2 \times \mathbb{R}^p \to \mathbb{R}^2$ of a tangential family f is *tangential* if $F_\lambda := F(\cdot; \lambda)$ is a tangential family for every λ. Remark that the supports of the deformed families form a smooth deformation of the support of the initial family.

Below we will consider tangential family germs. Note that graphs of tangential family germs are smooth. In [7] we proved that there are exactly two tangential family singularities which are stable under small tangential deformations (for the Left-Right equivalence relation). These singularities, denoted by I and II, are represented by $(\xi + t, t^2)$ and $(\xi + t, t^2\xi)$. Their envelopes are respectively smooth and have an order 2 self-tangency.

Consider the projectivized cotangent bundle $PT^*\mathbb{R}^2$, endowed with the standard contact structure and the standard Legendre fibration $\pi_L : PT^*\mathbb{R}^2 \to \mathbb{R}^2$.

Definition. The *Legendrian graph* of a tangential family is the surface in $PT^*\mathbb{R}^2$ formed by the Legendrian lifts of the family curves.

We remark that the envelope of a tangential family is the π_L-apparent contour of its Legendrian graph.

We say that a Legendrian graph germ is of *first type* (resp., *second type*) if it is generated by a tangential family germ of the same type, i.e., having a singularity I (resp., II).

Let us recall that a surface germ has a singularity of type $A_n^\pm$ (resp., H_n) if it is diffeomorphic to the surface locally parameterized by the map germ $(\xi, t^2, t^3 \pm t\xi^{n+1})$ (resp., by

$(\xi, \xi t + t^{3n-1}, t^3))$; these singularities are simple. The singularities A_n^+ and A_n^- coincide if and only if n is even. The singularities $A_1^\pm$, shown in Figure 1, are called Double Whitney Umbrellas.

Theorem 1. *The Legendrian graph germs of first type are smooth, while those of second type have generically a Double Whitney Umbrella singularity* $A_1^\pm$. *The other second type Legendrian graph germs have* $A_n^\pm$ *or* H_n *singularities for* $n \geq 2$ *or* $n = \infty$.

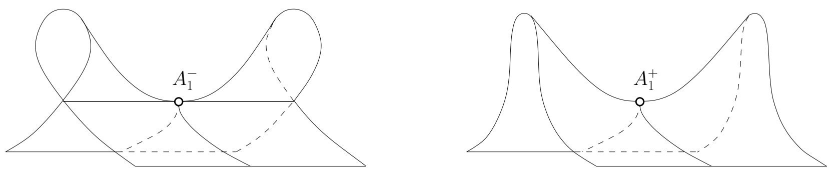

Figure 1: Double Whitney Umbrellas.

In the statement, "generically" means that the second type Legendrian graph germs for which the claim does not hold form a (non-connected) codimension 1 submanifold in the manifold formed by the second type Legendrian graph germs.

Remark. The singularities of map germs from $\mathbb{R}^2$ to $\mathbb{R}^3$ usually denoted by $B_n^\pm$, $C_n^\pm$, F_4 (see [3], [13]) appear as singularities of Legendrian graph generated by non typical tangential families (i.e., which are neither of first nor second type). For example, the Legendrian graphs of S-type tangential family germs have $B_n^\pm$ singularities. Simple tangential family germs are classified in [8].

A Legendrian graph is *stable* (under small tangential deformations) if for every small enough tangential deformation of the tangential family generating it, the initial and the deformed graphs are diffeomorphic. A similar definition holds for germs.

Theorem 2. *The Double Whitney Umbrellas* $A_1^\pm$ *are, in addition to smooth graphs, the only stable Legendrian graph singularities.*

We point out that in Mond's general theory of maps from the space to the plane [13], the Double Whitney Umbrellas are not stable.

