Boundary Value Problems in Some Ramified Domains with a Fractal Boundary: Analysis and Numerical Methods. Part I: Diffusion and Propagation problems.

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Boundary Value Problems in Some Ramified Domains with a Fractal Boundary: Analysis and Numerical Methods. Part I: Diffusion and Propagation problems.

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Abstract

This paper is devoted to numerical methods for solving boundary value problems in self-similar ramified domains of $\mathbb{R}^2$ with a fractal boundary. Homogeneous Neumann conditions are imposed on the fractal part of the boundary, and Dirichlet conditions are imposed on the remaining part of the boundary. Several partial differential equations are considered. For the Laplace equation, the Dirichlet to Neumann operator is studied. It is shown that it can be computed as the unique fixed point of a rational map. From this observation, a self-similar finite element method is proposed and tested. For the Helmholtz equation, it is shown that the Dirichlet to Neumann operator can also be computed as the limit of an inductive sequence of operators. Here too, a finite element method is designed and tested. It permits to compute numerically the spectrum of the Laplace operator in the irregular domain with Neumann boundary conditions, as well as the eigenmodes. The repartition of the eigenvalues is investigated. The eigenmodes are normalized by means of a perturbation method and the spectral decomposition of a compactly supported function is carried out. This permits to solve numerically the wave equation in the self-similar ramified domain.

1 Introduction

In this paper, we deal with the numerical simulation of diffusion and propagation phenomena in a self-similar ramified domain of $\mathbb{R}^2$ with a fractal boundary. This work was motivated by a wider and very challenging project aiming at simulating the diffusion of medical sprays in the lungs. Our ambitions here are more modest, since the geometry of the problems (two dimensions only) and the underlying physical phenomena are much simpler, but we hope that giving rigorous results and methods will prove useful. The geometry under consideration is that of a self-similar ramified bidimensional domain, see Figure 1 below. It can be seen as a simple model for a tree or for lungs. This domain can be obtained by glueing together dilated/translated copies of a simple polygonal domain of $\mathbb{R}^2$, called ω^0 below.

Partial differential equations in domain with fractal boundaries or fractal interfaces is a relatively new topic: variational techniques have been developed, involving new results on fonctional analysis, see [12, 9, 10]. A very nice theory on variational problems in fractal media is given in [13].

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The difficulty of solving boundary value problems with partial differential equations in this kind of domains comes essentially from the multiscale character of the boundary. Yet, when the equation is homogeneous, it is possible to make use of the geometric self-similarity in order to compute very accurately the restrictions of the solutions to subdomains obtained by interrupting the fractal construction after a finite number of generations.
We consider first Poisson problems (with the Laplace operator) with homogeneous Neumann conditions on the fractal part of the boundary. On the remaining part of the boundary, noted Γ^0 below, we impose a Dirichlet condition. In other words, we aim at computing the harmonic lifting of a function defined in Γ^0. For that, it is possible to solve an equivalent boundary value problem in a subdomain obtained by interrupting the fractal construction after a finite number of generations: this equivalent problem involves a non local Dirichlet to Neumann operator, which maps a function defined on Γ^0 to the normal derivative of its harmonic lifting in the whole domain. It turns out that the Dirichlet to Neumann operator on Γ^0 can be computed very accurately by making use of the geometric self-similarity. The Dirichlet to Neumann operator is approximated as the limit of an inductive sequence, see (41) (42) below. When discretizing the problem with finite elements with self similar meshes, the same procedure can be implemented.

Next, turning to vibration problems in the domain described above leads to consider boundary value problems with Helmholtz equation. Here again, it is very natural to study the Dirichlet to Neumann operators (depending on the pulsation of the related harmonic wave), which, thanks to the self-similar structure of the set, can be approximated by iterations of a renormalization operator, see (77) (78) below. The discrete counterpart of this can be implemented with finite elements as soon as the mesh is self-similar. A related problem arises in the analysis of the spectrum of fractal domains such as Sierpinski gasket (the present paper does not consider a fractal domain, but a domain with a fractal boundary). The numerical method developed in this paper is very reminiscent of some of the techniques involved in the theoretical analysis of finitely ramified fractals (see [15],[20], [17], [16], and [2, 14, 5] for numerical simulations). The simple structure of these sets allows to do an explicit analysis of the spectral properties. This involves the dynamics of a renormalization map which acts on the Dirichlet to Neumann operator on the boundary (which for finitely ramified fractal consists only on a finite number of points). In this paper, the natural boundary is not so simple, but the numerical method is based on a similar strategy.

Once we know how to solve the boundary value problems with Helmholtz equation, it is natural to turn our attention to the spectral analysis of the Laplace operator in the domain under consideration. The above mentioned Dirichlet to Neumann operators contain a lot of information on the eigenvalues and their eigenfunctions. In particular, their construction permits to compute numerically the spectrum, and the eigenmodes of the domain. Several important problems concerning spectral analysis on domains with fractal boundaries motivate our numerical computations. The first one concerns the eigenvalue repartition. Rigorous results about the eigenvalue repartition have been obtained when the domain has a smooth boundary, but when the boundary of the domain is fractal, we only have some bounds and conjectures (the so-called Weyl-Berry formula, see [11]). In § 6.4.2, we present the numerical results relative to this problem. Another important problem concerns the shape of the eigenfunctions: physicists believe that they exhibit strong localization (cf [19], [18]), close to the fractal boundary. These kind of properties are important to understand the geometry of the set, but from the numerical point of view, they are difficult to analyze, due to the multiscale character of the fractal boundary. The methods presented here, using self-similar meshes, takes into account fine scales in the

ramifications.
In order to solve time dependent problems in the irregular domain, the spectral information can be used, but for that, one needs to normalize the eigenmodes: in this paper, we propose a perturbation method for normalizing the eigenmodes. This permits to compute the spectral decomposition of any function compactly supported in the domain, and finally to solve numerically time dependent equations like the wave equation.
It is also possible to develop numerical methods for boundary value problems with nonzero Neumann data on the fractal part of the boundary, by making use of the Dirichlet to Neumann operator. This is the topic of a forthcoming work, [1].

2 Geometrical setting of the model problem

Hereafter, we use the notation

$$s_n = \sum_{i=0}^{n} 2^{-i}. \tag{1}$$

Consider the following T-shaped subset of $\mathbb{R}^2$

$$Q^0 = \big((-1,1) \times (0,2]\big) \cup ((-2,2) \times (2,3)) \cup (((-2,-1) \cup (1,2)) \times \{3\}) .$$

The self-similar ramified domain Ω^0 is constructed as an infinite union of subsets of $\mathbb{R}^2$ obtained by translating/dilating Q^0; at a first stage, two copies of $1/2 \cdot Q^0$ are translated respectively on top-left and on top-right of Q^0 and are glued to Q^0: more precisely, let F_1 and F_2 be the affine mappings

$$F_i(x) = \xi_i^1 + \frac{1}{2}x, \text{ where } \xi_1^1 = (-\frac{3}{2}, 3) \text{ and } \xi_2^1 = (\frac{3}{2}, 3), \tag{2}$$

and let Q^1 be the set $Q^1 = F_1(Q^0) \cup F_2(Q^0)$. Next, the construction is recursive: the points ξ_i^n for $i = 1, \dots, 2^n$ are defined by the relation: for $j = 1, \dots, 2^{n-1}$, $\xi_{2j-1}^n = \xi_j^{n-1} + \frac{1}{2^{n-1}}\xi_1^1$ and $\xi_{2j}^n = \xi_j^{n-1} + \frac{1}{2^{n-1}}\xi_2^1$, and the following sets are introduced:

$$Q^n = \cup_{i=1}^{2^n} Q_i^n, \quad \text{with} \quad Q_i^n = \xi_i^n + \frac{1}{2^n} \cdot Q^0. \tag{3}$$

For an integer n, $n \geq 1$, calling $\mathcal{A}_n$ the set containing all the mappings from $\{1, \dots, 2^{n-1}\}$ to $\{1, 2\}$, and for $\sigma \in \mathcal{A}_n$, $\mathcal{M}_\sigma(F_1, F_2) = F_{\sigma(1)} \circ F_{\sigma(2)} \circ \cdots \circ F_{\sigma(2^n)}$, (3) can also be written

$$Q^n = \cup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(F_1, F_2)(Q^0).$$

It will sometimes be convenient to agree that $\mathcal{A}_0 = \{0\}$ and that $\mathcal{M}_0(F_1, F_2)$ is the identity.
Finally, the self-similar ramified Ω^0 is defined by

$$\Omega^0 = \cup_{n=0}^{\infty} Q^n. \tag{4}$$

The construction of Ω^0 is displayed on Figure 1. It is straightforward to see that $\Omega^0 \subset (-3,3) \times (0,6)$. Note that Ω^0 may also be obtained as a union of overlapping open subsets of $\mathbb{R}^2$, thus Ω^0 is an open set.
It will be useful to define the truncated domain Ω^N, which has also a fractal boundary:

$$\Omega^N = \cup_{n=N}^{\infty} Q^n. \tag{5}$$

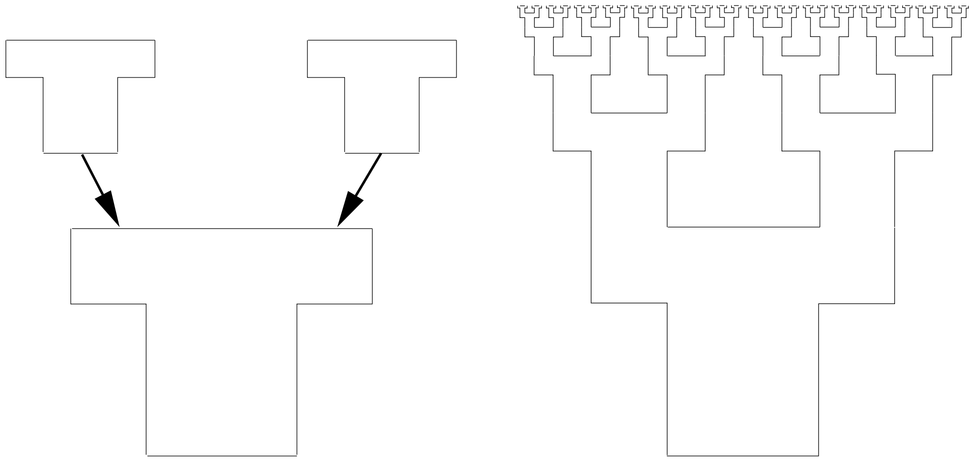

Figure 1: Left: the first step of the construction. Right: the self-similar ramified domain (only a few generations are displayed)

The following self-similarity property is true: Ω^N is the union of 2^N translated copies of $\frac{1}{2^N} \cdot \Omega^0$, i.e.

$$\Omega^N = \cup_{\sigma \in \mathcal{A}_N} \Omega^\sigma, \tag{6}$$

where

$$\Omega^\sigma = \mathcal{M}_\sigma(F_1, F_2)(\Omega^0). \tag{7}$$

Also, $\Omega^N \backslash \Omega^{N+1} = Q^N$ for any $N \geq 0$.
We define the bottom boundary of Ω^0 by $\Gamma^0 = ((-1;1) \times \{0\})$ and $\Sigma^0 = \partial\Omega^0 \cap \{(x,y); x \in \mathbb{R}, 0 < y < 6\}$. We have

$$\partial\Omega^0 \cap \{(x,y); x \in \mathbb{R}, y < 6\} = \Gamma^0 \cup \Sigma^0. \tag{8}$$

Similarly, the bottom boundary of Ω^N is $\Gamma^N = \cup_{i=1}^{2^N} \Gamma_i^N$, $\Gamma_i^N = \xi_i^N + \frac{1}{2^N} \cdot \Gamma^0$. In an equivalent manner,

$$\Gamma^N = \cup_{\sigma \in \mathcal{A}_N} \Gamma^\sigma, \tag{9}$$

where

$$\Gamma^\sigma = \mathcal{M}_\sigma(F_1, F_2)(\Gamma^0). \tag{10}$$

For $N > 0$, Γ^N is contained in the line $y = y_N = 3s_{N-1}$, see (1). We define also $\Sigma^N = \partial\Omega^N \cap \{(x,y); x \in \mathbb{R}, 3s_{N-1} < y < 6\}$.
For what follows, it is also useful to introduce the open domains ω^N, for $N \geq 0$:

$$\omega^N = \mathrm{Int}\left(\Omega^0 \backslash \Omega^{N+1}\right). \tag{11}$$

Remark 1 *Note that it is also possible to construct similar domains using dilations with ratii α^n with $\alpha \in]0; 1/2]$; here we have chosen $\alpha = 1/2$.*

3 A Poincaré inequality

Consider the function space $H^1(\Omega^n) = \{v \in L^2(\Omega^n) \text{ s.t. } \nabla v \in (L^2(\Omega^n))^2\}$. Similarly, for all positive integer p, it is possible to define $H^p(\Omega^n)$ as the space of functions whose partial derivatives up to order p belong to $L^2(\Omega)$, and for all positive real number $s \notin \mathbb{N}$, $H^s(\Omega^n)$ is defined

by interpolation between $H^p(\Omega^n)$ and $H^{p+1}(\Omega^n)$, where p is the integer such that $p \le s < p+1$. Likewise, it is possible to define the Sobolev spaces $H^s(\omega^n)$ for all nonnegative integers n.
Of course, for all $n \ge 0$, the restriction of a function $v \in H^1(\Omega^0)$ to ω^n belongs to $H^1(\omega^n)$, so it is possible to define the trace of v on Γ^n. The trace operator on Γ^n is bounded from $H^1(\Omega^0)$ to $L^2(\Gamma^n)$, so one can define the closed subspace of $H^1(\Omega^n)$:

$$\mathcal{V}(\Omega^n) = \{v \in H^1(\Omega^n) \text{ s.t. } v|_{\Gamma^n} = 0\}. \tag{12}$$

In what follows, for a function u integrable on Γ^σ, the notation $\langle u \rangle_{\Gamma^\sigma}$ will be used for the mean value of u on Γ^σ.

Theorem 1 (Poincaré's inequality) *For any* $u \in \mathcal{V}(\Omega^0)$,

$$\|u\|_{L^2(\Omega^0)} \le \sqrt{32}\|\nabla u\|_{L^2(\Omega^0)}. \tag{13}$$

Proof. We proceed by proving first the Poincaré inequality for functions in the space $\mathcal{V}(\omega^N) = \{v \in H^1(\omega^N) \text{ s.t. } v|_{\Gamma^0} = 0\}$, with a constant independent of N. Since the function space $\{v \in \mathcal{C}^\infty(\overline{\omega^N}) \text{ s.t. } v|_{\Gamma^0} = 0\}$ is dense in $\mathcal{V}(\omega^N)$, it is enough to prove the inequality for functions in that space.
The idea of the proof is to construct explicitly a change of variables which maps Ω^0 onto a fractured set contained in the rectangle $(-1,1) \times (0,8)$.
We define first a continuous and piecewise affine change of variables γ^0 mapping $\widehat{\omega^0} = ((-1,1) \times (0,4]) \backslash (\{0\} \times [3,4])$ onto Q^0 by

$$\text{if } x > 0, \qquad \gamma^0(x,t) = \begin{cases} (x,t) & \text{for } t \in (0, 3-x] \\ (t-3+2x, 3-x) & \text{for } t \in [3-x, 4-x] \\ (x+1, t-1) & \text{for } t \in [4-x, 4] \end{cases},$$

$$\text{if } x < 0, \qquad \gamma^0(x,t) = (-\gamma^0_1(-x,t), \gamma^0_2(-x,t)) \quad \text{for } t \in (0,4],$$

$$\gamma^0(0,t) = (0,t) \quad \text{for } t \in (0,3)$$

It is easy to check that γ^0 is one to one. The set $\widehat{\omega^0}$ is fractured in the sense that it does not lye locally on one side of its boundary.
Note also that for each $x \in (-1,1)$, the trajectory $\{\gamma^0(x,t), t \in (0,4]\}$ is made of at most three straight lines parallel to the axes, and that for $x \in (0,1)$, $\gamma^0(x,4) = x+1$ so $\{\gamma(x,4), x \in (0,1)\} = (1,2) \times \{3\}$. Similarly, one can check that $\nabla\gamma^0$ is piecewise constant and can only take the values

$$\nabla\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \nabla\gamma^0 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus, at the points where it makes sense, $\det(\nabla\gamma^0) = 1$. Therefore the mapping γ^0 preserves the measure.
It is possible to define a one to one continuous and piecewise affine mapping γ^1 from $\widehat{\omega^1} = ((-1,1) \times (0,6]) \backslash \left((\{0\} \times [3,6]) \bigcup (\{\frac{1}{2}\} \times [\frac{11}{2}, 6]) \bigcup (\{-\frac{1}{2}\} \times [\frac{11}{2}, 6])\right)$ onto $Q^0 \cup Q^1$, by

$$\begin{aligned} \gamma^1(x,t) &= \gamma^0(x,t) && \text{for } t \le 4, \\ \gamma^1(x,t) &= (\tfrac{3}{2}, 3) + \tfrac{1}{2}\gamma^0(2x-1, 2(t-4)) && \text{for } t > 4, x > 0, \\ \gamma^1(x,t) &= (-\tfrac{3}{2}, 3) + \tfrac{1}{2}\gamma^0(2x+1, 2(t-4)) && \text{for } t > 4, x < 0. \end{aligned}$$

It is very easy to check that $\det(\nabla\gamma^1) = 1$, at all the points where $\nabla\gamma^1$ is defined.
Call $y_N = 4s_N - 2^{-N}$ and consider the doubly-indexed sequence (x^n_j) for $n \geq 0$ and $0 \leq j < 2^n$ defined by the recursion

$$\begin{array}{rcl} x^0_0 & = & 0, \\ x^1_j & = & -\frac{1}{2} + j, \quad j = 0, 1, \\ x^n_j & = & x^{n-1}_{\frac{j}{2}} + 2^{-n+1} x^1_{j\%2}, \quad j = 0, \ldots, 2^n - 1, \end{array}$$

where $\frac{j}{2}$ and $j\%2$ are respectively the quotient and remainder of the Euclidean division of j by 2. By proceeding recursively, we can define a one to one continuous and piecewise affine mapping

$$\gamma^N : \quad \widehat{\omega^N} = ((-1,1) \times (0, 4s_N]) \setminus \bigcup_{n=0}^{N} \left(\left(\bigcup_{j=0}^{2^n-1} \{x^n_j\} \right) \times [y_n, 4s_N] \right) \mapsto \cup_{n=0}^{N} Q^n,$$

which preserve the measure. Finally, introducing the open set

$$\widehat{\Omega^0} = ((-1,1) \times (0,8)) \setminus \bigcup_{n=0}^{\infty} \left(\left(\bigcup_{j=0}^{2^n-1} \{x^n_j\} \right) \times [y_n, 8) \right),$$

we also have a one to one continuous and piecewise affine mapping from $\widehat{\Omega^0}$ onto Ω^0. The sets $\widehat{\omega^0}$ and $\widehat{\Omega^0}$ are displayed on Figure 2.

