No-arbitrage in discrete-time markets with proportional transaction costs and general information structure

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Abstract

We discuss the no-arbitrage conditions in a general framework for discrete-time models of financial markets with proportional transaction costs and general information structure. We extend the results of Kabanov and al. (2002), Kabanov and al. (2003) and Schachermayer (2004) to the case where bid-ask spreads are not known with certainty. In the "no-friction" case, we retrieve the result of Kabanov and Stricker (2003).

Key words: Absence of arbitrage, proportional transaction costs, imperfect information, optional projection.

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1 Introduction

While the "insider trading" problem, where the agent's filtration $\mathbb{H}$ is strictly bigger than the asset's filtration $\mathbb{F}^S$, has been widely studied in the recent literature, see e.g. [1], [5],

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[8] and the references therein, less care has been given to the imperfect information case where $\mathbb{H}$ does not contain $\mathbb{F}^S$. Such situations may arise for instance if the small investor has not a direct access to the market. In this case, his orders can be executed with a delay and therefore at a price which is not known in advance, see [4] and [12]. From the point of view of the arbitrage theory, the "insider trading" case is well known. Indeed, all the necessary and sufficient condition for the absence of arbitrage opportunities available in the case $\mathbb{H} = \mathbb{F}^S$ apply to the general case $\mathbb{F}^S \subset \mathbb{H}$. In particular, the usual "no-arbitrage" conditions imply that, roughly speaking (see [6] and [7] for precise results), prices must be semi-martingales in the filtration $\mathbb{H}$ and that there must be an equivalent probability measure $\mathbb{Q}$ under which they are $(\mathbb{Q}, \mathbb{H})$-local martingales.

The case of imperfect information where $\mathbb{H} \subset \mathbb{F}^S$ and $\mathbb{H} \neq \mathbb{F}^S$ is much more difficult to handle. In particular, the arguments of [6] do not work in this situation. Even in the case of infinite discrete time, the proof of [18] does not apply. However, in finite discrete time, it was noticed in [12] that the proof of the Dallang-Morton-Willinger theorem reported in [10] still holds up to minor modifications for any given filtration $\mathbb{H}$. In this case, the no-arbitrage condition is equivalent to the existence of a probability measure $\mathbb{Q}$ such that the optional projection under $\mathbb{Q}$, $(\mathbb{E}^{\mathbb{Q}}[S_t \mid \mathcal{H}_t])_t$, of the asset prices $(S_t)_t$ on $\mathbb{H} = (\mathcal{H}_t)_t$ is a $(\mathbb{Q}, \mathbb{H})$-martingale.

The aim of this paper is to extend this result to the case where exchanges are subject to proportional transaction costs. In the recent literature, such models have been widely studied, from the seminal work of [9] to the recent papers [11], [13], [14], [17], [16], [15] and [2] among others. The recent abstract formulation consists in introducing a sequence of random closed convex cones $(K_t)_t$ and describing the wealth process as $V_t = \sum_{s \le t} \xi_s$ with $\xi_s \in -K_s$ a.s. The "usual" example is given by

$$- K_t(\omega) = \{x \in \mathbb{R}^d \ : \ \exists \, a \in \mathbb{M}^d_+, \ x^i \le \sum_{j \le d} a^{ji} - a^{ij}\pi^{ij}_t(\omega) \ , \ i \le d\} \ , \tag{1.1}$$

where $\mathbb{M}^d_+$ denotes the set of square d-dimensional matrices with non-negative entries. Here π^{ij} should be interpreted as the costs in units of asset i one has to pay to obtain one unit of asset j. If we allow to throw out money, an exchange ξ_t at time t is then affordable if $\xi_t \in -K_t$ a.s.

In the case of imperfect information, i.e. π is not $\mathbb{H}$-adapted, this approach cannot be used since K is no longer $\mathbb{H}$-adapted. Hence, we have to change the modelisation. Instead of considering the ξ's as the controls, we have to rewrite them as $\xi^i_t = \sum_{j \le d} \eta^{ji}_t - \eta^{ij}_t \pi^{ij}_t$, where η is an $\mathbb{H}$-adapted process with values in the set of square d-dimensional matrices with non-negative entries $\mathbb{M}^d_+$. Here, η^{ji}_t is the number of physical units of i we obtain, at time t, against $\eta^{ji}_t \pi^{ji}_t$ units of j. Because the ξ may not be adapted the proofs of [13], [14] and [17] does not apply to this setting and, in contrast to [12], we have to work a bit more to extend their results.

In the above model, we fix the number of units η^{ij}_t of asset j we want to buy and the number of units of asset i one has to sell is given by $\eta_t^{ij}\pi_t^{ij}$. In the case of perfect information, i.e. π is $\mathbb{H}$-adapted, one can also fix the amount $\tilde{\eta}_t^{ij}$ of units of asset i one wants to sell and compute η_t^{ij} accordingly by using the formula $\tilde{\eta}_t^{ij} = \eta_t^{ij}\pi_t^{ij}$. But in the case where π is not $\mathbb{H}$-adapted this is no more possible and one cannot control exactly $\tilde{\eta}_t^{ij}$. This means that orders can be formulated only in terms of the quantity of units of the asset we want to buy and we shall see in Subsection 3.3 that, in such a situation, orders may be non-reversible even in the case of no-friction where $\pi^{ij} = \pi^{ji}$ for all $i, j \le d$. Clearly, this is not reasonable and in practice one should also be able to fix $\tilde{\eta}_t^{ij}$. To pertain for such orders, one can slightly modify the above model by taking ξ in the form $\xi_t^i = \sum_{j\le d} \eta_t^{ji}(1 + \lambda_t^{ij}(\omega)\mathbb{1}_{\eta_t^{ji}<0}) - \eta_t^{ij}\tau_t^{ij}(\omega)(1 + \lambda_t^{ij}(\omega)\mathbb{1}_{\eta_t^{ij}>0})$ where η is $\mathbb{H}$-adapted process with values in the set $\mathbb{M}^d$ of square d-dimensional matrices. Here, τ^{ij} stands for the costs in units of asset i one has to pay to obtain one unit of asset j, before to pay the transaction costs. The transaction costs $\eta_t^{ij}\tau_t^{ij}\lambda^{ij}(\omega)\mathbb{1}_{\eta_t^{ij}>0} + \eta_t^{ji}\lambda_t^{ji}(\omega)\mathbb{1}_{\eta_t^{ji}<0}$ are paid in units of the sold asset i. With the above notation, one has $\pi_t^{ij} = \tau_t^{ij}(1 + \lambda^{ij})$. This corresponds to

$$
\begin{aligned}
-K_t(\omega) \;=\; & \{x \in \mathbb{R}^d \;:\; \exists\, a \in \mathbb{M}^d, \qquad (1.2)\\
& x^i \le \sum_{j\le d} a^{ji}(1 + \lambda_t^{ij}(\omega)\mathbb{1}_{a^{ij}<0}) - a^{ij}\tau_t^{ij}(\omega)(1 + \lambda_t^{ij}(\omega)\mathbb{1}_{a^{ij}>0}) \;,\; i \le d\} \;,
\end{aligned}
$$

Contrary to the model (1.1), we can now fix the number of units of asset i we want to sell against units of asset j by fixing $\eta_t^{ji} < 0$ so that $|\eta_t^{ji}|$ coincides with the amount of exchanged units of i. Once again, in the case of perfect information both models are equivalent, but this is no more true if τ and/or λ are not $\mathbb{H}$-adapted. One could also argue that paying the transaction costs in units of the asset which is sold, as in (1.2), is not the same thing than paying these costs in units of the asset which is bought. Here again one could consider a more general model which pertains for different costs structures.

In order to take into account all these different situations, we propose a general formalism where the wealth process V is written as $V_t = \sum_{s\le t} F_s(\eta_s)$, for some sequence of random maps $F = (F_t)_t$. Here, η is $\mathbb{H}$-adapted process with values in a closed convex cone $\mathcal{A}$ of $\mathbb{M}^d$ (we have in mind to take $\mathcal{A} = \mathbb{M}^d$, however, in order to take also the model (1.1) into account it is more convenient to allow for the possibility of having $\mathcal{A} = \mathbb{M}_+^d$). We make no assumption on the filtration under which F is adapted. Thus, this approach pertains for the cases of "insider trading" or imperfect information and for all other mixed cases (for instance, we can imagine that we do not observe the price of the assets but have some extra information which is not contained in the filtration induced by the processes of exchange rates. Observe that, if we know that the price of some asset will double between today and tomorrow, we can make an arbitrage without knowing this price - assuming that transaction costs are reasonable).

In Section 2, we study the no-arbitrage conditions considered in [13], [14] and [17] in this abstract setting. Examples of application are provided in Section 3

2 The abstract formulation

Throughout this paper, we fix a finite time horizon $T \in \mathbb{N}$ and consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a filtration $\mathbb{H} = (\mathcal{H}_t)_{t\in\mathbb{T}}$ with $\mathbb{T} = \{0, \ldots, T\}$. Importantly, we only assume that $\mathcal{H}_T \subset \mathcal{F}$. In particular, most of the processes considered in this paper need not be $\mathbb{H}$-adapted. In all this paper, inequalities involving random variables must be understood in the $\mathbb{P}-$a.s. sense, if it is clear from the context, and inclusive relations between elements of $\mathcal{F}$ are assumed to hold up to $\mathbb{P}$-null sets.

2.1 The model

We consider a closed convex cone $\mathcal{A}$ of $\mathbb{M}^d$, $d \geq 1$, and denote by $\mathbb{F}$ the set of continuous maps F from $\mathbb{M}^d$ into $\mathbb{R}^d$ such that

$\mathbf{HF}_1$: For $\lambda \geq 0$ and $a \in \mathbb{M}^d$, $\lambda F(a) = F(\lambda a)$.

$\mathbf{HF}_2$: For $\lambda \geq 0$, $\beta \geq 0$ and $a,\ a' \in \mathcal{A}$, $F(\lambda a + \beta a') - (\lambda F(a) + \beta F(a')) \in \mathbb{R}^d_+$.

We then define $\mathbf{F}$ as the set of $\mathcal{F}$-measurable sequences $F = (F_t)_{t\in\mathbb{T}}$ such that F_t takes a.s. values in $\mathbb{F}$, for each $t \in \mathbb{T}$. Observe that $\mathbf{HF}_1$ implies that $F(0) = 0$.