A Legendrian graph singularity L is said to be *adjacent* to a Legendrian graph singularity K $(L \to K)$, if every Legendrian graph in L can be deformed into a Legendrian graph in K by an arbitrary small tangential deformation. If $L \to K \to K'$, the class L is also adjacent to K'. In this case we omit the arrow $L \to K'$. The adjacencies of the typical Legendrian graph singularities are as follows (E means embedding).

$$\begin{array}{ccccccccccc} E & \longleftarrow & A_1^\pm & \longleftarrow & A_2 & \longleftarrow & A_3^\pm & \longleftarrow & \cdots & \longleftarrow & A_\infty^\pm \\ & & & \nwarrow & & & & & & & \\ & & & & H_2 & \longleftarrow & H_3 & \longleftarrow & \cdots & \longleftarrow & H_\infty^\pm \end{array}$$

3 Normal forms of Legendrian graph projections

In this section we study how Legendrian graphs project into the envelopes of their generating tangential families. In other terms, we find normal forms of typical Legendrian graphs with respect to an equivalence relation preserving the fiber structure of $PT^*\mathbb{R}^2$.

Definition. The projections of two Legendrian graphs Λ_1 and Λ_2 by π_L are said to be *equivalent* if there exists a commutative diagram

$$\begin{array}{ccccc} \Lambda_1 & \xrightarrow{i_1} & PT^*\mathbb{R}^2 & \xrightarrow{\pi_L} & \mathbb{R}^2 \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda_2 & \xrightarrow{i_2} & PT^*\mathbb{R}^2 & \xrightarrow{\pi_L} & \mathbb{R}^2 \end{array}$$

in which the vertical arrows are diffeomorphisms and i_1, i_2 are inclusions.

Such an equivalence is a pair of a diffeomorphism between the two Legendrian graphs and a diffeomorphism of $PT^*\mathbb{R}^2$ fibered over the base $\mathbb{R}^2$ (the diffeomorphism is not presumed to be a contactomorphism). A similar definition holds for germs.

Let $\mathscr{A}^*$ be the subgroup of $\mathscr{A} := \mathrm{Diff}(\mathbb{R}^2, 0) \times \mathrm{Diff}(\mathbb{R}^3, 0)$, formed by the pairs (φ, ψ) such that ψ is fibered with respect to π. This subgroup inherits the standard action of $\mathscr{A}$ on the maximal ideal $(\mathfrak{m}_{\xi,t})^3$: $(\varphi, \psi) \cdot f := \psi \circ f \circ \varphi^{-1}$. Projections of Legendrian graphs are locally equivalent if and only if their local parameterizations are $\mathscr{A}^*$-equivalent.

Theorem 3. *The projection germs of the typical Legendrian graphs are equivalent to the projection germs of the surfaces parameterized by the map germs f in the 3-space $\{x, y, z\}$ by a pencil of lines parallel to the z-axis, where f is the normal form in the following table, according to the graph type.*

Type	Singularity	Normal form	Restrictions
I	Fold	(ξ, t^2, t)	$\varnothing$
II	$A_1^\pm$	$(\xi, t^3 + t^2\xi + at\xi^2, t^2 + bt^3)$	$a \neq -1, 0,\ a < 1/3$

Moreover, a Legendrian graph germ of second type, parameterized by the above normal form, has a singularity A_1^+ (resp., A_1^-) if and only if $0 < a < 1/3$ (resp., $-1 \neq a < 0$).

Typical Legendrian graphs are those having only stable singularities. In Theorem 3, "generically" means that the second type Legendrian graph germs for which the claim does not hold form a non-connected codimension 1 submanifold in the manifold of all the second type Legendrian graph germs.

Typical Legendrian graph projections are depicted in figure 2.

Corollary. *The Fold is the only stable and the only simple singularity of Legendrian graphs (with respect to tangential deformations and Left-Right equivalence relation).*

Let us denote by $F_{a,b}$ the $A_1^\pm$ normal form in Theorem 3 and by z its third coordinate.

