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FISHER'S INFORMATION FOR DISCRETELY SAMPLED LEVY PROCESSES

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This paper studies the asymptotic behavior of the Fisher information for a Lévy process discretely sampled at an increasing frequency. We show that it is possible to distinguish not only the continuous part of the process from its jumps part, but also different types of jumps, and derive the rates of convergence of efficient estimators.

1. Introduction. Models allowing for sample path discontinuities are of considerable interest in mathematical finance, for instance in option pricing [see e.g., Eberlein and Jacod (1997), Chan (1999), Boyarchenko and Levendorskii (2002), Mordecki (2002) and Carr and Wu (2004)], testing for the presence of jumps in asset prices [see Aït-Sahalia (2002) and Carr and Wu (2003)], interest rate modelling [see e.g., Eberlein and Raible (1999)], risk management [see e.g., Eberlein et al. (1998) and Khindanova et al. (2001)], optimal portfolio choice [see e.g., Kallsen (2000), Rachev and Han (2000) and Emmer and Klüppelberg (2004)], stochastic volatility modelling [see e.g., Barndorff-Nielsen (1997), Barndorff-Nielsen (1998), Leblanc and Yor (1998), Carr et al. (2003) and Klüppelberg et al. (2004)] or for the purpose of better describing asset returns data [see e.g., Mandelbrot (1963), Fama and Roll (1965), Mittnik and Rachev (2001), Carr et al. (2002)].

While these theoretical models are commonly used in mathematical finance, relatively little is known about the corresponding inference problem, which is a difficult one. A string of the literature focuses on the tail properties of stable processes to estimate the stable index [see e.g., Fama and Roll (1968), Fama and Roll (1971), de Haan and Resnick (1980), Dumouchel (1983), McCulloch (1997)]. Since Lévy processes have known characteristic functions, given by the Lévy-Khintchine formula, a method often proposed is based on the empirical characteristic function as an estimating equation [see e.g., Press (1972), Fenech (1976), Feuerverger and McDunnough (1981b), Chapter 4 in Zolotarev (1986) and Singleton (2001)], maximum likelihood by Fourier inversion of the characteristic

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function [see Feuerverger and McDunnough (1981a)], or a regression based on the explicit form of the characteristic function [see Koutrouvelis (1980)]. Some of these methods were compared in Akgiray and Lamoureux (1989).

Fairly little is known in most cases as to the optimality of statistical procedures in the presence of jumps. So we consider in this paper the behavior of the Fisher information when the observations are generated by a Lévy process X whose law depends on a parameter vector η to be estimated. In light of the Cramer-Rao bound, our objective is to establish the optimality of potential estimators of η , and the rate at which they will converge. While we focus on its implications for the classical likelihood inference problem, Fisher's information also plays as usual an important role in Bayesian inference or in determining the form of asymptotically most powerful tests.

The essential difficulty in this class of problems is the fact that the density of most discretely sampled Lévy processes, hence the corresponding likelihood function and Fisher's information, are not known in closed form. Representations in terms of special functions are available [see Zolotarev (1995) in terms of Meijer G -functions and Hoffmann-Jørgensen (1993) in terms of incomplete hypergeometric functions] although they do not appear to lead to practical formulae. One must therefore rely on numerical methods as the maximum likelihood estimator cannot be computed exactly [see Dumouchel (1971) for a multinomial approximation to the likelihood function, and Nolan (1997) and Nolan (2001)]. Therefore, there is potential value in considering alternative estimators which can both be computed explicitly and be rate-efficient. Indeed, in a companion paper [Aït-Sahalia and Jacod (2004)], we propose estimators designed to achieve the efficient rate that we identify in this paper based on the convergence properties of the Fisher information.

Let us be more specific. The Lévy process X is observed at n times $\Delta, 2\Delta, \dots, n\Delta$. Recalling that $X_0 = 0$, this amounts to observing the n increments $X_{i\Delta} - X_{(i-1)\Delta}$. So when $\Delta > 0$ is fixed, we observe n i.i.d. variables distributed as $X_\Delta - X_0$ and having a density which depends smoothly on the parameter η , and we are on known grounds: the Fisher information at stage n has the form $I_{n,\Delta}(\eta) = nI_\Delta(\eta)$, where $I_\Delta(\eta) > 0$ is the Fisher information (an invertible matrix if η is multi-dimensional) of the model based upon the observation of the single variable $X_\Delta - X_0$; we have the LAN property with rate \sqrt{n} ; the asymptotically efficient estimators $\hat{\eta}_n$ are those for which $\sqrt{n}(\hat{\eta}_n - \eta)$ converges in law to the normal distribution $\mathcal{N}(0, I_\Delta(\eta)^{-1})$, and the MLE does the job [see e.g., Dumouchel (1973a)].

Things become more complicated when the time interval separating successive observations, Δ , varies, and more specifically in the limit when it becomes small. This type of asymptotics corresponds to a situation which is increasingly common in financial applications, where high frequency data are available with sampling intervals measured in seconds for many stocks or currencies. At stage n we have n observations, recorded at times $i\Delta_n$ for some time lag Δ_n going to 0. Equivalently, we observe the n increments $\chi_i^n = X_{i\Delta_n} - X_{(i-1)\Delta_n}$ which are i.i.d. for any given n . These increments are known as the log-returns when X is the log of the price of a financial asset. However the law of χ_1^n depends on n , and indeed weakly converges to the Dirac mass at 0. The Fisher information at stage n still has the form $I_{n,\Delta_n}(\eta) = nI_{\Delta_n}(\eta)$, but the behavior of the information $I_{\Delta_n}(\eta)$ is far from obvious.

In order to say more about the behavior of Fisher's information, we need of course to parametrize the model. For the same reason that the computation of the MLE is hindered by the absence of an explicit density, the analysis of the Fisher information matrix is difficult. Dumouchel (1973b) and Dumouchel (1975) computed the information by numerical approximation of the densities and their derivatives. Such direct computation is numerically cumbersome because the series expansion for the density converges slowly, especially when the order of the stable process is near one. Brockwell and Brown (1980) propose an alternative numerical computation of the information based on a Fourier series for the derivatives of the density.

In this paper, we are able to explicitly describe, in closed form, the limiting behavior of the Fisher information when we restrict attention to a special kind of Lévy process that is relevant to applications in financial statistics. While our form of the process is undoubtedly restrictive, it is nevertheless sufficiently rich to exhibit a surprising range of different asymptotic behaviors for the Fisher information. In fact, we will show that different rates of convergence are achieved for different parameters, and for different types of Lévy processes. Rates depart from the standard \sqrt{n} in a number of different and often unexpected ways.

Specifically, we split X into the sum of two independent Lévy processes, with possibly one or two scale parameters. That is, we suppose that

$$(1) \quad X_t = \sigma W_t + \theta Y_t.$$

Here, we have $\sigma > 0$ and $\theta \in \mathbb{R}$, and W is a standard symmetric stable process with index $\beta \in (0, 2]$, and we are often interested in the situation where $\beta = 2$ and so W is a Wiener process (hence the notation used). As for Y , it is another Lévy process, viewed as a perturbation of W . In some

applications, Y may represent frictions that are due to the mechanics of the trading process, or in the case of compound Poisson jumps it may represent the infrequent arrival of relevant information related to the asset. In the latter case, W is then the driving process for the ordinary fluctuations of the asset value. Y is independent of W , and its law is either known or is a nuisance parameter, and is *dominated* by W in a sense stated below. For example, when W is a Brownian motion, this just means that Y has no Brownian part; when $\beta < 2$, then Y could for example be another stable process with index $\alpha < \beta$, or a compound Poisson process. The parameter vector we then consider is $\eta = (\sigma, \beta, \theta)$.

If Y is viewed as a perturbation of W , then our interest in studying the Fisher information lies in deciding whether we can estimate the parameter σ , and also in some cases the index β (the only two parameters on which the law of the process σW depends) with the same degree of accuracy as when the process Y is absent, at least asymptotically. The answer to this question is “yes”. When W is a Brownian motion this means that one can distinguish between the jumps due to Y and the continuous part of X , and this fact was already known in the specific example of a Brownian motion coupled with either a Poisson or Cauchy process [see Aït-Sahalia (2004)]. It comes more as a surprise when $\beta < 2$: we can then discriminate between the jumps due to W and those due to Y , despite the fact that both processes jump and we only have discrete observations.

The paper is organized as follows. In Section 2, we set up the problem and define in particular the class of processes Y that are dominated by W . In Section 3, we study the baseline case where $X_t = \sigma W_t$ and establish the properties of the Fisher information in the absence of the perturbation process Y . In Section 4, we characterize the set of processes Y whose presence does not affect the estimation of the base parameters (σ, β) . Then we study in Section 5 the estimation problem for the dominated scale parameter θ . In this case, the results vary substantially according to the structure of the process Y , and we illustrate the versatility of the situation by displaying the variety of convergence rates that arise.

Finally, we also briefly consider in Section 6 a slightly different model, where

$$(2) \quad X_t = \sigma(W_t + Y_t).$$

Here it is natural to consider the law of Y and the index β as known, and σ to be the only parameter to be estimated. The results are again a bit unexpected, namely one can do as well as when Y is

absent, and in some instances (when the law of Y is sufficiently singular) the presence of Y can in fact help us improve the estimation of σ .

All proofs are in Section 7.

2. Setup. The characteristic function of W_t is

$$(3) \quad \mathbb{E}(e^{iuW_t}) = e^{-t|u|^\beta/2}$$

The factor 2 above is unusual for stable processes when $\beta < 2$, but we put it here to ensure continuity between the stable and the Gaussian cases. As is well known, when $\beta < 2$ we have $\mathbb{E}(|W_t|^\rho) < \infty$ if and only if $0 < \rho < \beta$, and the tails of W_1 behave according to $\mathbb{P}(W_1 > w) \sim c_\beta/\beta w^\beta$ as $w \rightarrow \infty$ (and symmetrically as $w \rightarrow -\infty$), where the constants c_β are given by

$$(4) \quad c_\beta = \begin{cases} \frac{\beta(1-\beta)}{4 \Gamma(2-\beta) \cos(\beta\pi/2)} & \text{if } \beta \neq 1 \\ \frac{1}{2\pi} & \text{if } \beta = 1 \end{cases}$$

This follows from the series expansion of the density due to Bergström (1952), the duality property of the stable densities of order β and $1/\beta$ [see e.g., Chapter 2 in Zolotarev (1986)], with an adjustment factor to reflect our definition of the characteristic function in (3).

The law of Y (as a process) is entirely specified by the law G_Δ of the variable Y_Δ for any given $\Delta > 0$. We write $G = G_1$, and we recall that the characteristic function of G_Δ is given by the Lévy-Khintchine formula

$$(5) \quad \mathbb{E}(e^{ivY_\Delta}) = \exp \Delta \left(ivb - \frac{cv^2}{2} + \int F(dx) (e^{ivx} - 1 - ivx1_{\{|x| \leq 1\}}) \right)$$

where (b, c, F) is the “characteristic triple” of G (or, of Y): $b \in \mathbb{R}$ is the drift of Y , and $c \geq 0$ the local variance of the continuous part of Y , and F is the Lévy jump measure of Y , which satisfies $\int (1 \wedge x^2) F(dx) < \infty$ [see e.g., Chapter II.2 in Jacod and Shiryaev (2003)].

The fact that Y is “dominated” by W is expressed by the property that G belongs to the class \mathcal{G}_β which we define as follows. Let first Φ be the class of all increasing and bounded functions $\phi: (0, 1] \rightarrow \mathbf{R}_+$ having $\lim_{x \downarrow 0} \phi(x) = 0$. Then we set

$$(6) \quad \mathcal{G}(\phi, \alpha) = \text{the set of all infinitely divisible distributions with } c = 0 \text{ and, for all } x \in (0, 1],$$

$$\begin{cases} x^\alpha F([-x, x]^c) \leq \phi(x) & \text{if } \alpha < 2 \\ x^2 F([-x, x]^c) \leq \phi(x) \quad \text{and} \quad \int_{\{|y| \leq x\}} |y|^2 F(dy) \leq \phi(x) & \text{if } \alpha = 2, \end{cases}$$

$$(7) \quad \mathcal{G}_\alpha = \cup_{\phi \in \Phi} \mathcal{G}(\phi, \alpha).$$

We have $\lim_{x \downarrow 0} x^\alpha F([-x, x]^c) = 0$ if and only if the function $\phi(y) = \sup_{x \in (0, y]} x^\alpha F([-x, x]^c)$ belongs to Φ , whereas $\int_{\{|y| \leq x\}} |y|^2 F(dy)$ always decreases to 0 as $x \downarrow 0$, so we also have another, simpler, description of \mathcal{G}_α for all $\alpha \in (0, 2]$:

$$(8) \quad \mathcal{G}_\alpha = \left\{ G \text{ is infinitely divisible, } c = 0, \lim_{x \downarrow 0} x^\alpha F([-x, x]^c) = 0 \right\}.$$

We also have for any $0 < x < y \leq 1$:

$$x^2 F([-x, x]^c) \leq x^2 F([-y, y]^c) + \int_{\{\|z\| \leq y\}} z^2 F(dz),$$

from which we deduce that $x^2 F([-x, x]^c) \rightarrow 0$ as $x \downarrow 0$ for any infinitely divisible G . Therefore, \mathcal{G}_2 is indeed the set of all infinitely divisible laws G such that $c = 0$. Obviously $\alpha < \alpha'$ implies $\mathcal{G}_\alpha \subset \mathcal{G}_{\alpha'}$. If G is a (non necessarily symmetric) stable law with index γ it belongs to \mathcal{G}_α for all $\alpha > \gamma$, but not to \mathcal{G}_γ . If Y is a compound Poisson process plus a drift, then G is in $\cup_{\alpha > 0} \mathcal{G}_\alpha$.

The variables under consideration have densities which depend smoothly on the parameters, so Fisher's information is an appropriate tool for studying the optimality of estimators. In the basic case of the model (1), the law of the observed process X depends on the three parameters σ , β , θ to be estimated, plus on the law of Y which is summarized by G . The law of the variable X_Δ has a density which depends smoothly on σ and θ , so that the 2×2 Fisher information matrix (relative to σ and θ) of our experiment exists; it also depends smoothly on β when $\beta < 2$, so in this case the 3×3 Fisher information matrix exists. In all cases we denote it by $I_{n, \Delta_n}(\sigma, \beta, \theta, G)$, and it has the form

$$I_{n, \Delta_n}(\sigma, \beta, \theta, G) = n I_{\Delta_n}(\sigma, \beta, \theta, G),$$

where $I_\Delta(\sigma, \beta, \theta, G)$ is the Fisher information matrix associated with the observation of a single variable X_Δ . We denote the elements of the matrix $I_\Delta(\sigma, \beta, \theta, G)$ as $I_\Delta^{\sigma\sigma}(\sigma, \beta, \theta, G)$, $I_\Delta^{\sigma\beta}(\sigma, \beta, \theta, G)$, etc. We may occasionally drop G , but at this stage it is mentioned because it may appear as a nuisance parameter in our model and we wish to have estimates for the Fisher information that are *uniform* in G , at least on some reasonable class of G 's. Let us also mention that in many cases the parameter β is indeed known: this is particularly true when W is a Brownian motion.

For the other model (2) the Fisher information for estimating σ (a positive number here) is now denoted by $I'_{n,\Delta_n}(\sigma, \beta, G)$, and it still has the form

$$I'_{n,\Delta_n}(\sigma, \beta, G) = n I'_{\Delta_n}(\sigma, \beta, G),$$

with $I'_{\Delta}(\sigma, \beta, G)$ being the Fisher information associated with the observation of a single variable X_{Δ} .

3. The baseline case: estimating the parameters of the stable process $X = \sigma W$. In this section we consider the base case $Y = 0$, that is we observe the stable process $X = \sigma W$ with scale parameter $\sigma > 0$ and index parameter $\beta \in (0, 2]$. In our general framework, this corresponds to the situation where $G = \delta_0$, a Dirac mass at 0, and we set the (now unidentified) parameter θ to 0, or for that matter any arbitrary value.

We have only the two parameters σ and β here, and our objective in this section is to compute the Fisher information matrix in this case:

$$I_{\Delta}(\sigma, \beta, 0, \delta_0) = \begin{pmatrix} I_{\Delta}^{\sigma\sigma}(\sigma, \beta, 0, \delta_0) & I_{\Delta}^{\sigma\beta}(\sigma, \beta, 0, \delta_0) \\ I_{\Delta}^{\sigma\beta}(\sigma, \beta, 0, \delta_0) & I_{\Delta}^{\beta\beta}(\sigma, \beta, 0, \delta_0) \end{pmatrix}.$$

In future sections, we will examine how the terms in $I_{\Delta}(\sigma, \beta, \theta, G)$ relate to those in $I_{\Delta}(\sigma, \beta, 0, \delta_0)$.