Given $F \in \mathbf{F}$, we define $N(F) = (N_t(F))_{t\in\mathbb{T}}$ and $N^0(F) = (N^0_t(F))_{t\in\mathbb{T}}$ by

$$N_t(F) \;=\; \left\{F_t(\eta),\ \eta \in L^0(\mathcal{A}; \mathcal{H}_t)\right\} \quad \text{and} \quad N^0_t(F) \;=\; N_t(F) \cap (-N_t(F)) \ .$$

Here, for $E \subset \mathbb{M}^d$ (or $E \subset \mathbb{R}^d$) and a σ-algebra $\mathcal{G}$ included in $\mathcal{F}$, $L^0(E; \mathcal{G})$ denotes the set of E-valued $\mathcal{G}$-measurable random variables. For a process ξ such that $\xi_t \in N_t(F)$ for all $t \in \mathbb{T}$, we shall simply write $\xi \in N(F)$. We shall similarly write $\xi \in N^0(F)$ if $\xi_t \in N^0_t(F)$ for all $t \in \mathbb{T}$.

Given a process ξ with values in $\mathbb{R}^d$, we finally define

$$V_t(\xi) \;=\; \sum_{s=0}^{t} \xi_s \quad \text{and} \quad A_t(F) \;:=\; \left\{V_t(\xi) - r,\ \xi \in N(F),\ r \in L^0(\mathbb{R}^d_+; \mathcal{F})\right\} \ , \quad t \in \mathbb{T} \ .$$

Observe that we do not impose that the above processes are $\mathbb{H}$-adapted: F_t, ξ_t and $V_t(\xi_t)$ need not be $\mathcal{H}_t$-measurable.

Remark 2.1 In financial applications, $F^i_t(\eta_t)$ will correspond to the change in the number of units of asset i held in the portfolio $V(\xi)$ at time t. This results from the different exchanges η^{ij}_t and η^{ji}_t made between the i-th asset and the other j-th assets, under the self-financing condition and after paying the transaction costs. In this case, $A_t(F)$ stands for the set of contingent claims, labeled in physical units, that can be super-hedged by trading up to time t and starting with an initial endowment equal to 0. This formalism applies to model (1.2) with $\mathcal{A} = \mathbb{M}^d$ and

$$F_t^i(\eta_t)(\omega) \;=\; \sum_{j \le d} \eta_t^{ji}(1+\lambda_t^{ij}(\omega)\mathbb{1}_{\eta_t^{ij}<0}) - \eta_t^{ij}\tau_t^{ij}(\omega)(1+\lambda_t^{ij}(\omega)\mathbb{1}_{\eta_t^{ij}>0}) \;\;,\; i \le d\,.$$

This model will be further discussed in Section 3.

In this section, we provide sufficient conditions under which $A_T(F)$ is closed in probability and study abstract versions of the no-arbitrage conditions of [13], [14] and [17].

2.2 Sufficient conditions for the closedness of $A_T(F)$

In all this subsection, we shall assume that the sequence of random maps F satisfies the following conditions:

KP: For each ξ and $\tilde{\xi}$ in $N(F)$, $V_T(\xi) + V_T(\tilde{\xi}) \in L^0(\mathbb{R}^d_+; \mathcal{F})$ implies that $\xi \in N^0(F)$ and $V_T(\xi) + V_T(\tilde{\xi}) = 0$.

HN^0: For $t \in \mathbb{T}$ and $\eta \in L^0(\mathcal{A}; \mathcal{H}_t)$, $F_t(\eta) \in N_t^0(F) \Rightarrow F_t(-\eta) = -F_t(\eta)$ and $-\eta \in L^0(\mathcal{A}; \mathcal{H}_t)$.

We call the first condition **KP** as "key property" as it results from what was called "key Lemma" in [14], see condition (iii) in [14] and Lemma 3 in [13]. In Subsection 2.4, we shall provide sufficient conditions for this property to hold.

In financial models with transaction costs, the second condition can be understood as follows: $\xi_t := F_t(\eta_t) \in N_t^0(F)$ means that the exchange ξ_t is reversible, i.e. starting with the endowment ξ_t we can make immediate exchanges so as to come back to 0. Intuitively, this means that η_t corresponds to exchanges between assets that can be exchanged freely, i.e. without paying transaction costs. In this case, we should be able to do the opposite operation, $-\eta_t$, to reverse these transactions. In the formalism of [13] and [14] such an assumption is not required and the only important property is that if $\xi_t \in N_t^0(F)$ then $-\xi_t \in N_t(F)$, which, in their setting, implies that $-\xi_t$ is also an admissible exchange. Since, in our case, $-\xi_t$ may not be $\mathcal{H}_t$-measurable, we need to rewrite it as some $F_t(\tilde{\eta}_t)$ for some suitable $\tilde{\eta}_t \in L^0(\mathcal{A}; \mathcal{H}_t)$. In view of the above discussion, it is natural to assume that such a $\tilde{\eta}_t$ should be simply given by $-\eta_t$.

The aim of this section is to show that it implies the closedness (in probability) of the set $A_T(F)$.

For the reader's convenience, we first recall the following Lemma whose proof can be found in [10].

Lemma 2.1 *Set $\mathcal{G} \subset \mathcal{F}$ and $E \subset \mathbb{R}^d$. Let $(\eta^n)_{n\geq 1}$ be a sequence in $L^0(E;\mathcal{G})$. Set $\tilde{\Omega}$ $:= \{\liminf_{n\to\infty} \|\eta^n\| < \infty\}$. Then, there is an increasing sequence of random variables $(\tau(n))_{n\geq 1}$ in $L^0(\mathbb{N};\mathcal{G})$ such that $\tau(n) \to \infty$ a.s. and $\eta^{\tau(n)}\mathbb{I}_{\tilde{\Omega}}$ converges a.s. to $\eta^*\mathbb{I}_{\tilde{\Omega}}$ for some $\eta^* \in L^0(E;\mathcal{G})$.*

In the following, we shall denote by $\mathbb{L}^0(\mathcal{A};\mathbb{H})$ the set of $\mathcal{A}$-valued $\mathbb{H}$-adapted processes.

Proposition 2.1 *Fix $F \in \mathbf{F}$ such that* **KP** *and* $\mathbf{HN}^0$ *hold. Then, $A_T(F)$ is closed in probability.*

Proof. Let us define $A_{t,T} := \{\sum_{s=t}^T \xi_s - r,\ \xi \in N(F),\ r \in L^0(\mathbb{R}^d_+;\mathcal{F})\}$, $t \in \mathbb{T}$. We claim that $A_{T,T}$ is closed in probability (see 3. below) and use an inductive argument. We assume that $A_{t+1,T}$ is closed in probability for some $t \leq T-1$ and show that $A_{t,T}$ is closed too. Let $(g^n)_{n\geq 1}$ be a sequence in $A_{t,T}$ which converges a.s. to some $g \in L^0(\mathbb{R}^d;\mathcal{F})$. We have to show that $g \in A_{t,T}$. Let $(\eta^n, r^n)_{n\geq 1}$ be a sequence in $\mathbb{L}^0(\mathcal{A};\mathbb{H}) \times L^0(\mathbb{R}^d_+;\mathcal{F})$ such that

$$V_T(\xi^n) - r^n = g^n \tag{2.1}$$

with $\xi^n := F(\eta^n)$ and $\eta^n = 0$ on $\{0,\ldots,t-1\}$. Set $\alpha^n := \|\eta^n_t\|$ and $B := \{\liminf_{n\to\infty} \alpha^n < \infty\}$. Since $B \in \mathcal{H}_t$, we can work separately on B and B^c, by considering the two sequences $(\eta^n\mathbb{I}_B, r^n\mathbb{I}_B)_{n\geq 1}$ and $(\eta^n\mathbb{I}_{B^c}, r^n\mathbb{I}_{B^c})_{n\geq 1}$, and therefore do as if either $\mathbb{P}[B] = 1$ or $\mathbb{P}[B] = 0$.

1. If $\mathbb{P}[B] = 1$, then, by Lemma 2.1, there is a random sequence $(\tau(n))_{n\geq 1}$ in $L^0(\mathbb{N};\mathcal{H}_t)$ such that $\tau(n) \to \infty$ a.s. and $\eta_t^{\tau(n)}$ converges a.s. to some $\eta^*_t \in L^0(\mathcal{A};\mathcal{H}_t)$. Then, by a.s. continuity of F_t, $F_t(\eta_t^{\tau(n)})$ converges to $F_t(\eta^*_t)$. Since by construction $g^{\tau(n)} - F_t(\eta_t^{\tau(n)}) \in A_{t+1,T}(F)$, and, by assumption, the later is closed in probability, we can find some $\tilde{\xi} \in N(F)$ such that $\tilde{\xi} = 0$ on $\{0,\ldots,t\}$ and $\sum_{s=t+1}^T \tilde{\xi}_s = g - F_t(\eta^*_t)$. Since $F_t(\eta^*_t) \in N_t(F)$, this shows that $g \in A_{t,T}$.