Theorem 4. *The map germ $F_{a,b} + (\mu_1 z, \lambda t + \mu_2 z, 0)$ is an $\mathscr{A}^*$-miniversal tangential deformation of the normal form $F_{a,b}$, provided that $b \neq 0$.*

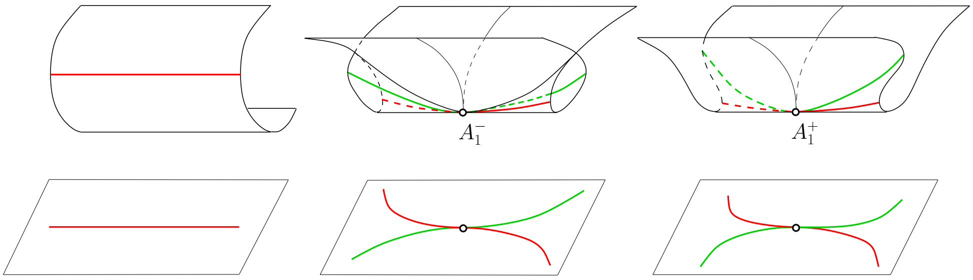

Figure 2: Typical Legendrian graph projections.

Remark. The above deformation is not the simpler possible, but it has the property that the parameters μ_1, μ_2 deform the direction of the projection, but leave fixed the Legendrian graph, while the parameter λ deforms the graph without changing the projection. In particular, the deformation restricted to $\mu_1 = \mu_2 = 0$ provides an $\mathscr{A}$-miniversal deformation of $F_{a,b}$.

The second order self-tangency of the envelope of a second type tangential family germ is not stable under non-tangential deformations (see [7]). Under such a deformation, the envelope experiences a beaks perestroika, that may be interpreted as the apparent contour in the plane of the perestroika occurring to its Legendrian graph, as shown in figure 3. We call it *Legendrian beaks perestroika*. Actually, there are two such perestroikas, according to the sign of $A_1^{\pm}$. Figure 3 has been obtained investigating the critical sets of the $\mathscr{A}$-miniversal deformation $F_{a,b} + (0, \lambda t, 0)$ of the projection normal form $F_{a,b}$, which leaves unchanged the direction of the projection.

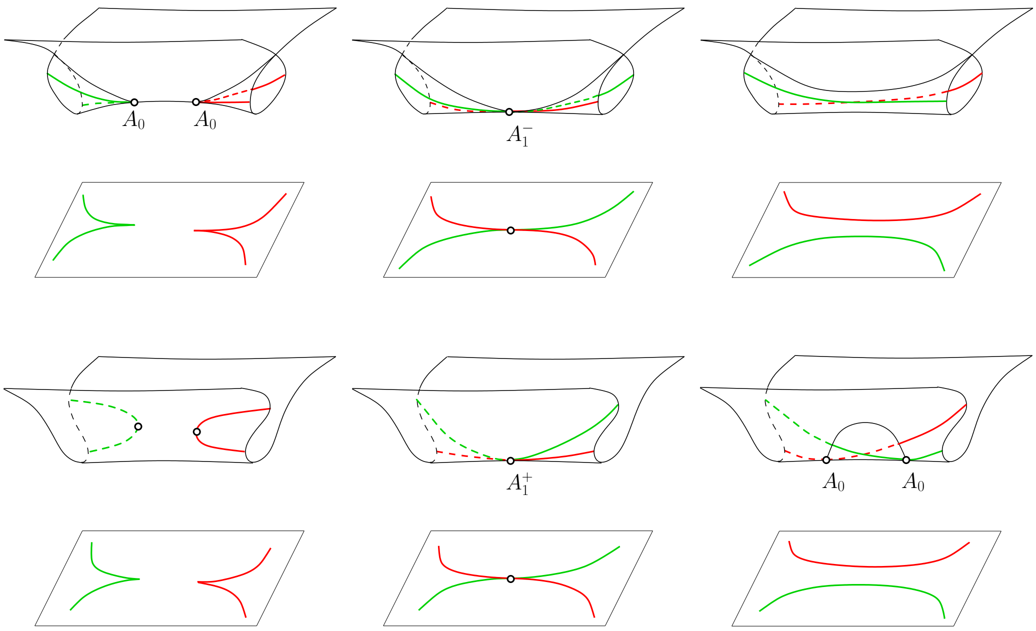

Figure 3: Legendrian beaks perestroikas.