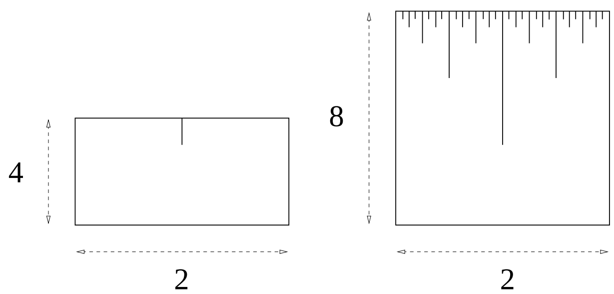

Figure 2: Left: the set $\widehat{\omega^0}$. Right: the open set $\widehat{\Omega^0}$ (only the longest fractures are displayed)

Let us define $I_N = 1 + \sum_{i=0}^{N} 2^i$ and call a_i, $0 \leq i \leq I_N$ the abscissa of the vertical boundaries of $\widehat{\omega^N}$, ordered increasingly. Consider a function $u \in C^\infty_0(\omega^N)$ such that $u|_{\Gamma^0} = 0$.

$$\int_{\omega^N} u^2 = \int_{\widehat{\omega^N}} u^2(\gamma^N(x,t)) = \sum_{i=0}^{I_N-1} \int_{a_i}^{a_{i+1}} dx \int_0^{4s_N} u^2(\gamma^N(x,t))dt$$

$$= \sum_{i=0}^{I_N-1} \int_{a_i}^{a_{i+1}} dx \int_0^{4s_N} \left(\int_0^t \frac{d}{ds}(u(\gamma^N(x,s)))\, ds \right)^2 dt$$

$$\leq \sum_{i=0}^{I_N-1} \int_{a_i}^{a_{i+1}} dx \int_0^{4s_N} dt\, t \int_0^t \left(\frac{\partial u}{\partial x}(\gamma^N(x,s)) \frac{\partial \gamma^N_1}{\partial t}(x,s) \right)^2 + \left(\frac{\partial u}{\partial y}(\gamma^N(x,s)) \frac{\partial \gamma^N_2}{\partial t}(x,s) \right)^2 ds$$

by Cauchy-Schwarz inequality and because $\frac{\partial \gamma_1^N}{\partial t}\frac{\partial \gamma_2^N}{\partial t} = 0$. Therefore

$$\begin{aligned}\int_{\omega^N} u^2 &\le 32 \sum_{i=0}^{I_N-1} \int_{a_i}^{a_{i+1}} dx \int_0^{4s_N} \left(\frac{\partial u}{\partial x}(\gamma^N(x,s))\frac{\partial \gamma_1^N}{\partial t}(x,s)\right)^2 + \left(\frac{\partial u}{\partial y}(\gamma^N(x,s))\frac{\partial \gamma_2^N}{\partial t}(x,s)\right)^2 ds \\ &\le 32 \sum_{i=0}^{I_N-1} \int_{a_i}^{a_{i+1}} dx \int_0^{4s_N} \left(\frac{\partial u}{\partial x}(\gamma^N(x,s))\right)^2 + \left(\frac{\partial u}{\partial y}(\gamma^N(x,s))\right)^2 ds\end{aligned}$$

because $|\frac{\partial \gamma_1^N}{\partial t}| \le 1$ and $|\frac{\partial \gamma_2^N}{\partial t}| \le 1$. Performing the inverse change of variables, we obtain that

$$\int_{\omega^N} u^2 \le 32 \int_{\omega^N} |\nabla u|^2. \tag{14}$$

By density, it is clear that (14) holds for $u \in \mathcal{V}(\omega^N)$. Since the constant in (14) does not depend of N, we obtain (13) by using Lebesgue's theorem. ■

In what follows, we will use the notation $\lesssim$ to indicate that there may arise constants in the estimates, which are independent of the index n in Ω^n or ω^n or on the mesh size when dealing with finite elements.

Corollary 1 *There exists a positive constant C such that for all $u \in H^1(\Omega^0)$,*

$$\|u\|^2_{L^2(\Omega^0)} \le C\left(\|\nabla u\|^2_{L^2(\Omega^0)} + \|u|_{\Gamma^0}\|^2_{L^2(\Gamma^0)}\right). \tag{15}$$

Proof. Define $H^{\frac{1}{2}}(\Gamma^0)$ as the space of the traces on Γ^0 of the functions belonging to $H^1(\omega^0)$, endowed with the norm

$$\|u\|_{H^{\frac{1}{2}}(\Gamma^0)} = \inf_{v\in H^1(\omega^0), v|_{\Gamma^0}=u} \|v\|_{H^1(\omega^0)}.$$

It is a classical result that for all $v \in H^1(\omega^0)$,

$$\|v|_{\Gamma^0}\|_{H^{\frac{1}{2}}(\Gamma^0)} \lesssim \left(\|\nabla v\|^2_{L^2(\omega^0)} + \|v|_{\Gamma^0}\|^2_{L^2(\Gamma^0)}\right)^{\frac{1}{2}}. \tag{16}$$

For $u \in H^1(\Omega^0)$, consider the function $\tilde{u} \in H^1(\omega^0)$ such that

$$\Delta \tilde{u} = 0 \text{ in } \omega^0, \quad \tilde{u}|_{\Gamma^0} = u|_{\Gamma^0}, \quad \tilde{u}|_{\Gamma^1} = 0, \quad \frac{\partial \tilde{u}}{\partial n} = 0 \text{ on } \partial\omega^0 \backslash (\Gamma^0 \cup \Gamma^1).$$

It can be checked that

$$\|\tilde{u}\|_{H^1(\omega^0)} \lesssim \|u|_{\Gamma^0}\|_{H^{\frac{1}{2}}(\Gamma^0)}. \tag{17}$$

Calling again $\tilde{u}$ the extension by 0 of $\tilde{u}$ in Ω^0, we have that $u - \tilde{u} \in \mathcal{V}(\Omega^0)$, and from (13), (16) and (17),

$$\begin{aligned}\|u-\tilde{u}\|^2_{L^2(\Omega^0)} &\le 32\|\nabla(u-\tilde{u})\|^2_{L^2(\Omega^0)} \le 64\left(\|\nabla u\|^2_{L^2(\Omega^0)} + \|\nabla \tilde{u}\|^2_{L^2(\Omega^0)}\right) \\ &\lesssim \|\nabla u\|^2_{L^2(\Omega^0)} + \|u|_{\Gamma^0}\|^2_{H^{\frac{1}{2}}(\Gamma^0)} \\ &\lesssim \|\nabla u\|^2_{L^2(\Omega^0)} + \|u|_{\Gamma^0}\|^2_{L^2(\Gamma^0)}.\end{aligned}$$

We obtain (15) by using again (16) and (17).

■

Remark 2 *Results similar to Corollary 1 can be proved, for instance: there exists a positive constant C such that for all $u \in H^1(\Omega^0)$,*

$$\|u\|^2_{L^2(\Omega^0)} \le C\left(\|\nabla u\|^2_{L^2(\Omega^0)} + \langle u\rangle^2_{\Gamma^0}\right). \tag{18}$$

By a simple scaling argument, we obtain from (15) the

Corollary 2 *There exists a positive constant C such that for all integer $n \ge 0$, and for all $i \in \{0, \dots, 2^n\}$, for all $u \in H^1(\Omega^n_i)$,*

$$\|u\|^2_{L^2(\Omega^n_i)} \le C\left(4^{-n}\|\nabla u\|^2_{L^2(\Omega^n_i)} + 2^{-n}\|u|_{\Gamma^n_i}\|^2_{L^2(\Gamma^n_i)}\right), \tag{19}$$

and for all $u \in H^1(\Omega^n)$

$$\|u\|^2_{L^2(\Omega^n)} \le C\left(4^{-n}\|\nabla u\|^2_{L^2(\Omega^n)} + 2^{-n}\|u|_{\Gamma^n}\|^2_{L^2(\Gamma^n)}\right). \tag{20}$$

Lemma 1 *There exists a positive constant C such that for all $u \in H^1(\Omega^0)$, for all $N \ge 0$,*

$$\|u\|^2_{L^2(\Omega^N)} \le C2^{-N}\left(\|\nabla u\|^2_{L^2(\Omega^0)} + \|u|_{\Gamma^0}\|^2_{L^2(\Gamma^0)}\right). \tag{21}$$

Proof. We use a trace inequality on Q^n_i: for a constant C independent on n, we have

$$2^{n+1}\left(\|u|_{\Gamma^{n+1}_{2i-1}}\|^2_{L^2(\Gamma^{n+1}_{2i-1})} + \|u|_{\Gamma^{n+1}_{2i}}\|^2_{L^2(\Gamma^{n+1}_{2i})}\right) \le C\|\nabla u\|^2_{L^2(Q^n_i)} + 2^{n+1}\|u|_{\Gamma^n_i}\|^2_{L^2(\Gamma^n_i)}. \tag{22}$$

Summing (22) over i, we obtain that

$$2^{n+1}\|u|_{\Gamma^{n+1}}\|^2_{L^2(\Gamma^{n+1})} \le C\|\nabla u\|^2_{L^2(Q^n)} + 2^{n+1}\|u|_{\Gamma^n}\|^2_{L^2(\Gamma^n)}. \tag{23}$$

Multiplying (23) by 2^{N-n-1} and summing up from $n = 0$ to $N-1$, we obtain that

$$2^N\|u|_{\Gamma^N}\|^2_{L^2(\Gamma^N)} \le C2^N\left(\|\nabla u\|^2_{L^2(\omega^{N-1})} + \|u|_{\Gamma^0}\|^2_{L^2(\Gamma^0)}\right)$$

Injecting this into (20), we obtain (21). ■

Theorem 2 (Compactness) *The imbedding from $H^1(\Omega^0)$ in $L^2(\Omega^0)$ is compact.*

Proof. From (21), we have

$$\|u - 1_{\omega^N}u\|_{L^2(\Omega^0)} \le C2^{-\frac{N}{2}}\|u\|_{H^1(\Omega^0)}.$$

On the other hand, the imbedding from $H^1(\omega^N)$ in $L^2(\omega^N)$ is compact. Combining the previous two remarks yields the desired result. ■

Remark 3 *In [1], we give several other theoretical results on the space $H^1(\Omega^0)$, among which extension theorems, density results, and trace theorems on the top boundary of Ω^0, namely $\Gamma^\infty = (-3, 3) \times \{6\}$.*

4 Diffusion problems

The aim of this section is to study some Poisson problems in Ω^0 with Neumann boundary conditions on Σ^0.

4.1 Harmonic lifting of functions defined on Γ^0

For a function $u \in H^{\frac{1}{2}}(\Gamma^0)$, we define the harmonic lifting $\mathcal{H}^0(u)$ of u by $\mathcal{H}^0(u) \in H^1(\Omega^0)$, the trace of $\mathcal{H}^0(u)$ on Γ^0 is u, and for all $v \in \mathcal{V}(\Omega^0)$,

$$\int_{\Omega^0} \nabla \mathcal{H}^0(u) \cdot \nabla v = 0. \tag{24}$$

This is the weak form of the following problem

$$\begin{array}{rcll} -\Delta \mathcal{H}^0(u) &=& 0, & \text{in } \Omega^0, \\ \mathcal{H}^0(u) &=& u, & \text{on } \Gamma^0, \\ \frac{\partial \mathcal{H}^0(u)}{\partial n} &=& 0, & \text{on } \Sigma^0. \end{array}$$

The existence and uniqueness of $\mathcal{H}^0(u)$, and the fact that that $\mathcal{H}^0$ is a bounded operator from $H^{\frac{1}{2}}(\Gamma^0)$ to $H^1(\Omega^0)$ are consequences of Theorem 1.

Remark 4 *All what follows holds when (24) is replaced by the more general problem*

$$\int_{\Omega^0} \chi \nabla \mathcal{H}^0(u) \cdot \nabla v = 0, \tag{25}$$

where χ is a symmetric and positive definite tensor.

Remark 5 *In [1], we study the boundary value problem with a nonzero Neumann data on Γ^∞.*

Similarly, for an integer $n > 0$, and for $\sigma \in \mathcal{A}_n$, one can define the lifting operator $\mathcal{H}^\sigma$ from $H^{\frac{1}{2}}(\Gamma^\sigma)$ to $H^1(\Omega^\sigma)$: for all $u \in H^{\frac{1}{2}}(\Gamma^\sigma)$, the trace of $\mathcal{H}^\sigma(u)$ on Γ^σ is u and for all $v \in \mathcal{V}(\Omega^\sigma)$, $\int_{\Omega^\sigma} \nabla \mathcal{H}^\sigma(u) \cdot \nabla v = 0$. It is easy to check that, for all $v \in H^{\frac{1}{2}}(\Gamma^\sigma)$,

$$\mathcal{H}^\sigma(v) \circ \mathcal{M}_\sigma(F_1, F_2) = \mathcal{H}^0(v \circ \mathcal{M}_\sigma(F_1, F_2)). \tag{26}$$

Lemma 2 *There exists a positive constant C such that, for all $u \in H^{\frac{1}{2}}(\Gamma^0)$,*

$$\|\nabla \mathcal{H}^0(u)\|_{L^2(\omega^0)} \geq C \|\nabla \mathcal{H}^0(u)\|_{L^2(\Omega^0)}. \tag{27}$$

Proof. It is enough to prove (27) for all $u \in H^{\frac{1}{2}}(\Gamma^0)$ such that $\int_{\Gamma^0} u = 0$, because $\mathcal{H}^0(1_{\Gamma^0}) = 1_{\Omega^0}$. From the analogue of (18) for functions of $H^1(\omega^0)$ with mean value 0 on Γ^0, we have that

$$\|u\|_{H^{\frac{1}{2}}(\Gamma^0)} \lesssim \|\nabla \mathcal{H}^0(u)\|_{L^2(\omega^0)}.$$

On the other hand, from the continuity of $\mathcal{H}^0$, we have that

$$\|\nabla \mathcal{H}^0(u)\|_{L^2(\Omega^0)} \lesssim \|u\|_{H^{\frac{1}{2}}(\Gamma^0)}.$$

The desired result follows from the previous two estimates. ∎

Lemma 3 *There exists a constant $\rho < 1$ such that for all $u \in H^{\frac{1}{2}}(\Gamma^0)$,*

$$\int_{\Omega^1} |\nabla \mathcal{H}^0(u)|^2 \leq \rho \int_{\Omega^0} |\nabla \mathcal{H}^0(u)|^2. \tag{28}$$

Proof. The result is a direct consequence of Lemma 2. ∎

Theorem 3 *For all $u \in H^{\frac{1}{2}}(\Gamma^0)$,*

$$\int_{\Omega^N} |\nabla \mathcal{H}^0(u)|^2 \leq \rho^N \int_{\Omega^0} |\nabla \mathcal{H}^0(u)|^2, \tag{29}$$

where the constant $\rho < 1$ has been introduced in Lemma 3.