2. If $\mathbb{P}[B] = 0$ then we set $\bar{\eta}^n := \eta^n/(\alpha^n \vee 1)$. Since $\liminf_{n\to\infty} \|\bar{\eta}^n_t\| < \infty$ a.s., we can assume (after possibly passing to a $\mathcal{H}_t$-measurable random subsequence as above) that $\bar{\eta}^n_t$ converges a.s. to some element of $L^0(\mathcal{A};\mathcal{H}_t)$. Arguing as above, using $\mathbf{HF}_1$ and observing that $g^n/(\alpha^n \vee 1)$ converges a.s. to 0, we can find some $\bar{\eta} \in \mathbb{L}^0(\mathcal{A};\mathbb{H})$, such that $\bar{\eta}^n_t \to \bar{\eta}_t$, and $\bar{r} \in L^0(\mathbb{R}^d_+;\mathcal{F})$ for which

$$\sum_{s=t}^T F_s(\bar{\eta}_s) - \bar{r} \;=\; 0\,. \tag{2.2}$$

From **KP**, we deduce that $\bar{r} = 0$ and $\bar{\xi}_s := F_s(\bar{\eta}_s) \in N^0_s(F)$ for all $s \geq t$. Since $\|\bar{\eta}_t\| = 1$, there is partition of B into (possibly empty) disjoint sets $(B_{ij})_{i,j\leq d}$ such that

$B_{ij} \subset \{\bar{\eta}_t^{ij} \neq 0\}$. We then define $\tilde{\eta}_s^n := \sum_{i,j \leq d} (\eta_s^n - \beta_n^{ij} \bar{\eta}_s) 1\!\!1_{B_{ij}} 1\!\!1_{s \geq t}$ where $\beta_n^{ij} = (\eta_t^n)^{ij} / \bar{\eta}_t^{ij}$ on B_{ij} and $\beta_n^{ij} = 0$ on B_{ij}^c. Set $C_{ij} = \{\beta_n^{ij} = |\beta_n^{ij}|\} \cap B_{ij}$ and $\tilde{C}_{ij} = \{\beta_n^{ij} = -|\beta_n^{ij}|\} \cap B_{ij}$. By $\mathbf{HF}_1$, $\mathbf{HF}_2$, $\mathbf{HN}^0$, (2.2) and the fact that $\bar{r} = 0$, we get

$$
\begin{aligned}
V_T(F(\tilde{\eta}^n)) &= \sum_{i,j \leq d} \sum_{s=t}^{T} F_s(\eta_s^n) + |\beta_n^{ij}| F_s(\bar{\eta}_s (1\!\!1_{\tilde{C}_{ij}} - 1\!\!1_{C_{ij}})) + \check{r}^n \\
&= \sum_{i,j \leq d} \sum_{s=t}^{T} F_s(\eta_s^n) + |\beta_n^{ij}| \left(F_s(\bar{\eta}_s) 1\!\!1_{\tilde{C}_{ij}} - F_s(\bar{\eta}_s) 1\!\!1_{C_{ij}} \right) + \check{r}^n \\
&= g^n + r^n + \check{r}^n ,
\end{aligned}
$$

for some sequence $(\check{r}^n)_{n \geq 1}$ in $L^0(\mathbb{R}^d_+; \mathcal{F})$. Hence, we have constructed a new sequence $(\tilde{\xi}^n := F(\tilde{\eta}^n), \tilde{r}^n := r^n + \check{r}^n)_{n \geq 1}$ for which (2.1) holds and $(\tilde{\eta}_t^n)^{ij} = 0$ on B_{ij}. Repeating this argument recursively on the different B_{ij}'s and arguing as in [13], we can finally obtain, in a finite number of operations, a sequence $(\hat{\eta}^n)_{n \geq 1}$ in $\mathbb{L}^0(\mathcal{A}; \mathbb{H})$ such that $\liminf_{n \to \infty} \|\hat{\eta}_t^n\| < \infty$ a.s. and $\sum_{s=t}^{T} F_s(\hat{\eta}_s^n) = g^n + \hat{r}^n$, for some sequence $(\hat{r}^n)_{n \geq 1}$ in $L^0(\mathbb{R}^d_+; \mathcal{F})$. Applying the argument of 1. above then concludes the proof.

3. The fact that $A_{T,T}$ is closed in probability is obtained by similar arguments. Given a sequence $(g^n)_{n \geq 1}$ in $A_{T,T}$ which converges a.s. to some $g \in L^0(\mathbb{R}^d; \mathcal{F})$, we consider a sequence $(\eta_T^n, r^n)_{n \geq 1}$ in $L^0(\mathcal{A}; \mathcal{H}_T) \times L^0(\mathbb{R}^d_+; \mathcal{F})$ such that $F_T(\eta_T^n) - r^n = g^n$. Considering separately the event sets $\{\liminf_{n \to \infty} \|\eta_T^n\| < \infty\}$ and $\{\liminf_{n \to \infty} \|\eta_T^n\| = \infty\}$ as in 1. and 2., we can construct a new sequence $(\hat{\eta}_T^n, \hat{r}^n)_{n \geq 1}$ such that $F_T(\hat{\eta}_T^n) - \hat{r}^n = g^n$ and $\liminf_{n \to \infty} \|\hat{\eta}_T^n\| < \infty$. By possibly passing to a random subsequence, we can then assume that $\hat{\eta}_T^n$ converges a.s. to some $\hat{\eta}_T \in L^0(\mathcal{A}; \mathcal{H}_T)$ and therefore $\hat{r}^n$ converges to some $\hat{r} \in L^0(\mathbb{R}^d_+; \mathcal{F})$ for which $F_T(\hat{\eta}_T) - \hat{r} = g$. □

2.3 Abstract weak no-arbitrage property

In this section, we use Proposition 2.1 to provide a dual characterization of the *weak no-arbitrage* condition studied in [11] and [13], see also the references therein,

$\mathbf{NA}^w$: $\quad A_T(F) \cap L^0(\mathbb{R}^d_+; \mathcal{F}) = \{0\}$.

As observed in [17], in financial models, it corresponds to the usual no-arbitrage condition. Here, we keep the notations of [11] and [13] to enhance the difference with the notions of *strict* no-arbitrage and *robust* no-arbitrage that we shall consider in Subsection 2.4.

We denote by e_{ij} the element of $\mathbb{M}^d_+$ whose component (i,j) is equal to one and all others are equal to 0, $i, j \leq d$. In addition to **KP**, we make the following assumption on $\mathcal{A}$.

$\mathbf{H}\mathcal{A}$: 1. $F(\delta e_{ij}) = 0$ if $\delta e_{ij} \notin \mathcal{A}$, $\delta \in \{-1, 1\}$, $i, j \leq d$.
2. For η in $L^0(\mathbb{M}^d; \mathcal{F})$, $F(\eta) = \sum_{i,j \leq d} (\eta^{ij})^+ F(e_{ij}) + (\eta^{ij})^- F(-e_{ij})$.

Here, x^+ and x^- stands for the positive and negative parts of x. Condition 1. can be viewed as a convention. The reason for imposing this assumption will be clear in Section 3. In the examples of Section 3, e_{ij} (resp. $-e_{ij}$) will correspond to a transfer of units of asset i so as obtain (resp. get rid of) one unit of asset j. Since an order, η, can be viewed as a composition of single transfers of the form e_{ij} or $-e_{ij}$, condition 2. simply means that the induced changes $F_t(\eta)$ in the portfolio should correspond to the combination of the changes $F_t(e_{ij})$ and $F_t(-e_{ij})$ associated to these single transfers.

Observe from $\mathbf{HF}_1$, $\mathbf{HF}_2$ that, for all $i, j, k \leq d$,

$$F^k(e_{ji}) \leq -F^k(-e_{ji}) \text{ if } (e_{ji}, -e_{ji}) \in \mathcal{A} \times \mathcal{A}\,, \tag{2.3}$$

since $F(e_{ji} - e_{ji}) = F(0) = 0$.

We shall also assume in the sequel that

$$F_t(e_{ij}) \text{ and } F_t(-e_{ij}) \in L^1(\mathbb{R}^d; \mathcal{F}) \quad \text{for all} \quad i, j \leq d \text{ and } t \in \mathbb{T}\,. \tag{2.4}$$

Here, $L^1(\mathbb{R}^d; \mathcal{F})$ denotes the set of $\mathbb{P}$-integrable elements of $L^0(\mathbb{R}^d; \mathcal{F})$.

Remark 2.2 Observe that we can always reduce to this case by passing to the equivalent probability measure whose density with respect to $\mathbb{P}$ is defined by $H/\mathbb{E}[H]$ with $H := \exp(-\sum_{i,j\leq d}\sum_{t\in\mathbb{T}} \|F_t(e_{ij})\| + \|F_t(-e_{ij})\|)$.

Remark 2.3 *If $F \in \mathbf{F}$ satisfies $\mathbf{H}\mathcal{A}$, then it is completely characterized by the family $\{F(e_{ij}), F(-e_{ij})\}_{i,j\leq d}$.*

2.3.1 Dual characterization of $\mathbf{NA}^w$ under KP

For $Z \in L^\infty(\mathbb{R}^d; \mathcal{F})$, the set of bounded random variables in $L^0(\mathbb{R}^d; \mathcal{F})$, and $\eta \in \mathbb{L}^0(\mathcal{A}; \mathbb{H})$, we define

$$\bar{F}_t(\eta_t; Z) \;:=\; \mathbb{E}\left[Z \cdot F_t(\eta_t) \mid \mathcal{H}_t\right] \;,\quad t \in \mathbb{T}\,.$$

Here "$\cdot$" denotes the natural scalar product of $\mathbb{R}^d$. By (2.4) and $\mathbf{H}\mathcal{A}$, these conditional expectations are well defined.

We then define $\mathcal{D}(F)$ as the set of elements Z of $L^\infty(\mathbb{R}^d; \mathcal{F})$ satisfying $Z^i > 0$ for all $i \leq d$ and such that for all $\eta \in \mathbb{L}^0(\mathcal{A}; \mathbb{H})$ and $t \in \mathbb{T}$

$\mathbf{D}_1$: $\bar{F}_t(\eta_t; Z) \leq 0$.

$\mathbf{D}_2$: $F_t(\eta_t)\mathbb{1}_{\bar{F}_t(\eta_t;Z)=0} \in N^0_t(F)$.

Theorem 2.1 *Let $F \in \mathbf{F}$ be such that $\mathbf{H}\mathcal{A}$ holds. Then, $\mathcal{D}(F) \neq \emptyset \Rightarrow \mathbf{NA}^w$. If moreover $\mathbf{HN}^0$ and $\mathbf{KP}$ hold, then $\mathbf{NA}^w \Rightarrow \mathcal{D}(F) \neq \emptyset$.*

The proof will be provided in the next subsection.

