Explicit characterization of the super-replication strategy in financial markets with partial transaction costs

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Abstract

We consider a multivariate financial market with transaction costs and study the problem of finding the minimal initial capital needed to hedge, without risk, European-type contingent claims. The model is similar to the one considered in Bouchard and Touzi (2000), except that some of the assets can be exchanged freely, i.e. without paying transaction costs. In this context, we generalize the result of the above paper and prove that the value of this stochastic control problem is given by the cost of the cheapest hedging strategy in which the number of non-freely exchangeable assets is kept constant over time.

Kew words: Non efficient transaction costs, hedging options, viscosity solutions.
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1 Introduction

Since the 90's, there has been many papers devoted to the proof of the conjecture of Davis and Clark (1994) : in the context of the Black and Scholes model with proportional transaction costs, the cheapest super-hedging strategy for a European call option is just the price (up to initial transaction costs) of the underlying asset. The first proofs of this result were obtained, independently, by Soner, Shreve and Cvitanić (1995) and Levental and Skorohod (1995). In a one-dimensional Markov diffusion model, the result was extended by Cvitanić, Pham and Touzi (1999) for general contingent claims. Their approach relies on the dual formulation of the super-replication cost (see Jouini and Kallal 1995 and Cvitanić and Karatzas 1996).

The multivariate case was then considered by Bouchard and Touzi (2000). In contrast to Cvitanić, Pham and Touzi (1999), they did not use the dual formulation but introduced a family of fictitious markets without transaction costs but with modified price processes evolving in the bid-ask spreads of the original market. Then, they defined the associated super-hedging problems and showed that they provide lower bounds for the original one. By means of a direct dynamic programming principle for stochastic targets problems, see e.g. Soner and Touzi (2002), they provided a PDE characterization for the upper bound of these auxiliary super-hedging prices. Using similar arguments as in Cvitanić, Pham and Touzi (1999), they were then able to show that the associated value function is concave in space and non-increasing in time. This was enough to show that it corresponds to the price of the cheapest buy-and-hold strategy in the original market. A different proof relying on the dual formulation for multivariate markets, see Kabanov (1999), was then proposed by Bouchard (2000).

It should be noticed that a crucial point of all this analysis is that transaction costs are efficient, i.e. there is no couple of freely exchangeable assets. In this paper, we propose a first attempt to characterize the super-replication strategy in financial markets with "partial" transaction costs, where some assets can be exchanged freely. As a first step, we follow the approach of Bouchard and Touzi (2000). We introduce a family of fictitious markets and provide a PDE characterization similar to the one obtained in this paper. However, in our context, one can only show that the corresponding value function is concave in some directions (the ones where transaction costs are effective), and this is not sufficient to provide a precise characterization of the super-hedging strategy. With the help of a comparison theorem for PDE's, we next obtain a new lower bound associated to an auxiliary control problem, written in standard from. This allows us to characterize the optimal hedging strategy : it consists in keeping constant the number of non-freely exchangeable assets held in the portfolio and hedging the remaining part of the claim by trading dynamically on the freely exchangeable ones.

The paper is organized as follows. After setting some notations in Section 2, we describe the model and the super-replication problem in Section 3. The main result of the paper is stated in Section 4. In Section 5, we introduce an auxiliary super-hedging problem similar to the one considered in Bouchard and Touzi (2000) and derive the PDE associated to the value function. Further properties of this value function are obtained in Section 6. The proof is concluded in Section 7.

2 Notations

For the reader's convenience, we first introduce the main notations of this paper.
Given a vector $x \in I\!R^n$, its i-th component is denoted by x^i. All elements of $I\!R^n$ are identified with column vectors and the scalar product is denoted by $\cdot$. $I\!M^{n,p}$ denotes the set of all real-valued matrices with n rows and p columns. Given a matrix $M \in I\!M^{n,p}$, we denote by M^{ij} the component corresponding to the i-th row and the j-th column. $I\!M_+^{n,p}$ denotes the subset of $I\!M^{n,p}$ whose elements have non-negative entries. If $n = p$, we simply denote $I\!M^n$ and $I\!M_+^n$ for $I\!M^{n,n}$ and $I\!M_+^{n,n}$. Since $I\!M^{n,p}$ can be identified with $I\!R^{np}$, we define the norm on $I\!M^{n,p}$ as the norm of the associated element of $I\!R^{np}$. In both cases, the norm is simply denoted by $|\cdot|$. Transposition is denoted by $'$. Given a square matrix $M \in I\!M^n$, we denote by $\mathrm{Tr}[M] := \sum_{i=1}^n M^{ii}$ the associated trace. For $x \in I\!R^p$ and $\eta > 0$, we denote by $B(x,\eta)$ the open ball with radius η centered in x, $\partial B(x,\eta)$ its boundary and $\bar{B}(x,\eta)$ its closure.
Given n scalars $x_1, \ldots, x_n$, we denote by $\mathrm{Vect}[x_i]_{i\le n}$ the vector of $I\!R^n$ defined by the components $x_1, \ldots, x_n$. For all $x \in I\!R^n$, $\mathrm{diag}[x]$ denotes the diagonal matrix of $I\!M^n$ whose i-th diagonal element is x^i.
We denote by $\mathbf{1}_i$ the vector of $I\!R^n$ defined by $\mathbf{1}_i^j = 1$ if $j = i$ and 0 otherwise.
Given a smooth function φ mapping $I\!R^n$ into $I\!R^p$, we denote by $D_z\varphi$ the (partial) Jacobian matrix of φ with respect to its z variable. In the case $p = 1$, we denote by $D^2_{zs}\varphi$ the matrix defined as $(D^2_{zs}\varphi)^{ij} = \partial^2\varphi/\partial z^i \partial s^j$. If φ depends only on z, we simply write $D\varphi$ and $D^2\varphi$ in place of $D_z\varphi$ and $D^2_{zz}\varphi$.