Proof. The desired result will be proved ounce we have established that

$$\int_{\Omega^{n+1}} |\nabla \mathcal{H}^0(u)|^2 \leq \rho \int_{\Omega^n} |\nabla \mathcal{H}^0(u)|^2.$$

For that, we make use of (28); we consider the two bounded operators in $H^{\frac{1}{2}}(\Gamma^0)$, $\mathcal{L}_i$, $i = 1, 2$:

$$\mathcal{L}_i(v) = \left((\mathcal{H}^0 v)|_{\Gamma_i^1}\right) \circ F_i, \tag{30}$$

where F_i are defined in (2). Calling $\Omega_i^1 = F_i(\Omega^0)$, $i = 1, 2$, it is easy to check that for all $u \in H^{\frac{1}{2}}(\Gamma^0)$,

$$(\mathcal{H}^0 \circ \mathcal{L}_i)(u) = \left((\mathcal{H}^0(u))|_{\Omega_i^1}\right) \circ F_i, \tag{31}$$

and that

$$\int_{\Omega^0} |\nabla(\mathcal{H}^0 \circ \mathcal{L}_i)(u)|^2 = \int_{\Omega_i^1} |\nabla \mathcal{H}^0(u)|^2. \tag{32}$$

Therefore, from (28),

$$\sum_{i=1}^{2} \int_{\Omega^0} |\nabla(\mathcal{H}^0 \circ \mathcal{L}_i)(u)|^2 \leq \rho \int_{\Omega^0} |\nabla \mathcal{H}^0(u)|^2. \tag{33}$$

For $\sigma \in \mathcal{A}_n$, we use the notation $\mathcal{M}_\sigma(\mathcal{L}_1, \mathcal{L}_2) = \mathcal{L}_{\sigma(1)} \circ \mathcal{L}_{\sigma(2)} \circ \cdots \circ \mathcal{L}_{\sigma(2^n)}$. For $n > 1$, we have, for all $u \in H^{\frac{1}{2}}(\Gamma^0)$,

$$\begin{aligned} \int_{\Omega^{n+1}} |\nabla \mathcal{H}^0(u)|^2 &= \sum_{\sigma \in \mathcal{A}_n} \int_{\Omega^0} \left|\nabla \left(\mathcal{H}^0 \circ \mathcal{M}_\sigma(\mathcal{L}_1, \mathcal{L}_2)\right)(u)\right|^2 \\ &= \sum_{i=1}^{2} \sum_{\sigma \in \mathcal{A}_{n-1}} \int_{\Omega^0} \left|\nabla \left(\mathcal{H}^0 \circ \mathcal{L}_i \circ \mathcal{M}_\sigma(\mathcal{L}_1, \mathcal{L}_2)\right)(u)\right|^2, \end{aligned}$$

and from (33),

$$\begin{aligned} \int_{\Omega^{n+1}} |\nabla \mathcal{H}^0(u)|^2 &\leq \rho \sum_{\sigma \in \mathcal{A}_{n-1}} \int_{\Omega^0} \left|\nabla \left(\mathcal{H}^0 \circ \mathcal{M}_\sigma(\mathcal{L}_1, \mathcal{L}_2)\right)(u)\right|^2 \\ &= \rho \int_{\Omega^n} |\nabla \mathcal{H}^0(u)|^2, \end{aligned}$$

which yields (29). ■

From the general theory of boundary value problem, see [6] for example, we have the following regularity:

Lemma 4 (Local Regularity) *For all $u \in H^{\frac{1}{2}}(\Gamma^0)$, for all open bounded domain $\mathcal{O}$ strictly contained in $\mathbb{R} \times (0, 6)$, and for all ϵ, $0 < \epsilon < \frac{5}{3}$, the restriction of $\mathcal{H}^0(u)$ to $\Omega^0 \cap \mathcal{O}$ belongs to $H^{\frac{5}{3}-\epsilon}(\Omega^0 \cap \mathcal{O})$.*

Orientation We will try to solve (24) numerically. Of course, it is not possible to represent completely the domain Ω^0 in numerical simulations, because this would imply an infinite memory and computing time. Rather, for some $n \in \mathbb{N}$, we aim at computing as well as possible the restriction of $\mathcal{H}^0(u)$ to ω^n, $n \in \mathbb{N}$. This turns out to be possible, but for that we need to use nonlocal operators on Γ^σ, $\sigma \in \mathcal{A}_{n+1}$. We will see later that these operators can be called Dirichlet to Neumann operators. They will be computed by using the geometric self-similarity.

4.2 The Dirichlet-Neumann operator

Call $\left(H^{\frac{1}{2}}(\Gamma^0)\right)'$ the topological dual space of $H^{\frac{1}{2}}(\Gamma^0)$ and consider the Dirichlet-Neumann operator $T^0 : H^{\frac{1}{2}}(\Gamma^0) \mapsto \left(H^{\frac{1}{2}}(\Gamma^0)\right)'$

$$\langle T^0 u, v\rangle = \int_{\Omega^0} \nabla \mathcal{H}^0(u) \cdot \nabla \mathcal{H}^0(v). \tag{34}$$

We remark that

$$\langle T^0 u, v\rangle = \int_{\Omega^0} \nabla \mathcal{H}^0(u) \cdot \nabla \tilde{v}, \tag{35}$$

for any function $\tilde{v} \in H^1(\Omega^0)$ such that $\tilde{v}|_{\Gamma^0} = v$.
When $\mathcal{H}^0(u)$ is regular enough, $T^0 u$ is the normal derivative of $\mathcal{H}^0(u)$ on Γ^0. This is why T^0 is called a Dirichlet-Neumann operator.
The operator T^0 is bounded, self-adjoint and positive semi-definite. It is clear that $T^0 1 = 0$. Call V the closed subspace of $H^{\frac{1}{2}}(\Gamma^0)$:

$$V = \{v \in H^{\frac{1}{2}}(\Gamma^0),\ \langle v\rangle_{\Gamma^0} = 0\}. \tag{36}$$

From the definition of the norm in $H^{\frac{1}{2}}(\Gamma^0)$ and from (18), we see that T^0 is coercive on V, i.e. there exists a positive constant α such that

$$\forall v \in V, \quad \langle T^0 v, v\rangle \geq \alpha \|v\|^2_{H^{\frac{1}{2}}(\Gamma^0)}. \tag{37}$$

Similarly, for $\sigma \in \mathcal{A}_n$, one can define the operators T^σ, from $H^{\frac{1}{2}}(\Gamma^\sigma)$ (see (7) and (10)) to their respective duals by $\langle T^\sigma u, v\rangle = \int_{\Omega^\sigma} \nabla \mathcal{H}^\sigma(u) \cdot \nabla \mathcal{H}^\sigma(v) = \int_{\Omega^\sigma} \nabla \mathcal{H}^\sigma(u) \cdot \nabla \tilde{v}$, for any function $\tilde{v} \in H^1(\Omega^\sigma)$ such that $\tilde{v}|_{\Gamma^\sigma} = v$. From the self-similarity of Ω^0, we have that

$$\forall u, v \in H^{\frac{1}{2}}(\Gamma^\sigma), \quad \langle T^\sigma u, v\rangle = \langle T^0(u \circ \mathcal{M}_\sigma(F_1, F_2)), (v \circ \mathcal{M}_\sigma(F_1, F_2))\rangle, \tag{38}$$

where the duality pairing in left (resp. right) hand side of (38) is the duality $\left(H^{\frac{1}{2}}(\Gamma^\sigma)\right)'$- $H^{\frac{1}{2}}(\Gamma^\sigma)$ (resp. $\left(H^{\frac{1}{2}}(\Gamma^0)\right)'$- $H^{\frac{1}{2}}(\Gamma^0)$).

Lemma 5 *For all $u \in H^{\frac{1}{2}}(\Gamma^0)$, for $n \geq 1$, the restriction of $\mathcal{H}^0(u)$ to ω^{n-1} is the solution to the following boundary value problem: find $\hat{u} \in H^1(\omega^{n-1})$ such that $\hat{u}|_{\Gamma^0} = u$ and $\forall v \in \mathcal{V}(\omega^{n-1})$,*

$$\int_{\omega^{n-1}} \nabla \hat{u} \cdot \nabla v + \sum_{\sigma \in \mathcal{A}_n} \left\langle T^0 \left(\hat{u}|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\right), v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\right\rangle = 0. \tag{39}$$

Furthermore, $\forall v \in H^1(\omega^{n-1})$,

$$\begin{aligned} \langle T^0 u, v|_{\Gamma^0}\rangle &= \int_{\omega^{n-1}} \nabla \hat{u} \cdot \nabla v + \sum_{\sigma \in \mathcal{A}_n} \langle T^\sigma \hat{u}|_{\Gamma^\sigma}, v|_{\Gamma^\sigma}\rangle \\ &= \int_{\omega^{n-1}} \nabla \hat{u} \cdot \nabla v + \sum_{\sigma \in \mathcal{A}_n} \left\langle T^0(\hat{u}|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)), v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\right\rangle. \end{aligned} \tag{40}$$

Proof. Follows from (26) and (38). ■

Remark 6 *Note that the boundary value problem (39) is well posed because the bilinear form in the left hand side is continuous, symmetric and coercive on* $\mathcal{V}(\omega^{n-1})$.

Orientation We see from (39) that once the nonlocal operator T^0 is known, the restriction of $\mathcal{H}^0(u)$ to ω^{n-1} can be computed exactly by solving a boundary value problem in ω^{n-1} with a boundary condition involving T^0. Thus, if T^0 or a good approximation of T^0 is available, then the restriction of $\mathcal{H}^0(u)$ to ω^{n-1} can be approximated by a standard discrete method for (39). There remains to compute T^0: for that, we will make use of (40), in the case $n = 1$.

Lemma 5, in the case $n = 1$, leads us to introduce the cone $\mathbb{O}$ of self adjoint, positive semi-definite, bounded linear operators from $H^{\frac{1}{2}}(\Gamma^0)$ to its dual, vanishing on the constants, and the mapping $\mathbb{M} : \mathbb{O} \mapsto \mathbb{O}$ defined as follows: for $Z \in \mathbb{O}$, define $\mathbb{M}(Z)$ by

$$\forall u \in H^{\frac{1}{2}}(\Gamma^0),\ \forall v \in H^1(\omega^0), \qquad \langle \mathbb{M}(Z)u, v|_{\Gamma^0}\rangle = \int_{\omega^0} \nabla\hat{u}\cdot\nabla v + \sum_{i=1}^{2} \left\langle Z(\hat{u}|_{\Gamma_i^1} \circ F_i), v|_{\Gamma_i^1} \circ F_i \right\rangle, \tag{41}$$

where $\hat{u} \in H^1(\omega^0)$ is such that $\hat{u}|_{\Gamma^0} = u$ and

$$\forall v \in \mathcal{V}(\omega^0), \qquad \int_{\omega^0} \nabla\hat{u}\cdot\nabla v + \sum_{i=1}^{2} \left\langle Z(\hat{u}|_{\Gamma_i^1} \circ F_i), v|_{\Gamma_i^1} \circ F_i \right\rangle = 0. \tag{42}$$

Lemma 5 tells that T^0 is a fixed point of $\mathbb{M}$. In fact, we have the

Theorem 4 *The operator* T^0 *is the unique fixed point of* $\mathbb{M}$. *Moreover, for all* $Z \in \mathbb{O}$, *there exists a positive constant* C *independent of* n *such that, for all* $n \geq 0$,

$$\|\mathbb{M}^n(Z) - T^0\| \leq C\rho^{\frac{n}{4}}, \tag{43}$$

where ρ, $0 < \rho < 1$ *is the constant appearing in Theorem 3.*

Proof. It is easy to check by induction that

$$\begin{aligned} &\forall u \in H^{\frac{1}{2}}(\Gamma^0),\ \forall v \in H^1(\omega^{n-1}),\\ &\langle \mathbb{M}^n(Z)u, v|_{\Gamma^0}\rangle = \int_{\omega^{n-1}} \nabla\hat{u}\cdot\nabla v + \sum_{\sigma\in\mathcal{A}_n} \langle Z(\hat{u}|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)), (v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2))\rangle, \end{aligned} \tag{44}$$

where $\hat{u} \in H^1(\omega^{n-1})$ is such that $\hat{u}|_{\Gamma^0} = u$ and

$$\forall v \in \mathcal{V}(\omega^{n-1}), \qquad \int_{\omega^{n-1}} \nabla\hat{u}\cdot\nabla v + \sum_{\sigma\in\mathcal{A}_n} \langle Z(\hat{u}|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)), (v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2))\rangle = 0. \tag{45}$$

Let $\mathcal{H}^n(\hat{u}) \in H^1(\Omega^0)$ be defined by

$$\mathcal{H}^n(\hat{u}) = \begin{cases} \hat{u} & \text{in } \omega^{n-1},\\ \text{the harmonic lifting of } \hat{u}|_{\Gamma^n} & \text{in } \Omega^n. \end{cases}$$

It can be proved that for a constant C independent of n, $\|\mathcal{H}^n(\hat{u})\|_{H^1(\Omega^0)} \leq C\|\hat{u}\|_{H^1(\omega^{n-1})}$. Then

$$\begin{aligned}
&\int_{\omega^{n-1}} |\nabla(\hat{u}-\mathcal{H}^0(u))|^2 + \sum_{\sigma\in\mathcal{A}_n} \left\langle Z((\hat{u}-\mathcal{H}^0(u))|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)), (\hat{u}-\mathcal{H}^0(u))|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)\right\rangle \\
&= -\int_{\omega^{n-1}} \nabla(\mathcal{H}^n(\hat{u})-\mathcal{H}^0(u))\cdot\nabla\mathcal{H}^0(u) - \sum_{\sigma\in\mathcal{A}_n} \left\langle Z((\mathcal{H}^0(u))|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)), (\hat{u}-\mathcal{H}^0(u))|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)\right\rangle \\
&= \int_{\Omega^n} \nabla(\mathcal{H}^n(\hat{u})-\mathcal{H}^0(u))\cdot\nabla\mathcal{H}^0(u) - \sum_{\sigma\in\mathcal{A}_n} \left\langle Z((\mathcal{H}^0(u))|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)), (\hat{u}-\mathcal{H}^0(u))|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)\right\rangle \\
&\lesssim \left(\int_{\Omega^n} |\nabla\mathcal{H}^0(u)|^2\right)^{\frac{1}{2}} \left(\int_{\Omega^n} |\nabla\mathcal{H}^n(\hat{u})-\nabla\mathcal{H}^0(u)|^2\right)^{\frac{1}{2}} \\
&\lesssim \rho^{\frac{n}{2}}\|\nabla\mathcal{H}^0(u)\|_{L^2(\Omega^0)}\left(\|\nabla(\mathcal{H}^0(u))\|_{L^2(\Omega^n)} + \|\nabla(\mathcal{H}_n(\hat{u}))\|_{L^2(\Omega^n)}\right) \\
&\lesssim \rho^{\frac{n}{2}}\|\nabla\mathcal{H}^0(u)\|_{L^2(\Omega^0)}\left(\|\mathcal{H}^0(u)\|_{H^1(\Omega^0)} + \|\hat{u}\|_{H^1(\omega^{n-1})}\right) \\
&\lesssim \rho^{\frac{n}{2}}\|u\|^2_{H^{\frac{1}{2}}(\Gamma^0)}
\end{aligned}$$

But

$$\begin{aligned}
\langle(\mathbb{M}^n(Z)-T^0)u, v\rangle &= I + II + III, \\
I &= \int_{\omega^{n-1}} \nabla(\hat{u}-\mathcal{H}^0u)\cdot\nabla v \\
II &= \sum_{\sigma\in\mathcal{A}_n} \left\langle Z((\hat{u}-\mathcal{H}^0(u))|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)\right\rangle \\
III &= \sum_{\sigma\in\mathcal{A}_n} \left\langle (Z-T^0)(\mathcal{H}^0(u))|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)\right\rangle
\end{aligned}$$

Thanks to the estimate above, $|I+II| \lesssim \rho^{\frac{n}{4}}\|u\|_{H^{\frac{1}{2}}(\Gamma^0)}\|v\|_{H^{\frac{1}{2}}(\Gamma^0)}$. Estimate (29) implies that $|III| \lesssim \rho^{\frac{n}{2}}\|u\|_{H^{\frac{1}{2}}(\Gamma^0)}\|v\|_{H^{\frac{1}{2}}(\Gamma^0)}$. Estimate (43) is proved. As a consequence, T^0 is the unique fixed point of $\mathbb{M}$. ■

4.3 A self similar finite element method

4.3.1 Description of the method

We consider a regular family of triangulations $\mathcal{T}_h^0$ of ω^0, (see [3]) with the special property that for $i=1,2$, the set of the nodes of $\mathcal{T}_h^0$ lying on Γ_i^1 is the image by F_i of the set the nodes of $\mathcal{T}_h^0$ lying on Γ^0. Thanks to this property, the set of triangles $\mathcal{T}_h^1 = \mathcal{T}_h^0 \cup \bigcup_{i=1}^2 F_i(\mathcal{T}_h^0)$ is a triangulation of ω^1. Similarly, let us call $\mathcal{T}_h^N$ the triangulation of ω^N: $\mathcal{T}_h^N = \bigcup_{n=0}^N \cup_{\sigma\in\mathcal{A}_n}\mathcal{M}_\sigma(F_1,F_2)(\mathcal{T}_h^0)$. Finally, it is possible to construct a self-similar mesh $\mathcal{T}_h^\infty = \bigcup_{n=0}^\infty \cup_{\sigma\in\mathcal{A}_n}\mathcal{M}_\sigma(F_1,F_2)(\mathcal{T}_h^0)$. Let us call $V_h(\omega^n)$ the set of piecewise linear functions:

$$V_h(\omega^n) = \{v_h \in \mathcal{C}^0(\overline{\omega^n}),\ \forall\tau\in\mathcal{T}_h^n,\ v_h|_\tau \text{ is linear}\}. \tag{46}$$

Similarly,

$$V_h(\Omega^0) = \{v_h \in H^1(\Omega^0),\ \forall\tau\in\mathcal{T}_h^\infty,\ v_h|_\tau \text{ is linear}\}. \tag{47}$$

It is clear that for all $v_h \in V_h(\Omega^0)$, the restriction of v_h to ω^n belongs to $V_h(\omega^n)$. Consider $V_{h,0}(\Omega^0) = \{v_h \in V_h(\Omega^0);\ v_h|_{\Gamma^0} = 0\}$, and let $V_h(\Gamma^n)$ (resp. $V_h(\Gamma_i^n)$, $1\leq i\leq 2^N$) be the space

of the traces of the functions of $V_h(\Omega^0)$ on Γ^n (resp. Γ_i^n). It is clear that for all $v_h \in V_h(\Gamma_i^1)$, $v_h \circ F_i \in V_h(\Gamma^0)$.
We have the approximation result, whose proof is skipped for brevity:

Lemma 6 *For all $u \in H^1(\Omega^0)$,*

$$\lim_{h\to 0} \inf_{u_h \in V_h(\Omega^0)} \|u - u_h\|_{H^1(\Omega^0)} = 0.$$

We are now ready to define the discrete harmonic lifting operator $\mathcal{H}_h^0 : V_h(\Gamma^0) \mapsto V_h(\Omega^0)$, $\forall u_h \in V_h(\Gamma^0)$, $\mathcal{H}_h^0(u_h)|_{\Gamma^0} = u_h$ and

$$\forall v_h \in V_{h,0}(\Omega^0), \qquad \int_{\Omega^0} \nabla \mathcal{H}_h^0(u_h) \cdot \nabla v_h = 0. \tag{48}$$

As a consequence of Lemma 6 and of Céa's lemma (see [3]), we can state the following result, whose proof is omitted for brevity:

Proposition 1 *Call I_h any linear operator from $L^2(\Gamma^0)$ to $V_h(\Gamma^0)$, such that, for all $u \in H^{\frac{1}{2}}(\Gamma^0)$, $\lim_{h\to 0} \|u - I_h u\|_{H^{\frac{1}{2}}(\Gamma^0)} = 0$, (take for example the Clément's local regularization operator, see [3, 4]). For all $u \in H^{\frac{1}{2}}(\Gamma^0)$,*

$$\lim_{h\to 0} \|\mathcal{H}^0(u) - \mathcal{H}_h^0(I_h u)\|_{H^1(\Omega^0)} = 0,$$

Remark 7 *It should be possible to obtain sharper results when u is more regular.*

We have the analogue of Theorem 3:

Theorem 5 *There exists a constant $\rho < 1$, independent of h such that for all $u_h \in V_h(\Gamma^0)$,*

$$\int_{\Omega^N} |\nabla \mathcal{H}_h^0(u_h)|^2 \le \rho^N \int_{\Omega^0} |\nabla \mathcal{H}_h^0(u_h)|^2. \tag{49}$$

Proof. Exactly similar to that of Theorem 3. ∎
We can also define the discrete Dirichlet-Neumann operator $T_h^0 : V_h(\Gamma^0) \mapsto \left(V_h(\Gamma^0)\right)'$

$$\langle T_h^0 u_h, v_h\rangle = \int_{\Omega^0} \nabla \mathcal{H}_h^0(u_h) \cdot \nabla \mathcal{H}_h^0(v_h) = \int_{\Omega^0} \nabla \mathcal{H}^0(u_h) \cdot \nabla \tilde{v}_h, \tag{50}$$

for any function $\tilde{v}_h \in V_h(\Omega^0)$ such that $\tilde{v}_h|_{\Gamma^0} = v_h$. Exactly as for the continuous problem, we introduce the cone $\mathbb{O}_h$ of self adjoint, positive semi-definite, bounded linear operators from $V_h(\Gamma^0)$ to its dual, vanishing on the constants, and the mapping $\mathbb{M}_h : \mathbb{O}_h \mapsto \mathbb{O}_h$ defined as follows: for $Z_h \in \mathbb{O}_h$, define $\mathbb{M}_h(Z_h)$ by

$$\forall u_h \in V_h(\Gamma^0), \forall v_h \in V_h(\omega^0), \qquad \langle \mathbb{M}_h(Z_h) u_h, v_h|_{\Gamma^0}\rangle = \int_{\omega^0} \nabla \hat{u}_h \cdot \nabla v_h + \sum_{i=1}^{2} \left\langle Z_h(\hat{u}_h|_{\Gamma_i^1} \circ F_i), v_h|_{\Gamma_i^1} \circ F_i \right\rangle, \tag{51}$$

where $\hat{u}_h \in V_h(\omega^0)$ is such that $\hat{u}_h|_{\Gamma^0} = u_h$ and

$$\forall v_h \in V_h(\omega^0) \text{ with } v_h|_{\Gamma_0} = 0, \qquad \int_{\omega^0} \nabla \hat{u}_h \cdot \nabla v_h + \sum_{i=1}^{2} \left\langle Z_h(\hat{u}_h|_{\Gamma_i^1} \circ F_i), v_h|_{\Gamma_i^1} \circ F_i \right\rangle = 0. \tag{52}$$

We have the analogue of Theorem 4:

Theorem 6 *The operator T_h^0 is the unique fixed point of $\mathbb{M}_h$ and for all $Z_h \in \mathbb{O}_h$, there exists a positive constant C independent of n such that, for all $n \geq 0$,*

$$\|\mathbb{M}_h^n(Z_h) - T_h^0\| \leq C\rho^{\frac{n}{4}}, \tag{53}$$

where ρ, $0 < \rho < 1$ is the constant appearing in Theorem 5.

Proof. Exactly similar to that of Theorem 4. ■

4.3.2 The linear algebra viewpoint

Let us call $N_h(\omega^0)$ (resp. N) the dimension of $V_h(\omega^0)$, (resp $V_h(\Gamma^0)$). Call $(x_i)_{i=1,\dots,N}$ the abscissa of the mesh-nodes lying on Γ^0, ordered increasingly. Let us introduce the nodal basis $(\phi_i)_{i=1,\dots,N_h(\omega^0)}$ of $V_h(\omega^0)$ ordered as follows:

1. for $j = 1, \dots, N$, ϕ_j corresponds to the node $(x_j, 0) \in \Gamma^0$.
2. for $i = 1, 2$ and $j = 1, \dots, N$ ϕ_{iN+j} corresponds to the node $F_i(x_j, 0) \in \Gamma_i^1$.
3. for $3N < j \leq N_h(\omega^0)$, the node corresponding to ϕ_j belongs to $\overline{\omega_0} \backslash (\Gamma^0 \cup \Gamma^1)$.

Consider the bilinear for $a_h : V_h(\omega^0) \times V_h(\omega^0) \mapsto \mathbb{R}$: $a_h(u_h, v_h) = \int_{\omega^0} \nabla u_h \cdot \nabla v_h$, and let A be the matrix of a_h in the nodal basis described above. We have the block decomposition

$$A = \begin{pmatrix} A_{\Gamma^0,\Gamma^0} & 0 & A_{\Gamma^0,I} \\ 0 & A_{\Gamma^1,\Gamma^1} & A_{\Gamma^1,I} \\ A_{\Gamma^0,I}^T & A_{\Gamma^1,I}^T & A_{I,I} \end{pmatrix}, \qquad \begin{array}{l} A_{\Gamma^0,\Gamma^0} \in \mathbb{R}^{N\times N} \\ A_{\Gamma^1,\Gamma^1} \in \mathbb{R}^{2N\times 2N} \end{array}. \tag{54}$$

The block $A_{I,I}$ is positive definite; it is the matrix arising when dealing with a Poisson problem with Dirichlet conditions on $\Gamma^0 \cup \Gamma^1$ and Neumann conditions on Σ^0. The Schur complement of A obtained by eliminating the degrees of freedom corresponding to the mesh nodes in $\omega^0 \cup \Sigma^0$ is $S \in R^{3N\times 3N}$:

$$S = \begin{pmatrix} S_{\Gamma^0,\Gamma^0} & S_{\Gamma^0,\Gamma^1} \\ S_{\Gamma^0,\Gamma^1}^T & S_{\Gamma^1,\Gamma^1} \end{pmatrix}, \qquad \begin{array}{l} S_{\Gamma^0,\Gamma^0} = A_{\Gamma^0,\Gamma^0} - A_{\Gamma^0,I}A_{I,I}^{-1}A_{\Gamma^0,I}^T \in \mathbb{R}^{N\times N} \\ S_{\Gamma^1,\Gamma^1} = A_{\Gamma^1,\Gamma^1} - A_{\Gamma^1,I}A_{I,I}^{-1}A_{\Gamma^1,I}^T \in \mathbb{R}^{2N\times 2N} \\ S_{\Gamma^0,\Gamma^1} = -A_{\Gamma^0,I}A_{I,I}^{-1}A_{\Gamma^1,I}^T \in \mathbb{R}^{N\times 2N} \end{array}. \tag{55}$$

The block S_{Γ^0,Γ^0} is the matrix in the nodal basis of $V_h(\Gamma^0)$ of the bilinear form mapping $(u_h, v_h) \in V_h(\Gamma^0) \times V_h(\Gamma^0)$ to $\int_{\omega^0} \nabla \hat{u}_h \nabla \hat{v}_h$, where $\hat{u}_h$ and $\hat{v}_h$ satisfy

$$\begin{array}{l} \hat{u}_h \in V_h(\omega^0), \quad \hat{u}_h|_{\Gamma_0} = u_h, \quad \hat{u}_h|_{\Gamma_1} = 0, \\ \hat{v}_h \in V_h(\omega^0), \quad \hat{v}_h|_{\Gamma_0} = v_h, \quad \hat{v}_h|_{\Gamma_1} = 0, \\ \forall w_h \in V_h(\omega^0) \text{ such that } w_h|_{\Gamma_0} = 0, \quad w_h|_{\Gamma_1} = 0, \quad \int_{\omega^0} \nabla \hat{u}_h \nabla w_h = \int_{\omega^0} \nabla \hat{v}_h \nabla w_h = 0. \end{array}$$

The block S_{Γ^1,Γ^1} is the matrix in the nodal basis of $V_h(\Gamma^1)$ of the bilinear form mapping $(u_h, v_h) \in V_h(\Gamma^1) \times V_h(\Gamma^1)$ to $\int_{\omega^0} \nabla \hat{u}_h \nabla \hat{v}_h$, where where $\hat{u}_h$ and $\hat{v}_h$ satisfy

$$\begin{array}{l} \hat{u}_h \in V_h(\omega^0), \quad \hat{u}_h|_{\Gamma_1} = u_h, \quad \hat{u}_h|_{\Gamma_0} = 0, \\ \hat{v}_h \in V_h(\omega^0), \quad \hat{v}_h|_{\Gamma_1} = v_h, \quad \hat{v}_h|_{\Gamma_0} = 0, \\ \forall w_h \in V_h(\omega^0) \text{ such that } w_h|_{\Gamma_1} = 0, \quad w_h|_{\Gamma_0} = 0, \quad \int_{\omega^0} \nabla \hat{u}_h \nabla w_h = \int_{\omega^0} \nabla \hat{v}_h \nabla w_h = 0. \end{array}$$

The block S_{Γ^0,Γ^1} is the matrix of the bilinear form mapping $(u_h, v_h) \in V_h(\Gamma^0) \times V_h(\Gamma^1)$ to $\int_{\omega^0} \nabla \hat{u}_h \nabla \hat{v}_h$, where where $\hat{u}_h$ and $\hat{v}_h$ satisfy

$$\begin{array}{l} \hat{u}_h \in V_h(\omega^0), \quad \hat{u}_h|_{\Gamma_0} = u_h, \quad \hat{u}_h|_{\Gamma_1} = 0, \\ \hat{v}_h \in V_h(\omega^0), \quad \hat{v}_h|_{\Gamma_1} = v_h, \quad \hat{v}_h|_{\Gamma_0} = 0, \\ \forall w_h \in V_h(\omega^0) \text{ such that } w_h|_{\Gamma_1} = 0, \quad w_h|_{\Gamma_0} = 0, \quad \int_{\omega^0} \nabla \hat{u}_h \nabla w_h = \int_{\omega^0} \nabla \hat{v}_h \nabla w_h = 0. \end{array}$$

Denoting O the cone of the positive semi-definite matrices $Z \in \mathbb{R}^{N\times N}$ such that for $i = 1, \dots, N$, $\sum_{j=1}^N Z_{ij} = 0$, it is clear from the interpretations of S_{Γ^0,Γ^0}, S_{Γ^1,Γ^1} and S_{Γ^0,Γ^1} given above that the matrix counterpart of the operator $\mathbb{M}_h$ defined in (51) (52) is the operator $M : O \mapsto O$:

$$M(Z) = S_{\Gamma^0,\Gamma^0} - S_{\Gamma^0,\Gamma^1} \left(S_{\Gamma^1,\Gamma^1} + \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix} \right)^{-1} S^T_{\Gamma^0,\Gamma^1}. \tag{56}$$

As a corollary to Theorem 6, we have the

Proposition 2 *For any $Z \in O$, the sequence $M^n(Z)$ converges geometrically to the unique fixed point T of M, and T is the matrix of the discrete Dirichlet-Neumann operator T_h^0 defined in (50) in the nodal basis of $V_h(\Gamma^0)$.*

4.3.3 Algorithmic issues

Proposition 2 tells us that, for obtaining an approximation of the matrix T with an accuracy ϵ (in a fixed matrix norm), one can depart from any operator $Z \in O$, ($Z = 0$ is possible) and repeat $M(Z) \leftarrow Z$, $O(|\log \epsilon|)$ times. Assuming that S_{Γ^0,Γ^0}, S_{Γ^1,Γ^1} and S_{Γ^0,Γ^1} are known, and performing a Cholesky factorization of

$$S_{\Gamma^1,\Gamma^1} + \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix},$$

the mapping $M(Z) \leftarrow Z$ requires $O(N^3)$ operations.
Thanks to the sparsity of the matrix A, and using a multifrontal algorithm for factorizing A_{II} (for example SuperLU), one may compute the matrices S_{Γ^0,Γ^0}, S_{Γ^1,Γ^1} and S_{Γ^0,Γ^1} in $O(N^3)$ operations. This has to be done once and for all. Finally, one may approach T with an accuracy ϵ with a work of $O(|\log \epsilon|)N^3$ operations.
Once T is computed, for any $u_h \in V_h(\Gamma^0)$, computing the restriction of $\mathcal{H}_h^0(u_h)$ to ω^0 by solving the discrete counterpart of (39) can be done by

1. finding the trace of $\mathcal{H}_h^0(u_h)$ on Γ^1 by solving a system of the kind

$$\left(S_{\Gamma^1,\Gamma^1} + \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \right) U^1 = S_{\Gamma^1,\Gamma^0} U^0,$$

2. solving a linear system with the matrix A_{II}.

4.4 Numerical Results

In the numerical tests, we have taken for Ω^0 a dilation by a factor π of the domain described in § 2. We are interested in computing the Dirichlet-Neumann operator corresponding to problem (25) for the Laplace operator ($\chi = Id$) and for an anisotropic diffusion operator

$$\chi = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}.$$

The mesh used for ω^0 is plotted on Figure 3. It has the property mentioned in § 4.3, which permits the construction of a self-similar mesh of Ω^0. We apply the fixed point method introduced above. On Figure 4, we plot the Frobenius norm of $M^{n+1}(0) - M^n(0)$ for the two cases above. We see that the norm of the increment $M^{n+1}(0) - M^n(0)$ decays exponentially as n grows and that the decay factor is quite small (of the order 10^{-6} in the case of the Laplace operator and 10^{-4} in the anisotropic case). We see that a few iterations are enough to have a very accurate

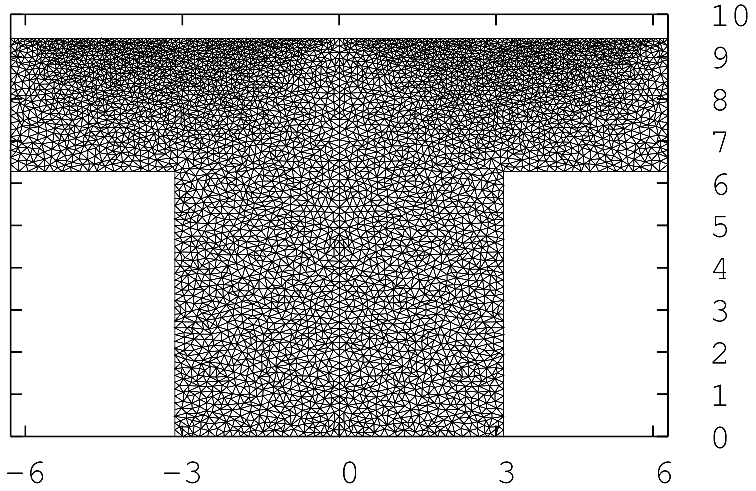

Figure 3: The mesh used for ω^0

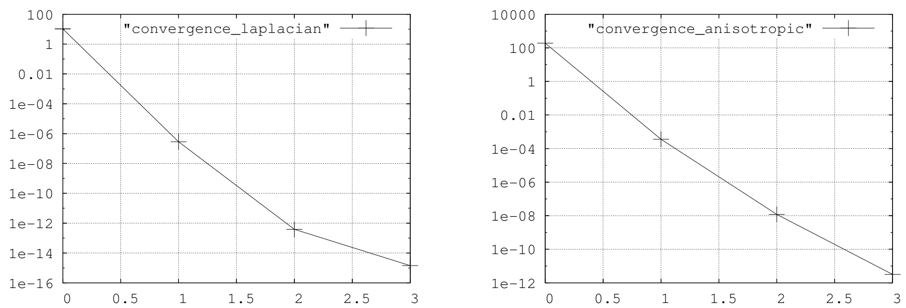

Figure 4: The Frobenius norms of the increments $M^{n+1}(0) - M^n(0)$ (in log-scale) vs. n

approximation of T^0. This says that (25) is well approximated by solving the same problem in Ω^n (with a Neumann condition on Γ^{n+1}) with n reasonably small, ($n = 4$ in most problems). As we shall see in the following, this will not be the case for Helmholtz equation.

Remark 8 *Let n be a positive integer number. Once T^0 is known, it is easy to compute the restriction to ω^n, by solving successively problems in ω^0, in $F_i(\omega^0)$, $i = 1, 2$, in $\mathcal{M}_\sigma(F_1, F_2)(\omega^0)$, $\sigma \in \mathcal{A}_2$, and so on... Note that all the linear systems to be solved involve the same matrix, which can be factorized and stored once and for all.*

5 Propagation problems

In this part, we wish to study time-harmonic waves in Ω^0.