3.1. *The scale parameter σ .* By the scaling property of symmetric stable processes, which says that W_{Δ} and $\Delta^{1/\beta}W_1$ have the same law, it is intuitively clear that $I_{\Delta}^{\sigma\sigma}(\sigma, \beta, 0, \delta_0)$ does not depend on Δ . Indeed, let us denote by h_{β} the density of W_1 , which is defined through (3). The density of $X_{\Delta} = \sigma W_{\Delta}$ is

$$p_{\Delta}(x|\sigma, \beta, 0, \delta_0) = \frac{1}{\sigma\Delta^{1/\beta}} h_{\beta}\left(\frac{x}{\sigma\Delta^{1/\beta}}\right).$$

It is well known that h_{β} is C^{∞} (by repeated integration of the characteristic function), even, and that its n -th derivative $h_{\beta}^{(n)}$ behaves as follows (the first two derivatives are denoted h' and h''):

$$(9) \quad \left| h_{\beta}^{(n)}(w) \right| \sim \begin{cases} \frac{c_{\beta}(1+\beta)(2+\beta)\dots(n-1+\beta)}{|w|^{n+1+\beta}} & \text{if } \beta < 2 \\ |w|^n e^{-w^2/2} / \sqrt{2\pi} & \text{if } \beta = 2 \end{cases} \quad \text{as } |w| \rightarrow \infty.$$

where c_{β} is given in (4); this result follows from the same series expansion as above). Let us also associate with h_{β} the following functions:

$$(10) \quad \check{h}_{\beta}(w) = h_{\beta}(w) + wh'_{\beta}(w), \quad \tilde{h}_{\beta}(w) = \frac{\check{h}_{\beta}(w)^2}{h_{\beta}(w)}.$$

Then \tilde{h}_β is positive, even, continuous, and $\tilde{h}_\beta(w) = 0(1/|w|^{1+\beta})$ as $|w| \rightarrow \infty$, hence \tilde{h}_β is Lebesgue-integrable.

Consider now

$$(11) \quad \mathcal{I}(\beta) = \int \tilde{h}_\beta(w) dw,$$

which is well defined and positive. Moreover if $\beta = 2$ (W is then Brownian motion), h_2 is Gaussian and we have $h_2'(w) = -wh_2(w)$, so $\tilde{h}_2(w) = (1 - w^2 + w^4)h_2(w)$ and

$$(12) \quad \beta = 2 \quad \Rightarrow \quad \mathcal{I}(\beta) = 2.$$

The Fisher information for σ associated with the observation of a single variable $X_\Delta = \sigma W_\Delta$ is

$$I_\Delta^{\sigma\sigma}(\sigma, \beta, 0, \delta_0) = \int \frac{(\partial_\sigma p_\Delta(x|\sigma, \beta, 0, \delta_0))^2}{p_\Delta(x|\sigma, \beta, 0, \delta_0)} dx = \frac{1}{\sigma^3 \Delta^{1/\beta}} \int \tilde{h}_\beta(x/\sigma \Delta^{1/\beta}) dx$$

which, in light of (11) and by a change of variable, reduces to:

$$(13) \quad I_\Delta^{\sigma\sigma}(\sigma, \beta, 0, \delta_0) = \frac{1}{\sigma^2} \mathcal{I}(\beta).$$

So, as said before, this does not depend on Δ . In fact, $\mathcal{I}(\beta)$ is simply the Fisher information at point $\sigma = 1$ for the statistical model in which we observe σW_1 and W_1 is a variable with density h_β .

3.2. The index parameter β . Consider now the estimation of β . This problem was studied by Dumouchel (1973a), who computed numerically the term $I_\Delta^{\beta\beta}(\sigma, \beta, 0, \delta_0)$, including also an asymmetry parameter. It is easily seen that $\beta \mapsto h_\beta(w)$ is differentiable on $(0, 2]$, and we denote by $\dot{h}_\beta(w)$ its derivative. However, instead of (9) one has

$$(14) \quad \left| \dot{h}_\beta(w) \right| \sim \begin{cases} \frac{c_\beta \log |w|}{|w|^{1+\beta}} & \text{if } \beta < 2 \\ \frac{1}{|w|^3} & \text{if } \beta = 2, \end{cases}$$

as $|w| \rightarrow \infty$, by differentiation of the series expansion for the stable density. Therefore the quantity

$$(15) \quad \mathcal{K}(\beta) = \int \frac{\dot{h}_\beta(w)^2}{h_\beta(w)} dw$$

is finite when $\beta < 2$ and infinite for $\beta = 2$. This is the Fisher information for estimating β , upon observing the single variable W_1 .

Instead of computing the information quantities numerically based on approximations of the stable density h_β , we study explicitly their asymptotic behavior as $\Delta \rightarrow 0$. Excluding the degenerate case where $\beta = 2$ [see Dumouchel (1983) for the behavior of the MLE for β when $\beta = 2$], the Fisher information for β associated with the observation of a single variable σW_Δ when $\beta < 2$ is

$$\begin{aligned} I_\Delta^{\beta\beta}(\sigma, \beta, 0, \delta_0) &= \int \frac{(\partial_\beta p_\Delta(x|\sigma, \beta, 0, \delta_0))^2}{p_\Delta(x|\sigma, \beta, 0, \delta_0)} dx = \int \frac{\left(\log(\Delta)\check{h}_\beta(w) + \beta^2\dot{h}_\beta(w)\right)^2}{\beta^4 h_\beta(w)} dw \\ &= \frac{(\log(\Delta))^2}{\beta^4} \mathcal{I}(\beta) + \frac{2\log(\Delta)}{\beta^2} \int \frac{\check{h}_\beta(w)\dot{h}_\beta(w)}{h_\beta(w)} dw + \mathcal{K}(\beta), \end{aligned}$$

and the middle integral in the last display is smaller than $\sqrt{\mathcal{I}(\beta)\mathcal{K}(\beta)}$ by Cauchy–Schwarz. Therefore, as $\Delta \rightarrow 0$, we have

$$(16) \quad \frac{I_\Delta^{\beta\beta}(\sigma, \beta, 0, \delta_0)}{(\log(1/\Delta))^2} \rightarrow \frac{1}{\beta^4} \mathcal{I}(\beta).$$

3.3. *The cross (σ, β) term.* As for the cross-term, when of course $\beta < 2$ again, we have

$$\begin{aligned} I_\Delta^{\sigma\beta}(\sigma, \beta, 0, \delta_0) &= \int \frac{\partial_\sigma p_\Delta(x|\sigma, \beta, 0, \delta_0)}{p_\Delta(x|\sigma, \beta, 0, \delta_0)} \frac{\partial_\beta p_\Delta(x|\sigma, \beta, 0, \delta_0)}{p_\Delta(x|\sigma, \beta, 0, \delta_0)} dx \\ &= -\frac{1}{\sigma\beta^2} \int \frac{\check{h}_\beta(w) \left(\log(\Delta)\check{h}_\beta(w) - \beta^2\dot{h}_\beta(w)\right)}{h_\beta(w)} dw \\ &= -\frac{\log(\Delta)}{\sigma\beta^2} \mathcal{I}(\beta) - \frac{1}{\sigma} \int \frac{\check{h}_\beta(w)\dot{h}_\beta(w)}{h_\beta(w)} dw. \end{aligned}$$

Therefore, as $\Delta \rightarrow 0$, we have

$$(17) \quad \frac{I_\Delta^{\sigma\beta}(\sigma, \beta, 0, \delta_0)}{\log(1/\Delta)} \rightarrow \frac{1}{\sigma\beta^2} \mathcal{I}(\beta).$$

3.4. *The information for a translation model.* We will see another information appear in some of the forthcoming formulas, namely the Fisher information associated with the estimation of the real number a for the model where one observes the single variable $W_1 + a$. This Fisher information is of course the following number:

$$(18) \quad \mathcal{J}(\beta) = \int \frac{h'_\beta(w)^2}{h_\beta(w)} dw.$$

Observe in particular that

$$(19) \quad \beta = 2 \quad \Rightarrow \quad \mathcal{J}(\beta) = 1.$$

3.5. *Some consequences for the estimation.* If now we come back to our setting where n values (or increments) of $X = \sigma W$ are observed along a time lag Δ_n , we see that when β is known we can hope for estimators $\hat{\sigma}_n$ for σ which are asymptotically efficient in the sense that $\sqrt{n}(\hat{\sigma}_n - \sigma)$ converges in law to $\mathcal{N}(0, \sigma^2/\mathcal{I}(\beta))$, whatever Δ_n behaves like as $n \rightarrow \infty$, and of course the MLE satisfies that.

When it comes to estimating β , things are different. When $\Delta_n \rightarrow 0$ and when the true value is $\beta < 2$, we can hope for estimators converging to β with the faster rate $\sqrt{n} \log(1/\Delta_n)$ (and we provide such estimators in Aït-Sahalia and Jacod (2004)). Some of these estimators are constructed using the jumps of the process of a size greater than some threshold, as in Höpfner and Jacod (1994). Note also that if we suspect that $\beta = 2$ we would rather perform a test, as advised by Dumouchel (1973a), and anyway in this case the behavior of the Fisher information does not provide much insight.

4. The general semiparametric case. The data generating process is now given by (1). We are interested in estimating (σ, β) , and in some instances θ as well, leaving the distribution $G \in \mathcal{G}_\beta$ unspecified.

4.1. *Estimation of (σ, β) .* We start by studying whether the limiting behavior of $I_\Delta^{\sigma\sigma}(\sigma, \beta, \theta, G)$ when $\beta = 2$ and of the (σ, β) block of the matrix $I_\Delta(\sigma, \beta, \theta, G)$ when $\beta < 2$ is affected by the presence of Y . First, we have the intuitively obvious majoration of Fisher's information in presence of Y by the one for which Y is absent. Note that in this result no assumption whatsoever is made on Y (except of course that it is independent of W):

Theorem 1. *For any $\Delta > 0$ we have*

$$(20) \quad I_\Delta^{\sigma\sigma}(\sigma, 2, \theta, G) \leq I_\Delta^{\sigma\sigma}(\sigma, 2, 0, \delta_0)$$

and, when $\beta < 2$, the difference

$$\begin{pmatrix} I_\Delta^{\sigma\sigma}(\sigma, \beta, 0, \delta_0) & I_\Delta^{\sigma\beta}(\sigma, \beta, 0, \delta_0) \\ I_\Delta^{\sigma\beta}(\sigma, \beta, 0, \delta_0) & I_\Delta^{\beta\beta}(\sigma, \beta, 0, \delta_0) \end{pmatrix} - \begin{pmatrix} I_\Delta^{\sigma\sigma}(\sigma, \beta, \theta, G) & I_\Delta^{\sigma\beta}(\sigma, \beta, \theta, G) \\ I_\Delta^{\sigma\beta}(\sigma, \beta, \theta, G) & I_\Delta^{\beta\beta}(\sigma, \beta, \theta, G) \end{pmatrix}$$

is a positive semi-definite matrix, and in particular we have:

$$(21) \quad I_\Delta^{\beta\beta}(\sigma, \beta, \theta, G) \leq I_\Delta^{\beta\beta}(\sigma, \beta, 0, \delta_0).$$

Next, how does the limit as $\Delta \rightarrow 0$ of $I_\Delta(\sigma, \beta, \theta, G)$ compare to that of $I_\Delta(\sigma, \beta, 0, \delta_0)$? For instance, given that in the absence of Y we can estimate σ with information $\mathcal{I}(\beta)/\sigma^2$, we would like to find

out what is the impact, if any, of the presence of Y on the information we can gather about that parameter from the discrete observations where W is perturbed by Y :

$$\chi_i^n = \sigma(W_{i\Delta_n} - W_{(i-1)\Delta_n}) + \theta(Y_{i\Delta_n} - Y_{(i-1)\Delta_n})$$

for $i = 1, \dots, n$.

The answer to that question is given by the following.

Theorem 2. a) If $G \in \mathcal{G}_\beta$ we have as $\Delta \rightarrow 0$:

$$(22) \quad I_\Delta^{\sigma\sigma}(\sigma, \beta, \theta, G) \rightarrow \frac{1}{\sigma^2} \mathcal{I}(\beta),$$

and also, when $\beta < 2$:

$$(23) \quad \frac{I_\Delta^{\beta\beta}(\sigma, \beta, \theta, G)}{(\log(1/\Delta))^2} \rightarrow \frac{1}{\beta^4} \mathcal{I}(\beta), \quad \frac{I_\Delta^{\sigma\beta}(\sigma, \beta, \theta, G)}{\log(1/\Delta)} \rightarrow \frac{1}{\sigma\beta^2} \mathcal{I}(\beta)$$

b) For any $\phi \in \Phi$ and $\alpha \in (0, \beta]$ and $K > 0$, we have as $\Delta \rightarrow 0$:

$$(24) \quad \left\{ \begin{array}{l} \sup_{G \in \mathcal{G}(\phi, \alpha), |\theta| \leq K} \left| I_\Delta^{\sigma\sigma}(\sigma, \beta, \theta, G) - \frac{\mathcal{I}(\beta)}{\sigma^2} \right| \rightarrow 0, \\ \beta < 2 \Rightarrow \left\{ \begin{array}{l} \sup_{G \in \mathcal{G}(\phi, \alpha), |\theta| \leq K} \left| \frac{I_\Delta^{\beta\beta}(\sigma, \beta, \theta, G)}{(\log(1/\Delta))^2} - \frac{\mathcal{I}(\beta)}{\beta^4} \right| \rightarrow 0, \\ \sup_{G \in \mathcal{G}(\phi, \alpha), |\theta| \leq K} \left| \frac{I_\Delta^{\sigma\beta}(\sigma, \beta, \theta, G)}{\log(1/\Delta)} - \frac{\mathcal{I}(\beta)}{\sigma\beta^2} \right| \rightarrow 0. \end{array} \right. \end{array} \right.$$

c) For each n , let G^n be the standard symmetric stable law of index α_n , with α_n a sequence strictly increasing to β . Then for any sequence $\Delta_n \rightarrow 0$ such that $(\beta - \alpha_n) \log \Delta_n \rightarrow 0$ (i.e. the rate at which $\Delta_n \rightarrow 0$ is slow enough), the sequence of numbers $I_{\Delta_n}^{\sigma\sigma}(\sigma, \beta, \theta, G^n)$ (resp. $I_{\Delta_n}^{\beta\beta}(\sigma, \beta, \theta, G^n)/(\log(1/\Delta_n))^2$ when further $\beta < 2$) converges to a limit which is strictly less than $\mathcal{I}(\beta)/\sigma^2$ (resp. $\mathcal{I}(\beta)/\beta^4$).

In other words, at their respective leading orders in Δ , the presence of Y has no impact on the information terms $I_\Delta^{\sigma\sigma}$, $I_\Delta^{\beta\beta}$ and $I_\Delta^{\sigma\beta}$, as soon as Y is “dominated” by W : so, in the limit where $\Delta \rightarrow 0$, the parameters σ and β can be estimated with the exact same degree of precision whether Y is present or not. Moreover, part (b) states the convergence of Fisher’s information is *uniform* on the set $\mathcal{G}(\phi, \alpha)$ and $|\theta| \leq K$ for all $\alpha \in [0, \beta]$; this settles the case where G and θ are considered as nuisance parameters when we estimate σ and β .

But as α tends to β , the *convergence disappears*, as stated in part (c). This shows that the class \mathcal{G}_β is effectively the largest one for which the presence of a Y process does not affect the estimation

of the parameters of the process σW . For example, if $\beta = 2$, in part (c) take G^n to be the symmetric stable law with index $\alpha_n \in (0, 2)$ and scale parameter s in the sense that its characteristic function is $u \mapsto \exp\left(-\frac{s^2}{2}|u|^{\alpha_n}\right)$. Then if $\alpha_n \rightarrow 2$, for all sequences $\Delta_n \rightarrow 0$ satisfying $(2 - \alpha_n) \log \Delta_n \rightarrow 0$ we have

$$I_{\Delta_n}^{\sigma\sigma}(\sigma, \beta, 1, G^n) \rightarrow \frac{2}{\sigma^2 + s^2}.$$

This is of course to be expected, since in the limit we are observing $\sqrt{\sigma^2 + s^2} W$, and we supposedly know s and wish to estimate σ .