In order to relate the above result to the literature, we now provide an alternative characterization of the set $\mathcal{D}(F)$. To $Z \in L^\infty((0,\infty)^d;\mathcal{F})$, we associate the $\mathbb{H}$-martingale $\bar{Z}$ defined by $\bar{Z}_t = \mathbb{E}[Z \mid \mathcal{H}_t]$, $t \in \mathbb{T}$. Then, to $F \in \mathbf{F}$ satisfying $\mathbf{H}\mathcal{A}$, we associate $\hat{F}(\cdot;Z)$ defined as the element of $\mathbf{F}$ satisfying $\mathbf{H}\mathcal{A}$ and

$$\hat{F}^k_t(\delta e_{ij};Z) \;=\; \mathbb{E}\left[Z^k F^k_t(\delta e_{ij}) \mid \mathcal{H}_t\right]/\bar{Z}^k_t \;\;,\;\; i,j,k \le d \;,\; t \in \mathbb{T} \;,\; \delta \in \{-1,1\} \;,$$

see Remark 2.3. Observe that $\bar{Z} \cdot \hat{F}(\cdot;Z) = \bar{F}(\cdot;Z)$. Given $i,j \le d$, we then introduce the sequence of random convex cones $\hat{K}^{ij}(F,Z) = (\hat{K}^{ij}_t(F,Z))_{t\in\mathbb{T}}$ defined by

$$\hat{K}^{ij}_t(F,Z)(\omega) \;=\; \mathrm{cone}\{-\hat{F}_t(e_{ij};Z)(\omega) \;,\; -\hat{F}_t(-e_{ij};Z)(\omega)\} + \mathbb{R}^d_+ \;,$$

where, for $E \subset \mathbb{R}^d$, $\mathrm{cone}\{E\}$ is the smallest closed convex cone that contains E. We also define the sequence $\hat{K}^{ij*}(F,Z) = (\hat{K}^{ij*}_t(F,Z))_{t\in\mathbb{T}}$ by

$$\hat{K}^{ij*}_t(F,Z)(\omega) \;=\; \{y \in \mathbb{R}^d \;:\; x \cdot y \ge 0, \;\text{ for all } x \in \hat{K}^{ij}_t(F,Z)(\omega)\} \;.$$

In the case of perfect information, i.e. F is $\mathbb{H}$-adapted, $\hat{K}(F,Z) := \sum_{i,j\le d} \hat{K}^{ij}(F,Z)$ coincides with the sequence of random "solvency" cones defined in [13] and [14]. The following proposition combined with Theorem 2.1 then extends the results of [13], [14] and [17] to our context, see also Remark 2.4 below.

Proposition 2.2 *Let $F \in \mathbf{F}$ be such that $\mathbf{HN}^0$ and $\mathbf{H}\mathcal{A}$ hold. Then, $\mathcal{D}(F)$ is the set of elements Z of $L^\infty((0,\infty)^d;\mathcal{F})$ such that $\bar{Z}_t \in \bigcap_{i,j\le d} \mathrm{ri}(\hat{K}^{ij*}_t(F,Z))$ a.s. for all $t \in \mathbb{T}$.*

Proof. Let $\mathcal{D}'(F)$ denote the set of elements Z of $L^\infty((0,\infty)^d;\mathcal{F})$ such that $\bar{Z}_t \in \bigcap_{i,j\le d} \mathrm{ri}(\hat{K}^{ij*}_t(F,Z))$, for all $t \in \mathbb{T}$.
1. We fix $t \in \mathbb{T}$. Since $\bar{Z} \cdot \hat{F}(\cdot;Z) = \bar{F}(\cdot;Z)$, it follows that $\bar{F}_t(\delta e_{ij};Z) \le 0$ for all $\delta \in \{-1,1\}$ is equivalent to $\bar{Z}_t \in \hat{K}^{ij*}_t(F,Z)$, for all $i,j \le d$.
2. Assume that $\mathcal{D}(F) \ne \emptyset$, fix $Z \in \mathcal{D}(F)$ and $i,j \le d$. Set $B := \{\bar{Z}_t \notin \mathrm{ri}(\hat{K}^{ij*}_t(F,Z))\}$. If $\mathbb{P}[B] > 0$, we can find some $\hat{\xi}$ in $L^0(\mathbb{R}^d;\mathcal{H}_t)$ with values in $(-\hat{K}^{ij}_t(F,Z)) \setminus \hat{K}^{ij}_t(F,Z)$ on B such that $\hat{\xi} \cdot \bar{Z}_t = 0$. Since $\hat{\xi} \in -\hat{K}^{ij}_t(F,Z)$, there is some $(\eta,r) \in L^0(\mathcal{A};\mathcal{H}_t)$ $\times L^0(\mathbb{R}^d_+;\mathcal{F})$ such that $\eta^{kl} = 0$ if $(k,l) \ne (i,j)$ and $\hat{\xi} = \hat{F}_t(\eta;Z) - r$, recall $\mathbf{HF}_2$. By $\mathbf{D}_1$, it satisfies $\bar{F}_t(\eta;Z) = 0$ and $r = 0$. Set $\xi := F_t(\eta)$. We claim that $\xi \notin N^0_t(F)$, which, in view of $\mathbf{D}_2$, leads to a contradiction. To see this observe from $\mathbf{HN}^0$ that $\hat{F}_t(\eta;Z)$ $= -\hat{F}_t(-\eta;Z)$ whenever $\xi \in N^0_t(F)$. By $\mathbf{H}\mathcal{A}$, this implies that $\hat{\xi} \in \hat{K}^{ij}_t(F,Z)$ on B, a contradiction too. Hence $\mathbb{P}[B] = 0$. This shows that $\mathcal{D}(F) \subset \mathcal{D}'(F)$.
3. Assume that $\mathcal{D}'(F) \ne \emptyset$ and fix $Z \in \mathcal{D}'(F)$. In view of 1. and $\mathbf{H}\mathcal{A}$, it remains to show that if $\xi \in N_t(F)$ is such that $\mathbb{E}[Z \cdot \xi \mid \mathcal{H}_t] = 0$, then $\xi \in N^0_t(F)$. Set $\eta \in L^0(\mathcal{A};\mathcal{H}_t)$ such that $\xi = F_t(\eta)$. Since $\bar{F}_t(\eta;Z) = 0$, it follows from $\mathbf{H}\mathcal{A}$ that

$$0 \;=\; \sum_{i,j\le d} (\eta^{ij})^+ \bar{F}_t(e_{ij};Z) + (\eta^{ij})^- \bar{F}_t(-e_{ij};Z) \;. \tag{2.5}$$

Since $\bar{Z}_t \in \bigcap_{i,j\le d} \hat{K}_t^{ij*}(F,Z)$, we deduce that

$$\eta^{ij} = 0 \quad \text{on } \{\bar{F}_t(e_{ij};Z) < 0\} \cap \{\bar{F}_t(-e_{ij};Z) < 0\} \ . \tag{2.6}$$

We claim that

$$\{\bar{F}_t(e_{ij};Z) = 0\} \ = \ \{\bar{F}_t(-e_{ij};Z) = 0\} \subset \{F_t(e_{ij}) = -F_t(-e_{ij})\} \ , \ i,j \le d \ . \tag{2.7}$$

In view of (2.6) and $\mathbf{H}\mathcal{A}$, this implies that $F_t(\eta) = -F_t(-\eta)$ and therefore $\xi \in N_t^0(F)$. It remains to prove (2.7). Fix $i, j \le d$. Since $\bar{Z}_t \in \mathrm{ri}(\hat{K}_t^{ij*}(F,Z))$, we must have $\{\bar{F}_t(e_{ij};Z) = 0\} = \{\bar{F}_t(-e_{ij};Z) = 0\} =: B_{ij}$. If $(e_{ij}, -e_{ij}) \in \mathcal{A} \times \mathcal{A}$, then (2.3) implies that $F_t(e_{ij}) + F_t(-e_{ij}) = 0$ on B_{ij}, recall that Z has a.s. positive components. If $e_{ij} \notin \mathcal{A}$, then $F_t(e_{ij}) = 0$, recall $\mathbf{H}\mathcal{A}$, and $\bar{F}_t(-e_{ij};Z) = 0$ implies $F_t(-e_{ij}) = 0$ since otherwise $\bar{Z}_t$ would not take values in $\mathrm{ri}(\bar{K}_t^{ij*}(F,Z))$ a.s. Similarly, if $-e_{ij} \notin \mathcal{A}$, then $F_t(-e_{ij}) = 0$ and $\bar{F}_t(e_{ij};Z) = 0$ implies $F_t(e_{ij}) = 0$. □

Remark 2.4 Under the assumptions of Proposition 2.2, $\bigcap_{i,j\le d} \ \mathrm{ri}(\hat{K}_t^{ij*}(F,Z))$ is a.s. non-empty whenever $\mathcal{D}(F) \neq \emptyset$. It follows that

$$\mathrm{ri}(\bigcap_{i,j\le d} \hat{K}_t^{ij*}(F,Z)) \ = \ \bigcap_{i,j\le d} \mathrm{ri}(\hat{K}_t^{ij*}(F,Z)) \ .$$

Recalling that $\hat{K}_t(F,Z) := \sum_{i,j\le d} \hat{K}_t^{ij}(F,Z)$, we then have

$$\mathrm{ri}(\hat{K}_t^*(F,Z)) \ = \ \bigcap_{i,j\le d} \mathrm{ri}(\hat{K}_t^{ij*}(F,Z)) \ ,$$

where

$$\hat{K}_t^*(F,Z)(\omega) \ = \ \{y \in \mathbb{R}^d \ : \ x \cdot y \ge 0, \ \text{ for all } x \in \hat{K}_t(F,Z)(\omega)\} \ .$$

2.3.2 Proof of Theorem 2.1

The two following Lemmas prepare for the proof of Theorem 2.1 which will be concluded at the end of this subsection.