5.1 The continuous problem

The aim of this section is to study weak solutions of the Helmholtz equation in the domain Ω^0. Our first tool is the compactness of the imbedding of $H^1(\Omega^0)$ in $L^2(\Omega^0)$, see Theorem 2, and as in the preceding part, we are going to make extensive use of the self-similarity in order to design an accurate finite element methods.
More precisely, let us consider the solutions to the variational problem:
for an integer $n \geq 0$, and for $\sigma \in \mathcal{A}_n$, given k a real number and $u \in H^{\frac{1}{2}}(\Gamma^\sigma)$, (with the notation $\Omega^\sigma = \mathcal{M}_\sigma(F_1, F_2)(\Omega^0)$, $\Gamma^\sigma = \mathcal{M}_\sigma(F_1, F_2)(\Gamma^0)$),

$$\text{find } \hat{u} \in H^1(\Omega^\sigma) \text{ such that } \hat{u}|_{\Gamma^\sigma} = u \text{ and for all } v \in \mathcal{V}(\Omega^\sigma),$$
$$\int_{\Omega^\sigma} \nabla \hat{u} \cdot \nabla v - k \int_{\Omega^\sigma} \hat{u} v = 0, \tag{57}$$

whose solution, if it exists, is a weak solution to the Helmholtz equation $\Delta \hat{u} + k\hat{u} = 0$ in Ω^σ.
Let us define the operator L_k^σ:

$$L_k^\sigma : \mathcal{V}(\Omega^\sigma) \mapsto (\mathcal{V}(\Omega^\sigma))', \quad \langle L_k^\sigma(w), v \rangle = \int_{\Omega^\sigma} \nabla w \cdot \nabla v - k \int_{\Omega^\sigma} wv. \tag{58}$$

Let us call $(\ker(L_k^\sigma))^\circ$ the closed space of the functions $u \in H^{\frac{1}{2}}(\Gamma^\sigma)$ such that, for all lifting $\tilde{u} \in H^1(\Omega^\sigma)$ of u,

$$\int_{\Omega^\sigma} \nabla \tilde{u} \cdot \nabla v - k\tilde{u}v = 0, \quad \forall v \in \ker(L_k^\sigma).$$

As a corollary to Theorem 2, we have the

Proposition 3 *For all $n \in \mathbb{N}$, there exists a countable set $Sp^{D,n} = \{\lambda_p, p \in \mathbb{N}\}$ of positive numbers, with $\lambda_p \leq \lambda_{p+1}$ and $\lim_{p\to\infty} \lambda_p = +\infty$ such that for $\sigma \in \mathcal{A}_n$,*

- *for $k \in \mathbb{R}\backslash Sp^{D,n}$, the operator L_k^σ is one to one, with a bounded inverse.*
- *for all $k \in Sp^{D,n}$, $\ker(L_k^\sigma)$ has a positive and finite dimension.*

One can obtain an Hilbertian basis of $\mathcal{V}(\Omega^\sigma)$ by assembling bases of $\ker(L_k^\sigma)$, $k \in Sp^{D,n}$.
We have

$$Sp^{D,n} = 4^n Sp^{D,0}. \tag{59}$$

For $u \in (\ker(L_k^\sigma))^\circ$, there exists $\hat{u} \in H^1(\Omega^\sigma)$ satisfying (57), and $\hat{u}$ is unique up to the addition of functions in $\ker(L_k^\sigma)$. Problem (57) defines an injective bounded operator $\mathcal{H}_k^\sigma$ from $(\ker(L_k^\sigma))^\circ$ to $H^1(\Omega^\sigma)/\ker(L_k^\sigma)$ by $\mathcal{H}_k^\sigma(u) = \hat{u}$.

Proof. The existence of a nondecreasing sequence of nonnegative eigenvalues, converging to $+\infty$, is a consequence of Theorem 2. The self-similarity of Ω^0 implies (59). We have seen in § 4 that $0 \notin \mathrm{Sp}^{D,n}$. The last statement of Proposition 3 is a consequence of the Fredholm alternative. ■

Remark 9 *In relation with Proposition 3, we know from (59) that for any $k \in \mathbb{R}$, there exists a nonnegative integer $N(k)$ such that for all $n \geq N(k)$, for all $\sigma \in \mathcal{A}_n$, the operator L_k^σ is coercive on $\mathcal{V}(\Omega^\sigma)$ which implies that $k \notin Sp^{D,n}$. We have $N(k) = 0$ if $k \leq 0$ and $N(k) \sim \log(k)$ as $k \to +\infty$.*

Remark 10 *Thanks to Remark 9, we have the analogue to Theorem 3: for all $u \in (\ker(L_k^\sigma))^\circ$, $\|\nabla \mathcal{H}_k^\sigma(u)\|_{L^2(\Omega^p)}$ decays exponentially with p as $p \to \infty$.*

Remark 11 *Similarly, for all $n \in \mathbb{N}$ and $\sigma \in \mathcal{A}_n$, the eigenvalues of the operator $\tilde{L}^\sigma$*

$$\tilde{L}^\sigma : \; H^1(\Omega^\sigma) \mapsto \left(H^1(\Omega^\sigma)\right)', \quad \left\langle \tilde{L}^\sigma(u), v \right\rangle = \int_{\Omega^\sigma} \nabla u \cdot \nabla v \tag{60}$$

form a nondecreasing sequence of nonnegative numbers $(\mu_p)_{p\in\mathbb{N}}$ with $\mu_0 = 0$, $\mu_1 > 0$, and $\lim_{p\to\infty} \mu_p = +\infty$. These numbers do not depend on σ. Calling $Sp^{N,n} = \{\mu_p, p \in \mathbb{N}\}$, we have $Sp^{N,n} = 4^n Sp^{N,0}$.

5.2 The Dirichlet-Neumann operator

The Dirichlet-Neumann operator $T_k^\sigma : (\ker(L_k^\sigma))^\circ \mapsto \left(H^{\frac{1}{2}}(\Gamma^\sigma)\right)'$ is defined by: $\forall u \in (\ker(L_k^\sigma))^\circ$, $\forall v \in H^{\frac{1}{2}}(\Gamma^\sigma)$,

$$\langle T_k^\sigma u, v\rangle = \int_{\Omega^\sigma} \nabla \mathcal{H}_k^\sigma(u) \cdot \nabla \tilde{v} - k \int_{\Omega^\sigma} \mathcal{H}_k^\sigma(u)\tilde{v}, \tag{61}$$

for any function $\tilde{v} \in H^1(\Omega^\sigma)$ such that $\tilde{v}|_{\Gamma^\sigma} = v$. For $k \notin \mathrm{Sp}^{D,n}$, T_k^σ is a bounded self-adjoint operator from $H^{\frac{1}{2}}(\Gamma^\sigma)$ to its dual.
For simplicity, since $\mathcal{A}_0$ has only one element, we use the notation $\mathcal{H}_k^0$ and T_k^0 if $n = 0$.

Lemma 7 *The operator T_k^σ is a perturbation of a bounded self-adjoint coercive operator from $(\ker(L_k^\sigma))^\circ$ to $\left(H^{\frac{1}{2}}(\Gamma^\sigma)\right)'$ by a compact operator.*

Proof. For a large positive constant α, one may write

$$\langle T_k^\sigma u, v\rangle = \int_{\Omega^\sigma} \nabla \mathcal{H}_k^\sigma(u) \cdot \nabla \tilde{v} + \alpha \int_{\Gamma^\sigma} uv - k \int_{\Omega^\sigma} \mathcal{H}_k^\sigma(u)\tilde{v} - \alpha \int_{\Gamma^\sigma} uv, \tag{62}$$

for any function $\tilde{v} \in H^1(\Omega^\sigma)$ such that $\tilde{v}|_{\Gamma^\sigma} = v$. But from (15), we know that for α large enough, the operator $\widehat{T}_k^\sigma$:

$$\langle \widehat{T}_k^\sigma u, v\rangle = \int_{\Omega^\sigma} \nabla \mathcal{H}_k^\sigma(u) \cdot \nabla \tilde{v} + \alpha \int_{\Gamma^\sigma} uv - k \int_{\Omega^\sigma} \mathcal{H}_k^\sigma(u)\tilde{v}$$

is coercive, whereas $T_k^\sigma - \widehat{T}_k^\sigma$ is clearly compact. ■

Lemma 8 *For all $\sigma \in \mathcal{A}_n$, for all $u \in (\ker(L_k^\sigma))^\circ$, $u \circ \mathcal{M}_\sigma(F_1, F_2) \in (\ker(L^0_{\frac{k}{4^n}}))^\circ$ and for all $v \in H^{\frac{1}{2}}(\Gamma^\sigma)$,*

$$\langle T_k^\sigma u, v\rangle = \left\langle T^0_{\frac{k}{4^n}} \left(u \circ \mathcal{M}_\sigma(F_1, F_2)\right), v \circ \mathcal{M}_\sigma(F_1, F_2)\right\rangle. \tag{63}$$

For all $u \in (\ker(L_k^0))^\circ$, the restriction $\hat{u}$ to ω^{n-1} of any function in the class $\mathcal{H}_k^0(u)$ satisfies, for all $\sigma \in \mathcal{A}_n$,

$$\hat{u}|_{\Gamma^\sigma} \in (\ker(L_k^\sigma))^\circ \tag{64}$$

and is a solution to the following boundary value problem: $\hat{u}|_{\Gamma^0} = u$, *and* $\forall v \in \mathcal{V}(\omega^{n-1})$,

$$\int_{\omega^{n-1}} \nabla \hat{u} \cdot \nabla v - k \int_{\omega^{n-1}} \hat{u} v + \sum_{\sigma \in \mathcal{A}_n} \langle T_k^\sigma \hat{u}|_{\Gamma^\sigma}, v|_{\Gamma^\sigma} \rangle = 0. \tag{65}$$

The last problem has a unique solution up to restrictions of functions of $\ker(L_k^0)$ *to* ω^{n-1}. *Furthermore,* $\forall v \in H^1(\omega^{n-1})$,

$$\begin{aligned} \langle T_k^0 u, v|_{\Gamma^0} \rangle &= \int_{\omega^{n-1}} \nabla \hat{u} \cdot \nabla v - k \int_{\omega^{n-1}} \hat{u} v + \sum_{\sigma \in \mathcal{A}_n} \langle T_k^\sigma \hat{u}|_{\Gamma^\sigma}, v|_{\Gamma^\sigma} \rangle \\ &= \int_{\omega^{n-1}} \nabla \hat{u} \cdot \nabla v - k \int_{\omega^{n-1}} \hat{u} v + \sum_{\sigma \in \mathcal{A}_n} \left\langle T^0_{\frac{k}{4^n}} (\hat{u}|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)), v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2) \right\rangle . \end{aligned} \tag{66}$$

Proof. Equation (63) is a consequence of the tree self-similarity. Then (65) (66) are proved by induction. ∎

Orientation We see from (63) (65) that, once the nonlocal operator $T^0_{\frac{k}{4^n}}$ is known, the restriction of $\mathcal{H}_k^0(u)$ to ω^{n-1} can be computed exactly by solving a boundary value problem in ω^{n-1}, with a boundary condition involving $T^0_{\frac{k}{4^n}}$. The equations (63) (65) (66) can be seen as a backward induction formula with respect to n, in order to compute T_k^0. Observing that $\lim_{n\to\infty} T^0_{\frac{k}{4^n}} = T^0$ enables to initialize the induction. The aim of what follows is to carry out this program in details.

5.3 Approximations of the Dirichlet-Neumann operator

For $\sigma \in \mathcal{A}_n$ and $p \geq n$, let us introduce $L_k^{\sigma,p}$ the operators

$$L_k^{\sigma,p} : \mathcal{V}(\Omega^\sigma) \mapsto (\mathcal{V}(\Omega^\sigma))', \quad \langle L_k^{\sigma,p}(u), v \rangle = \int_{\Omega^\sigma} \nabla u \cdot \nabla v - k \int_{\omega^{p-1} \cap \Omega^\sigma} uv.$$

Note that for $u \in H^{\frac{1}{2}}(\Gamma^\sigma)$, a solution to the problem

$$\begin{aligned} &\text{find } \hat{u} \in H^1(\Omega^\sigma) \text{ such that } \hat{u}|_{\Gamma^\sigma} = u \text{ and for all } v \in \mathcal{V}(\Omega^\sigma), \\ &\qquad\qquad \langle L_k^{\sigma,p} \hat{u}, v \rangle = 0, \end{aligned} \tag{67}$$

is a weak solution to the Helmholtz equation $\Delta \hat{u} + k 1_{\omega^{p-1} \cap \Omega^\sigma} \hat{u} = 0$ in Ω^σ.
Call $(\ker(L_k^{\sigma,p}))^\circ$ the closed space of the functions $u \in H^{\frac{1}{2}}(\Gamma^\sigma)$ satisfying, for all lifting $\tilde{u} \in H^1(\Omega^\sigma)$ of u,

$$\int_{\Omega^\sigma} \nabla \tilde{u} \cdot \nabla v - k \int_{\omega^{p-1} \cap \Omega^\sigma} \tilde{u} v = 0, \quad \forall v \in \ker(L_k^{\sigma,p}).$$

We have the analogue of Proposition 3:

Proposition 4 *For all* $n \in \mathbb{N}$ *and* $p \geq n$, *there exists a countable set* $Sp^{D,n,p} = \{\lambda_q, q \in \mathbb{N}\}$ *of positive numbers, with* $\lambda_q \leq \lambda_{q+1}$ *and* $\lim_{q\to\infty} \lambda_q = +\infty$ *such that, for all* $\sigma \in \mathcal{A}_n$,

- *for all* $k \in \mathbb{R} \backslash Sp^{D,n,p}$, *the operator* $L_k^{\sigma,p}$ *is one to one, with a bounded inverse.*
- *for all* $k \in Sp^{D,n,p}$, $\ker(L_k^{\sigma,p})$ *has a positive and finite dimension.*

We have

$$Sp^{D,n,p} = 4^n Sp^{D,0,p-n}. \tag{68}$$

If $u \in (\ker(L_k^{\sigma,p}))^\circ$*, then there exists* $\hat{u} \in H^1(\Omega^\sigma)$ *satisfying*

$$\begin{aligned} &\text{find } \hat{u} \in H^1(\Omega^\sigma) \text{ such that } \hat{u}|_{\Gamma^\sigma} = u \text{ and for all } v \in \mathcal{V}(\Omega^\sigma), \\ &\int_{\Omega^\sigma} \nabla \hat{u} \cdot \nabla v - k \int_{\omega^{p-1}\cap\Omega^\sigma} \hat{u} v = 0, \end{aligned} \tag{69}$$

and $\hat{u}$ *is unique up to functions in* $\ker(L_k^{\sigma,p})$*. Problem (69) defines an injective bounded operator* $\mathcal{H}_k^{\sigma,p}$ *from* $(\ker(L_k^{\sigma,p}))^\circ$ *to* $H^1(\Omega^\sigma)/\ker(L_k^{\sigma,p})$ *by* $\mathcal{H}_k^\sigma(u) = \hat{u}$*.*

The following result will be useful for approximating T_k^0:

Lemma 9 *For* $\sigma \in \mathcal{A}_n$ *and* $k \notin Sp^{D,n}$*, there exists a positive integer* $P(k,n) \geq n$ *such that for all* $p \geq P(k,n)$*, the operator* $L_k^{\sigma,p}$ *is one to one, and there exists a constant* $C > 0$*, (depending of* k *but not of* n *and* p*), such that, for* $p \geq P(k,n)$*,*

$$\|(L_k^{\sigma,p})^{-1} - (L_k^\sigma)^{-1}\| \leq C 2^{-n-p}. \tag{70}$$

Proof. Since $k \notin \mathrm{Sp}^{D,n}$, L_k^σ is one to one. From (21), we have that $\|L_k^{\sigma,p} - L_k^\sigma\| \lesssim 2^{-n-p}$ and therefore $\lim_{p\to\infty} \|L_k^{\sigma,p} - L_k^\sigma\| = 0$. It is a standard matter to deduce the desired results from the last two observations. ■
The Dirichlet-Neumann operator $T_k^{\sigma,p} : (\ker(L_k^{\sigma,p}))^\circ \mapsto \left(H^{\frac{1}{2}}(\Gamma^\sigma)\right)'$ is defined by: $\forall u \in (\ker(L_k^{\sigma,p}))^\circ$, $\forall v \in H^{\frac{1}{2}}(\Gamma^\sigma)$,

$$\langle T_k^{\sigma,p} u, v \rangle = \int_{\Omega^\sigma} \nabla \mathcal{H}_k^{\sigma,p}(u) \cdot \nabla \tilde{v} - k \int_{\omega^{p-1}\cap\Omega^\sigma} \mathcal{H}_k^{\sigma,p}(u) \tilde{v}, \tag{71}$$

for any function $\tilde{v} \in H^1(\Omega^\sigma)$ such that $\tilde{v}|_{\Gamma^\sigma} = v$.
We have the analogues of Lemmas 7 and 8:

Lemma 10 *The operator* $T_k^{\sigma,p}$ *is the perturbation of a bounded and coercive self-adjoint operator from* $(\ker(L_k^{\sigma,p}))^\circ$ *to* $\left(H^{\frac{1}{2}}(\Gamma^\sigma)\right)'$ *by a compact operator.*

Lemma 11 *For all* $p > 0$*,* $0 \leq n \leq p$*,* $\sigma \in \mathcal{A}_n$*,* $u \in (\ker(L_k^{\sigma,p}))^\circ$*, we have* $u \circ \mathcal{M}_\sigma(F_1, F_2) \in (\ker(L_{\frac{k}{4^n}}^{0,p-n}))^\circ$ *and for all* $v \in H^{\frac{1}{2}}(\Gamma^\sigma)$*,*

$$\langle T_k^{\sigma,p} u, v \rangle = \left\langle T_{\frac{k}{4^n}}^{0,p-n} \left(u \circ \mathcal{M}_\sigma(F_1, F_2)\right), v \circ \mathcal{M}_\sigma(F_1, F_2) \right\rangle. \tag{72}$$

For all $u \in (\ker(L_k^{0,p}))^\circ$*, the restriction* $\hat{u}$ *to* ω^{n-1} *of any function in the class* $\mathcal{H}_k^{0,p}(u)$ *satisfies, for all* $\sigma \in \mathcal{A}_n$*,*

$$\hat{u}|_{\Gamma^\sigma} \in (\ker(L_k^{\sigma,p}))^\circ \tag{73}$$

and is a solution to the following boundary value problem: $\hat{u}|_{\Gamma^0} = u$*, and* $\forall v \in \mathcal{V}(\omega^{n-1})$*,*

$$\int_{\omega^{n-1}} \nabla \hat{u} \cdot \nabla v - k \int_{\omega^{n-1}} \hat{u} v + \sum_{\sigma \in \mathcal{A}_n} \left\langle T_k^{\sigma,p}(\hat{u}|_{\Gamma^\sigma}), v|_{\Gamma^\sigma} \right\rangle = 0. \tag{74}$$

((74) can be written in terms of $T^{0,p-n}_{\frac{k}{4^n}}$ *thanks to (72)) Problem (74) has a unique solution up to restrictions of functions of* $\ker(L^{0,p}_k)$ *to* ω^{n-1}*. Furthermore,* $\forall v \in H^1(\omega^{n-1})$*,*

$$\begin{aligned}\langle T^{0,p}_k u, v|_{\Gamma^0}\rangle &= \int_{\omega^{n-1}} \nabla\hat{u}\cdot\nabla v - k\int_{\omega^{n-1}} \hat{u}v + \sum_{\sigma\in\mathcal{A}_n} \langle T^{\sigma,p}_k \hat{u}|_{\Gamma^\sigma}, v|_{\Gamma^\sigma}\rangle \\ &= \int_{\omega^{n-1}} \nabla\hat{u}\cdot\nabla v - k\int_{\omega^{n-1}} \hat{u}v + \sum_{\sigma\in\mathcal{A}_n} \left\langle T^{0,p-n}_{\frac{k}{4^n}}(\hat{u}|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)\right\rangle\end{aligned} \tag{75}$$

As a corollary to Lemma 9, we have the

Lemma 12 *For* $\sigma \in \mathcal{A}_n$ *and* $k \notin Sp^{D,n}$*, there exists* $P(k,n) \geq n$ *and a constant* $C > 0$ *(independent of p) such that for all* $p \geq P(k,n)$*,* $k \notin Sp^{D,n,p}$ *and therefore* $T^{\sigma,p}_k$ *is bounded from* $H^{\frac{1}{2}}(\Gamma^\sigma)$ *to its dual, and*

$$\|T^{\sigma,p}_k - T^\sigma_k\| \leq C2^{-n-p}. \tag{76}$$

One can then construct $T^{0,p}_k$ by the following induction:

Recursive construction of $T^{0,p}_k$ Let us construct the operators $(Z^j)_{0\leq j\leq p}$ by

- $Z^0 = T^0$.