Another interesting feature, due to the fact that $\Delta \rightarrow 0$, is that the limiting behavior of $I_{\Delta}^{\beta\beta}$ and $I_{\Delta}^{\sigma\beta}$ when $\beta < 2$ involves $\mathcal{I}(\beta)$ but not $\mathcal{K}(\beta)$, as one could have guessed at first glance.

4.2. Estimation of θ . For the entries of Fisher's information matrix involving the parameter θ , things are more complicated. First, observe that $I_{\Delta}^{\theta\theta}(0, \beta, \theta, G)$ (that is the Fisher information for the model $X = \theta Y$) does not necessarily exist, but of course if it does we have an inequality similar to (20) for all σ :

$$(25) \quad I_{\Delta}^{\theta\theta}(\sigma, \beta, \theta, G) \leq I_{\Delta}^{\theta\theta}(0, \beta, \theta, G).$$

Contrary to (20), however, this is a very rough estimate, which does not take into account the properties of W . The (θ, θ) -Fisher information is usually much smaller than what the right side above suggests, and we give below a more accurate estimate when Y has second moments, but without the "domination" assumption that $G \in \mathcal{G}_{\beta}$. Recall the notation (18).

Theorem 3. *If Y_1 has a finite variance v and a mean m , we have*

$$(26) \quad I_{\Delta}^{\theta\theta}(\sigma, \beta, \theta, G) \leq \frac{\mathcal{J}(\beta)}{\sigma^2} \left(m^2 \Delta^{2-2/\beta} + v \Delta^{1-2/\beta} \right).$$

This estimate holds for all $\Delta > 0$. The asymptotic variant, which says that

$$(27) \quad \limsup_{\Delta \rightarrow 0} \Delta^{2/\beta-1} I_{\Delta}^{\theta\theta}(\sigma, \beta, \theta, G) \leq \frac{\delta \mathcal{J}(\beta)}{\sigma^2},$$

is sharp in some cases and not in others, as we will see in the examples below. These examples will also enlighten the fact that the "translation" Fisher information $\mathcal{J}(\beta)$ comes into the picture here.

5. Examples. The calculations of the previous section involving the parameter θ can be made fully explicit if we specify the distribution of the process Y , in some cases at least. We will always suppose that β is known in these examples.

5.1. *Stable process plus drift.* Here we assume that $Y_t = t$, so $G_\Delta = \delta_\Delta$ and $G = \delta_1$ (recall that the notation δ means a Dirac mass):

Theorem 4. *The 2×2 Fisher information matrix for estimating (σ, θ) is*

$$(28) \quad \begin{pmatrix} I_\Delta^{\sigma\sigma}(\sigma, \beta, \theta, \delta_1) & I_\Delta^{\sigma\theta}(\sigma, \beta, \theta, \delta_1) \\ I_\Delta^{\sigma a}(\sigma, \beta, \theta, \delta_1) & I_\Delta^{aa}(\sigma, \beta, \theta, \delta_1) \end{pmatrix} = \frac{1}{\sigma^2} \begin{pmatrix} \mathcal{I}(\beta) & 0 \\ 0 & \Delta^{2-2/\beta} \mathcal{J}(\beta) \end{pmatrix}.$$

This has several interesting consequences (we will denote by $T_n = n\Delta_n$ the length of the *observation window*):

1. If θ is known, one may hope for estimators $\hat{\sigma}_n$ for σ such that $\sqrt{n}(\hat{\sigma}_n - \sigma) \xrightarrow{d} \mathcal{N}(0, \sigma^2/\mathcal{I}(\beta))$ (that is, *asymptotically efficient* in the Cramer–Rao sense). As a matter of fact, in this setting, observing χ_i^n is equivalent to observing $\chi_i'^n = \chi_i^n - a\Delta_n$, so we are in the situation of Section 3.
2. If σ is known, one may hope for estimators $\hat{\theta}_n$ for θ such that $\sqrt{n}\Delta_n^{1-1/\beta}(\hat{\theta}_n - \theta)$ converges in law to $\mathcal{N}(0, \sigma^2/\mathcal{J}(\beta))$. If $\beta = 2$ the rate is thus $\sqrt{T_n}$: this is in accordance with the well known fact that for a diffusion the rate for estimating the drift coefficient is the square root of the total observation window, that is $\sqrt{T_n}$ here; moreover in this case, the variable X_{T_n}/T_n is $\mathcal{N}(\theta, \sigma^2/T_n)$; so $\hat{\theta}_n = X_{T_n}/T_n$ is an asymptotically efficient estimator for θ (recall that $\mathcal{J}(\beta) = 1$ when $\beta = 2$). When $\beta < 2$ we have $1 - 1/\beta < 1/2$, so the rate is bigger than $\sqrt{T_n}$, and it increases when β decreases; when $\beta < 1$ this rate is even bigger than \sqrt{n} .

Observe that here Y_1 has mean $m = 1$ and variance $v = 0$: so the estimate (26) is indeed an equality. The fact that the translation Fisher information $\mathcal{J}(\beta)$ appears here is transparent.

3. If both σ and θ are unknown, one may hope for estimators $\hat{\sigma}_n$ and $\hat{\theta}_n$ such that the pairs $(\sqrt{n}(\hat{\sigma}_n - \sigma), \sqrt{n}\Delta_n^{1-1/\beta}(\hat{\theta}_n - \theta))$ converge in law to the product $\mathcal{N}(0, \sigma^2/\mathcal{I}(\beta)) \otimes \mathcal{N}(0, \sigma^2/\mathcal{J}(\beta))$.

5.2. *Stable process plus Poisson process.* Here we assume that Y is a standard Poisson process (jumps of size 1, intensity 1), whose law we write as $G = P$. We can describe the limiting behavior of the (σ, θ) block of the matrix $I_\Delta(\sigma, \beta, \theta, P)$ as $\Delta \rightarrow 0$.

Theorem 5. *If Y is a standard Poisson process we have, as $\Delta \rightarrow 0$:*

$$(29) \quad I_{\Delta}^{\sigma\sigma}(\sigma, \beta, \theta, P) \rightarrow \frac{1}{\sigma^2} \mathcal{I}(\beta)$$

$$(30) \quad \Delta^{1/\beta-1/2} I_{\Delta}^{\sigma\theta}(\sigma, \beta, \theta, P) \rightarrow 0$$

$$(31) \quad \Delta^{2/\beta-1} I_{\Delta}^{\theta\theta}(\sigma, \beta, \theta, P) \rightarrow \frac{1}{\sigma^2} \mathcal{J}(\beta)$$

Since $P \in \mathcal{G}_{\beta}$, (29) is nothing else than the first part of (22). One could prove more than (30), namely that $\sup_{\Delta} \Delta^{1/\beta-1} |I_{\Delta}^{\sigma\theta}(\sigma, \beta, \theta, P)| \leq \infty$. Here again, we deduce some interesting consequences:

1. If σ is known, one may hope for estimators $\hat{\theta}_n$ for θ such that $\sqrt{n\Delta_n^{1-2/\beta}} (\hat{\theta}_n - \theta)$ converge in law to $\mathcal{N}(0, \sigma^2/\mathcal{J}(\beta))$. So the rate is *bigger* than \sqrt{n} , except when $\beta = 2$. More generally, if both σ and θ are unknown, one may hope for estimators $\hat{\sigma}_n$ and $\hat{\theta}_n$ such that the pairs $\left(\sqrt{n} (\hat{\sigma}_n - \sigma), \sqrt{n\Delta_n^{1-2/\beta}} (\hat{\theta}_n - \theta) \right)$ converge in law to the product $\mathcal{N}(0, \sigma^2/\mathcal{I}(\beta)) \otimes \mathcal{N}(0, \sigma^2/\mathcal{J}(\beta))$.
2. However, the above-described behavior of any estimator $\hat{\theta}_n$ cannot be true when $T_n = n\Delta_n$ does not go to infinity, because in this case there is a positive probability that Y has no jump on the biggest observed interval, and so no information about θ can be drawn from the observations in that case. It is true, though, when $T_n \rightarrow \infty$, because Y will eventually have infinitely many jumps on the observed intervals. This discrepancy between the asymptotic behavior of Fisher information and of estimators shows that some care must be taken when the Fisher information is used as a measure of the quality of estimators.
3. Observe that here Y_1 has mean $m = 1$ and variance $v = 1$. So in view of (31) the asymptotic estimate (27) is sharp.

5.3. *Stable process plus compound Poisson process.* Here we assume that Y is a compound Poisson process with arrival rate λ and law of jumps μ : that is, the characteristics of G are $b = \lambda \int_{\{|x| \leq 1\}} x \mu(dx)$ and $c = 0$ and $F = \lambda \mu$. We then write $G = P_{\lambda, \mu}$, which belongs to \mathcal{G}_{β} .

We will further assume that μ has a density f satisfying:

$$(32) \quad \lim_{|u| \rightarrow \infty} u f(u) = 0, \quad \sup_u (|f'(u)|(1 + |u|)) < \infty.$$

We also suppose that the “multiplicative” Fisher information associated with μ (that is, the Fisher information for estimating θ in the model when one observes a single variable θU with U distributed according to μ) exists. It then has the form

$$(33) \quad \mathcal{L} = \int \frac{(uf'(u) + f(u))^2}{f(u)} du.$$

We can describe the limiting behavior of the (σ, θ) block of the matrix $I_\Delta(\sigma, \beta, \theta, P_{\lambda, \mu})$ as $\Delta \rightarrow 0$.

Theorem 6. *If Y is a compound Poisson process satisfying (32) and such that \mathcal{L} in (33) is finite, we have as $\Delta \rightarrow 0$:*

$$(34) \quad I_\Delta^{\sigma\sigma}(\sigma, \beta, \theta, P_{\lambda, \mu}) \rightarrow \frac{1}{\sigma^2} \mathcal{I}(\beta)$$

$$(35) \quad \frac{1}{\sqrt{\Delta}} I_\Delta^{\sigma\theta}(\sigma, \beta, \theta, P_{\lambda, \mu}) \rightarrow 0$$

and

$$(36) \quad \frac{\lambda^2}{\theta^2} \int \frac{(xf'(x) + f(x))^2}{\lambda f(x) + \frac{c_\beta \sigma^\beta}{\theta^\beta |x|^{1+\beta}}} dx \leq \liminf \frac{1}{\Delta} I_\Delta^{\theta\theta}(\sigma, \beta, \theta, P_{\lambda, \mu}) \leq \limsup \frac{1}{\Delta} I_\Delta^{\theta\theta}(\sigma, \beta, \theta, G) \leq \frac{1}{\theta^2} \mathcal{L}$$

when $\beta < 2$ (c_β is the constant defined in (4)), and also, when $\beta = 2$:

$$(37) \quad \frac{1}{\Delta} I_\Delta^{\theta\theta}(\sigma, \beta, \theta, P_{\lambda, \mu}) \rightarrow \frac{1}{\theta^2} \mathcal{L}.$$

As for the previous theorem, (34) is nothing else than the first part of (22). We could prove more than (35), namely that $\sup_\Delta \frac{1}{\Delta} |I_\Delta^{\sigma\theta}(\sigma, \beta, \theta, P_{\lambda, \mu})| < \infty$. Here again, we deduce some interesting consequences:

1. One may hope for estimators $\hat{\theta}_n$ for θ such that $\sqrt{T_n}(\hat{\theta}_n - \theta)$ is tight (the rate is the same as for the case $Y_t = t$), and is even asymptotically normal when $\beta = 2$.
2. However, this is not true when T_n does not go to infinity, for the same reason as for the previous theorem.
3. When the measure μ has a second order moment, the right side of (26) is larger than the result of the previous theorem, so the estimate in Theorem 3 is not sharp.

The rates for estimating θ in the two previous theorems, and the limiting Fisher information as well, can be explained as follows (supposing that σ is known and that we have n observations and that $T_n \rightarrow \infty$):

1. For Theorem 5: θ comes into the picture whenever the Poisson process has a jump. On the interval $[0, T_n]$ we have an average of T_n jumps, most of them being isolated in an interval $(i\Delta_n, (i+1)\Delta_n]$. So it essentially amounts to observing T_n (or rather the integer part $[T_n]$) independent variables, all distributed as $\sigma\Delta_n^{1/\beta}W_1 + \theta$. The Fisher information for each of those (for estimating θ) is $J(\beta)/\sigma^2\Delta_n^{2/\beta}$, and the “global” Fisher information, namely $nI_{\Delta_n}^{\theta\theta}$, is approximately $T_n J(\beta)/\sigma^2\Delta_n^{2/\beta} \sim J(\beta)/\sigma^2\Delta_n^{2/\beta-1}$.
2. For Theorem 6: Again θ comes into the picture whenever the compound Poisson process has a jump. We have an average of λT_n jumps, so it essentially amounts to observing λT_n independent variables, all distributed as $\sigma\Delta_n^{1/\beta}W_1 + \theta V$ where V has the distribution μ . The Fisher information for each of those (for estimating θ) is approximately L/θ^2 (because the variable $\sigma\Delta_n^{1/\beta}W_1$ is negligible), and the “global” Fisher information $nI_{\Delta_n}^{\theta\theta}$ is approximately $\lambda T_n L\theta^2 \sim n\Delta_n L/\theta^2$. This explains the rate in (36), and is an indication that (37) may be true even when $\beta < 2$, although we have been unable to prove it thus far.

5.4. *Two stable processes.* Our last example is about the case where Y is also a symmetric stable process with index α , $\alpha < \beta$. We write $G = S_\alpha$. Surprisingly, the results are quite involved, in the sense that for estimating θ we have different situations according to the relative values of α and β . We obviously still have (34), so we concentrate on the term $I_\Delta^{\theta\theta}$ and ignore the cross term in the statement of the following theorem:

Theorem 7. *If Y is a standard symmetric stable process with index $\alpha < \beta$, we have as $\Delta \rightarrow 0$:*

$$(38) \quad \beta = 2 \quad \Rightarrow \quad \frac{(\log(1/\Delta))^{\alpha/2}}{\Delta^{\frac{\beta-\alpha}{\beta}}} I_\Delta^{\theta\theta}(\sigma, \beta, \theta, S_\alpha) \rightarrow \frac{2\alpha c_\alpha \beta^{\alpha/2}}{\theta^{2-\alpha} \sigma^\alpha (2(\beta-\alpha))^{\alpha/2}}$$

$$\beta < 2, \alpha > \frac{\beta}{2} \quad \Rightarrow \quad \frac{1}{\Delta^{\frac{2(\beta-\alpha)}{\beta}}} I_\Delta^{\theta\theta}(\sigma, \beta, \theta, S_\alpha)$$

$$(39) \quad \rightarrow \frac{\alpha^2 c_\alpha^2 \theta^{2\alpha-2}}{\sigma^{2\alpha}} \int \frac{\left(\int_{\mathbf{R}} |y|^{1-\alpha} dy \int_0^1 (1-v) h_\beta''(x-yv) dv \right)^2}{h_\beta(y)} dx$$

$$(40) \quad \beta < 2, \alpha = \frac{\beta}{2} \quad \Rightarrow \quad \frac{1}{\Delta^{\frac{2(\beta-\alpha)}{\beta}} \log(1/\Delta)} I_\Delta^{\theta\theta}(\sigma, \beta, \theta, S_\alpha) \rightarrow \frac{2\alpha(\beta-\alpha) c_\alpha^2 \theta^{2\alpha-2}}{\beta c_\beta \sigma^{2\alpha}}$$

$$(41) \quad \beta < 2, \alpha < \frac{\beta}{2} \quad \Rightarrow \quad \frac{1}{\Delta} I_\Delta^{\theta\theta}(\sigma, \beta, \theta, S_\alpha) \rightarrow \frac{\alpha^2 c_\alpha^2 \theta^{2\alpha-2}}{\sigma^{2\alpha}} \int \frac{1}{c_\beta |z|^{1+2\alpha-\beta} + c_\alpha \theta^\alpha |z|^{1+\alpha} / \sigma^\alpha} dz$$

Then if σ is known one may hope to find estimators $\widehat{\theta}_n$ for θ such that $u_n(\widehat{\theta}_n - \theta)$ converges in law to $N(0, V)$, with

$$\begin{aligned} u_n &= \frac{\sqrt{n} \Delta_n^{\frac{\beta-\alpha}{2\beta}}}{(\log(1/\Delta_n))^{\alpha/4}} & \text{if } \beta = 2 \\ u_n &= \sqrt{n} \Delta_n^{\frac{\beta-\alpha}{\beta}} & \text{if } \beta < 2, \alpha > \beta/2 \\ u_n &= \sqrt{n} \Delta_n^{\frac{\beta-\alpha}{\beta}} \sqrt{\log(1/\Delta_n)} & \text{if } \beta < 2, \alpha = \beta/2 \\ u_n &= \sqrt{T_n} & \text{if } \beta < 2, \alpha < \beta/2. \end{aligned}$$

and of course the asymptotic variance V should be the inverse of the right hand sides in (38)–(41).