Lemma 2.2 *Let $F \in \mathbf{F}$ be such that $\mathbf{H}\mathcal{A}$ holds. Assume that $\mathcal{D}(F) \neq \emptyset$. Then, for all $g \in A_T(F)$ and $Z \in \mathcal{D}(F)$ such that $\mathbb{E}\left[Z \cdot g \mid \mathcal{H}_T\right]^- \in L^1(\mathbb{R};\mathcal{F})$, $\mathbb{E}\left[Z \cdot g\right] \le 0$.*

Proof. We use a resursive agument as in [13]. If $g \in A_0(F)$ then $g = F_0(\eta_0) - r$ for some $\eta_0 \in L^0(\mathcal{A};\mathcal{H}_0)$ and $r \in L^0(\mathbb{R}^d_+;\mathcal{F})$. By $\mathbf{D}_1$, we have $\mathbb{E}\left[Z \cdot g \mid \mathcal{H}_0\right] \le \bar{F}_0(\eta_0;Z)$ ≤ 0. Next assume that for $g \in A_{t-1}(F)$ such that $\mathbb{E}\left[Z \cdot g \mid \mathcal{H}_{t-1}\right]^- \in L^1(\mathbb{R};\mathcal{F})$ we have $\mathbb{E}\left[Z \cdot g\right] \le 0$, for some $0 < t \le T$. Then, if $g = \sum_{s=0}^t F_s(\eta_s) - r$ for some $\eta \in L^0(\mathcal{A};\mathbb{H})$ and $r \in L^0(\mathbb{R}^d_+;\mathcal{F})$, we have $Z \cdot g \le Z \cdot \sum_{s=0}^t F_s(\eta_s)$ and, by $\mathbf{D}_1$, $\mathbb{E}\left[Z \cdot \sum_{s=0}^{t-1} F_s(\eta_s) \mid \mathcal{H}_t\right] \ge$ $-\mathbb{E}\left[Z \cdot \sum_{s=0}^{t} F_s(\eta_s) \mid \mathcal{H}_t\right]^-$. It follows that $\mathbb{E}\left[Z \cdot \sum_{s=0}^{t-1} F_s(\eta_s) \mid \mathcal{H}_{t-1}\right]^- \in L^1(\mathbb{R};\mathcal{F})$ and therefore $\mathbb{E}\left[Z \cdot \sum_{s=0}^{t-1} F_s(\eta_s)\right] \le 0$. Since by $\mathbf{D}_1$, $\bar{F}_t(\eta_t;Z) \le 0$, it follows that $\mathbb{E}\left[Z \cdot g\right] \le$ 0. Observe that we have no problem in defining the above conditional expectations thanks to (2.4) and $\mathbf{H}\mathcal{A}$. □

Lemma 2.3 *Let $F \in \mathbf{F}$ be such that $\mathbf{NA}^w$, $\mathbf{KP}$, $\mathbf{HN}^0$ and $\mathbf{H}\mathcal{A}$ hold. Then, for all $t \in \mathbb{T}$ and $\mu \in L^0(\mathcal{A};\mathcal{H}_t)$, there is $Z^\mu \in L^\infty(\mathbb{R}^d;\mathcal{F})$ with $(Z^\mu)^i > 0$ for all $i \le d$ such that*

(i) $\bar F_s(\eta_s; Z^\mu) \le 0$ *for all* $\eta \in \mathbb{L}^0(\mathcal{A};\mathbb{H})$ *and* $s \in \mathbb{T}$

(ii) $F_t(\mu)\mathbb{1}_{\bar F_t(\mu;Z^\mu)=0} \in N_t^0(F)$.

Proof. We follow the argument of Lemma 4 in [14]. Observe from $\mathbf{HF}_1$ and $\mathbf{HF}_2$ that $A_T^1(F) := A_T(F) \cap L^1(\mathbb{R}^d;\mathcal{F})$ is a convex cone which contains $-L^1(\mathbb{R}^d_+;\mathcal{F})$. Since it is closed in $L^1(\mathbb{R}^d;\mathcal{F})$, see Proposition 2.1, and satisfies $A_T^1(F) \cap L^1(\mathbb{R}^d_+;\mathcal{F}) = \{0\}$, see $\mathbf{NA}^w$, we deduce from the Hahn-Banach separation theorem together with a classical exhaustion argument, see e.g. Section 3 in [18], that there is some $Z \in L^\infty(\mathbb{R}^d;\mathcal{F})$ with $Z^i > 0$ for all $i \le d$ such that $\mathbb{E}[Z \cdot g] \le 0$ for all $g \in A_T^1(F)$. Let $\mathcal{Z}$ denote the set of such random variables Z.

1. It is clear that (i) holds for all $Z \in \mathcal{Z}$. Indeed, assume that for some $\eta \in \mathbb{L}^0(\mathcal{A};\mathbb{H})$ and $s \in \mathbb{T}$, $B := \{\bar F_s(\eta_s; Z) > 0\}$ has positive probability. Set $\tilde g := H_s F_s(\eta_s)\mathbb{1}_B$ with $H_s := \exp(-\|\eta_s\|) \in L^0((0,\infty);\mathcal{H}_s)$. By $\mathbf{HF}_1$, $H_s F_s(\eta_s)\mathbb{1}_B = F_s(H_s\eta_s\mathbb{1}_B)$ so that, by (2.4) and $\mathbf{H}\mathcal{A}$, $\tilde g \in A_T^1(F)$. Since $\mathbb{E}[Z \cdot \tilde g] > 0$, we get a contradiction to the definition of $\mathcal{Z}$.

2. By the same argument as in Lemma 4 in [14], we can find some Z^μ such that $\mathbb{P}\left[\bar F_t(\mu;Z^\mu) < 0\right] = \max_{Z\in\mathcal{Z}} \mathbb{P}\left[\bar F_t(\mu;Z) < 0\right]$. Set $B := \{\bar F_t(\mu;Z^\mu) = 0\}$ and $B_k := B \cap \{\|\mu\| \le k\}$, $k \in \mathbb{N}$. We claim that if (ii) fails for (μ, Z^μ) then $-F_t(\mu\mathbb{1}_{B_k}) \notin A_T^1(F)$ for some $k > 0$. Indeed, otherwise, for all $k > 0$, we could find some $\eta_k \in \mathbb{L}^0(\mathcal{A};\mathbb{H})$ and $r_k \in L^0(\mathbb{R}^d_+;\mathcal{F})$ such that $V_T(F(\eta_k)) = -F_t(\mu\mathbb{1}_{B_k}) + r_k \in A_T^1(F)$, so that $V_T(F(\eta_k)) + F_t(\mu\mathbb{1}_{B_k}) \in L^0(\mathbb{R}^d_+;\mathcal{F})$. By $\mathbf{KP}$, this would imply that $F_t(\mu\mathbb{1}_{B_k}) \in N_t^0(F)$, so that, by $\mathbf{HN}^0$, $F_t(\mu\mathbb{1}_{B_k}) = -F_t(-\mu\mathbb{1}_{B_k})$. Sending $k \to \infty$, we would then get $F_t(\mu\mathbb{1}_B) = -F_t(-\mu\mathbb{1}_B)$, showing that $F_t(\mu\mathbb{1}_B) \in -N_t(F)$, a contradiction. Hence, if (ii) fails $-F_t(\mu\mathbb{1}_{B_k}) \notin A_T^1(F)$ for some $k > 0$. Repeating the argument of 1., we can then find some $Z \in L^\infty(\mathbb{R}^d_+;\mathcal{F})$ such that $\mathbb{E}[Z \cdot g] \le 0 < \mathbb{E}[Z \cdot (-F_t(\mu\mathbb{1}_{B_k}))]$ for all $g \in A_T^1(F)$. Taking $\tilde Z = Z + Z^\mu$, we obtain $\mathbb{P}\left[\bar F_t(\mu;\tilde Z) < 0\right] > \mathbb{P}\left[\bar F_t(\mu;Z^\mu) < 0\right]$, a contradiction to the definition of Z^μ. This shows that (ii) must hold. □

Proof of Theorem 2.1. 1. The first implication follows from Lemma 2.2 since the elements of $\mathcal{D}(F)$ have a.s. positive entries.

2. We now prove the converse implication. Let $Z_+^{ij,t}$ (resp. $Z_-^{ij,t}$) be an element of $L^\infty((0,\infty)^d;\mathcal{F})$ such that (i) and (ii) of Lemma 2.3 hold for the process $(e_{ij}\mathbb{1}_{s=t})_{s\in\mathbb{T}}$ (resp. $(-e_{ij}\mathbb{1}_{s=t})_{s\in\mathbb{T}}$), $i,j \le d$ and $t \in \mathbb{T}$. We claim that $\hat Z := \sum_{t\in\mathbb{T}}\sum_{i,j\le d} Z_+^{ij,t} + Z_-^{ij,t}$ belongs to $\mathcal{D}(F)$. Clearly, it satisfies $\mathbf{D}_1$. Fix $\eta \in \mathbb{L}^0(\mathcal{A};\mathcal{H}_t)$ for some $t \in \mathbb{T}$, and recall from $\mathbf{H}\mathcal{A}$ that

$$F_t(\eta) \quad = \quad \sum_{i,j\le d} (\eta^{ij})^+ F_t(e_{ij}) + (\eta^{ij})^- F_t(-e_{ij}) \,. \tag{2.8}$$

Set $B := \{\bar{F}_t(\eta; \hat{Z}) = 0\}$. From the definition of $(Z_+^{ij,t}, Z_-^{ij,t})_{i,j,t}$, we deduce that $(\eta^{ij})^+ F_t(e_{ij})\mathbb{1}_B$ and $(\eta^{ij})^- F_t(-e_{ij})\mathbb{1}_B$ belongs to $N_t^0(F)$ for all $i, j \leq d$. By $\mathbf{HN}^0$, $\mathbf{H}\mathcal{A}$ and (2.8), we then deduce that

$$
\begin{aligned}
-F_t(\eta \mathbb{1}_B) &= \sum_{i,j \leq d} -(\eta^{ij})^+ F_t(e_{ij}) - (\eta^{ij})^- F_t(-e_{ij}) \\
&= \sum_{i,j \leq d} (\eta^{ij})^+ F_t(-e_{ij}) + (\eta^{ij})^- F_t(e_{ij}) \\
&= F_t(-\eta \mathbb{1}_B)
\end{aligned}
$$

so that $F_t(\eta \mathbb{1}_B) = -F_t(-\eta \mathbb{1}_B) \in -N_t(F)$ and therefore $F_t(\eta \mathbb{1}_B) \in N_t^0(F)$. Hence, $\hat{Z}$ satisfies $\mathbf{D}_2$. □

2.4 Strict and robust no-arbitrage conditions

In this section, we study the other no-arbitrage conditions considered in [13], [14] and [17].

Following [13], we say that $F \in \mathbf{F}$ satisfies the *strict no-arbitrage condition* if one has

$$
\mathbf{NA}^s : \quad A_t(F) \cap (-N_t(F) + L^0(\mathbb{R}_+^d; \mathcal{F})) \subset N_t^0(F) \quad \text{for all } \; t \in \mathbb{T} \,,
$$

and that the model has "efficient frictions" if

$$
\mathbf{EF} : \quad N_t^0(F) = \{0\} \quad \text{for all } \; t \in \mathbb{T} \,.
$$

As in [17], we also define a *robust* version of the no-arbitrage property. We say that $F \in \mathbf{F}$ satisfies the *robust no-arbitrage condition*, $\mathbf{NA}^r$, if there is some sequence $G \in \mathbf{F}$ such that for all $\eta \in \mathbb{L}^0(\mathcal{A}; \mathbb{H})$, $t \in \mathbb{T}$ and $i \leq d$:

1. $G_t^i(\eta_t) \geq F_t^i(\eta_t)$
2. $F_t(\eta_t) \notin N_t^0(F) \;\Rightarrow\; \{\exists\, k \leq d \text{ such that } G_t^k(\eta_t) > F_t^k(\eta_t)\} \neq \emptyset$
3. $\mathbf{NA}^w$ holds for G.