• **Induction formula (I.F.)** Suppose that after j steps, $j < p$, we have constructed a possibly unbounded operator Z^j, from $H^{\frac{1}{2}}(\Gamma^0)$ to its dual, whose domain $\mathcal{D}^j$ is closed and has a finite codimension, and such that the restriction of Z^j to $\mathcal{D}^j$ is a perturbation of a coercive self-adjoint operator on $\mathcal{D}^j$ by a compact operator. Then, from the Fredholm alternative, we know that there exists a closed subspace $\mathcal{D}^{j+1}$ of $H^{\frac{1}{2}}(\Gamma^0)$ with a finite codimension, such that the problem

$$\begin{aligned}&\text{find } \hat{u} \in H^1(\omega^0), \text{ such that } \hat{u}|_{\Gamma^0} = u \text{ and } \forall\sigma\in\mathcal{A}_1,\ \hat{u}|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)\in\mathcal{D}^j,\\ &\forall v\in\mathcal{V}(\omega^0) \text{ such that } \forall\sigma\in\mathcal{A}_1,\ v|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)\in\mathcal{D}^j,\\ &\int_{\omega^0}\nabla\hat{u}\cdot\nabla v - \frac{k}{4^{p-j-1}}\int_{\omega^0}\hat{u}v + \sum_{\sigma\in\mathcal{A}_1}\langle Z^j(\hat{u}|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)\rangle = 0\end{aligned} \tag{77}$$

has a solution if $u \in \mathcal{D}^{j+1}$, which is unique up to functions belonging to a finite dimensional space. Then we can define the operator Z^{j+1} whose domain is $\mathcal{D}^{j+1}$, by: $\forall u \in \mathcal{D}^{j+1}$, $\forall v \in H^1(\omega^0)$,

$$\langle Z^{j+1}u, v|_{\Gamma^0}\rangle = \int_{\omega^0}\nabla\hat{u}\cdot\nabla v - \frac{k}{4^{p-j-1}}\int_{\omega^0}\hat{u}v + \sum_{\sigma\in\mathcal{A}_1}\langle Z^j(\hat{u}|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)\rangle, \tag{78}$$

where $\hat{u}$ is a solution to (77). It is easy to see that Z^{j+1} has the same properties as Z^j.

Proposition 5 *With the construction above, we have for* $j \leq p$*,*

$$Z^j = T^{0,j}_{\frac{k}{4^{p-j}}} \quad \text{and} \quad \mathcal{D}^j = \left(\ker(L^{0,j}_{\frac{k}{4^{p-j}}})\right)^\circ.$$

Proof. By induction. ■

Remark 12 *In fact, for k belonging to a dense subset in $\mathbb{R}$, the domains $\mathcal{D}^j$, $0 \leq j \leq p$ all coincide with $H^{\frac{1}{2}}(\Gamma^0)$.*

Proposition 5 says that $T_k^{0,p}$ can be constructed recursively, departing from T^0. In practise, T^0 is not available, and one has to use approximations of T^0 instead. Therefore,in order to approximate $T_k^{0,p}$, we need to study the stability of the recursive construction above with respect to Z^0:

Proposition 6 *Let X_q be a sequence of operators in $\mathbb{O}$ converging to T^0 as $q \to \infty$. For an integer p, and $0 \leq j \leq p$, let us call $Z^j_{q,p}$ the operators such that*

- *$Z^0_{q,p} = X_q$.*
- *for $0 \leq j < p$, $Z^{j+1}_{q,p}$ is obtained from $Z^j_{q,p}$ by the induction (I.F.) above.*

Assume that $k \notin Sp^{D,0,p}$. Then, there exists an integer Q such that for all $q > Q$, $Z^p_{q,p}$ is a bounded operator from $H^{\frac{1}{2}}(\Gamma^0)$ to its dual, and for $q > Q$, for a constant C independent on q,

$$\|Z^p_{q,p} - T_k^{0,p}\| \leq C\|X_q - T^0\|. \tag{79}$$

Proof. Since $k \notin \mathrm{Sp}^{D,0,p}$, the problem: find $u_g \in \mathcal{V}(\omega^{p-1})$, such that for all $v \in \mathcal{V}(\omega^{p-1})$,

$$\int_{\omega^{p-1}} \nabla u_g \cdot \nabla v - k \int_{\omega^{p-1}} u_g v + \sum_{\sigma \in \mathcal{A}_p} \left\langle T^0(u_g|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)\right\rangle = \langle g, v\rangle$$

defines an isomorphism $\Psi : g \mapsto u_g$ from $\mathcal{V}'(\omega^{p-1})$ onto $\mathcal{V}(\omega^{p-1})$.
For $u \in H^{\frac{1}{2}}(\Gamma^0)$, consider the problem: find $\tilde{u} \in H^1(\omega^{p-1})$ such that $\tilde{u}|_{\Gamma^0} = u$ and for all $v \in \mathcal{V}(\omega^{p-1})$,

$$\int_{\omega^{p-1}} \nabla \tilde{u} \cdot \nabla v - k \int_{\omega^{p-1}} \tilde{u} v + \sum_{\sigma \in \mathcal{A}_p} \langle X_q(\tilde{u}|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)\rangle = 0. \tag{80}$$

Let $\hat{u} = \mathcal{H}_k^{0,p} u$. Problem (80) is equivalent to finding $e = \tilde{u} - \hat{u} \in \mathcal{V}(\omega^{p-1})$ such that for all $v \in \mathcal{V}(\omega^{p-1})$,

$$\begin{aligned}&\int_{\omega^{p-1}} \nabla e \cdot \nabla v - k \int_{\omega^{p-1}} e v + \sum_{\sigma \in \mathcal{A}_p} \left\langle T^0(e|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)\right\rangle \\ &= \sum_{\sigma \in \mathcal{A}_p} \left\langle (T^0 - X_q)((e+\hat{u})|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)\right\rangle,\end{aligned}$$

so (80) can be reformulated as a fixed point problem with a linear operator involving Ψ. For q large enough, the operator in the fixed point is a contraction, so (80) has a unique solution and there exists $Q > 0$ and a constant $C > 0$ such that for all $q > Q$

$$\|\hat{u} - \tilde{u}\|_{H^1(\omega^{p-1})} \leq C\|T^0 - X_q\|\|u\|_{H^{\frac{1}{2}}(\Gamma^0)}. \tag{81}$$

Now, one can check that, for all $v \in H^1(\omega^{p-1})$,

$$\langle Z^p_{q,p} u, v|_{\Gamma^0}\rangle = \int_{\omega^{p-1}} \nabla \tilde{u} \cdot \nabla v - k \int_{\omega^{p-1}} \tilde{u} v + \sum_{\sigma \in \mathcal{A}_p} \langle X_q(\tilde{u}|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)), v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)\rangle, \tag{82}$$

so from (75) and (74) in the case $n = p$, (82) and (81), one can deduce (79). ■
As a consequence of Theorem 4, Lemma 12 and Proposition 6, we have the

Theorem 7 *For all $Y \in \mathbb{O}$, $p, q \in \mathbb{N}$, consider the sequence $Z^n_{q,p}$, $0 \le n \le p$:*

- $Z^0_{q,p} = \mathbb{M}^q(Y)$.
- *for* $0 \le n < p$, $Z^{n+1}_{q,p}$ *is obtained from* $Z^n_{q,p}$ *by the induction (I.F.) above,*

where $\mathbb{M}$ has been introduced in (41) (42). Assume that $k \notin Sp^{D,0}$. Then there exist two integers $P(k)$ and $Q(k)$ such that for all $p > P(k)$, for all $q > Q(k)$ $Z^p_{q,p}$ is a bounded operator from $H^{\frac{1}{2}}(\Gamma^0)$ to its dual, and there exists a constant C such that for all $p > P(k)$, $q > Q(k)$,

$$\|Z^p_{q,p} - T^0_k\| \le C(\rho^{\frac{q}{2}} + 2^{-p}), \tag{83}$$

where $0 < \rho < 1$ is the constant introduced in (28).

5.4 A self similar finite element method

We construct the discrete version of (57) by using the same self-similar finite element method described for the Poisson problem in § 4.3. All the results proved in §5.1 have their discrete counterparts.
For brevity, we do not discuss here the convergence of the discrete method when the step h tends to 0. This can be done by using the results contained in [3].
With the same notations as in § 4.3, let us take directly the linear algebra viewpoint. The matrix of the bilinear form $V_h(\omega^0) \times V_h(\omega^0) \mapsto \mathbb{R}$: $(u_h, v_h) \mapsto \int_{\omega^0} \nabla u_h \cdot \nabla v_h - k \int_{\omega^0} u_h v_h$ in the nodal basis is $A - kB$ where A is the stiffness matrix introduced in § 4.3.2 and where B is the mass matrix. Both A and B have the block decomposition described in (54). Let us give the matrix counterpart of the induction formula (I.F.) described above: The counterpart of problem (77) is: given $U \in \mathbb{R}^{N_h(\Gamma^0)}$, to find $\hat{U}_I$ and $\hat{U}_{\Gamma^1}$ such that

$$\begin{aligned}&\left(\begin{pmatrix} A_{\Gamma^1,\Gamma^1} & A_{\Gamma^1,I} \\ A^T_{\Gamma^1,I} & A_{I,I}\end{pmatrix} - k4^{n-p+1}\begin{pmatrix} B_{\Gamma^1,\Gamma^1} & B_{\Gamma^1,I} \\ B^T_{\Gamma^1,I} & B_{I,I}\end{pmatrix} + \begin{pmatrix} \tilde{Z}^n & 0 \\ 0 & 0\end{pmatrix}\right)\begin{pmatrix} \hat{U}_{\Gamma^1} \\ \hat{U}_I\end{pmatrix} \\ &= -\begin{pmatrix} 0 \\ (A^T_{\Gamma^0,I} - k4^{n-p+1}B^T_{\Gamma^0,I})U\end{pmatrix}\end{aligned} \tag{84}$$

where

$$\tilde{Z}^n = \begin{pmatrix} Z^n & 0 \\ 0 & Z^n\end{pmatrix}.$$

Let us assume that the real number k is such that, for all n, $0 \le n \le p$, the matrix

$$G^n_k = \begin{pmatrix} A_{\Gamma^1,\Gamma^1} - k4^{n-p+1}B_{\Gamma^1,\Gamma^1} + \tilde{Z}^n & A_{\Gamma^1,I} - k4^{n-p+1}B_{\Gamma^1,I} \\ A^T_{\Gamma^1,I} - k4^{n-p+1}B^T_{\Gamma^1,I} & A_{I,I} - k4^{n-p+1}B_{I,I}\end{pmatrix}$$

in the left hand side of (84) is invertible. This occurs for k in a dense subset of $\mathbb{R}$. Then the discrete counterpart of the induction formula (I.F.) is

$$\begin{aligned} Z^{N+1} =& A_{\Gamma^1,\Gamma^1} - k4^{n-p+1}B_{\Gamma^1,\Gamma^1} \\ &- \begin{pmatrix} 0 & , & (A_{\Gamma^0,I} - k4^{n-p+1}B_{\Gamma^0,I})\end{pmatrix}(G^n_k)^{-1}\begin{pmatrix} 0 \\ (A^T_{\Gamma^0,I} - k4^{n-p+1}B^T_{\Gamma^0,I})\end{pmatrix}.\end{aligned} \tag{85}$$

We use the following algorithm to approximate the discrete version of T^0_k: for all $Y \in \mathcal{O}$, $p, q \in \mathbb{N}$, we consider the sequence $Z^n_{q,p}$, $0 \le n \le p$:

- $Z^0_{q,p} = M^q(Y)$.
- for $0 \le n < p$, $Z^{n+1}_{q,p}$ is obtained from $Z^n_{q,p}$ by the induction (85),

where M has been introduced in (56).

5.5 Numerical Results

We use the domain and mesh displayed on Figure 3. We take $k = 1$, and we approximate the discrete version of T_k^0 by the construction described in § 5.4. We choose $q = 4$ because the numerical tests in § 4.4 show that four iterations of the fixed point algorithm described in § 4.3.1 are enough for computing T_h^0. Then we test the method for $p \leq 27$. There is no point in taking larger values of p when working in double precision, because 4^{-27} is of the order of the machine smallest double precision number. On Figure 5 we plot the Frobenius norm of the increments $Z_{4,p}^p - Z_{4,p-2}^{p-2}$ as a function of p. We see that these increments decay exponentially in n, and the decay exponent is very close to $\frac{1}{2}$ (in log-scale, the graph is very close to a straight line, with a slope close to $-\log(2)$). Figure 5 shows that for approximating T_k^0 with an error of order 10^{-6}, we need approximately 25 iterations of the construction above. We have used this

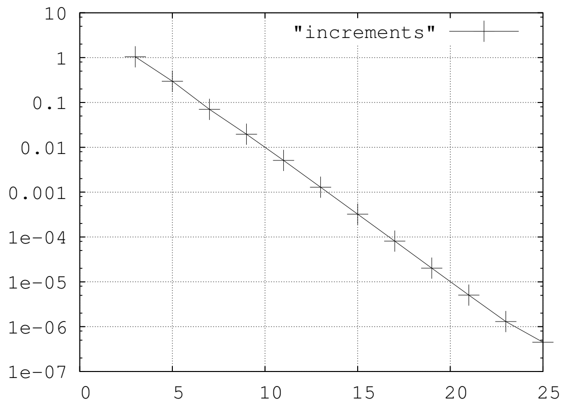

Figure 5: The Frobenius norms of the increments $Z_{4,p}^p - Z_{4,p-2}^{p-2}$ (in log-scale) as a function of p

approximation of T_k^0 in order to compute numerically $\mathcal{H}_k^0 u$ where $u = \frac{(x^2-\pi^2)^2}{\pi^4}$, in ω^2: the result is plotted on Figure 6.

6 The vibration modes

6.1 Characterization of the eigenvalues of the Neumann problem

The goal here is the computation of the eigenvalues and normalized eigenmodes of the Neumann operator $\tilde{L}^0$ introduced in (60). The following Lemmas will be useful for computing the eigenvalues in $\mathrm{Sp}^{N,0}$.

Lemma 13 *For any real number k,*

$$u \in \ker(\tilde{L}_k^0) \Rightarrow u|_{\Gamma^0} \in (\ker(L_k^0))^\circ. \tag{86}$$

Proof. We have $\int_{\Omega^0} \nabla u \cdot \nabla v - kuv = 0$ for all $v \in H^1(\Omega^0)$. Let $\tilde{u} \in H^1(\Omega^0)$ be another lifting of $u|_{\Gamma^0}$, then $e = u - \tilde{u} \in \mathcal{V}(\Omega^0)$ and for all $v \in \ker(L_k^0)$, $\int_{\Omega^0} \nabla e \cdot \nabla v - kev = 0$. Subtracting the two identities, we obtain that for all $v \in \ker(L_k^0)$, $\int_{\Omega^0} \nabla \tilde{u} \cdot \nabla v - k\tilde{u}v = 0$. This says exactly that $u|_{\Gamma^0} \in (\ker(L_k^0))^\circ$. ∎

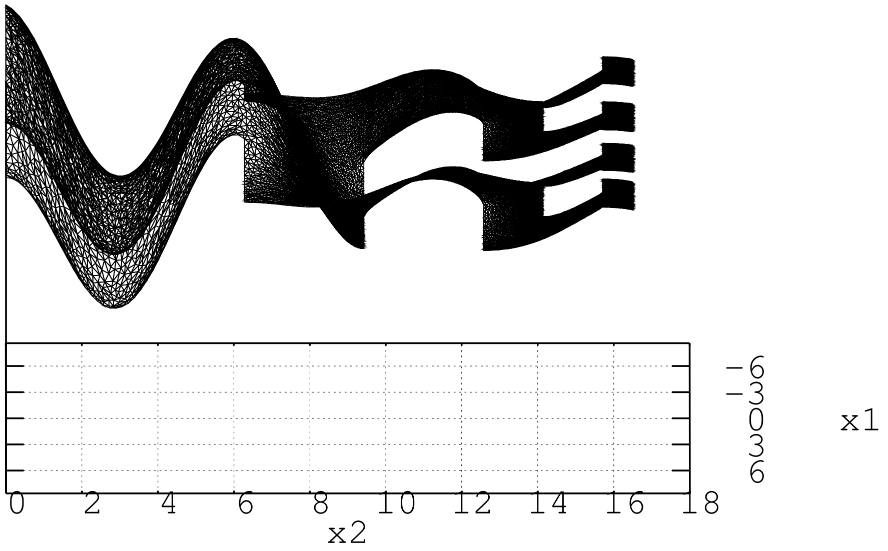

Figure 6: $\mathcal{H}_k^0 u$ with $k = 1$ and $u = \frac{(x_1^2 - \pi^2)^2}{\pi^4}$

Lemma 14 *For any real number k,*

$$Sp^{N,0} = \left\{ k \in \mathbb{R}, \ \text{such that } ker(T_k^0) \neq \{0\} \right\}, \tag{87}$$

and

$$\ker(\tilde{L}_k^0) = \mathcal{H}_k^0(\ker(T_k^0)). \tag{88}$$

One can obtain an Hibertian basis of $H^1(\Omega^0)$ by assembling bases of $\mathcal{H}_k^0(\ker(T_k^0))$ for $k \in Sp^{N,0}$.