6. The multiplicative model. Another interesting situation is the model (2), when β is known. If we observe a single variable X_Δ , the corresponding Fisher information is obviously

$$I'_\Delta(\sigma, \beta, G) = I_\Delta^{\sigma\sigma}(\sigma, \beta, \sigma, G) + 2I_\Delta^{\sigma\theta}(\sigma, \beta, \sigma, G) + I_\Delta^{\theta\theta}(\sigma, \beta, \sigma, G).$$

We will not develop a full theory here, but we translate the examples of the previous section in this setting. In view of the previous results, the proofs of the next three theorem follow, and these results show the variety of situations we may encounter for this multiplicative model. The problems dealt with in Theorems 9-10 below have been solved previously by Far (2001) and Jedidi (2001).

Theorem 8. *If $Y_t = t$ the Fisher information for estimating σ in the model (2) satisfies, as $\Delta \rightarrow 0$:*

$$\begin{aligned} I'_\Delta(\sigma, \beta, \delta_1) &\rightarrow \frac{1}{\sigma^2} \mathcal{I}(\beta) & \text{if } \beta \in (1, 2] \\ I'_\Delta(\sigma, \beta, \delta_1) &\rightarrow \frac{1}{\sigma^2} (\mathcal{I}(\beta) + \mathcal{J}(\beta)) & \text{if } \beta = 1 \\ \Delta^{2/\beta-2} I'_\Delta(\sigma, \beta, \delta_1) &\rightarrow \frac{1}{\sigma^2} \mathcal{J}(\beta) & \text{if } \beta \in (0, 1). \end{aligned}$$

So in this situation we may hope for estimators $\widehat{\sigma}_n$ for σ such that

$$\begin{aligned} \sqrt{n}(\widehat{\sigma}_n - \sigma) &\xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\mathcal{I}(\beta)}\right) & \text{if } \beta \in (1, 2] \\ \sqrt{n}(\widehat{\sigma}_n - \sigma) &\xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\mathcal{I}(\beta) + \mathcal{J}(\beta)}\right) & \text{if } \beta = 1 \\ \sqrt{n}\Delta_n^{1-1/\beta}(\widehat{\sigma}_n - \sigma) &\xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\mathcal{J}(\beta)}\right) & \text{if } \beta \in (0, 1). \end{aligned}$$

Theorem 9. *If Y is a standard Poisson process, the Fisher information for estimating σ in the model (2) satisfies, as $\Delta \rightarrow 0$:*

$$\begin{aligned} I'_\Delta(\sigma, \beta, P) &\rightarrow \frac{1}{\sigma^2} (\mathcal{I}(\beta) + \mathcal{J}(\beta)) && \text{if } \beta = 2 \\ \Delta^{2/\beta-1} I'_\Delta(\sigma, \beta, P) &\rightarrow \frac{1}{\sigma^2} \mathcal{J}(\beta) && \text{if } \beta \in (0, 2). \end{aligned}$$

So in this situation, and as soon as $T_n \rightarrow \infty$, we may hope for estimators $\hat{\sigma}_n$ for σ such that

$$\begin{aligned} \sqrt{n}(\hat{\sigma}_n - \sigma) &\xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\mathcal{I}(\beta) + \mathcal{J}(\beta)}\right) && \text{if } \beta = 2 \\ \sqrt{n\Delta_n^{1-2/\beta}}(\hat{\sigma}_n - \sigma) &\xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\mathcal{J}(\beta)}\right) && \text{if } \beta \in (0, 2). \end{aligned}$$

Theorem 10. *If Y is a compound Poisson process satisfying (32) and such that L in (33) is finite, and also if Y is a symmetric stable process with index $\alpha < \beta$, the Fisher information for estimating σ in the model (2) satisfies, as $\Delta \rightarrow 0$:*

$$I'_\Delta(\sigma, \beta, G) \rightarrow \frac{1}{\sigma^2} \mathcal{I}(\beta).$$

So in this situation, and as soon as $T_n \rightarrow \infty$, we may hope for estimators $\hat{\sigma}_n$ for σ such that

$$\sqrt{n}(\hat{\sigma}_n - \sigma) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\mathcal{J}(\beta)}\right).$$

7. Proofs.

7.1. *Preliminaries about the class $G(\phi, \alpha)$.* In the sequel, we denote by C_γ a constant depending only on the parameter γ , and which may change from line to line.

Lemma 1. *Let $\phi \in \Phi$ and $\alpha \in (0, 2]$. There is an increasing function $\phi_\alpha : (0, 1) \rightarrow R_+$ having $\lim_{x \downarrow 0} \phi_\alpha = 0$ and $\phi \leq \phi_\alpha$ on $(0, 1]$, such that for all $G \in G(\phi, \alpha)$ and $\varepsilon \in (0, 1]$ we have*

$$(42) \quad \int_{\{|x| \leq \varepsilon\}} |x|^q F(dx) \leq \begin{cases} \frac{q}{q-\alpha} \varepsilon^{q-\alpha} \phi_\alpha(\varepsilon) & \text{if } q > \alpha \\ \phi_\alpha(\varepsilon) & \text{if } q = \alpha = 2, \end{cases}$$

$$(43) \quad \int_{\{\varepsilon < |x| \leq 1\}} |x| F(dx) \leq \begin{cases} \phi_\alpha(1) & \text{if } \alpha < 1 \\ \phi_\alpha(\varepsilon) \log(1/\varepsilon) & \text{if } \alpha = 1 \\ \phi_\alpha(\varepsilon) \varepsilon^{1-\alpha} & \text{if } \alpha > 1. \end{cases}$$

Proof. First we define ϕ_α as follows, for $x \in (0, 1)$:

$$\phi_\alpha(x) = \begin{cases} \frac{\phi(x)}{1-\alpha} & \text{if } \alpha < 1 \\ \phi(x) + \frac{\phi(x)}{\sqrt{\log(1/x)}} + \phi\left(1 \wedge e^{-\sqrt{\log(1/x)}}\right) & \text{if } \alpha = 1 \\ \phi(x) + \frac{\phi(\sqrt{x})}{\alpha-1} + \frac{\phi(1)}{\alpha-1} x^{\frac{\alpha-1}{2}} & \text{if } \alpha > 1. \end{cases}$$

It is clear that $\phi_\alpha(x) \rightarrow 0$ as $x \downarrow 0$, and that $\phi \leq \phi_\alpha$ on $(0, 1)$.

(42) when $q = \alpha = 2$ is trivial because $\phi \leq \phi_\alpha$. When $q > \alpha$, Fubini Theorem and (6) yield:

$$\begin{aligned} \int_{\{|x| \leq \varepsilon\}} |x|^q F(dx) &= \int_{\{|x| \leq \varepsilon\}} F(dx) q \int_0^{|x|} y^{q-1} dy = q \int_0^\varepsilon y^{q-1} F(|x| > y) dy \\ &\leq q \int_0^\varepsilon \phi(y) y^{q-1-\alpha} dy \leq \frac{q}{q-\alpha} \phi(\varepsilon) \varepsilon^{q-\alpha} \end{aligned}$$

because ϕ is increasing: so we get (42) again.

In a similar way, for every $z \in [\varepsilon, 1]$ we get

$$\begin{aligned} \int_{\{\varepsilon < |x| \leq 1\}} |x| F(dx) &= \int_{\{\varepsilon < |x| \leq 1\}} F(dx) \int_0^{|x|} dy \\ &= \int_0^\varepsilon F(\varepsilon < |x| \leq 1) dy + \int_\varepsilon^z F(y < |x| \leq 1) dy + \int_z^1 F(y < |x| \leq 1) dy \\ &\leq \phi(\varepsilon) \varepsilon^{1-\alpha} + \phi(z) \int_\varepsilon^z y^{-\alpha} dy + \phi(1) \int_z^1 y^\alpha dy, \end{aligned}$$

Then in view of our definition of ϕ_α , a simple calculation allows to deduce (43), upon taking $z = 1$ when $\alpha < 1$, and $z = 1$ when $\alpha = 1$ and $\varepsilon \geq 1/e$, and $z = \exp -\sqrt{\log(1/\varepsilon)}$ if $\alpha = 1$ and $\varepsilon < 1/e$, and $z = \sqrt{\varepsilon}$ when $\alpha > 1$. \square

In view of (42), for any pair (G, α) such that $G \in G_\alpha$ we can introduce the following notation:

$$(44) \quad b'(G, \alpha) = \begin{cases} b - \int_{\{|x| \leq 1\}} x F(dx) & \text{if } \alpha < 1 \\ b & \text{if } \alpha \geq 1, \end{cases} \quad Z_\Delta(\alpha, \beta) := \Delta^{-1/\beta} (Y_\Delta - b'(G, \alpha)\Delta).$$

and we let $G'_{\Delta, \alpha, \beta}$ denote the law of $Z_\Delta(\alpha, \beta)$. Part (c) of the forthcoming lemma is not fully used here, but will be in the companion paper on estimation.

Lemma 2. *a) If $G \in G_\beta$ then $G'_{\Delta, \beta, \beta}$ converges to the Dirac mass δ_0 as $\Delta \rightarrow 0$.*

b) If $\alpha \leq \beta$ and $\phi \in \Phi$ and G^n is a sequence of measures in $G(\phi, \alpha)$ and $\Delta_n \rightarrow 0$, then the associated sequence $G'_{\Delta_n, \alpha, \beta}$ converges to the Dirac mass δ_0 as $n \rightarrow \infty$.

c) If $\alpha \leq \beta$ and $\phi \in \Phi$ there is a constant $C = C_\alpha$ such that for all functions g with $|g(x)| \leq K(1 \wedge |x|)$ and all $\Delta \in (0, 1]$, we have (with ϕ_α like in the previous lemma):

$$(45) \quad G \in \mathcal{G}(\phi, \alpha) \quad \implies \quad \mathbf{E}(|g(Z_\Delta(\alpha, \beta))|) \leq CK \Delta^{\frac{2(\beta-\alpha)}{\beta(2+\alpha)}} \phi_\alpha \left(\Delta^{\frac{2+\beta}{\beta(2+\alpha)}} \right).$$

Proof. Observe that (c) \implies (b) \implies (a), so we prove (c) only.

Let $\eta \in (0, 1/2]$ to be chosen later. For any given $G \in G(K, \alpha)$ we associate the Lévy process Y and the characteristics $(b, 0, F)$. Let F' and F'' be the restrictions of F to the sets $[-\eta, \eta]$ and $[-\eta, \eta]^c$ respectively. We can decompose Y into the sum $Y_t = at + Y'_t + Y''_t$, where Y' is a Lévy process with characteristics $(0, 0, F')$ and Y'' is a compound Poisson process with Lévy measure F'' , and $a = b - \int_{\{\eta < |x| \leq 1\}} xF(dx)$. Then $a' = a - b'(G, \alpha)$ is (recall (44)):

$$a' = \begin{cases} \int_{\{|x| \leq \eta\}} xF(dx) & \text{if } \alpha < 1 \\ - \int_{\{\eta < |x| \leq 1\}} xF(dx) & \text{if } \alpha \geq 1. \end{cases}$$

Therefore (42) and (43) yield (for a constant $C = C_\alpha$ not depending on $G \in G(\phi, \alpha)$):

$$(46) \quad |a'| \leq \begin{cases} C\eta^{1-\alpha} \phi_\alpha(\eta) & \text{if } \alpha \neq 1 \\ C \log(1/\eta) \phi_\alpha(\eta) & \text{if } \alpha = 1. \end{cases}$$

Also, since Y' has no drift, no Wiener part, and no jump bigger than 1, one knows (by differentiating (5) for example) that $E((Y'_t)^2) = t \int x^2 F'(dx)$. Then (42) again yields for some $C = C_\alpha$:

$$(47) \quad \mathbf{E}(|Y'_\Delta|^2) \leq C\Delta \eta^{2-\alpha} \phi_\alpha(\eta).$$

We set $Z_\Delta = Z_\Delta(\alpha, \beta)$. Since $|g| \leq K$ we have $|g(Z_\Delta)| \leq K$. If further $Y''_\Delta = 0$, we have also $Y_\Delta = a\Delta + Y'_\Delta$, hence $Z_\Delta = \Delta^{-1/\beta}(Y'_\Delta + a'\Delta)$, hence $|g(Z_\Delta)| \leq K\Delta^{-1/\beta}(|Y'_\Delta| + \Delta|a'|)$. Now, we have $P(Y''_\Delta \neq 0) \leq \Delta F''(\mathbb{R}) \leq \Delta \phi_\alpha(\eta)/\eta^\alpha$ because $G \in G(\phi, \alpha)$. Therefore we deduce from (46) and (47) that for some constant $C = C_{K, \alpha}$:

$$\mathbf{E}(|g(Z_\Delta)|) \leq \begin{cases} CK (\Delta\eta^{-1} + \Delta^{1/2-1/\beta}\eta^{1/2} + \Delta^{1-1/\beta} \log(1/\eta)) \phi_1(\eta) & \text{if } \alpha = 1 \\ CK (\Delta\eta^{-\alpha} + \Delta^{1/2-1/\beta}\eta^{1-\alpha/2} + \Delta^{1-1/\beta}\eta^{1-\alpha}) \phi_\alpha(\eta) & \text{otherwise} \end{cases}$$

as soon as $G \in G(\phi, \alpha)$. Then take $\eta = \Delta^{(2+\beta)/\beta(2+\alpha)}$ to get (45). \square

7.2. *Fisher's information when $X = \sigma W + \theta Y$.* From independence of W and Y , the density of X_Δ in (1) is the convolution (recall $G_\Delta = L(Y_\Delta)$):

$$(48) \quad p_\Delta(x|\sigma, \beta, \theta, G) = \frac{1}{\sigma\Delta^{1/\beta}} \int G_\Delta(dy) h_\beta\left(\frac{x-\theta y}{\sigma\Delta^{1/\beta}}\right).$$

We now seek to characterize the entries of the full Fisher information matrix. Since h'_β and \check{h}_β and \dot{h}_β are continuous and bounded, we can differentiate under the integral in (48) to get

$$(49) \quad \partial_\sigma p_\Delta(x|\sigma, \beta, \theta, G) = -\frac{1}{\sigma^2\Delta^{1/\beta}} \int G_\Delta(dy) \check{h}_\beta\left(\frac{x-\theta y}{\sigma\Delta^{1/\beta}}\right),$$

$$(50) \quad \partial_\beta p_\Delta(x|\sigma, \beta, \theta, G) = v_\Delta(x|\sigma, \beta, \theta, G) - \frac{\sigma \log \Delta}{\beta^2} \partial_\sigma p_\Delta(x|\sigma, \beta, \theta, G),$$

$$(51) \quad \partial_\theta p_\Delta(x|\sigma, \beta, \theta, G) = -\frac{1}{\sigma^2\Delta^{2/\beta}} \int G_\Delta(dy) y h'_\beta\left(\frac{x-\theta y}{\sigma\Delta^{1/\beta}}\right),$$

where

$$(52) \quad v_\Delta(x|\sigma, \beta, \theta, G) = \frac{1}{\sigma\Delta^{1/\beta}} \int G_\Delta(dy) \dot{h}_\beta\left(\frac{x-\theta y}{\sigma\Delta^{1/\beta}}\right).$$

The entries of the (σ, θ) block of the Fisher information matrix are (leaving implicit the dependence on $(\sigma, \beta, \theta, G)$):

$$(53) \quad I_\Delta^{\sigma\sigma} = \int \frac{\partial_\sigma p_\Delta(x)^2}{p_\Delta(x)} dx, \quad I_\Delta^{\sigma\theta} = \int \frac{\partial_\sigma p_\Delta(x) \partial_\theta p_\Delta(x)}{p_\Delta(x)} dx, \quad I_\Delta^{\theta\theta} = \int \frac{\partial_\theta p_\Delta(x)^2}{p_\Delta(x)} dx.$$