In financial models, the last condition can be interpreted as the existence of a model with slightly lower transaction costs (for those that are not already equal to 0) in which the *weak no-arbitrage condition* still holds, see [17].

In this section, we first show that these properties imply the condition **KP** used above. We will then be able to use Theorem 2.1 to provide a dual characterization of the absence of arbitrage opportunities in the spirit of [13], [14] and [17], see Theorem 2.2 below.

Lemma 2.4 *Let $F \in \mathbf{F}$ be such that one of the above conditions holds:*
(i) $\mathbf{NA}^r$
(ii) $\mathbf{NA}^s$ *and* **EF**.
Then, **KP** *holds.*

Proof. Set ξ and $\tilde{\xi}$ in $N(F)$ such that $V_T(\xi) + V_T(\tilde{\xi}) \in L^0(\mathbb{R}^d_+;\mathcal{F})$. Let η and $\tilde{\eta}$ be elements of $\mathbb{L}^0(\mathcal{A};\mathbb{H})$ such that $\xi = F(\eta)$ and $\tilde{\xi} = F(\tilde{\eta})$, set $\bar{\eta} := \eta + \tilde{\eta}$ and $\bar{\xi} := F(\bar{\eta})$.
1. We start with $\mathbf{NA}^r$. Let G be as in the definition of $\mathbf{NA}^r$ and define $\bar{\xi}' := G(\bar{\eta})$. By 1. and 2. of $\mathbf{NA}^r$, if for some $t \in \mathbb{T}$ $F_t(\eta) \notin N^0_t(F)$, then we can find $i \leq d$ and $B \in \mathcal{F}$ with positive measure such that $V^i_T(G(\bar{\eta})) > V^i_T(\bar{\xi})$ on B. By 1. of $\mathbf{NA}^r$ and $\mathbf{HF}_2$, we then have $V_T(\bar{\xi}') - V_T(\xi) - V_T(\tilde{\xi}) \in L^0(\mathbb{R}^d_+;\mathcal{F}) \setminus \{0\}$. Since $V_T(\xi) + V_T(\tilde{\xi}) \in L^0(\mathbb{R}^d_+;\mathcal{F})$, this leads to a contradiction to the fact that $\mathbf{NA}^w$ holds for G. Hence, $F(\eta) \in N^0(F)$ and we must have $V_T(\bar{\xi}') = 0$ so that $V_T(\xi) + V_T(\tilde{\xi}) = 0$.
2. We now assume that $\mathbf{NA}^s$ and $\mathbf{EF}$ hold. Assume that, for some $t \in \mathbb{T}$, $\xi_t \notin N^0_t(F)$ or $\tilde{\xi}_t \notin N^0_t(F)$ and set $t^* := \max\{t \in \mathbb{T} \ : \ \xi_t \notin N^0_t(F) \text{ or } \tilde{\xi}_t \notin N^0_t(F)\}$. Then, by $\mathbf{EF}$ and $\mathbf{HF}_2$, $V_{t^*-1}(\bar{\xi}) = \sum_{s=0}^{t^*-1} \xi_s + \tilde{\xi}_s + r = -\xi_{t^*} - \tilde{\xi}_{t^*} + r = -\bar{\xi}_{t^*} + r + r'$ for some r, r' in $L^0(\mathbb{R}^d_+;\mathcal{F})$. This shows that $V_{t^*-1}(\bar{\xi}) \in (-N_{t^*}(F) + L^0(\mathbb{R}^d_+;\mathcal{F})) \cap A_{t^*}(F)$. By $\mathbf{NA}^s$ and $\mathbf{EF}$, we must have $\bar{\xi}_{t^*} \in N^0_{t^*}(F) = \{0\}$ and $r = r' = 0$. Hence, $\xi_{t^*} = -\tilde{\xi}_{t^*} \in N^0_{t^*}(F)$, thus providing a contradiction to the definition of t^*. □

Observe that $\mathbf{NA}^r$ implies $\mathbf{NA}^w$ and that $\mathbf{NA}^s$ also implies $\mathbf{NA}^w$ whenever $N^0_T = \{0\}$. In view of Lemma 2.4, we can then apply Proposition 2.1 and Theorem 2.1 to deduce that, under $\mathbf{HN}^0$ and $\mathbf{H}\mathcal{A}$, $\mathbf{NA}^r$ and ($\mathbf{NA}^s$ and $\mathbf{EF}$) both imply that $A_T(F)$ is closed in probability and that $\mathcal{D}(F)$ is non-empty. Conversely, if $\mathcal{D}(F) \neq \emptyset$, on can show that $\mathbf{NA}^s$ and $\mathbf{NA}^r$ hold.

Theorem 2.2 *Let $F \in \mathbf{F}$ be such that $\mathbf{HN}^0$ and $\mathbf{H}\mathcal{A}$ hold. Then,*
(i) *If either $\mathbf{NA}^r$ or ($\mathbf{NA}^s$ and $\mathbf{EF}$) hold, then $\mathcal{D}(F) \neq \emptyset$ and $A_T(F)$ is closed in probability.*
(ii) *If $\mathcal{D}(F) \neq \emptyset$ then $\mathbf{NA}^s$ and $\mathbf{NA}^r$ hold.*

Proof. 1. Since $\mathbf{NA}^r$ implies $\mathbf{NA}^w$ and $\mathbf{NA}^s$ also implies $\mathbf{NA}^w$ whenever $\mathbf{EF}$ holds, combining Lemma 2.4 with Proposition 2.1 and Theorem 2.1 leads to (i) . To show that $\mathbf{NA}^s$ holds under $\mathcal{D}(F) \neq \emptyset$, we set $V_t \in A_t(F)$ such that $V_t = -F_t(\tilde{\eta}) + r$ for some $\tilde{\eta} \in L^0(\mathcal{A};\mathcal{H}_t)$ and $r \in L^0(\mathbb{R}^d_+;\mathcal{F})$. By $\mathbf{D}_1$, $\mathbb{E}[Z \cdot V_t \mid \mathcal{H}_t] \geq -\bar{F}_t(\tilde{\eta};Z) \geq 0$, and therefore, by Lemma 2.2, we must have $\mathbb{E}[Z \cdot V_t \mid \mathcal{H}_t] = 0$, $r = 0$ and $\bar{F}_t(\tilde{\eta};Z) = 0$ for all $Z \in \mathcal{D}(F)$. Then $\mathbf{D}_2$ implies that $F_t(\tilde{\eta}) \in N^0_t(F)$.
2. We now prove that $\mathcal{D}(F) \neq \emptyset$ implies $\mathbf{NA}^r$. To avoid unnecessary complications, we first consider the case where $(e_{ji}, -e_{ji}) \in \mathcal{A} \times \mathcal{A}$ for all $i, j \leq d$. We shall explain in 2.d. how to adapt our arguments to the general case.
Fix $Z \in \mathcal{D}(F)$ and consider the random variables

$$\delta^+_{ji,t} \ := \ -\bar{F}_t(e_{ji};Z) \quad \text{and} \quad \delta^-_{ji,t} \ := \ -\bar{F}_t(-e_{ji};Z) \ \ , \ \ i,j \leq d \ , \ t \in \mathbb{T} \ .$$

It follows from $\mathbf{D}_1$ that

$$\delta^+_{ji,t} \geq 0 \quad \text{and} \quad \delta^-_{ji,t} \geq 0 \ \ , \ \ i,j \leq d \ , \ t \in \mathbb{T} \ . \tag{2.9}$$

We claim that, for all $i, j \le d$ and $t \in \mathbb{T}$,

$$\delta^{+}_{ji,t} > 0 \ \text{ and } \ \delta^{-}_{ji,t} > 0 \ \text{ on } \{\bar{F}_t(e_{ji}; Z) < 0\} = \{\bar{F}_t(-e_{ji}; Z) < 0\} \,. \tag{2.10}$$

Indeed, by construction, we have $\delta^{+}_{ji,t} > 0$ on $\{\bar{F}_t(e_{ji}; Z) < 0\}$ and $\delta^{-}_{ji,t} > 0$ on $\{\bar{F}_t(-e_{ji}; Z) < 0\}$. Now, set $B_+ := \{\bar{F}_t(e_{ji}; Z) = 0\}$ and $B_- := \{\bar{F}_t(-e_{ji}; Z) = 0\}$. From $\mathbf{D}_2$ and $\mathbf{HN}^0$, we deduce that $F_t(e_{ji}\mathbb{1}_{B_+}) = -F_t(-e_{ji}\mathbb{1}_{B_+})$ so that $\bar{F}_t(-e_{ji}; Z) = 0$ on B_+. This shows that $B_+ \subset B_-$. Similarly, we can show the converse inclusion, which implies (2.10).

We can now construct G. For all $i, j, k \le d$, we set

$$\begin{aligned} G^k(e_{ji}) &= \left(F^k(e_{ji}) + \delta^{+}_{ji,t}/(d\,\bar{Z}^k_t)\right) \wedge (-F^k(-e_{ji})) \\ G^k(-e_{ji}) &= \left(F^k(-e_{ji}) + \delta^{-}_{ji,t}/(d\,\bar{Z}^k_t)\right) \wedge (-G^k(e_{ji})) \,. \end{aligned} \tag{2.11}$$

For $x \in \mathbb{M}^d$, we then set

$$G(x) = \sum_{i,j \le d} (x^{ji})^+ G(e_{ji}) + (x^{ji})^- G(-e_{ji}) \,.$$

It satisfies $\mathbf{HF}_1$. By (2.3), it also satisfies the condition 1. of $\mathbf{NA}^r$, recall (2.9). It remains to check that $\mathbf{HF}_2$, 2. and 3. of $\mathbf{NA}^r$ hold.