Proof. We know that $k \in \mathrm{Sp}^{N,0}$ if and only if there exists $\hat{u} \in H^1(\Omega^0)$ such that

$$\int_{\Omega^0} \nabla \hat{u} \cdot \nabla v = k \int_{\Omega^0} \hat{u} v, \quad \forall v \in H^1(\Omega^0). \tag{89}$$

Call $u \in H^{\frac{1}{2}}(\Gamma^0)$ the trace of $\hat{u}$ on Γ^0, then $\hat{u} = \mathcal{H}_k^0(u)$ and Lemma 13 tells us that $u \in (\ker(L_k^0))^\circ$. So $T_k^0(u)$ can be computed and by the definition (61) of T_k^0, we see that $T_k^0 u = 0$.
Conversely, if $u \in \ker(T_k^0)$, $\mathcal{H}_k^0(u)$ satisfies (89). We have proved (87) and (88) and the last statement of the Lemma follows. ■
From Lemma 14, one can compute the eigenmodes of $\tilde{L}^0$ by searching the numbers k such that T_k is noninjective, and by taking the harmonic lifting $\mathcal{H}_k^0$ of the vectors belonging to the kernel of T_k^0. Of course, it is possible to carry out this program with the self-similar finite element discretization introduced above, because Lemma 13 and 14 have their discrete counterparts.

Remark 13 *Conversely, it is possible to compute the eigenmodes of the Dirichlet operator L^0 by studying the Neumann-Dirichlet operators related to the Helmholtz equation.*

6.2 Normalization of the eigenmodes of the Neumann problem

A more difficult point is to obtain eigenmodes with unit $L^2(\Omega^0)$-norm, in order to construct an orthonormal basis of $L^2(\Omega^0)$. The next result says that, when $k \notin \mathrm{Sp}^{D,0}$, it is possible to normalize the eigenmodes thanks to a perturbation method.

Proposition 7 *Consider $k \in Sp^{N,0}$ such that $k > 0$ and $k \notin Sp^{D,0}$, $u \in \ker(T_k^0)$, $u \neq 0$, and $\hat{u} = \mathcal{H}_k^0(u)$. Let δk be a small variation of k, such that $k + \delta k \notin Sp^{D,0}$:*

$$\delta k \|\hat{u}\|^2_{L^2(\Omega^0)} = -\langle T^0_{k+\delta k} u, u \rangle + o(\delta k). \tag{90}$$

For δk small enough, $\delta k \langle T^0_{k+\delta k} u, u\rangle < 0$ and

$$\|\hat{u}\|^2_{L^2(\Omega^0)} = -\lim_{\delta k \to 0} \frac{\langle T^0_{k+\delta k} u, u\rangle}{\delta k}. \tag{91}$$

Proof. Let $\delta\hat{u} \in \mathcal{V}(\Omega^0)$ be such that $\hat{u} + \delta\hat{u} = \mathcal{H}^0_{k+\delta k}(u)$. Since $u \in \ker(T_k^0)$ and $\hat{u} = \mathcal{H}_k^0(u)$, we know that for all $v \in H^1(\Omega^0)$, $\int_{\Omega^0} \nabla\hat{u} \cdot \nabla v - k\hat{u}v = 0$. In particular, for $v = \hat{u} + \delta\hat{u}$,

$$\int_{\Omega^0} \nabla\hat{u} \cdot \nabla(\hat{u} + \delta\hat{u}) - k \int_{\Omega^0} \hat{u}(\hat{u} + \delta\hat{u}) = 0. \tag{92}$$

On the other hand, we know that for all $v \in \mathcal{V}(\Omega^0)$, $\int_{\Omega^0} \nabla(\hat{u}+\delta\hat{u})\cdot\nabla v - (k+\delta k)\int_{\Omega^0}(\hat{u}+\delta\hat{u})v = 0$. In particular, for $v = \delta\hat{u}$,

$$\int_{\Omega^0} \nabla\delta\hat{u} \cdot \nabla(\hat{u} + \delta\hat{u}) - (k + \delta k) \int_{\Omega^0} \delta\hat{u}(\hat{u} + \delta\hat{u}) = 0. \tag{93}$$

Adding (92) and (93) yields

$$\int_{\Omega^0} |\nabla(\hat{u} + \delta\hat{u})|^2 - (k + \delta k) \int_{\Omega^0} |\hat{u} + \delta\hat{u}|^2 = -\delta k \int_{\Omega^0} \hat{u}(\hat{u} + \delta\hat{u}). \tag{94}$$

But, since $\hat{u} + \delta\hat{u} = \mathcal{H}^0_{k+\delta k}(u)$,

$$\int_{\Omega^0} |\nabla(\hat{u} + \delta\hat{u})|^2 - (k + \delta k) \int_{\Omega^0} |\hat{u} + \delta\hat{u}|^2 = \langle T^0_{k+\delta k}(\hat{u} + \delta\hat{u})|_{\Gamma^0}, (\hat{u} + \delta\hat{u})|_{\Gamma^0}\rangle = \langle T^0_{k+\delta k} u, u\rangle.$$

So

$$\delta k \|\hat{u}\|^2_{L^2(\Omega^0)} = -\langle T^0_{k+\delta k} u, u\rangle - \delta k \int_{\Omega^0} \hat{u}\delta\hat{u}.$$

Finally, since $k \notin \mathrm{Sp}^{D,0}$, $\|\delta\hat{u}\|_{L^2(\Omega^0)} = O(\delta k)$, which completes the proof. ■

Proposition 7 permits to scale the vibration modes obtained by the characterization in § 6.1 so that their $L^2(\Omega^0)$ norm is close to one (not exactly one, since the scaling factor is obtained by a perturbation method). This will permit to project a function accurately enough on the eigenspace.

Remark 14 *We do not know how to normalize the vibration mode when $k \in Sp^{N,0} \cap Sp^{D,0}$. However, we have not observed this situation in our computations.*

6.3 The projection of a compactly supported function on the space spanned by the first N eigenmodes

Let $(e_n)_{n=0,\dots,\infty}$ be a Hilbertian basis of $L^2(\Omega^0)$ made of eigenmodes of $\tilde{L}^0$ with unit $L^2(\Omega^0)$ norm. In the following, we call k_n the eigenvalue of $\tilde{L}^0$ corresponding to e_n. We also call Λ^N the subspace of $H^1(\Omega^0)$: $\Lambda^N = span(e_n)_{n=0,\dots,N}$.

Assume that with the method described in § 6.2, we have obtained eigenmodes $\tilde{e}_n$, $n = 0, \dots, N$, whose $L^2(\Omega^0)$ norm are close to one: $\tilde{e}_n = \mu_n e_n$ and $|\mu_n|$ is close to one. More precisely, assume

that there exists ϵ, $0 < \epsilon < 1$, such that, for all n, $0 \le n \le N$, $|\mu_n^2 - 1| \le \epsilon$.
Consider a function $u \in H^1(\Omega^0)$ supported for example in ω^0. Call $\pi^N(u)$ the projection of u onto Λ^n:

$$\pi^N(u) = \sum_{n=0}^{N} (u, e_n) e_n = \sum_{n=0}^{N} \left(\int_{\omega^0} u e_n \right) e_n.$$

The function $\pi^N(u)$ cannot be computed directly since e_n are not available. What can be computed is $\tilde{\pi}^N(u) = \sum_{n=0}^{N} (u, \tilde{e}_n)\tilde{e}_n = \sum_{n=0}^{N} \left(\int_{\omega^0} u\tilde{e}_n \right) \tilde{e}_n$. It is easy to check that

$$\pi^N(u) - \tilde{\pi}^N(u) = \sum_{n=0}^{N} (1 - \mu_n^2)(u, e_n) e_n = \sum_{n=0}^{N} (1 - \mu_n^2)(\pi^N(u), e_n) e_n,$$

therefore

$$\|\pi^N(u) - \tilde{\pi}^N(u)\|_{L^2(\Omega^0)} \le \epsilon \|u\|_{L^2(\Omega^0)}.$$

The numerical test below will confirm the fact that the method described in § 6.2 permits to approximate correctly the projection of a compactly supported function on Λ^N.

6.4 Numerical tests

6.4.1 The Weyl-Berry formula

Asymptotics of the density of states is an old problem which started with the well-known Weyl formula: if Ω is an open subset of $\mathbb{R}^d$ then $\aleph^D(k)$, the number of Dirichlet eigenvalues smaller than k behaves like

$$\aleph^D(k) \sim (2\pi)^{-d} \mathcal{B}_d |\Omega|_d k^{d/2} \quad \text{when } k \to \infty,$$

where $\mathcal{B}_d$ is the volume of the unit ball of dimension d, and $|\Omega|_d$ is the volume of the domain Ω. When Ω has a smooth boundary (and under some extra conditions), a second term in the expansion can be obtained: it is of the form $c_d k^{(d-1)/2}$, with a constant c_d depending on the length of the boundary (cf Ivrii, [8]). When the boundary is irregular then the second term depends on the Minkowski dimension of the boundary. The Minkowski dimension is related to the volume of the ϵ-neighborhoods $\partial\Omega^\epsilon$ of the boundary $\partial\Omega$. More precisely, the Minkowski measurability can be defined relative to a gauge function $h : \mathbb{R}^+ \to \mathbb{R}^+$ increasing and with some extra properties (cf [7], part 2). The boundary is said to be h-Minkowski measurable if

$$\begin{aligned} 0 < \quad & \liminf_{\epsilon \to 0} \epsilon^{-d} h(\epsilon) |\partial\Omega^\epsilon \cap \Omega|_d \\ = \quad & \limsup_{\epsilon \to 0} \epsilon^{-d} h(\epsilon) |\partial\Omega^\epsilon \cap \Omega|_d < \infty. \end{aligned}$$

In this case, the remainder term

$$\aleph^D(k) - (2\pi)^{-d} \mathcal{B}_d |\Omega|_d k^{d/2}$$

is proved to be of order (cf [7], theorem 2.12)

$$O(1/h(\frac{1}{\sqrt{k}})),$$

(and it is expected to be comparable with this value). The same results hold for the Neumann boundary condition provided that an extra regularity condition (the "C' condition") is satisfied.

6.4.2 Computation of $\mathrm{Sp}^{N,0}$

We use the domain and mesh displayed on Figure 3. We have computed numerically the first part of the spectrum $\mathrm{Sp}^{N,0}$ by the method proposed in § 6.1. More precisely, we have chosen a subdivision of the interval $[0, 40]$, and we have computed the discrete version of T_k^0, for k at the nodes of this subdivision. The subdivision was piecewise uniform, with step size 10^{-3} for $k \le 10$ and 10^{-2} for $k > 10$. When the signature of T_k^0 changes from one node to the next, we run a dichotomy method in order to compute precisely a singular value k_{sing} between the two nodes, which may be either an eigenvalue (in $\mathrm{Sp}^{N,0}$) of the Neumann problem, if $\ker(T^0_{k_{\mathrm{sing}}}) \neq \{0\}$, or an eigenvalue in $\mathrm{Sp}^{D,0}$. Of course, this method is crude enough, and we may miss an eigenvalue if there are more than one singular values between two successive mesh nodes.

In the present case, it is easy to check that $|\Omega^0|_2 = 16\pi^2$ and that $\mathcal{B}_2 = \pi$. The Weyl formula is

$$\aleph^D(k) \sim 4\pi k \quad \text{as } k \to \infty. \tag{95}$$

It is also easy to check that

$$|(\partial\Omega^0)^\epsilon \cap \Omega^0|_2$$

is equivalent to

$$\epsilon \log(1 + \frac{1}{\epsilon})/\log 2,$$

which means that the boundary is h-Minkowski measurable, with $h(x) = \frac{x}{\log(1+\frac{1}{x})}$. Hence, the remainder term for the counting function of the Dirichlet eigenvalues $\aleph^D(k) - 4\pi k$ is of order $O(\sqrt{k}\log k)$.

Remark 15 *Note that the Dirichlet problem mentioned here consists of imposing a Dirichlet condition on all $\partial\Omega^0$, so it is not problem (57).*

In the present case, we compute numerically the counting function $\aleph$ of the Neumann eigenvalues. The "C' condition" is not satisfied, so the previous estimates are not known to be true for $\aleph$.
On Figure 7, we have plotted $\aleph_h(k)$, the number of computed eigenvalues smaller than k vs. k, for $k < 40$, and the graph of $k \mapsto 4\pi k$. We see that the Weyl estimate (95) is very well satisfied by $\aleph_h(k)$. This indicates that the Weyl estimate is true for the $\aleph$.
We go further and plot the remainder term $\aleph_h(k) - 4\pi k$. We have tried to fit this function by a function of the type $f(k) = (a\log(k) + b)\sqrt{k} + c$. The parameters a, b, c have been computing by using a least square algorithm in the interval $k = [0, 10]$. On figure , we plot the function $\aleph_h(k) - 4\pi k$ and $f(k)$, for $k \in (0, 40)$. Although, the least square algorithm has been used to fit the function in the region $(0, 10)$, we see that $f(k)$ approaches $\aleph_h(k) - 4\pi k$ well, for $k \in (10, 40)$.

On Figures 9 and 10, we have plotted the restrictions of the third and sixth eigenmodes (not normalized yet) to ω^2. On Figure 11, we have plotted the restriction of the eigenmodes 5, 10, 15 and 20 to ω^0.

6.4.3 Modal decomposition of a compactly supported function

To test the normalization of the eigenmodes, we choose a compactly supported function u and we compare u with $\tilde{\pi}^N u$ for $1 \le N \le 300$. We choose

$$u = \left(\frac{x_1^2 - \pi^2}{\pi^2}\right)^4 \left(\frac{x_2(3\pi - x_2)}{\frac{9\pi^2}{4}}\right)^3 1_{-\pi<x_1<\pi} 1_{0<x_2<3\pi} e^{-x_1^2-(x_2-\frac{3\pi}{2})^2}. \tag{96}$$

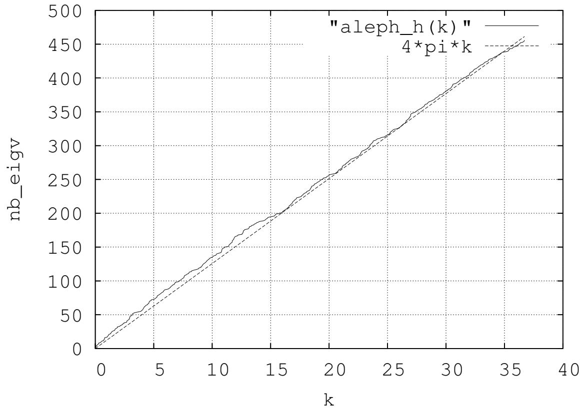

Figure 7: The computed spectrum $\mathrm{Sp}^{N,0}$: the number $\aleph_h(k)$ of eigenvalues smaller than k vs k, for $k < 40$, and the graph of $k \mapsto 4\pi k$.

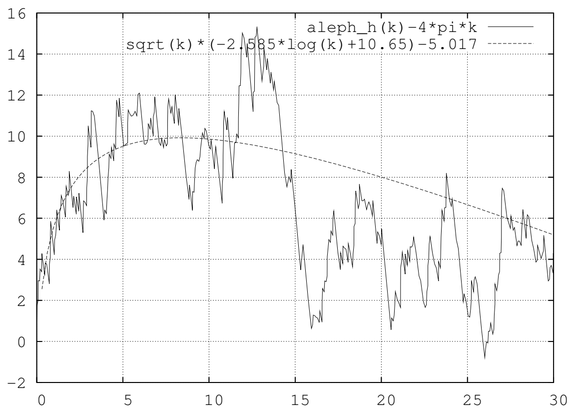

Figure 8: The remainder $\aleph_h(k) - 4\pi k$ and $f(k)$.

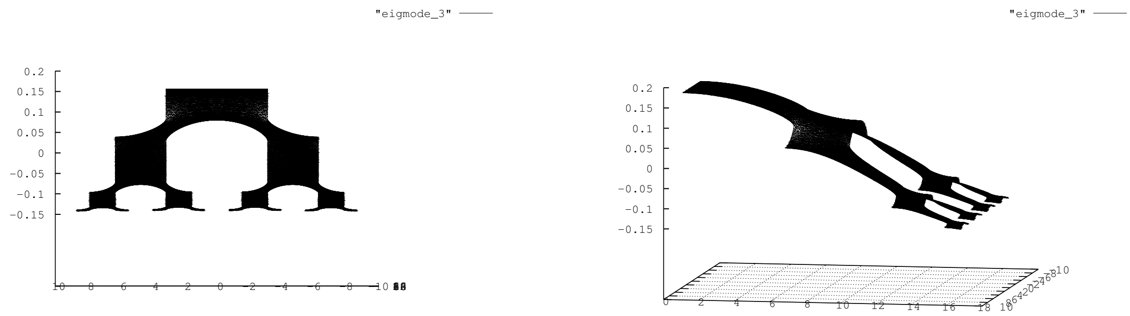

Figure 9: Two views of the third eigenmode, restricted to ω^2.