4 Proof of Theorems 1 and 2

We start constructing explicit parameterizations of Legendrian graph germs.

Lemma 1. *Every local parameterization of a Legendrian graph is* $\mathscr{A}^*$*-equivalent to a map germ of the form*

$$\Big(\xi, k_0t^2 + (\alpha - k_1)t^3 + k_1t^2\xi + \delta(3), 2k_0t + (3\alpha - 2k_1)t^2 + 2k_1t\xi + \delta(2)\Big)\ , \tag{1}$$

where $\delta(n)$ *denotes any function of* ξ, t *with zero* n*-jet at the origin. Moreover, the Legendrian graph germ is of first type (resp., of second type) if and only if if* $k_0 \neq 0$ *(resp.,* $k_0 = 0$ *and* $k_1 \neq 0, \alpha$*).*

Proof. Consider a tangential family germ at the origin. Up to a coordinate change, preserving the $\mathscr{A}^*$-singularity of the graph, we may assume that the family support is locally the x-axis. For every x small enough, denote by $K(x)$ the curvature at $(x, 0)$ of the corresponding family curve. Now, for $x \to 0$, let $k_0 + k_1x + o(x)$ and $k_0x^2 + \alpha x^3 + o(x^3)$ be the expansions of $K(x)/2$ and of the function whose graph (near the origin) is the curve associated to $(0, 0)$.

Then, one easily verifies that the Legendrian graph of such a tangential family is parameterized by $(\xi + t, u(\xi, t) + \delta(3), \partial_t u(\xi, t) + \delta(2))$, where $u(\xi, t) := k_0t^2 + \alpha t^3 + k_1t^2\xi$. This germ can be brought to the required form by $(\xi, t) \mapsto (\xi - t, t)$.

Finally, we proved in [7] that a tangential family germ is of first type (resp., of second type) if and only if $k_0 \neq 0$ (resp., $k_0 = 0$ and $k_1 \neq 0, \alpha$). □

We can prove now Theorems 1 and 2.

Proof of Theorem 1. If $k_0 \neq 0$, then the 1-jet of (1) is $\mathscr{A}$-equivalent to $(\xi, 0, t)$, which is $\mathscr{A}$-sufficient. Therefore, Legendrian graph germs of first type are smooth.

We consider now second type Legendrian graph germs (so from now on $k_0 = 0$). First, assume k_1 different from the four values 0, α, $3\alpha/2$ and 3α. Then (1) is $\mathscr{A}$-equivalent to $(\xi, t^3 \pm t\xi^2, t^2)$, where $\pm$ is the sign of $(k_1 - 3\alpha)(\alpha - k_1)/k_1^2$. Indeed, The 3-jet of (1) is $\mathscr{A}$-equivalent to $(\xi, t^3 \pm t\xi^2, t^2)$, which is $\mathscr{A}$-sufficient (see [13], Theorem 1:2).

Hence, the Legendrian graph germs of second type have an $A_1^\pm$ singularity whenever $k_1 \neq 3\alpha/2, 3\alpha$. Let us denote by $\hat{\text{II}}$ the manifold, formed by all the second type Legendrian graph local parameterizations. The remaining second type graphs belong to the union of the two submanifolds of $\hat{\text{II}}$, defined by $2k_1 = 3\alpha$ and $3\alpha = k_1$ (dropping the intersection $\alpha = k_1 = 0$, whose elements are not of second type). It remains to consider the germs belonging to these two submanifolds.