When $\beta < 2$, the other entries are

$$(54) \quad I_\Delta^{\sigma\beta} = J_\Delta^{\sigma\beta} - \frac{\sigma \log \Delta}{\beta^2} I_\Delta^{\sigma\sigma}, \quad I_\Delta^{\beta\theta} = J_\Delta^{\beta\theta} - \frac{\sigma \log \Delta}{\beta^2} I_\Delta^{\sigma\theta},$$

$$(55) \quad I_\Delta^{\beta\beta} = J_\Delta^{\beta\beta} - \frac{2\sigma \log \Delta}{\beta^2} J_\Delta^{\sigma\beta} + \frac{\sigma^2 (\log \Delta)^2}{\beta^4} I_\Delta^{\sigma\sigma},$$

where

$$(56) \quad J_\Delta^{\sigma\beta} = \int \frac{\partial_\sigma p_\Delta(x) v_\Delta(x)}{p_\Delta(x)} dx, \quad J_\Delta^{\beta\beta} = \int \frac{v_\Delta(x)^2}{p_\Delta(x)} dx, \quad J_\Delta^{\beta\theta} = \int \frac{v_\Delta(x) \partial_\theta p_\Delta(x)}{p_\Delta(x)} dx.$$

7.3. *Proof of Theorem 1.* The proof is standard, and given for completeness, and given only in the case where $\beta < 2$ (when $\beta = 2$ take $v = 0$ below). What we need to prove is that, for any $u, v \in R$, we have

$$(57) \quad \begin{aligned} & \int \frac{(u\partial_\sigma p_\Delta(x|\sigma, \beta, \theta, G) + v\partial_\beta p_\Delta(x|\sigma, \beta, \theta, G))^2}{p_\Delta(x|\sigma, \beta, \theta, G)} dx \\ & \leq \int \frac{(u\partial_\sigma p_\Delta(x|\sigma, \beta, 0, \delta_0) + v\partial_\beta p_\Delta(x|\sigma, \beta, 0, \delta_0))^2}{p_\Delta(x|\sigma, \beta, 0, \delta_0)} dx. \end{aligned}$$

We set

$$q(x) = p_\Delta(x|\sigma, \beta, \theta, G), \quad q_0(x) = p_\Delta(x|\sigma, \beta, 0, \delta_0),$$

$$r(x) = u\partial_\sigma p_\Delta(x|\sigma, \beta, \theta, G) + v\partial_\beta p_\Delta(x|\sigma, \beta, \theta, G), \quad r_0(x) = u\partial_\sigma p_\Delta(x|\sigma, \beta, 0, \delta_0) + v\partial_\beta p_\Delta(x|\sigma, \beta, 0, \delta_0).$$

Observe that by (48),

$$q(x) = \int G_\Delta(dy) q_0(x - \theta y),$$

hence

$$r(x) = \int G_\Delta(dy) r_0(x - \theta y)$$

as well. Apply Cauchy–Schwarz inequality to G_Δ with $r_0 = \sqrt{q_0} (r_0/\sqrt{q_0})$ to get

$$r(x)^2 \leq q(x) \int G_\Delta(dy) \frac{r_0(x - \theta y)^2}{q_0(x - \theta y)}.$$

Then

$$\int \frac{r(x)^2}{q(x)} dx \leq \int dx \int G_\Delta(dy) \frac{r_0(x - \theta y)^2}{q_0(x - \theta y)} = \int \frac{r_0(z)^2}{q_0(z)} dz$$

by Fubini and a change of variable: this is exactly (57).

7.4. Proof of Theorem 2. Clearly (b) implies (a). If (b) fails, one can find $\phi \in \Phi$ and $\alpha \in (0, \beta]$ and $\varepsilon > 0$, and also a sequence $\Delta_n \rightarrow 0$ and a sequence G^n of measures in $G(\phi, \alpha)$ and a sequence of numbers θ_n converging to a limit θ , such that

$$\beta = 2 \Rightarrow \left| I_{\Delta_n}^{\sigma\sigma}(\sigma, \beta, \theta_n, G^n) - \frac{\mathcal{I}(\beta)}{\sigma^2} \right| \geq \varepsilon,$$

$$\beta < 2 \Rightarrow \left| I_{\Delta_n}^{\sigma\sigma}(\sigma, \beta, \theta_n, G^n) - \frac{\mathcal{I}(\beta)}{\sigma^2} \right| + \left| \frac{I_{\Delta_n}^{\beta\beta}(\sigma, \beta, \theta_n, G^n)}{(\log(1/\Delta))^2} - \frac{\mathcal{I}(\beta)}{\beta^4} \right| + \left| \frac{I_{\Delta_n}^{\sigma\sigma}(\sigma, \beta, \theta_n, G^n)}{\log(1/\Delta)} - \frac{\mathcal{I}(\beta)}{\sigma\beta^2} \right| \geq \varepsilon$$

for all n .

In other words, to prove (a) and (b) it is enough to prove the following: let $\phi \in \Phi$ and $\alpha \in (0, \beta]$ and $\Delta_n \rightarrow 0$ and $\theta_n \rightarrow \theta$ and G^n be a sequence in $G(\phi, \alpha)$; then we have:

$$(58) \quad I_{\Delta_n}^{\sigma\sigma}(\sigma, \beta, \theta_n, G^n) \rightarrow \frac{\mathcal{I}(\beta)}{\sigma^2},$$

$$(59) \quad \beta < 2 \Rightarrow \frac{I_{\Delta_n}^{\beta\beta}(\sigma, \beta, \theta_n, G^n)}{(\log(1/\Delta))^2} \rightarrow \frac{\mathcal{I}(\beta)}{\beta^4}, \quad \frac{I_{\Delta_n}^{\sigma\beta}(\sigma, \beta, \theta_n, G^n)}{\log(1/\Delta)} \rightarrow \frac{\mathcal{I}(\beta)}{\sigma\beta^2}.$$

Let us proceed to proving (58). The change of variable $x \leftrightarrow (x - \theta b'(G^n, \alpha))/\sigma \Delta^{1/\beta}$ in (53) leads to $I_{\Delta}^{\sigma\sigma}(\sigma, \beta, \theta, G) = \frac{1}{\sigma^2} \int s_{\Delta, \theta, G}(x) dx$, where

$$(60) \quad s_{\Delta, \theta, G}(x) = \frac{\left(\int G'_{\Delta, \alpha, \beta}(du) \check{h}_{\beta}(x - u\theta/\sigma) \right)^2}{\int G'_{\Delta, \alpha, \beta}(du) h_{\beta}(x - u\theta/\sigma)}.$$

Since h_{β} and \check{h}_{β} are continuous and bounded, we deduce from Lemma 2 that

$$\int G'^n n_{\Delta_n, \alpha, \beta}(du) \check{h}_{\beta}(x - u\theta_n/\sigma) \rightarrow \check{h}_{\beta}(x), \quad \int G'^n n_{\Delta_n, \alpha, \beta}(du) h(x - u\theta_n/\sigma) \rightarrow h(x).$$

Thus $s_{\Delta_n, \theta_n, G^n}(x) \rightarrow \check{h}_{\beta}(x)$ for all x , and Fatou's Lemma yields

$$\liminf_n I_{\Delta_n}^{\sigma\sigma}(\sigma, \beta, \theta_n, G^n) \geq \frac{1}{\sigma^2} \int \check{h}_{\beta}(x) dx = \frac{\mathcal{I}(\beta)}{\sigma^2}.$$

This, combined with (11) and (13) and (20), gives (58).

Now suppose that $\beta < 2$ and recall (54) and (55): with the notation $J_{\Delta}^{\sigma\beta}(\sigma, \beta, \theta, G)$, etc... of (56), we see that

$$(61) \quad J_{\Delta}^{\sigma\beta}(\sigma, \beta, \theta, G) \leq \sqrt{I_{\Delta}^{\sigma\sigma}(\sigma, \beta, \theta, G) J_{\Delta}^{\beta\beta}(\sigma, \beta, \theta, G)}$$

by a first application of Cauchy–Schwarz inequality. A second application of the same plus (48) and (52) yield

$$v_{\Delta}(x|\sigma, \beta, \theta, G)^2 \leq \frac{p_{\Delta}(x|\sigma, \beta, \theta, G)}{\sigma \Delta^{1/\beta}} \int G_{\Delta}(dy) \frac{\check{h}_{\beta}^2}{h_{\beta}} \left(\frac{x - \theta y}{\sigma \Delta^{1/\beta}} \right).$$

Then Fubini and the change of variable $x \leftrightarrow (x - \theta y)/\sigma \Delta^{1/\beta}$ in (56) leads to

$$(62) \quad J_{\Delta}^{\beta\beta}(\sigma, \beta, \theta, G) \leq \mathcal{K}(\beta).$$

Then (59) readily follow from (58), (61), (62) and also (54) and (55).

It remains to prove (c). If we put together the majorations (20), (61) and (62) and also (54) and (55), we see that it is enough to prove that

$$(63) \quad \limsup_n I_{\Delta_n}^{\sigma\sigma}(\sigma, \beta, \theta, G^n) < \frac{\mathcal{I}(\beta)}{\sigma^2}.$$

Let $\rho_n = \Delta_n^{1/\alpha_n - 1/\beta}$, which by our assumption on Δ_n goes to 1. The measure $G'_{\Delta_n, \alpha_n, \beta}$ admits the density $x \mapsto g_n(x) = h_{\alpha_n}(x\rho_n)/\rho_n$, which converges to $h_{\beta}(x)$; so $G'_{\Delta_n, \alpha_n, \beta}$ weakly converges to the stable law with density h_{β} . Then, exactly as in the previous proof, we get that

$$(64) \quad s_{\Delta_n, \theta, G^n}(x) \rightarrow s(x) := \frac{\left(\int h_{\beta}(u) \check{h}_{\beta}(x - u\theta/\sigma) du \right)^2}{\int h_{\beta}(u) h_{\beta}(x - u\theta/\sigma) du}.$$

On the other hand $|\tilde{h}_\beta(y)| \leq C(1 \wedge 1/|y|^{1+\beta})$ for some constant C , and also $g'_n(y) \leq C(1 \wedge 1/|y|^{1+\alpha_n})$ with C not depending on n . Using once more Cauchy–Schwarz, we deduce from (60) that

$$(65) \quad \begin{aligned} s_{\sigma, \Delta_n, G^n}(x) &\leq \int g_n(u) \tilde{h}_\beta(x - u\theta/\sigma) du \\ &\leq s'(x) := C \int \left(1 \wedge \frac{1}{|u|^{1+\beta-\varepsilon}}\right) \left(1 \wedge \frac{1}{|x - u\theta/\sigma|^{1+\beta}}\right) du. \end{aligned}$$

for still another constant C , as soon as $\alpha_n > \beta - \varepsilon$ for some fixed $\varepsilon \in (0, \beta)$. But $\int s'(x)dx$ is finite, so (64) and the dominated convergence theorem yield

$$(66) \quad I_{\Delta_n}^{\sigma\sigma}(\sigma, \beta, \theta, G^n) \rightarrow \frac{1}{\sigma^2} \int s(x)dx.$$

Finally, exactly as for (65) we deduce from the Cauchy–Schwarz inequality and from the fact that the functions $\sqrt{\bar{h}_\beta}$ and $\check{h}_\beta/\sqrt{\bar{h}_\beta}$ are not Lebesgue–almost surely multiple one from the other, while $h_\beta > 0$ identically, that in fact $s(x) < \int h_\beta(u) \tilde{h}_\beta(x - u\theta/\sigma) du$ for all x . Therefore

$$\int s(x)dx < \int dx \int h_\beta(u) \tilde{h}_\beta(x - u\theta/\sigma) du = \int h_\beta(u) du \int \tilde{h}_\beta(y) dy = \int \tilde{h}_\beta(y) dy = \mathcal{I}(\beta),$$

and (66) yields that (63) holds, hence (c).

7.5. *Proof of Theorem 3.* Cauchy–Schwarz inequality gives us, by (48) and (51):

$$|\partial_\theta p_\Delta(x|\Delta, \beta, \theta, G)|^2 \leq \frac{1}{\sigma^3 \Delta^{3/\beta}} p_\Delta(x|\Delta, \beta, \theta, G) \int G_\Delta(dy) y^2 \tilde{h}_\beta\left(\frac{x - \theta y}{\sigma \Delta^{1/\beta}}\right).$$

Plugging this into (53), applying Fubini and doing the change of variable $x \leftrightarrow (x - \theta y)/\sigma \Delta^{1/\beta}$ leads to

$$I_\Delta^{\theta\theta} \leq \frac{1}{\sigma^2 \Delta^{2/\beta}} \int G_\Delta(dy) y^2 \int \frac{h'_\beta(x)^2}{h_\beta(x)} dx.$$

Since $E(Y_\Delta^2) = m\Delta^2 + \delta\Delta$, we readily deduce (26).

7.6. *Proof of Theorem 4.* When $Y_t = t$ we gave $G_\Delta = \delta_\Delta$. Then (28) follows directly from applying the formulae (49) and (51) and from the change of variable $x \leftrightarrow (x - \theta)/\sigma \Delta^{1/\beta}$ in (53), after observing that the function $\check{h}_\beta h'_\beta/h_\beta$ is integrable and odd, hence has a vanishing Lebesgue integral.

7.7. *Proof of Theorem 5.* a) We first introduce some notation to be used also for the proof of Theorem 6. We suppose that Y is a compound Poisson process with arrival rate λ and law of jumps

μ , and μ_k is the k th fold convolution of μ . So we have

$$G_\Delta = e^{-\lambda\Delta} \sum_{k=0}^{\infty} \frac{(\lambda\Delta)^k}{k!} \mu_k.$$

Set

$$(67) \quad \gamma_\Delta^{(1)}(k, x) = \frac{1}{\sigma\Delta^{1/\beta}} \int \mu_k(du) h_\beta \left(\frac{x - \theta u}{\sigma\Delta^{1/\beta}} \right),$$

$$(68) \quad \gamma_\Delta^{(2)}(k, x) = \frac{1}{\sigma^2\Delta^{1/\beta}} \int \mu_k(du) \check{h}_\beta \left(\frac{x - \theta u}{\sigma\Delta^{1/\beta}} \right),$$

$$(69) \quad \gamma_\Delta^{(3)}(k, x) = \frac{1}{\sigma^2\Delta^{2/\beta}} \int \mu_k(du) u h'_\beta \left(\frac{x - \theta u}{\sigma\Delta^{1/\beta}} \right).$$

We have (recall that $\mu_0 = \delta_0$):

$$(70) \quad p_\Delta(x|\sigma, \beta, \theta, G) = e^{-\lambda\Delta} \sum_{k=0}^{\infty} \frac{(\lambda\Delta)^k}{k!} \gamma_\Delta^{(1)}(k, x),$$

$$(71) \quad \partial_\sigma p_\Delta(x|\sigma, \beta, \theta, G) = -e^{-\lambda\Delta} \sum_{k=0}^{\infty} \frac{(\lambda\Delta)^k}{k!} \gamma_\Delta^{(2)}(k, x),$$

$$(72) \quad \partial_\theta p_\Delta(x|\sigma, \beta, \theta, G) = -e^{-\lambda\Delta} \sum_{k=1}^{\infty} \frac{(\lambda\Delta)^k}{k!} \gamma_\Delta^{(3)}(k, x),$$

Omitting the mention of $(\sigma, \beta, \theta, G)$, we also set

$$(73) \quad i = 2, 3 : \quad \Gamma_\Delta^{(i)}(k, k') = \int \frac{\gamma_\Delta^{(i)}(k, x) \gamma_\Delta^{(i)}(k', x)}{p_\Delta(x)} dx, \quad \Gamma_\Delta^{(4)}(k, k') = \int \frac{\gamma_\Delta^{(2)}(k, x) \gamma_\Delta^{(3)}(k', x)}{p_\Delta(x)} dx.$$

By Cauchy–Schwarz inequality, we have

$$(74) \quad i = 2, 3 : \quad \left| \Gamma_\Delta^{(i)}(k, k') \right| \leq \sqrt{\Gamma_\Delta^{(i)}(k, k) \Gamma_\Delta^{(i)}(k', k')}, \quad \left| \Gamma_\Delta^{(4)}(k, k') \right| \leq \sqrt{\Gamma_\Delta^{(2)}(k, k) \Gamma_\Delta^{(3)}(k', k')}.$$

For any $k \geq 0$ we have $p_\Delta(x) \geq e^{-\lambda\Delta} \frac{(\lambda\Delta)^k}{k!} \gamma_\Delta^{(1)}(k, x)$. Therefore

$$(75) \quad i = 2, 3 : \quad \Gamma_\Delta^{(i)}(k, k) \leq \frac{e^{\lambda\Delta} k!}{(\lambda\Delta)^k} \int \frac{\gamma_\Delta^{(i)}(k, x)^2}{\gamma_\Delta^{(1)}(k, x)} dx.$$