2.a. We first check $\mathbf{HF}_2$. We fix $i, j, k \le d$, $\alpha \ge \beta \ge 0$. Then, $G^k(\alpha e_{ji} - \beta e_{ji}) = (\alpha - \beta) G^k(e_{ji})$. By (2.11), it follows that $G^k(\alpha e_{ji} - \beta e_{ji}) \ge \alpha G^k(e_{ji}) + \beta G^k(-e_{ji})$. In the case where $\beta \ge \alpha \ge 0$, we obtain the same result. Since G satisfies $\mathbf{H}\mathcal{A}$, this shows that it also satisfies $\mathbf{HF}_2$.

2.b. We now check 2. of $\mathbf{NA}^r$. Set $\eta \in L^0(\mathcal{A}; \mathcal{H}_t)$ and $t \in \mathbb{T}$ such that $F_t(\eta) \notin N^0_t(F)$. We must show that, with positive probability, we can find $k \le d$ such that $G^k_t(\eta) > F^k_t(\eta)$. First observe that we cannot have $\{\eta^{ji} \neq 0\} \subset \{F_t(e_{ji}) = -F_t(-e_{ji})\}$ for all $i, j \le d$ since this would imply that $F_t(\eta) \in N^0_t(F)$. Hence, there is (i, j) and $k \le d$ such that $B := \{\eta^{ji} \neq 0\} \cap \{F^k_t(e_{ji}) < -F^k_t(-e_{ji})\} \neq \emptyset$, recall (2.3). Since, by $\mathbf{D}_1$ and (2.3), $\{F^k_t(e_{ji}) < -F^k_t(-e_{ji})\} \subset \{\bar{F}_t(e_{ji}; Z) < 0\} \cup \{\bar{F}_t(-e_{ji}; Z) < 0\}$, (2.10) and (2.11) imply that $G^k_t(\eta) > F^k_t(\eta)$ on B.

2.c. To check 3. of $\mathbf{NA}^r$, it suffices to observe that, for $\eta \in \mathbb{L}^0(\mathcal{A}; \mathbb{H})$ and $Z \in \mathcal{D}(F)$, we have $\bar{G}(\eta; Z) \le 0$. Since Z has a.s. positive entries, the same arguments as in Lemma 2.2 imply $\mathbf{NA}^w$ for G.

2.d. We now explain how to consider the case where some e_{ji} or $-e_{ji}$ do not belong to $\mathcal{A}$. We assume that, for some (i, j), e_{ji} or $-e_{ji} \in \mathcal{A}$, otherwise there is nothing to prove. We keep the definition of G as above except that in the right hand-sides of (2.11), we replace $-F^k(-e_{lm})$ by $+\infty$ if $-e_{lm} \notin \mathcal{A}$ and $-G^k(e_{lm})$ by $+\infty$ if $e_{lm} \notin \mathcal{A}$. Using the convention 1. of $\mathbf{H}\mathcal{A}$, we see that G satisfies $\mathbf{HF}_1$ and 1. of $\mathbf{NA}^r$. The arguments of 2.a. and 2.c. still hold, so that it also satisfies $\mathbf{HF}_2$ and 3. of $\mathbf{NA}^r$. To obtain 2. of $\mathbf{NA}^r$, we just recall that $F^k_t(e_{lm}) < 0$, for some $k \le d$, implies $\bar{F}_t(e_{lm}; Z) < 0$ whenever $-e_{lm} \notin \mathcal{A}$, see $\mathbf{D}_2$, $\mathbf{HN}^0$ and recall 1. of $\mathbf{H}\mathcal{A}$. With this in mind, adapting the arguments of 2.b. is straightforward. □

3 Applications to financial markets with proportional transaction costs

In this section, we apply the above results to three examples of discrete time financial markets with proportional transaction costs. The first one corresponds to a "security market" where it is possible to make transactions only between a "non-risky asset" and some "risky" ones, direct transactions between the "risky assets" being prohibited. The two other ones correspond to "currency markets" where transactions between all assets (interpreted as currencies) are possible. The information of the financial agent is modeled by the filtration $\mathbb{H}$ and a strategy is a process $\eta \in \mathbb{L}^0(\mathcal{A}; \mathbb{H})$.

3.1 Security market

We take the first asset as a numéraire and consider an $\mathbb{M}^d_+$-valued process π such that $\pi^{1i} \geq \pi^{i1} > 0$ for all $i, j \leq d$ and $\pi^{ii} = 1$ for all $i \leq d$. Here, π^{i1} must be interpreted as the number of physical units of asset 1 one receives when selling one unit of i, and π^{1i} as the number of units of asset 1 one pays to buy one unit of i. The condition $\pi^{1i}_t \geq \pi^{i1}_t$ is natural since otherwise their would be trivial arbitrages. The case $\pi^{1i}_t = \pi^{i1}_t$ (resp. $\pi^{1i}_t > \pi^{i1}_t$) corresponds to the situation with no-friction (resp. with frictions) between the assets i and 1.
We construct the sequence of random maps F as follows. To $\rho \in \mathbb{M}^d_+$ such that $\rho^{1i} \geq \rho^{i1} > 0$, we associate the map $f(\cdot; \rho)$ from $\mathbb{M}^d$ into $\mathbb{R}^d$ defined by

$$f^1(a;\rho) \; = \; \sum_{i \leq d} a^{1i} \left(\rho^{i1} \mathbb{1}_{a^{1i}>0} + \rho^{1i} \mathbb{1}_{a^{1i}<0} \right) \quad \text{and} \quad f^i(a;\rho) \; = \; -a^{1i} \quad \text{for } i > 1 \; .$$

Then, we set $F_t(\cdot) = f(\cdot; \pi_t)$ for $t \in \mathbb{T}$. For the sake of simplicity, we take $\mathcal{A} = \mathbb{M}^d$. Observe that $\mathbf{H}\mathcal{A}$ and $\mathbf{HF}_1$ trivially holds, and that the condition $\pi^{1i} \geq \pi^{i1}$, $i \leq d$, implies $\mathbf{HF}_2$.

If positive, the quantity η^{1i}_t corresponds to the number of units of asset i which are sold in exchange of $\eta^{1i}_t \pi^{i1}_t$ units of asset 1. Otherwise $|\eta^{1i}_t|$ corresponds to the number of units of asset i which are obtained by converting $|\eta^{1i}_t \pi^{1i}_t|$ units of asset 1. The other components of η play no role in this model.

In order to apply the result of the previous section, we first check that $\mathbf{HN}^0$ holds in this model.

Lemma 3.1 *Let F be defined as above, then $\mathbf{HN}^0$ holds.*

Proof. Fix $t \in \mathbb{T}$ and $\eta \in L^0(\mathbb{M}^d; \mathcal{H}_t)$ such that $F_t(\eta) \in N^0_t(F)$. We have to show that $F_t(-\eta) = -F_t(\eta)$. By definition, there is $\tilde{\eta} \in L^0(\mathbb{M}^d; \mathcal{H}_t)$ such that $F_t(\eta) = -F_t(\tilde{\eta})$.

Define $S \in L^0((0,\infty)^d;\mathcal{F})$ by $S^i = (\pi^{1i}_t + \pi^{i1}_t)/2$, $i \le d$. Recalling that $\pi^{11} = 1$, direct computation shows that

$$
\begin{aligned}
0 \;=\; S\cdot(F_t(\eta)+F_t(\tilde\eta)) \;&=\; \sum_{i=1}^{d} \eta^{1i}\left(-(\pi^{1i}_t+\pi^{i1}_t)/2 + \pi^{i1}_t \mathbb{I}_{\eta^{1i}>0} + \pi^{1i}_t \mathbb{I}_{\eta^{1i}<0}\right) \\
&+\; \sum_{i=1}^{d} \tilde\eta^{1i}\left(-(\pi^{1i}_t+\pi^{i1}_t)/2 + \pi^{i1}_t \mathbb{I}_{\tilde\eta^{1i}>0} + \pi^{1i}_t \mathbb{I}_{\tilde\eta^{1i}<0}\right) \\
&=\; \sum_{i=1}^{d} \left(|\eta^{1i}| + |\tilde\eta^{1i}|\right)\left(\pi^{i1}_t - \pi^{1i}_t\right)/2\ .
\end{aligned}
$$

Since $\pi^{1i}_t \ge \pi^{i1}_t$ for all $i,j \le d$, this shows that η^{1i} is equal to 0 on $\{\pi^{1i}_t - \pi^{i1}_t > 0\}$ and therefore $F_t(-\eta) = -F_t(\eta)$. □

Then, it follows from Theorem 2.2 that $\mathbf{NA}^r \Leftrightarrow \mathcal{D}(F) \neq \emptyset \Rightarrow \mathbf{NA}^s$ and that the last implication is an equivalence if **EF** holds. We then assume that $\mathbf{NA}^r$ or ($\mathbf{NA}^s$ and **EF**) hold, fix $Z \in \mathcal{D}(F)$, and define the process $\bar\pi$ by

$$
\bar\pi^{ij}_t := \mathbb{E}\left[Z^1 \pi^{ij}_t \mid \mathcal{H}_t\right]/\bar Z^1_t \ \ ,\ \ i,j \le d\ .
$$

With this notation, one easily checks that $\bar Z_t \in \mathrm{ri}(\hat K^{1i*}_t(Z,F))$ if and only if

$$
\bar Z^1_t \bar\pi^{i1}_t \le \bar Z^i_t \le \bar Z^1_t \bar\pi^{1i}_t \ \ ,
$$

with strict inequalities on $\{\bar\pi^{1i}_t > \bar\pi^{i1}_t\}$.

Let $\mathbb{Q}$ be the equivalent probability measure defined by $d\mathbb{Q}/d\mathbb{P} = Z^1/\mathbb{E}[Z^1]$. Then, $\bar\pi$ is the optional projection under $\mathbb{Q}$ of π on $\mathbb{H}$, i.e. $\bar\pi_t = \mathbb{E}^{\mathbb{Q}}[\pi_t \mid \mathcal{H}_t]$, and there is a $(\mathbb{Q},\mathbb{H})$-martingale $\bar Z/\bar Z^1$ such that each component i evolves in the relative interior of the "estimated" bid-ask spread $[\bar\pi^{i1}_t, \bar\pi^{1i}_t]$. This extends the discrete-time version of the result of [9].
In the "no frictions" case, i.e. $\pi^{i1} = \pi^{1i}$, then $\bar Z^1_t \bar\pi^{i1}_t = \bar Z^i_t = \bar Z^1_t \bar\pi^{1i}_t$ and we deduce that there is an equivalent probability measure under which the optional projection $\bar\pi$ of the discounted price processes π on $\mathbb{H}$ are $(\mathbb{Q},\mathbb{H})$-martingales. This is the result of [12].