If $3\alpha = 2k_1 \neq 0$, the 3-jet of (1) is $\mathscr{A}$-equivalent to $(\xi, t^3, t\xi)$. Then, Mond's classification (see [13], § 4.2.1) implies that the map germs (1), except those belonging to an infinite codimension submanifold of $\hat{\text{II}}$, are $\mathscr{A}$-equivalent to $(\xi, t^3, t\xi + t^{3n-1})$ for some $n \geq 2$. On the other hand, when $k_1 = 3\alpha \neq 0$, the 2-jet of (1) is $\mathscr{A}$-equivalent to $(\xi, 0, t^2)$; Mond's classification (see [13], §4.1) implies that the map germs of the form (1), except those belonging to an infinite codimension submanifold of $\hat{\text{II}}$, are $\mathscr{A}$-equivalent to $(\xi, t^3 \pm t\xi^{n+1}, t^2)$ for $n \geq 2$. □

Proof of Theorem 2. We first show that $A_1^\pm$ singularities are stable. It is well known that $(\xi, \lambda t + t^3 \pm t^2\xi, t^2)$ is a miniversal deformation of $A_1^\pm$ (the singularity being of codimension 1, see [13]). This deformation is not tangential, since it induces a beaks perestroika on the corresponding envelope. Therefore, every tangential deformation of the singularity is trivial, due to the envelope stability.

On the other hand, the non typical Legendrian graphs of second type are not stable, due to the adjacencies $A_{n+1}^\pm \to A_n^\pm$, $A_\infty \to A_n^\pm$, $H_{n+1}^\pm \to H_n^\pm$ and $H_\infty \to H_n^\pm$ (these adjacencies are obtained by small tangential deformations).

Finally, as proven in [7], a tangential family germ nether of first nor second type can be deformed into a second type tangential family germ via an arbitrary small tangential deformation. Hence, its Legendrian graph singularity is adjacent to $A_1^\pm$. □

5 Proof of Theorems 3 and 4

In this section we prove Theorems 3 and 4 (for computation details we refer to [6]). In order to follow the usual scheme for this reduction, we recall that a Finite Determinacy Theorem for the $\mathscr{A}_3^*$-equivalence relation has been proven by V. V. Goryunov in [10]; this result follows also from Damon's theory about *nice geometric subgroups of* $\mathscr{A}$ (see e.g. [9]) .

A map germ $f \in (\mathfrak{m}_{\xi,t})^3$ defines, by $f^*g := g \circ f$, a homomorphism from the ring $\mathscr{E}_{x,y,z}$ of the function germs in the target to the ring $\mathscr{E}_{\xi,t}$ of the function germs in the source. Hence, every $\mathscr{E}_{\xi,t}$-module has a structure of $\mathscr{E}_{x,y,z}$-module via this homomorphism. We define the *extended tangent space* of f as usual by

$$T_e\mathscr{A}^*(f) := \langle \partial_\xi f, \partial_t f\rangle_{\mathscr{E}_{\xi,t}} + f^*(\mathscr{E}_{x,y}) \times f^*(\mathscr{E}_{x,y}) \times f^*(\mathscr{E}_{x,y,z}) \ .$$

Note that $T_e\mathscr{A}^*(f)$ is an $\mathscr{E}_{x,y}$-module, being in general neither an $\mathscr{E}_{\xi,t}$-module nor an $\mathscr{E}_{x,y,z}$-module. The *reduced tangent space* $T_r\mathscr{A}^*(f)$ of f is by definition the $\mathscr{E}_{x,y}$-submodule of $T_e\mathscr{A}^*(f)$ defined by $\mathfrak{g}_+ + \mathcal{M}^*$, where $\mathfrak{g}_+$ is the space of all the vector field germs having positive order (see [4] for definitions) and $\mathcal{M}^*$ is the following $\mathscr{E}_{x,y}$-module:

$$f^*\left(\mathfrak{m}_{x,y}^2 \oplus \langle y\rangle_{\mathbb{R}}\right) \times f^*\left(\mathfrak{m}_{x,y}^2 \oplus \langle x\rangle_{\mathbb{R}}\right) \times f^*\left(\mathfrak{m}_{x,y,z}^2 \oplus \langle x, y\rangle_{\mathbb{R}}\right) \ .$$

The main tool in the proof of Theorems 3 and 4 is the following easy fact.