Finally, if we plug (71) and (72) into (53), we get

$$(76) \quad I_\Delta^{\sigma\theta} = e^{-2\lambda\Delta} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \frac{(\lambda\Delta)^{k+l}}{k! l!} \Gamma_\Delta^{(4)}(k, l),$$

$$(77) \quad I_\Delta^{\theta\theta} = e^{-2\lambda\Delta} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{(\lambda\Delta)^{k+l}}{k! l!} \Gamma_\Delta^{(3)}(k, l).$$

b) Now we can proceed to the proof of Theorem 5. When Y is a standard Poisson process, we have $\lambda = 1$ and $\mu_k = \varepsilon_k$. Therefore we get

$$(78) \quad \gamma_{\Delta}^{(1)}(k, x) = \frac{1}{\sigma \Delta^{1/\beta}} h_{\beta} \left(\frac{x - \theta k}{\sigma \Delta^{1/\beta}} \right),$$

$$(79) \quad \gamma_{\Delta}^{(2)}(k, x) = \frac{1}{\sigma^2 \Delta^{1/\beta}} \check{h}_{\beta} \left(\frac{x - \theta k}{\sigma \Delta^{1/\beta}} \right),$$

$$(80) \quad \gamma_{\Delta}^{(3)}(k, x) = \frac{k}{\sigma^2 \Delta^{2/\beta}} h'_{\beta} \left(\frac{x - \theta k}{\sigma \Delta^{1/\beta}} \right).$$

Plugging this into (75) yields

$$(81) \quad \Gamma_{\Delta}^{(2)}(k, k) \leq \frac{e^{\Delta} k!}{\sigma^2 \Delta^k} \mathcal{I}(\beta), \quad \Gamma_{\Delta}^{(3)}(k, k) \leq \frac{e^{\Delta} k^2 k!}{\sigma^2 \Delta^{k+2/\beta}} \mathcal{J}(\beta).$$

Recall that (29) follows from Theorem 2, so we need to prove (30) and (31). In view of (76) and (77), this amounts to proving the following two properties:

$$\sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \frac{\Delta^{k+l+1/\beta-1/2}}{k! l!} \Gamma_{\Delta}^{(4)}(k, l) \rightarrow 0, \quad \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\Delta^{k+l+2/\beta-1}}{k! l!} \Gamma_{\Delta}^{(3)}(k, l) \rightarrow \frac{1}{\sigma^2} \mathcal{J}(\beta).$$

If we use (74) and (81), it is easily seen that the sum of all summands in the first (resp. second) left side above, except the one for $k = 0$ and $l = 1$ (resp. $k = l = 1$) goes to 0. So we are left to prove

$$(82) \quad \Delta^{1/2+1/\beta} \Gamma_{\Delta}^{(4)}(0, 1) \rightarrow 0, \quad \Delta^{1+2/\beta} \Gamma_{\Delta}^{(3)}(1, 1) \rightarrow \frac{1}{\sigma^2} \mathcal{J}(\beta).$$

Let $\omega = \frac{1}{2(1+\beta)}$, so $(1 + \frac{1}{\beta})(1 - \omega) = 1 + 1/2\beta$. Assume first that $\beta < 2$. Then if $|x| \leq |\theta/\sigma \Delta^{1/\beta}|^{\omega}$, we have for some constant $C \in (0, \infty)$, possibly depending on θ , σ and β , and which changes from an occurrence to the other, and provided $\Delta \leq (2\theta/\sigma)^{\beta}$:

$$h_{\beta}(x) \geq C \Delta^{\omega(1+1/\beta)}, \quad h_{\beta}(x + r\theta/\sigma \Delta^{1/\beta}) \leq C \Delta^{1+1/\beta}$$

when $r \in \mathbb{Z} \setminus \{0\}$, and thus

$$|x| \leq |\theta/\sigma \Delta^{1/\beta}|^{\omega} \Rightarrow h_{\beta}(x + r\theta/\sigma \Delta^{1/\beta}) \leq C h_{\beta}(x) \Delta^{(1+1/\beta)(1-\omega)} = C h_{\beta}(x) \Delta^{1+1/2\beta}.$$

When $\beta = 2$, a simple computation on the normal density shows that the above property also holds.

Therefore in view of (70) and (78) we deduce that in all cases,

$$\begin{aligned} \left| \frac{x - \theta}{\sigma \Delta^{1/\beta}} \right| \leq \left(\frac{\theta}{\sigma \Delta^{1/\beta}} \right)^{\omega} &\Rightarrow p_{\Delta}(x) \leq \frac{e^{-\Delta}}{\sigma \Delta^{1/\beta}} h_{\beta} \left(\frac{x - \theta}{\sigma \Delta^{1/\beta}} \right) \left(\Delta + C \Delta^{1+\beta/2} \left(1 + \sum_{k=2}^{\infty} \frac{\Delta^k}{k!} \right) \right) \\ &\leq \frac{e^{-\Delta}}{\sigma \Delta^{1/\beta-1}} h_{\beta} \left(\frac{x - \theta}{\sigma \Delta^{1/\beta}} \right) \left(1 + C \Delta^{\beta/2} \right). \end{aligned}$$

By (73) it follows that

$$\begin{aligned} \Gamma_{\Delta}^{(3)}(1, 1) &\geq \frac{e^{\Delta}}{\sigma^3 \Delta^{1+3/\beta}} \int_{\{|(x-\theta)/\sigma\Delta^{1/\beta}| \leq (\theta/\sigma\Delta^{1/\beta})^{\omega}\}} \frac{h'_{\beta}((x-\theta)/\sigma\Delta^{1/\beta})^2}{h_{\beta}((x-\theta)/\sigma\Delta^{1/\beta})} dx \\ &\geq \frac{e^{\Delta}}{\sigma^2 \Delta^{1+2/\beta}} \int_{\{|x| \leq (\theta/\sigma\Delta^{1/\beta})^{\omega}\}} \frac{h'_{\beta}(x)^2}{h_{\beta}(x)} dx \end{aligned}$$

We readily deduce that $\liminf_{\Delta \rightarrow 0} \Delta^{1+2/\beta} \Gamma_{\Delta}^{(3)}(1, 1) \geq J(\beta)/\sigma^2$. On the other hand, (81) yields $\limsup_{\Delta \rightarrow 0} \Delta^{1+2/\beta} \Gamma_{\Delta}^{(3)}(1, 1) \leq J(\beta)/\sigma^2$, and thus the second part of (82) is proved.

Finally $\check{h}_{\beta}/h_{\beta}$ is bounded, so (79) and (80) and the fact that $p_{\Delta}(x) \geq e^{-\Delta} h_{\beta}(x/\sigma\Delta^{1/\beta})/\sigma\Delta^{1/\beta}$ yield

$$\left| \Gamma_{\Delta}^{(4)}(0, 1) \right| \leq \frac{e^{\Delta}}{\sigma^3 \Delta^{2/\beta}} \int \left| h'_{\beta} \left(\frac{x-\theta}{\sigma\Delta^{1/\beta}} \right) \right| dx \leq \frac{e^{\Delta}}{\sigma^2 \Delta^{1/\beta}} \int |h'_{\beta}(x)| dx,$$

and the first part of (82) readily follows.

7.8. Proof of Theorem 6. We use the same notation than in the previous proof, but here the measure μ_k has a density f_k for all $k \geq 1$, which further is differentiable and satisfies (32) uniformly in k , while we still have $\mu_0 = \delta_0$. Exactly as in (10), we set

$$\check{f}_k(u) = u f'_k(u) + f_k(u).$$

Recall the Fisher information \mathcal{L} defined in (33), which corresponds to estimating θ in a model where one observes a variable θU , with U having the law μ . Now if we have n independent variables U_i with the same law μ , the Fisher information associated with the observation of θU_i for $i = 1, \dots, n$ is of course $n\mathcal{L}$, and if instead one observes only $\theta(U_1 + \dots + U_n)$, one gets a smaller Fisher information $\mathcal{L}_n \leq n\mathcal{L}$. In other words, we have

$$(83) \quad \mathcal{L}_n := \int \frac{\check{f}_n(u)^2}{f_n(u)} du \leq n\mathcal{L}.$$

Taking advantage of the fact that μ_k has a density, for all $k \geq 1$ we can rewrite $\gamma_{\Delta}^{(i)}(k, x)$ as follows (using further an integration by parts when $i = 3$ and the fact that each f_k satisfies (32)):

$$(84) \quad \gamma_{\Delta}^{(1)}(k, x) = \frac{1}{\theta} \int h_{\beta}(y) f_k \left(\frac{x - y\sigma\Delta^{1/\beta}}{\theta} \right) dy$$

$$(85) \quad \gamma_{\Delta}^{(2)}(k, x) = \frac{1}{\sigma\theta} \int \check{h}_{\beta}(y) f_k \left(\frac{x - y\sigma\Delta^{1/\beta}}{\theta} \right) dy$$

$$(86) \quad \gamma_{\Delta}^{(3)}(k, x) = \frac{1}{\theta^2} \int h_{\beta}(y) \check{f}_k \left(\frac{x - y\sigma\Delta^{1/\beta}}{\theta} \right) dy.$$

Since the f_k 's satisfy (32) uniformly in k , we readily deduce that

$$(87) \quad k \geq 1, \quad i = 1, 2, 3 \quad \Rightarrow \quad |\gamma_{\Delta}^{(i)}(k, x)| \leq C, \quad \gamma_{\Delta}^{(1)}(1, x) \rightarrow \frac{1}{\theta} f \left(\frac{x}{\theta} \right), \quad \gamma_{\Delta}^{(3)}(1, x) \rightarrow \frac{1}{\theta^2} \check{f} \left(\frac{x}{\theta} \right),$$

Let us start with the lower bound. Since $\gamma_{\Delta}^{(1)}(0, x) = \frac{1}{\sigma\Delta^{1/\beta}} h_{\beta} \left(\frac{x}{\sigma\Delta^{1/\beta}} \right)$, we deduce from (9) and (87) and (70) that

$$(88) \quad \frac{1}{\Delta} p_{\Delta}(x) \rightarrow \frac{c_{\beta} \sigma^{\beta}}{|x|^{1+\beta}} + \frac{\lambda}{\theta} f \left(\frac{x}{\theta} \right)$$

as soon as $x \neq 0$, and with the convention $c_2 = 0$. In a similar way, we deduce from (87) and (72):

$$(89) \quad \frac{1}{\Delta} \partial_{\theta} p_{\Delta}(x) \rightarrow -\frac{\lambda}{\theta^2} \check{f} \left(\frac{x}{\theta} \right).$$

Then plugging (88) and (89) into the last equation in (53), we conclude by Fatou's Lemma and after a change of variable, that

$$\liminf_{\Delta \rightarrow 0} \frac{I_{\Delta}^{\theta\theta}}{\Delta} \geq \frac{\lambda^2}{\theta^2} \int \frac{\check{f}(x)^2}{\lambda f(x) + c_{\beta} \sigma^{\beta} / \theta^{\beta} |x|^{1+\beta}} dx.$$

It remains to prove (35) and the upper bound in (36) (including when $\beta = 2$). By Cauchy–Schwarz, we get (using successively the two equivalent versions for $\gamma_{\Delta}^{(i)}(k, x)$):

$$\gamma_{\Delta}^{(2)}(k, x)^2 \leq \frac{1}{\sigma^3} \gamma_{\Delta}^{(1)}(k, x) \int \mu_k(du) \check{h}_{\beta}((x - \theta u) / \sigma\Delta^{1/\beta}).$$

$$\gamma_{\Delta}^{(3)}(k, x)^2 \leq \frac{1}{\theta^3} \gamma_{\Delta}^{(1)}(k, x) \int h_{\beta}(y) \frac{\check{f}_k((x - y\sigma\Delta^{1/\beta}) / \theta)^2}{f_k((x - y\sigma\Delta^{1/\beta}) / \theta)} dy.$$

Then it follows from (75) and (83) that

$$(90) \quad \Gamma_{\Delta}^{(2)}(k, k) \leq \frac{e^{\lambda\Delta} k!}{\sigma^2(\lambda\Delta)^k} \mathcal{I}(\beta), \quad \Gamma_{\Delta}^{(3)}(k, k) \leq \frac{e^{\lambda\Delta} k k!}{\theta^2(\lambda\Delta)^k} \mathcal{L}.$$

We also need an estimate for $\Gamma_{\Delta}^{(4)}(0, 1)$. By (67) and (68) and (70) we obtain $p_{\Delta}(x) \geq e^{-\lambda\Delta} h_{\beta}(x/\sigma\Delta^{1/\beta})/\sigma\Delta^{1/\beta}$ and $\gamma_{\Delta}^{(2)}(0, x) = \check{h}_{\beta}(x/\sigma\Delta^{1/\beta})/\sigma^2\Delta^{1/\beta}$. Then use (86) and the definition (73) to get

$$\begin{aligned} \left| \Gamma_{\Delta}^{(4)}(0, 1) \right| &\leq \frac{1}{\sigma\theta^2} \int \left| \frac{\check{h}_{\beta}}{h_{\beta}} \left(\frac{x}{\sigma\Delta^{1/\beta}} \right) \right| dx \int h_{\beta}(y) \left| \check{f} \left(\frac{x - y\sigma\Delta^{1/\beta}}{\theta} \right) \right| dy \\ (91) \quad &= \frac{1}{\sigma\theta^2} \int h_{\beta}(y) dy \int \left| \frac{\check{h}_{\beta}}{h_{\beta}} \left(\frac{x}{\sigma\Delta^{1/\beta}} \right) \check{f} \left(\frac{x - y\sigma\Delta^{1/\beta}}{\theta} \right) \right| dx \leq C, \end{aligned}$$

where the last inequality comes from the facts that $\check{h}_{\beta}/h_{\beta}$ is bounded and that \check{f} is integrable (due to (32)).

At this stage we use (74), together with (76) and (77) and the fact that $2|xy| \leq ax^2 + y^2/a$ for all $a > 0$. Taking arbitrary constants $a_{kl} > 0$, we deduce from (90) that

$$\begin{aligned} I_{\Delta}^{\theta\theta} &\leq \frac{\mathcal{L}}{\theta^2} e^{-\lambda\Delta} \left(\sum_{k=1}^{\infty} \frac{(\lambda\Delta)^k k}{k!} + 2 \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \frac{(\lambda\Delta)^{k+l}}{k! l!} \left(\frac{k k!}{(\lambda\Delta)^k} \frac{l l!}{(\lambda\Delta)^l} \right)^{1/2} \right) \\ &\leq \frac{\mathcal{L}}{\theta^2} \left(\lambda\Delta + \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \frac{(\lambda\Delta)^l k}{l!} a_{kl} + \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \frac{(\lambda\Delta)^k l}{k!} \frac{1}{a_{kl}} \right) \end{aligned}$$

Then if we take $a_{kl} = (\lambda\Delta)^{\frac{k-l}{2}}$ for $l > k$, a simple computation shows that indeed

$$I_{\Delta}^{\theta\theta} \leq \frac{\mathcal{L}}{\theta^2} \left(\lambda\Delta + C\Delta^{3/2} \right)$$

for some constant C , and thus we get the upper bound in (36). In a similar way, and replacing L/θ^2 above by the supremum between L/θ^2 and $I(\beta)/\sigma^2$, we see that in (76) the sum of the absolute values of all summands except the one for $k = 0$ and $l = 1$ is smaller than a constant times $\Lambda\Delta$. Finally, the same holds for the term for $k = 0$ and $l = 1$, because of (91), and this proves (35).