3.2 Currency market #1

We now consider a $(0,\infty)^d$-valued process S which models the price of the different currencies, before transaction costs. Then $\tau^{ji}_t = S^i_t/S^j_t$ is the number of units of asset j that one can exchange at time t against one unit of asset i, before to pay the transaction costs. Transaction costs are modeled by a process λ with values in $\mathbb{M}^d_+$, i.e. λ^{ji}_t is the proportional costs to pay in units of j for an exchange at time t between j and i.

To construct the sequence of random maps F, we first define the maps $f(\cdot;\rho,\ell)$ from $\mathbb{M}^d$ into $\mathbb{R}^d$ by

$$f^i(a;\rho,\ell) \;=\; \sum_{j=1}^d a^{ji}\left(1+\ell^{ij}\mathbb{1}_{a^{ji}<0}\right) - a^{ij}\rho^{ij}\left(1+\ell^{ij}\mathbb{1}_{a^{ij}\geq 0}\right)\;,\;\rho,\;\ell\in\mathbb{M}^d_+\;.$$

Then, we set $F_t(\cdot)=f(\cdot;\tau_t,\lambda_t)$, $t\in\mathbb{T}$. For $\mathcal{A}=\mathbb{M}^d$, this corresponds to the model (1.2) described in the introduction. Clearly $\mathbf{H}\mathcal{A}$, $\mathbf{HF}_1$ and $\mathbf{HF}_2$ hold.
The quantity η^{ij}_t corresponds to number of units of asset j which are obtained by converting units of asset i. For such an exchange, the transaction costs are paid in units of asset i.

Here again, we need to check that $\mathbf{HN}^0$ holds in this model. For sake of simplicity, we assume that $\{\lambda^{ij}_t>0\}=\{\lambda^{ji}_t>0\}$ for all $i,j\leq d$ and $t\in\mathbb{T}$.

Lemma 3.2 *Let F be defined as above, then $\mathbf{HN}^0$ holds.*

Proof. Fix $t\in\mathbb{T}$ and $\eta\in L^0(\mathbb{M}^d;\mathcal{H}_t)$ such that $F_t(\eta)\in N^0_t(F)$. We have to show that $F_t(-\eta)=-F_t(\eta)$. By definition, there is $\tilde\eta\in L^0(\mathbb{M}^d;\mathcal{H}_t)$ such that $F_t(\eta)=-F_t(\tilde\eta)$. Since $F_t(\eta)+F_t(\tilde\eta)=0$, direct computation shows that

$$\begin{aligned} 0 \;=\; S_t\cdot(F_t(\eta)+F_t(\tilde\eta)) \;&=\; -\sum_{i,j=1}^d |\eta^{ij}|S^j_t\left(\lambda^{ji}_t\mathbb{1}_{\eta^{ij}<0}+\lambda^{ij}_t\mathbb{1}_{\eta^{ij}>0}\right)\\ &\quad -\sum_{i,j=1}^d |\tilde\eta^{ij}|S^j_t\left(\lambda^{ji}_t\mathbb{1}_{\tilde\eta^{ij}<0}+\lambda^{ij}_t\mathbb{1}_{\tilde\eta^{ij}>0}\right)\;. \end{aligned}$$

This shows that $(\eta^{ij})^+=(\eta^{ij})^-=0$ on $\{\lambda^{ij}_t>0\}=\{\lambda^{ji}_t>0\}$ and therefore $F_t(-\eta)=-F_t(\eta)$. □

It then follows from Theorem 2.2 that $\mathbf{NA}^r\Leftrightarrow\mathcal{D}(F)\neq\emptyset\Rightarrow\mathbf{NA}^s$ and that the last implication is an equivalence if $\mathbf{EF}$ holds. We then assume that $\mathbf{NA}^r$ or ($\mathbf{NA}^s$ and $\mathbf{EF}$) hold, fix $Z\in\mathcal{D}(F)$, and define the processes $\bar\tau$ and $\bar\lambda$ by

$$\bar\tau^{ij}_t := \mathbb{E}\left[Z^i\tau^{ij}_t\left(1+\lambda^{ij}_t\right)\;|\;\mathcal{H}_t\right]/(\bar Z^i_t(1+\bar\lambda^{ij}_t))\quad\text{and}\quad\bar\lambda^{ij}_t := \mathbb{E}\left[Z^i\lambda^{ij}_t\;|\;\mathcal{H}_t\right]/\bar Z^i_t\;.$$

With this notation, one easily checks that $\bar Z_t\in\mathrm{ri}(\hat K^{ij*}_t(Z,F))$ if and only if

$$\bar Z^j_t\bar\tau^{ji}_t/(1+\bar\lambda^{ij}_t)\leq\bar Z^i_t\leq\bar Z^j_t\bar\tau^{ji}_t(1+\bar\lambda^{ji}_t)\;\;,$$

with strict inequalities on $\{\bar\tau^{ji}_t(1+\bar\lambda^{ji}_t)>\bar\tau^{ji}_t/(1+\bar\lambda^{ij}_t)\}$.

3.3 Currency market #2

The model (1.1) discussed in the introduction corresponds to the one presented in the previous subsection with f defined by

$$f^i(a;\rho,\ell) \;=\; \sum_{j=1}^{d} a^{ji}\mathbb{1}_{a^{ji}>0} - a^{ij}\rho^{ij}\left(1+\ell^{ij}\right)\mathbb{1}_{a^{ij}>0}\;,\;\; \rho,\;\ell\in\mathbb{M}^d_+\;,$$

i.e. F is defined by $F_t(\cdot)=f(\cdot;\tau_t,\lambda_t)$, $t\in\mathbb{T}$.

For $\mathcal{A}=\mathbb{M}^d_+$, the conditions $\mathbf{HF}_1$, $\mathbf{HF}_2$ and $\mathbf{H}\mathcal{A}$ hold (this is a case where $-e_{ji}\notin\mathcal{A}$). However, by construction, $\mathbf{HN}^0$ does not hold except when $N^0(F)=\{0\}$. As in perfect information models, this is the case if $\lambda^{ij}+\lambda^{ji}>0$ for all $i,j\le d$.

Lemma 3.3 *Fix $t\in\mathbb{T}$ and assume that for all $B\in\mathcal{H}_t$ there is $B'\subset B$ with positive probability such that, for all $i,j,k\le d$, $(1+\lambda^{ij}_t)\le(1+\lambda^{ik}_t)(1+\lambda^{kj}_t)$ and $\lambda^{ij}_t+\lambda^{ji}_t>0$ on B'. Then $N^0_t(F)=\{0\}$.*

Proof. Fix $\eta\in L^0(\mathbb{M}^d_+;\mathcal{H}_t)$ and set $B:=\{\eta\neq 0\}$. Under the above conditions, on easily checks that the random cone

$$K_t \;=\; \{x\in\mathbb{R}^d\;:\;a\in\mathbb{M}^d_+,\;x+\sum_{j\le d}a^{ji}-a^{ij}\tau^{ij}_t\left(1+\lambda^{ij}_t\right)\ge 0\;,\;\forall\, i\le d\}$$

satisfies $K_t\cap(-K_t)=\{0\}$ on B', see e.g. [3] or [13]. Since $N^0_t(F)\subset\{\xi\in L^0(\mathbb{R}^d;\mathcal{F})\;:\;\xi\in K_t\cap(-K_t)\}$, this shows that $F_t(\eta)=0$. □

Remark 3.1 The condition $\mathbb{P}\left[(1+\lambda^{ij}_t)\le(1+\lambda^{ik}_t)(1+\lambda^{kj}_t)\mid\mathcal{H}_t\right]>0$ is natural since otherwise it would be a.s. cheaper to transfer money from i to j by passing through k than directly. In this case, any "optimal" strategy would induce an effective transaction cost corresponding to $\tilde\lambda^{ij}_t:=(1+\lambda^{ik}_t)(1+\lambda^{kj}_t)-1$.

As argued in the introduction, if τ or λ are not $\mathbb{H}$-adapted, transactions may be non-reversible even when transaction costs are equal to zero.

Lemma 3.4 *Assume that for some $t\in\mathbb{T}$ and $i\le d$, $\tau^{ij}_t(1+\lambda^{ij}_t)$ is not $\mathcal{H}_t$-measurable for all $j\le d$. Then, for all $\eta\in L^0(\mathbb{M}^d_+;\mathcal{H}_t)$, $F_t(\eta)\in N^0_t(F)$ implies $\sum_{j\le d}\eta^{ji}+\eta^{ij}=0$.*

Proof. Fix $\eta\in L^0(\mathbb{M}^d_+;\mathcal{H}_t)$ such that $F_t(\eta)\in N^0_t(F)$. By definition, there is $\tilde\eta\in L^0(\mathbb{M}^d_+;\mathcal{H}_t)$ such that $F_t(\eta)=-F_t(\tilde\eta)$. Hence,

$$\sum_{j\le d}\eta^{ji}+\tilde\eta^{ji} \;=\; \sum_{j\le d}(\eta^{ij}+\tilde\eta^{ij})\tau^{ij}_t\;(1+\lambda^{ij}_t)\;.$$

If $\sum_{j\le d}(\eta^{ij}+\tilde\eta^{ij})\neq 0$, then the left hand-side term is $\mathcal{H}_t$-measurable while the right hand-side is not. It follows that both terms must be equal to 0, so that $\sum_{j\le d}\eta^{ji}+\eta^{ij}=0$.

□

Since the conditions $\mathbf{HF}_1$ and $\mathbf{HF}_2$ hold, one can argue as in the above subsection to obtain that, if $N^0(F) = \{0\}$, then $\mathbf{NA}^r$ and $\mathbf{NA}^s$ are equivalent to the existence of some $Z \in \mathcal{D}(F)$ which must satisfy

$$\bar{Z}_t^i < \bar{Z}_t^j \, \bar{\tau}_t^{ji} \, (1 + \bar{\lambda}_t^{ji}) \quad , \quad i, j \leq d \; , \; t \in \mathbb{T} \; .$$

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