Lemma 2. *Consider $f \in (\mathfrak{m}_{\xi,t})^3$ and R a triple of homogeneous polynomials of degree p, q and r, such that $R \in T_r\mathscr{A}^*(f)/(\mathfrak{m}_{\xi,t}^{p+1} \times \mathfrak{m}_{\xi,t}^{q+1} \times \mathfrak{m}_{\xi,t}^{r+1})$. Then the (p, q, r)-jets of f and $f + R$ are $\mathscr{A}^*$-equivalent.*

We can start now the proof of Theorem 3.

Proof of Theorem 3. We consider first Legendrian graphs of first type tangential family germs ($k_0 \neq 0$). Then the 2-jet of (1) is $\mathscr{A}^*$-equivalent to (ξ, t^2, t), which is $\mathscr{A}^*$-sufficient, since its reduced tangent space contains $\mathfrak{m}_{\xi,t}^2 \times \mathfrak{m}_{\xi,t}^3 \times \mathfrak{m}_{\xi,t}^2$ (Lemma 2).

We consider now Legendrian graphs of second type tangential family germs ($k_0 = 0$). In this case, every map germ (1) is $\mathscr{A}^*$-equivalent to $(\xi, t^3 + t^2\xi + at\xi^2 + \delta(3), t^2 + \delta(2))$, where $a := (\alpha - k_1)(k_1 - 3\alpha)/k_1^2$. We remark that $a < 1/3$; indeed, we have $1 - 3a = (3\alpha - 2k_1)^2/k_1^2 >$

0, since $3\alpha \neq 2k_1$. Actually, a further computation shows that its $(2,4,3)$-jet is $\mathscr{A}^*$-equivalent to $F_{a,b}$, for a suitable $b \in \mathbb{R}$ ($F_{a,b}$ is the normal for defined in Section 3). Hence, the statement follows from Lemma 2 and the next inclusion, which holds for $a \neq -1, 0, 1/3$:

$$\mathfrak{m}^3_{\xi,t} \times \mathfrak{m}^5_{\xi,t} \times \mathfrak{m}^4_{\xi,t} \subset T_r\mathscr{A}^*(F_{a,b}) \ . \tag{2}$$

When the Legendrian graph has an $A_1^\pm$ singularity, the conditions $a \neq 0, 1/3$ are automatically fulfilled. On the other hand, $a \neq -1$ is a new condition, equivalent to $\alpha = 0$, giving rise to the submanifold for which Theorem 3 does not hold. □

Proof of Theorem 4. Since $\langle \xi, t^3 + t^2\xi + at\xi^2 \rangle_{\mathscr{E}_{\xi,t}} = \langle \xi, t^3 \rangle_{\mathscr{E}_{\xi,t}}$ and

$$\langle \xi, t^3 + t^2\xi + at\xi^2, t^2 + bt^3 \rangle_{\mathscr{E}_{\xi,t}} = \langle \xi, t^2 \rangle_{\mathscr{E}_{\xi,t}} \ ,$$

the well known Preparation Theorem of Mather–Malgrange (see e.g. [4]) implies that $\mathscr{E}_{\xi,t}$ is generated by $\{t, t^2\}$ as $\mathscr{E}_{x,y}$-module and by t as $\mathscr{E}_{x,y,z}$-module. Hence, we have:

$$\mathscr{E}^3_{\xi,t} = F^*_{a,b}(\mathscr{E}_{x,y}) \cdot \left\{ \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}, \begin{pmatrix} t^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t^2 \\ 0 \end{pmatrix} \right\} + F^*_{a,b}(\mathscr{E}_{x,y,z}) \cdot \left\{ \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \right\} \ .$$

For $b \neq 0$, we obtain

$$\mathscr{E}^3_{\xi,t} = T_e\mathscr{A}^*(F_{a,b}) \oplus \mathbb{R} \cdot \left\{ \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}, \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z \\ 0 \end{pmatrix} \right\} \ .$$

This proves the Theorem. □

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