7.9. Proof of Theorem 7. In the setting of Theorem 7 the measure G_{Δ} admits the density $y \mapsto h_{\alpha}(y/\Delta^{1/\alpha})/\Delta^{1/\alpha}$. For simplicity we set

$$(92) \quad u = \Delta^{\frac{1}{\alpha} - \frac{1}{\beta}},$$

(so $u \rightarrow 0$ as $\Delta \rightarrow 0$), and a change of variable allows to write (48) as

$$p_{\Delta}(x|\sigma, \beta, \theta, G) = \frac{1}{\theta\Delta^{1/\alpha}} \int h_{\beta} \left(\frac{x}{\sigma\Delta^{1/\beta}} - y \right) h_{\alpha} \left(\frac{\sigma y}{\theta u} \right).$$

Therefore

$$\partial_{\theta} p_{\Delta}(x|\sigma, \beta, \theta, G) = -\frac{1}{\theta^2\Delta^{1/\alpha}} \int h_{\beta} \left(\frac{x}{\sigma\Delta^{1/\beta}} - y \right) \check{h}_{\alpha} \left(\frac{\sigma y}{\theta u} \right),$$

and another change of variable in (53) leads to

$$(93) \quad I_{\Delta}^{\theta\theta} = \frac{\theta^{2\alpha-2}u^{2\alpha}}{\sigma^{2\alpha}} J_u,$$

where

$$(94) \quad \begin{cases} J_u = \int \frac{R_u(x)^2}{S_u(x)} dx, \\ R_u(x) = \left(\frac{\sigma}{\theta u}\right)^{1+\alpha} \int h_{\beta}(x-y) \check{h}_{\alpha}\left(\frac{\sigma y}{\theta u}\right) dy = \left(\frac{\sigma}{\theta u}\right)^{\alpha} \int h_{\beta}\left(x - \frac{y\theta u}{\sigma}\right) \check{h}_{\alpha}(y) dy, \\ S_u(x) = \frac{\sigma}{\theta u} \int h_{\beta}(x-y) h_{\alpha}\left(\frac{\sigma y}{\theta u}\right) dy = \int h_{\beta}\left(x - \frac{y\theta u}{\sigma}\right) h_{\alpha}(y) dy. \end{cases}$$

Below, we denote by K a constant and by ϕ a continuous function on R_+ with $\phi(0) = 0$, both of them changing from line to line and possibly depending on the parameters $\alpha, \beta, \sigma, \theta$; if they depend on another parameter η we write them as K_{η} or ϕ_{η} . Recalling (9), we have

$$(95) \quad h_{\alpha}(x) \sim \frac{c_{\alpha}}{|x|^{1+\alpha}}, \quad \check{h}_{\alpha}(x) \sim -\frac{\alpha c_{\alpha}}{|x|^{1+\alpha}}, \quad \int_{\{|y|>|x|\}} h_{\alpha}(y) dy \sim \frac{2c_{\alpha}}{\alpha|x|^{\alpha}}, \quad \text{as } |x| \rightarrow \infty.$$

Another application of (9) when $\beta < 2$ and of the explicit form of h_2 gives

$$(96) \quad |y| \leq 1 \implies |h_{\beta}''(x-y)| \leq \bar{h}_{\beta}(x) := \begin{cases} Kh_{\beta}(x) & \text{if } \beta < 2 \\ K(1+x^2)e^{-x^2/2} & \text{if } \beta = 2. \end{cases}$$

In order to obtain estimates on R_u and S_u , we split the first two integrals defining these functions into sums of integrals on the two domains $\{|y| \leq \eta\}$ and $\{|y| > \eta\}$, for some $\eta \in (0, 1]$ to be chosen later. We have $|h_{\beta}(x-y) - h_{\beta}(x) + h'_{\beta}(x)y| \leq \bar{h}_{\beta}(x)|y|^2$ as soon as $|y| \leq 1$, so the fact that both $f = \check{h}_{\alpha}$ and $f = h_{\alpha}$ are even functions gives

$$\left| \int_{\{|y| \leq \eta\}} h_{\beta}(x-y) f\left(\frac{\sigma y}{\theta u}\right) dy - h_{\beta}(x) \int_{\{|y| \leq \eta\}} f\left(\frac{\sigma y}{\theta u}\right) dy \right| \leq \bar{h}_{\beta}(x) \int_{\{|y| \leq \eta\}} y^2 \left| f\left(\frac{\sigma y}{\theta u}\right) \right| dy.$$

On the one hand we have with $f = \check{h}_{\alpha}$ or $f = h_{\alpha}$, and in view of (95):

$$\int_{\{|y| \leq \eta\}} y^2 \left| f\left(\frac{\sigma y}{\theta u}\right) \right| dy = \frac{\theta^3 u^3}{\sigma^3} \int_{\{|z| \leq \sigma\eta/\theta u\}} z^2 |f(z)| dz \leq Ku^{1+\alpha} \eta^{2-\alpha}.$$

On the other hand, the integrability of h_{α} and \check{h}_{α} and the fact that $\int \check{h}_{\alpha}(y) dy = 0$ yield

$$\begin{aligned} \int_{\{|y| \leq 1\}} h_{\alpha}\left(\frac{\sigma y}{\theta u}\right) dy &= \frac{\theta u}{\sigma} \int_{\{|z| \leq \sigma/\theta u\}} h_{\alpha}(z) dz = \frac{\theta u}{\sigma} (1 + \phi(u)), \\ \int_{\{|y| \leq \eta\}} \check{h}_{\alpha}\left(\frac{\sigma y}{\theta u}\right) dy &= \frac{\theta u}{\sigma} \int_{\{|z| \leq \sigma\eta/\theta u\}} \check{h}_{\alpha}(z) dz = -\frac{\theta u}{\sigma} \int_{\{|z| > \sigma\eta/\theta u\}} \check{h}_{\alpha}(z) dz = 2c_{\alpha} \frac{(\theta u)^{1+\alpha}}{\sigma^{1+\alpha} \eta^{\alpha}} (1 + \phi_{\eta}(u)), \end{aligned}$$

Putting all these facts together yields

$$(97) \quad \begin{cases} \left| \int_{\{|y| \leq 1\}} h_\beta(x-y) h_\alpha\left(\frac{\sigma y}{\theta u}\right) dy - \frac{\theta u}{\sigma} h_\beta(x) \right| \leq u \phi(u) h_\beta(x) + K u^{1+\alpha} \bar{h}_\beta(x), \\ \left| \int_{\{|y| \leq \eta\}} h_\beta(x-y) \check{h}_\alpha\left(\frac{\sigma y}{\theta u}\right) dy - 2c_\alpha \frac{(\theta u)^{1+\alpha}}{\sigma^{1+\alpha} \eta^\alpha} h_\beta(x) \right| \leq u^{1+\alpha} \left(h_\beta(x) \frac{\phi_\eta(u)}{\eta^\alpha} + K \bar{h}_\beta(x) \eta^{2-\alpha} \right). \end{cases}$$

For the integrals on $\{|y| > \eta\}$ we observe that by (95) we have $|h_\alpha(\sigma y / \theta u) - c_\alpha(\theta u / \sigma |y|)^{1+\alpha}| \leq (\theta u / \sigma y)^{1+\alpha} \phi(u)$, and the same for \check{h}_α except that c_α is substituted with $-\alpha c_\alpha$. Then if

$$(98) \quad D_\eta(x) = \int_{\{|y| > \eta\}} h_\beta(x-y) \frac{1}{|y|^{1+\alpha}} dy,$$

we readily get

$$(99) \quad \begin{cases} \left| \int_{\{|y| > 1\}} h_\beta(x-y) h_\alpha\left(\frac{\sigma y}{\theta u}\right) dy - c_\alpha \frac{(\theta u)^{1+\alpha}}{\sigma^{1+\alpha}} D_1(x) \right| \leq D_1(x) u^{1+\alpha} \phi(u), \\ \left| \int_{\{|y| > \eta\}} h_\beta(x-y) \check{h}_\alpha\left(\frac{\sigma y}{\theta u}\right) dy + \alpha c_\alpha \frac{(\theta u)^{1+\alpha}}{\sigma^{1+\alpha}} D_\eta(x) \right| \leq D_\eta(x) u^{1+\alpha} \phi(u). \end{cases}$$

At this stage, if we put together (97) and (99), we obtain

$$(100) \quad \begin{cases} \left| S_u(x) - \left(h_\beta(x) + c_\alpha \frac{(\theta u)^\alpha}{\sigma^\alpha} D_1(x) \right) \right| \leq (h_\beta(x) + u^\alpha D_1(x)) \phi(u) + K u^\alpha \bar{h}_\beta(x), \\ \left| R_u(x) - c_\alpha \left(\frac{2h_\beta(x)}{\eta^\alpha} - \alpha D_\eta(x) \right) \right| \leq \left(\frac{h_\beta(x)}{\eta^\alpha} + D_\eta(x) \right) \phi_\eta(u) + K \bar{h}_\beta(x) \eta^{2-\alpha}. \end{cases}$$

Our next step is to study the behavior at infinity of the continuous bounded and positive function D_η . We split the integral in (98) into the sum of the integrals, say $D_\eta^{(1)}$ and $D_\eta^{(2)}$, over the two domains $\{|y-x| \leq |x|^\gamma\}$ and $\{|y| > \eta, |y-x| > |x|^\gamma\}$, where $\gamma = 4/5$ if $\beta = 2$ and $\gamma \in ((1+\alpha)/(1+\beta), 1)$ if $\beta < 2$. On the one hand, $D_\eta^{(2)}(x) \leq K h_\beta(|x|^\gamma)$, so with our choice of γ we obviously have $|x|^{1+\alpha} D_\eta^{(2)}(x) \rightarrow 0$. On the other hand $|x|^{1+\alpha} D_\eta^{(1)}(x)$ is clearly equivalent, as $|x| \rightarrow \infty$, to $\int_{\{|y-x| \leq |x|^\gamma\}} h_\beta(x-y) dy$, which equals $\int_{\{|z| \leq |x|^\gamma\}} h_\beta(z) dz$, which in turns goes to 1. Hence we get for all $\eta > 0$:

$$(101) \quad D_\eta(x) \sim \frac{1}{|x|^{1+\alpha}} \quad \text{as } |x| \rightarrow \infty.$$

At this stage, we can obtain the behavior of R_u and S_u as $u \rightarrow 0$. First, an application of (95) to the last formula in (94) and Lebesgue theorem readily give

$$(102) \quad \lim_{u \rightarrow 0} S_u(x) = h_\beta(x).$$

Also, by (95), (96), (100) and (101), we get

$$(103) \quad S_u(x) \geq \begin{cases} \frac{C}{1+|x|^{1+\beta}} & \text{if } \beta < 2 \\ C \left(e^{-x^2/2} + \frac{u^\alpha}{1+|x|^{1+\alpha}} \right) & \text{if } \beta = 2 \end{cases}$$

for some $C > 0$ depending on the parameters, and all u small enough. In the same way, we see that $u^\alpha R_u(x) \rightarrow 0$, but this not enough. However, if $r_\eta = \frac{2h_\beta}{\eta^\alpha} - D_\eta$ we deduce from (100) that for all $\eta \in (0, 1]$,

$$(104) \quad \limsup_{u \rightarrow 0} |R_u(x) - c_\alpha r_\eta(x)| \leq K \bar{h}_\beta(x) \eta^{2-\alpha}.$$

A simple computation and the second order Taylor expansion with integral remainder for h_β yield

$$r_\eta(x) = \alpha \int_{\{|y|>\eta\}} \frac{h_\beta(x) - h_\beta(x-y)}{|y|^{1+\alpha}} dy = -\alpha \int_{\{|y|>\eta\}} |y|^{1-\alpha} dy \int_0^1 (1-v) h''_\beta(x-yv) dv.$$

By Lebesgue theorem, r_η converges pointwise as $\eta \rightarrow 0$ to the function r given by

$$r(x) = -\alpha \int_{\mathcal{R}} |y|^{1-\alpha} dy \int_0^1 (1-v) h''_\beta(x-yv) dv.$$

Then, taking into account (104), and using once more (100) together with (95), (96) and (101) and also $\alpha < \beta$, we get

$$(105) \quad \lim_{u \rightarrow 0} R_u(x) = c_\alpha r(x), \quad |R_u(x)| \leq \frac{K}{1 + |x|^{1+\alpha}}.$$

We are now in a position to prove (39), so $\beta < 2$ and $\alpha > \beta/2$ (and of course $\alpha < \beta$). Since $\alpha > \beta/2$, we see that (102), (103) and (105) allow to apply Lebesgue theorem in the definition (94) to get that $J_u \rightarrow \int \frac{(c_\alpha r(x))^2}{h_\beta(x)} dx$: this is (39) (obviously $|r(x)| \leq K/(1 + |x|^{1+\alpha})$, while $h_\beta(x) > C/(1 + |x|^{1+\beta})$ for some $C > 0$, so the integral in (39) converges).

The other cases are a bit more involved, because Lebesgue theorem does not apply and we will see that J_u goes to infinity. First, we introduce the following functions:

$$R'(x) = \frac{\alpha c_\alpha}{|x|^{1+\alpha}}, \quad S'_u(x) = \begin{cases} \frac{c_\beta}{|x|^{1+\beta}} + \frac{c_\alpha \theta^\alpha u^\alpha}{\sigma^\alpha |x|^{1+\alpha}} & \text{if } \beta < 2 \\ \frac{e^{-x^2/2}}{\sqrt{2\pi}} + \frac{c_\alpha \theta^\alpha u^\alpha}{\sigma^\alpha |x|^{1+\alpha}} & \text{if } \beta = 2 \end{cases}$$

Below we denote by $\psi(u, \Gamma)$ for $u \in (0, 1]$ and $\Gamma \geq 1$ the sum $\phi'(u) + \phi''(1/\Gamma)$ for any two functions ϕ' and ϕ'' like ϕ above (changing from line to line). We deduce from (95), (96), (100) for $\eta = 1$, and (101), that

$$(106) \quad |x| > \Gamma \quad \Rightarrow \quad |R_u(x) + R'(x)| \leq \psi(u, \Gamma) R'(x) + \begin{cases} \frac{K}{|x|^{1+\beta}} & \text{if } \beta < 2 \\ K x^2 e^{-x^2/2} & \text{if } \beta = 2. \end{cases}$$

In a similar way, we get

$$|x| > \Gamma \quad \Rightarrow \quad |S_u(x) - S'_u(x)| \leq \begin{cases} \psi(u, \Gamma) S'_u(x) & \text{if } \beta < 2 \\ \psi(u, \Gamma) S'_u(x) + K_0 u^\alpha x^2 e^{-x^2/2} & \text{if } \beta = 2, \end{cases}$$

where K_0 is some constant. Then for any $\varphi > 0$ we denote by Γ_φ the smallest number bigger than 1, such that $K_0 x^2 e^{-x^2/2} \leq \varphi \frac{c_\alpha \theta^\alpha}{\sigma^\alpha |x|^{1+\alpha}}$ for all $|x| > \Gamma_\varphi$. The last estimate above for $\beta = 2$ reads as $|S_u - S'_u| \leq S'_u(\psi(u, \Gamma) + \varphi)$, so in all cases we have for some fixed function ψ_0 as above:

$$(107) \quad S_u(x) = S'_u(x)(1 + \rho_u(x)), \quad \text{where} \quad |\rho_u(x)| \leq \begin{cases} \psi_0(u, \Gamma) & \text{if } \beta < 2, \quad |x| > \Gamma \\ \psi_0(u, \Gamma) + \varphi & \text{if } \beta = 2, \quad |x| > \Gamma > \Gamma_0. \end{cases}$$

At this stage, we set

$$J_{u, \Gamma} = \int_{\{|x| > \Gamma\}} \frac{R'(x)^2}{S'_u(x)} dx.$$

Observe that $J_u = J_{u, \Gamma} + \sum_{i=1}^4 J_{u, \Gamma}^{(i)}$, where

$$\begin{aligned} J_{u, \Gamma}^{(1)} &= \int_{\{|x| \leq \Gamma\}} \frac{R_u(x)^2}{S_u(x)} dx, & J_{u, \Gamma}^{(2)} &= \int_{\{|x| > \Gamma\}} \frac{(R_u(x) + R'(x))^2}{S_u(x)} dx, \\ J_{u, \Gamma}^{(3)} &= -2 \int_{\{|x| \leq \Gamma\}} \frac{R'(x)(R_u(x) + R'(x))}{S_u(x)} dx, & J_{u, \Gamma}^{(4)} &= \int_{\{|x| > \Gamma\}} \frac{R'(x)^2}{S_u(x)} dx - \int_{\{|x| > \Gamma\}} \frac{R'(x)^2}{S'_u(x)} dx. \end{aligned}$$

From (103), (105) and (106) we get for some $u_0 > 0$:

$$\sup_{u \in (0, u_0]} J_{u, \Gamma}^{(1)} < \infty, \quad u \in (0, u_0] \quad \Rightarrow \quad J_{u, \Gamma}^{(2)} \leq K + \psi(u, \Gamma) (J_{u, \Gamma}^{(4)} + J_{u, \Gamma}).$$

Cauchy–Schwarz inequality yields

$$|J_{u, \Gamma}^{(3)}| \leq 2 \left(J_{u, \Gamma}^{(2)} (J_{u, \Gamma}^{(4)} + J_{u, \Gamma}) \right)^{1/2}.$$