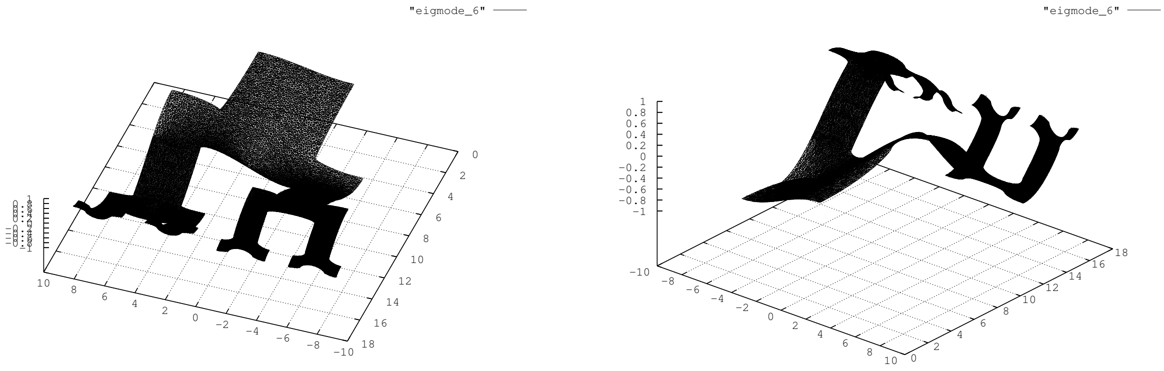

Figure 10: Two views of the sixth eigenmode, restricted to ω^2.

On Figure 12, we plot the error $\|u - \tilde{\pi}^N u\|_{L^2(\omega^0)}$ as a function of N. We see that the error decays as N tends to infinity. This shows that the family $(\tilde{e}_n)$ introduced in § 6.2 is close to orthonormal. The normalization of the eigenmodes by the perturbation method is accurate enough. On Figure 13, we plot the reconstructed function $\tilde{\pi}^N u$ and the error $u - \tilde{\pi}^N u$, for $N = 300$. There is no visible difference between u and $\tilde{\pi}^{300}(u)$. To more accurately test the normalization procedure, we compute $\alpha_n = \int_{\omega^0} (u - \tilde{\pi}^{300} u)\tilde{e}_n$ for $0 \leq n \leq 300$; If $\tilde{e}_n$ matched e_n for all n, $0 \leq n \leq 300$, then the numbers α_n would be exactly 0. On Figure 14, we plot α_n as a function of n. We see that these numbers never exceed 3.10^{-3}, which confirms the fact that the family $\tilde{e}_n$ is very close to being orthonormal.

6.4.4 Wave propagation in Ω^0

Finally, it is possible to use the above mentioned modal decomposition of a compactly supported function in order to solve a Cauchy problem for the wave equation in Ω^0 with compactly supported initial data: assume that we wish to solve the following problem

$$\begin{array}{rcll} \frac{\partial^2 w}{\partial t^2} - \Delta w &=& 0 & \text{in } (0,T) \times \Omega^0, \\ \frac{\partial w}{\partial n} &=& 0 & \text{on } (0,T) \times \left(\Gamma^0 \cup \Sigma^0\right), \\ w|_{t=0} &=& u_0 & \text{in } \Omega^0, \\ \frac{\partial w}{\partial t}|_{t=0} &=& u_1 & \text{in } \Omega^0, \end{array} \tag{97}$$

where u_0 and u_1 are two functions supported for example in ω^0. It is possible to write the modal decompositions of u_0 and u_1: $\tilde{\pi}^N u_0 = \sum_{n=0}^N \beta_n \tilde{e}_n$ and $\tilde{\pi}^N u_1 = \sum_{n=0}^N \gamma_n \tilde{e}_n$ and to solve exactly a Cauchy problem close to (97):

$$\begin{array}{rcll} \frac{\partial^2 \tilde{w}}{\partial t^2} - \Delta \tilde{w} &=& 0 & \text{in } (0,T) \times \Omega^0, \\ \frac{\partial \tilde{w}}{\partial n} &=& 0 & \text{on } (0,T) \times \left(\Gamma^0 \cup \Sigma^0\right), \\ \tilde{w}|_{t=0} &=& \tilde{\pi}^N u_0 & \text{in } \Omega^0, \\ \frac{\partial \tilde{w}}{\partial t}|_{t=0} &=& \tilde{\pi}^N u_1 & \text{in } \Omega^0, \end{array} \tag{98}$$

by

$$\tilde{w}(t,x) = (\beta_0 + \gamma_0 t)\tilde{e}_0(x) + \sum_{n=1}^N \left(\beta_n \cos(\sqrt{k_n} t) + \frac{\gamma_n}{\sqrt{k_n}} \sin(\sqrt{k_n} t) \right) \tilde{e}_n(x). \tag{99}$$

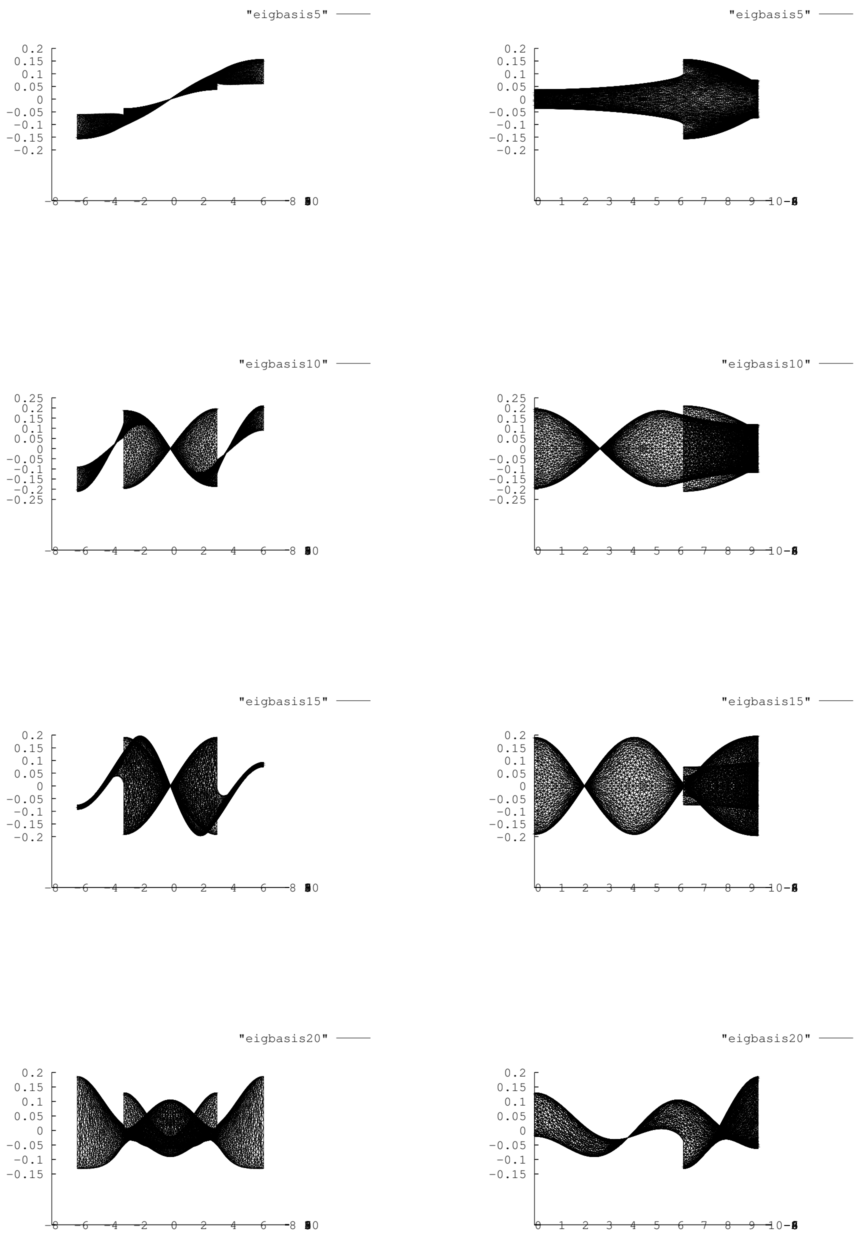

Figure 11: The restrictions of the eigenmodes 5,10,15 and 20 to ω^0, viewed from south and east

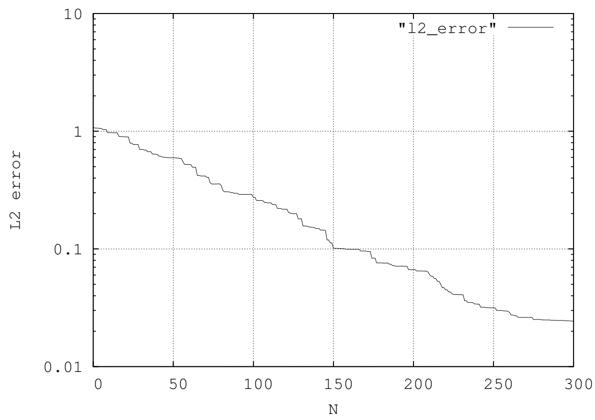

Figure 12: The error $\|u - \tilde{\pi}^N u\|_{L^2(\omega^0)}$ vs. N.

On Figure 15, we have plotted the value of $\tilde{w}(t,a)$ as a function of time, for $a = (\frac{3\pi}{2}, 3\pi)$, for $u_0 = u$ given by (96), and $u_1 = 0$.

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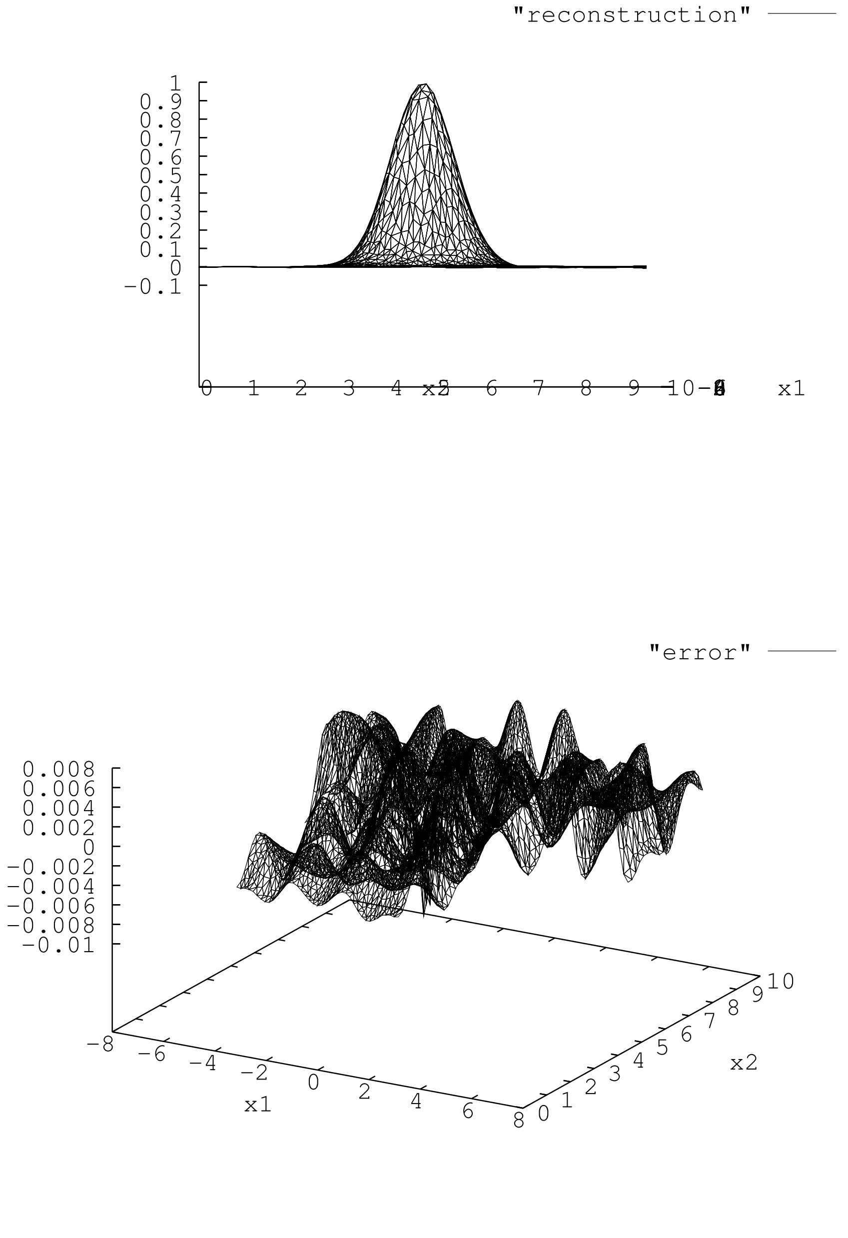

Figure 13: The function $\tilde{\pi}^{300}u$ (viewed from the east) and the error $u - \tilde{\pi}^{300}u$.

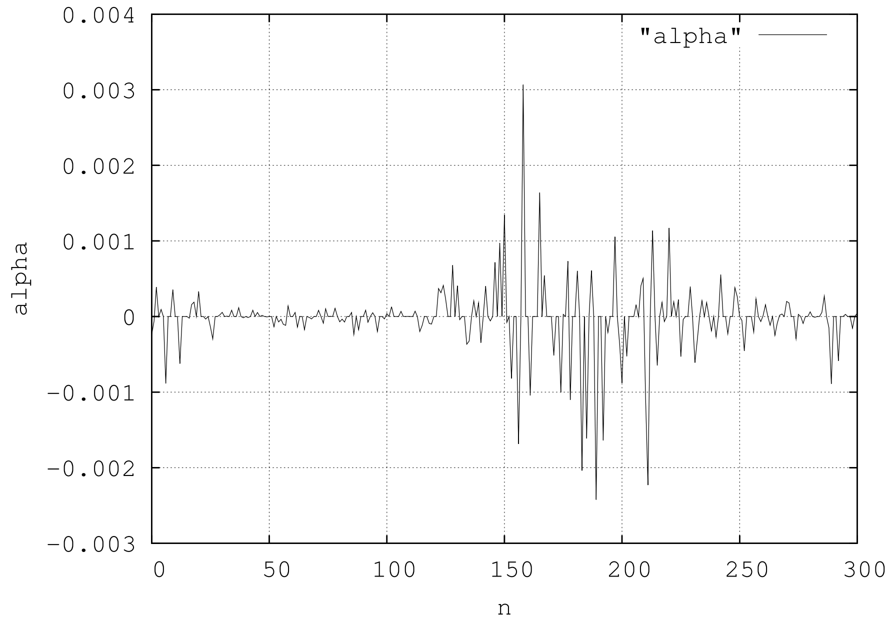

Figure 14: $\alpha_n = \int_{\omega^0}(u - \tilde{\pi}^{300}u)\tilde{e}_n$ vs. n.

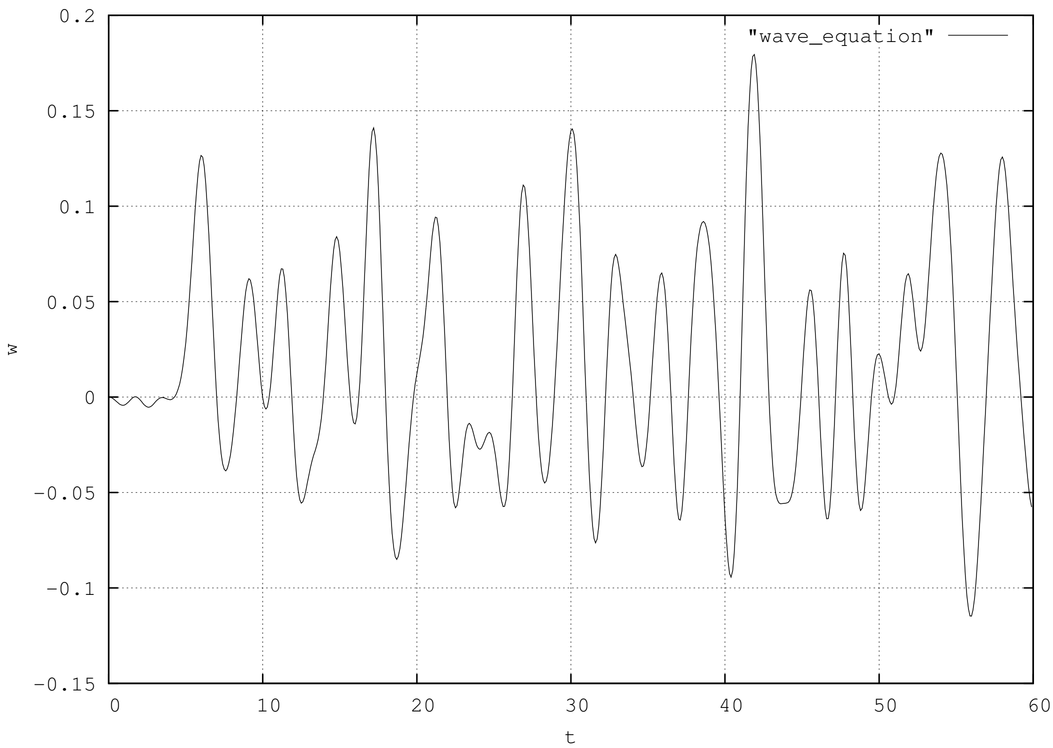

Figure 15: $\tilde{w}(t, a)$ vs. t, where $\tilde{w}$ is the solution to the Cauchy problem (98) with u_0 given by (96), and $u_1 = 0$.

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