Finally, (103) and (107) and the definition of R' yield (with ψ_0 as in (107)):

$$|J_{u, \Gamma}^{(4)}| \leq \begin{cases} 2\psi_0(u, \Gamma) J_{u, \Gamma} & \text{if } \beta < 2, \quad \psi_0(u, \Gamma) < 1/2 \\ 2(\psi_0(u, \Gamma) + \varphi) J_{u, \Gamma} & \text{if } \beta = 2, \quad \Gamma > \Gamma_\varphi, \quad \psi_0(u, \Gamma) + \varphi < 1/2 \end{cases}$$

Therefore we get

$$\left| \frac{J_u}{J_{u, \Gamma}} - 1 \right| \leq \frac{K}{J_{u, \Gamma}} + \psi(u, \Gamma) + 2(\psi_0(u, \Gamma) + \varphi),$$

as soon as $\psi_0(u, \Gamma) + \varphi < 1/2$ and $\Gamma > \Gamma_\varphi$, and with the convention that $\varphi = 0$ and $\Gamma_0 = 1$ when $\beta < 2$. Then, remembering that $\lim_{u \rightarrow 0} \lim_{\Gamma \rightarrow \infty} \psi(u, \Gamma) = 0$, and the same for ψ_0 , and that $\varphi = 0$

when $\beta < 2$ and φ is arbitrarily small when $\beta = 2$, we readily deduce the following fact: Suppose that for some function $u \mapsto \gamma(u)$ going to $+\infty$ as $u \rightarrow 0$, and independent of Γ , we have proved that

$$(108) \quad J_{u,\Gamma} \sim \gamma(u) \quad \text{as } u \rightarrow 0, \quad \forall \Gamma > 1;$$

then $J_u \sim \gamma(u)$, and therefore by (93) we get

$$(109) \quad I_{\Delta}^{\theta\theta} \sim \frac{\theta^{2\alpha-2} u^{2\alpha} \gamma(u)}{\sigma^{2\alpha}}.$$

The simplest case is when $\beta < 2$ and $\alpha < \beta/2$. Indeed the change of variables $z = xu^{\alpha/(\beta-\alpha)}$ yields

$$(110) \quad \begin{aligned} J_{u,\Gamma} &= \alpha^2 c_{\alpha}^2 \int_{\{|x|>\Gamma\}} \frac{1}{c_{\beta}|x|^{1+2\alpha-\beta} + c_{\alpha}\theta^{\alpha}u^{\alpha}|x|^{1+\alpha}/\sigma^{\alpha}} dx \\ &= \frac{\alpha^2 c_{\alpha}^2}{u^{\frac{\alpha(\beta-2\alpha)}{\beta-\alpha}}} \int_{\{|z|>\Gamma u^{\alpha/(\beta-\alpha)}\}} \frac{1}{c_{\beta}|z|^{1+2\alpha-\beta} + c_{\alpha}\theta^{\alpha}|z|^{1+\alpha}/\sigma^{\alpha}} dz, \end{aligned}$$

and we have (108) with

$$\gamma(u) = \frac{\alpha^2 c_{\alpha}^2}{u^{\frac{\alpha(\beta-2\alpha)}{\beta-\alpha}}} \int \frac{1}{c_{\beta}|z|^{1+2\alpha-\beta} + c_{\alpha}\theta^{\alpha}|z|^{1+\alpha}/\sigma^{\alpha}} dz.$$

So if we combine this with (109), we get (41).

Suppose now that $2\alpha = \beta < 2$. Then

$$J_{u,\Gamma} = 2\alpha^2 c_{\alpha}^2 \int_{\Gamma}^{\infty} \frac{1}{x(c_{\beta} + c_{\alpha}\theta^{\alpha}u^{\alpha}x^{\alpha}/\sigma^{\alpha})} dx.$$

For $v > 0$ we let H_v be the unique point $x > 0$ such that $c_{\alpha}\theta^{\alpha}v^{\alpha}x^{\alpha}/\sigma^{\alpha} = c_{\beta}$, so in fact $H_v = \rho/v$ for some $\rho > 0$. We have

$$J_{u,\Gamma} \leq \frac{2\alpha^2 c_{\alpha}^2}{c_{\beta}} \int_{\Gamma}^{H_u} \frac{1}{x} dx + \frac{K}{u^{\alpha}} \int_{H_u}^{\infty} \frac{1}{x^{1+\alpha}} dx \leq \frac{2\alpha^2 c_{\alpha}^2}{c_{\beta}} \log(1/u) + K.$$

On the other hand, if $\mu > 0$ and $\Gamma < x < H_{u\mu}$ we have $c_{\beta} + c_{\alpha}\theta^{\alpha}u^{\alpha}x^{\alpha}/\sigma^{\alpha} < c_{\beta}(1 + 1/\mu^{\alpha})$, hence

$$J_{u,\Gamma} > \frac{2\alpha^2 c_{\alpha}^2}{c_{\beta}} \int_{\Gamma}^{H_{u\varphi}} \frac{1}{x(1 + 1/\mu^{\alpha})} dx \geq \frac{2\alpha^2 c_{\alpha}^2}{c_{\beta}} \frac{1}{1 + 1/\mu^{\alpha}} \log(1/u) - K_{\varphi}.$$

Putting together these two estimates and choosing μ big give (108) with $\gamma(u) = \frac{2\alpha^2 c_{\alpha}^2}{c_{\beta}} \log(1/u)$, and we readily deduce (40).

The case $\beta = 2$ is treated in pretty much the same way. We have

$$J_{u,\Gamma} = 2\alpha^2 c_{\alpha}^2 \int_{\Gamma}^{\infty} \frac{1}{x^{1+\alpha}(x^{1+\alpha}e^{-x^2/2}/\sqrt{2\pi} + c_{\alpha}\theta^{\alpha}u^{\alpha}/\sigma^{\alpha})} dx.$$

We suppose that Γ is big enough for $x \mapsto x^{2+2\alpha} e^{-x^2/2}$ to be decreasing on $[\Gamma, \infty)$. For $v > 0$ small enough, there is a unique number $H_v = x > \Gamma$ such that $c_\alpha \theta^\alpha v^\alpha / \sigma^\alpha = x^{1+\alpha} e^{-x^2/2} / \sqrt{2\pi}$, so in fact $H_v \sim \sqrt{2\alpha \log(1/v)}$ when $v \rightarrow 0$. Then

$$\begin{aligned} J_{u,\Gamma} &\leq K \int_\Gamma^{H_u} \frac{e^{x^2/2}}{x^{2+2\alpha}} dx + \frac{2\alpha^2 c_\alpha \sigma^\alpha}{\theta^\alpha u^\alpha} \int_{H_u}^\infty \frac{1}{x^{1+\alpha}} dx \\ &\leq \frac{K}{H_u^\alpha} + \frac{2\alpha c_\alpha \sigma^\alpha}{\theta^\alpha u^\alpha H_u^\alpha} \leq \frac{2\alpha c_\alpha \sigma^\alpha}{\theta^\alpha u^\alpha (2\alpha)^{\alpha/2} (\log(1/u))^{\alpha/2}} (1 + o(1)). \end{aligned}$$

On the other hand, if $\mu > 1$ and $x > H_{u\mu}$ we have $x^{1+\alpha} e^{-x^2/2} / \sqrt{2\pi} + c_\alpha \theta^\alpha u^\alpha / \sigma^\alpha < +c_\alpha \theta^\alpha u^\alpha (1 + \mu^\alpha) / \sigma^\alpha$, hence

$$J_{u,\Gamma} > \frac{2\alpha^2 c_\alpha \sigma^\alpha}{\theta^\alpha u^\alpha} \int_\Gamma^{H_{u\mu}} \frac{1}{x^{1+\alpha}(1 + \mu^\alpha)} dx \geq \frac{2\alpha c_\alpha \sigma^\alpha}{\theta^\alpha u^\alpha (2\alpha)^{\alpha/2} (\log(1/u))^{\alpha/2} (1 + \mu^\alpha)} (1 + o(1)).$$

So again we see, by choosing μ close to 1, that the desired result holds with

$$\gamma(u) = \frac{2\alpha c_\alpha \sigma^\alpha}{\theta^\alpha u^\alpha (2\alpha)^{\alpha/2} (\log(1/u))^{\alpha/2}},$$

and we deduce (38).

REFERENCES

- AÏT-SAHALIA, Y. (2002). Telling from discrete data whether the underlying continuous-time model is a diffusion. *Journal of Finance* **57** 2075–2112.
- AÏT-SAHALIA, Y. (2003). Disentangling diffusion from jumps. *Journal of Financial Economics* **74** 487–528.
- AÏT-SAHALIA, Y. and JACOD, J. (2004). Estimators for discretely sampled Lévy processes. Tech. rep., Princeton University and Université de Paris VI.
- AKGIRAY, V. and LAMOUREUX, C. G. (1989). Estimation of stable-law parameters: A comparative study. *Journal of Business and Economic Statistics* **7** 85–93.
- BARNDORFF-NIELSEN, O.E. (1997). Normal inverse Gaussian distribution and stochastic volatility modelling. *Scandinavian Journal of Statistics* **24** 24–13.
- BARNDORFF-NIELSEN, O.E. (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics* **2** 41–68.
- BERGSTRØM, H. (1952). On some expansions of stable distribution functions. *Arkiv für Mathematik* **2** 375–378.
- BOYARCHENKO, S. and LEVENDORSKIÏ, S. (2002). Perpetual American options under Lévy processes. *SIAM Journal of Control and Optimization* **40** 1663–1696.
- BROCKWELL, P. and BROWN, B. (1980). High-efficiency estimation for the positive stable laws. *Journal of the American Statistical Association* **76** 626–631.
- CARR, P., GEMAN, H., MADAN, D. B. and YOR, M. (2002). The fine structure of asset returns: An empirical investigation. *Journal of Business* **75** 305–332.
- CARR, P., GEMAN, H., MADAN, D. B. and YOR, M. (2003). Stochastic volatility for Lévy processes. *Mathematical Finance* **13** 345–382.
- CARR, P. and WU, L. (2003). What type of process underlies options? A simple robust test. *Journal of Finance* **58** 2581–2610.
- CARR, P. and WU, L. (2004). Time-changed Lévy processes and option pricing. *Journal of Financial Economics* **71** 113–141.
- CHAN, K. (1999). Pricing contingent claims on stocks driven by Lévy processes. *Annals of Applied Probability* **9** 504–528.

- DE HAAN, L. and RESNICK, S. (1980). A simple asymptotic estimate for the index of a stable distribution. *Journal of the Royal Statistical Society - Series B* **42** 83–87.
- DUMOUCHEL, W. H. (1971). *Stable Distributions in Statistical Inference*. Ph.D. thesis, Department of Statistics, Yale University.
- DUMOUCHEL, W. H. (1973a). On the asymptotic normality of the maximum-likelihood estimator when sampling from a stable distribution. *Annals of Statistics* **1** 948–957.
- DUMOUCHEL, W. H. (1973b). Stable distributions in statistical inference: 1. Symmetric stable distributions compared to other symmetric long-tailed distributions. *Journal of the American Statistical Association* **68** 469–477.
- DUMOUCHEL, W. H. (1975). Stable distributions in statistical inference: 2. Information from stably distributed samples. *Journal of the American Statistical Association* **70** 386–393.
- DUMOUCHEL, W. H. (1983). Estimating the stable index α in order to measure tail thickness: A critique. *Annals of Statistics* **11** 1019–1031.
- EBERLEIN, E. and JACOD, J. (1997). On the range of option prices. *Finance and Stochastics*. **1** 131–140.
- EBERLEIN, E., KELLER, U. and PRAUSE, K. (1998). New insights into smile, mispricing and Value at Risk: The hyperbolic model. *Journal of Business* **71** 371–405.
- EBERLEIN, E., RAIBLE, S. (1999). Term structure models driven by general Lévy processes. *Mathematical Finance* **9** 31–53.
- EMMER, S., KLÜPPELBERG, C. (2004). Optimal portfolios when stock prices follow an exponential Lévy process. *Finance and Stochastics* **8** 17–44.
- FAMA, E. F. and ROLL, R. (1965). The behavior of stock market prices. *Journal of Business* **38** 34–105.
- FAMA, E. F. and ROLL, R. (1968). Some properties of symmetric stable distributions. *Journal of the American Statistical Association* **63** 817–835.
- FAMA, E. F. and ROLL, R. (1971). Parameter estimates for symmetric stable distributions. *Journal of the American Statistical Association* **66** 331–338.
- FAR, H. (2001). Propriétés asymptotiques de modèles paramétriques associés à l’observation discrétisée de processus de sauts. *PhD thesis, Université Paris-6*.
- FENECH, A. P. (1976). Asymptotically efficient estimation of location for a symmetric stable law. *Annals of Statistics* **4** 1088–1100.
- FEUERVERGER, A. and MCDUNNOUGH, P. (1981a). On efficient inference in symmetric stable laws and processes. In *Statistics and Related Topics* (M. Csörgö, D. A. Dawson, J. Rao and A. Saleh, eds.). North Holland, Amsterdam, The Netherlands, 109–122.
- FEUERVERGER, A. and MCDUNNOUGH, P. (1981b). On the efficiency of empirical characteristic function procedures. *Journal of the Royal Statistical Society - Series B* **43** 20–27.
- HOFFMANN-JØRGENSEN, J. (1993). Stable densities. *Theory of Probability and its Applications* **38** 350–355.
- HÖPFNER, R. and JACOD, J. (1994). Some remarks on the joint estimation of the index and the scale parameters for stable processes. In *Proceedings of the Fifth Prague Symposium on Asymptotic Statistics* (P. Mandl and M. Huskova, eds). Physica Verlag, Heidelberg, Germany, 273–284.
- JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*. 2nd ed. Springer-Verlag, New York.
- JEDIDI, W. (2001). Local asymptotic normality of statistical models associated with discrete observations of Lévy processes *Preprint, Johannes Gutenberg Universität Mainz*.
- KALLSEN, J. (2000). Optimal portfolios for exponential Lévy processes. *Mathematical Methods of Operations Research* **51** 357–374.
- KHINDANOVA, I. and RACHEV, S. and SCHWARTZ, E. (2001). Stable modeling of value at risk. *Mathematical and Computer Modelling* **34** 1223–1259.
- KLÜPPELBERG, C. and LINDNER, A. and MALLER, R. (2004). A continuous-time GARCH process driven by a Lévy process: stationarity and second-order behaviour. *Journal of Applied Probability* **41** 601–622.
- KOUTROVELIS, I. A. (1980). Regression-type estimation of the parameters of stable laws. *Journal of the American Statistical Association* **75** 918–928.
- LEBLANC, B. and YOR, M. (1998). Lévy processes in finance: a remedy to the non-stationarity of continuous martingales. *Finance and Stochastics* **2** 399–408.
- MANDELBROT, B. (1963). The variation of certain speculative prices. *Journal of Business* **36** 394–419.
- MCCULLOCH, J. H. (1997). Measuring tail thickness to estimate the stable index α : A critique. *Journal of Business and Economic Statistics* **15** 74–81.
- MITTNIK, S. and RACHEV, S.T. (2001). Modeling asset returns with alternative stable distributions. *Econometric Review* **12** 261–330.

- MORDECKI, E. (2002). Optimal stopping and perpetual options for Lévy processes. *Finance and Stochastics* **6** 473–493.
- NOLAN, J. P. (1997). Numerical computation of stable densities and distribution functions. *Communications in Statistics - Stochastic Models* **13** 759–774.
- NOLAN, J. P. (2001). Maximum likelihood estimation and diagnostics for stable distributions. In *Lévy Processes: Theory and Applications* (O. E. Barndorff-Nielsen, T. Mikosch and S. I. Resnick, eds.). Birkhäuser, Boston, 379–400.
- PRESS, S. J. (1972). Estimation in univariate and multivariate stable distributions. *Journal of the American Statistical Association* **67** 842–846.
- RACHEV, S.T. and HAN, S. (2000). Portfolio management with stable distributions. *Mathematical Methods of Operations Research* **512** 341–352.
- SINGLETON, K. (2001). Estimation of affine asset pricing models using the empirical characteristic function. *Journal of Econometrics* **102** 111–141.
- ZOLOTAREV, V. M. (1986). *One-dimensional Stable Distributions*, vol. 65 of *Translations of Mathematical Monographs*. American Mathematical Society.
- ZOLOTAREV, V. M. (1995). On representation of densities of stable laws by special functions. *Theory of Probability and its Applications* **39** 354–362.

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