In this paper, we shall consider $I\!R^d-$valued variables, $d \geq 1$, and we shall often write d as the sum of two positive integers $d_f + d_c$. For $x \in I\!R^d$, we will then write x as $x = (x^f, x^c)$ where x^f (resp. x^c) stands for the vector of $I\!R^{d_f}$ (resp. $I\!R^{d_c}$) formed by the d_f first (resp. d_c last) components of x.
All inequalities involving random variables have to be understood in the $\mathbb{P}$ − a.s. sense.

3 The model

Let T be a finite time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting a d-dimensional Brownian motion $\{W(t),\, 0 \leq t \leq T\}$. We shall denote by $I\!F = \{\mathcal{F}_t,\, 0 \leq t \leq T\}$ the $\mathbb{P}$-augmentation of the filtration generated by W.

3.1 The financial market

We consider a financial market which consists of one bank account, with constant price process, normalized to unity, and d risky assets $S := \{S^1, \ldots, S^d\}'$. The price process $S = \{S(t),\, 0 \leq t \leq T\}$ is an $I\!R^d$-valued stochastic process defined by the following stochastic differential system

$$dS(t) = \text{diag}[S(t)]\sigma(t, S(t))dW(t), \qquad 0 < t \leq T\,. \tag{3.1}$$

Here $\sigma(.,.)$ is an $I\!M^d$-valued function. We shall assume all over the paper that the function $\text{diag}[s]\sigma(t,s)$ satisfies the usual Lipschitz and linear growth conditions in order for the process S to be well-defined and that $\sigma(t,s)$ is invertible with $\sigma(t,s)^{-1}$ locally bounded, for all $(t,s) \in [0,T] \times I\!R^d_+$.

Remark 3.1 As usual, the assumption that the interest rate of the bank account is zero could be easily dispensed with by discounting. Also, there is no loss of generality in defining S as a martingale since we can always reduce the model to this context by an appropriate change of measure (under mild conditions on the initial coefficients).

We write d as $d = d_f + d_c$ with $d_f,\ d_c \geq 1$. The subscript f stands for "free" (of costs) while c means "costs". We assume that transactions in the market formed by the first d_f assets and the numéraire are free of costs. This means that a portfolio process associated to a trading strategy ϕ only based on the first d_f assets can be

written in the usual form $x + \int_0^\cdot \phi(t) \cdot dS^f(t)$, where, for $z \in I\!\!R^d$, we recall that $z^f = (z^1, \ldots, z^{d_f})'$ and $z^c = (z^{d_f+1}, \ldots, z^d)'$. On the other hand, we assume that any transaction involving the last d_c assets is subject to proportional costs.
Then, a trading strategy is described by a pair (ϕ, L) where ϕ is a $I\!\!R^{d_f}$-valued predictable process satisfying $\int_0^T |\phi(t)|^2 dt < \infty$ and L is an $I\!\!M_+^{1+d_c}$-valued process with initial value $L(0-) = 0$, such that L^{ij} is $I\!\!F-$adapted, right-continuous, and nondecreasing for all $i, j = 1, \ldots, 1 + d_c$.
To such a pair (ϕ, L) and $x \in I\!\!R^{1+d_c}$, we then associate the wealth process $X_x^{\phi,L}$ defined as the solution on $[0, T]$ of

$$
\begin{aligned}
X_x^i(0-) &= x^i \qquad \text{for } 1 \le i \le 1 + d_c \\
dX_x^{1+i}(t) &= X_x^{1+i}(t) \, \frac{dS^{d_f+i}(t)}{S^{d_f+i}(t)} + \sum_{j=1}^{1+d_c} \left[dL^{j(1+i)}(t) - (1 + \lambda^{(1+i)j}) dL^{(1+i)j}(t) \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \le i \le d_c \, , \\
dX_x^1(t) &= \phi(t) \cdot dS^f(t) + \sum_{j=1}^{1+d_c} \left[dL^{j1}(t) - (1 + \lambda^{1j}) dL^{1j}(t) \right] \, .
\end{aligned}
$$

The first component $(X_x^{\phi,L})^1$ stands for the amount of money which is invested on the market formed by the d_f first assets and the numéraire. For $i > 1$, $(X_x^{\phi,L})^i$ is the amount invested in the $(d_f + i - 1)$-th asset. The coefficient λ^{ij} are assumed to be non-negative and stands for the transaction costs which are paid when transacting on one of the d_c last assets.
Observe that we can always assume that

$$
(1 + \lambda^{ij}) \;\; \le \;\; (1 + \lambda^{ik})(1 + \lambda^{kj}) \;\; , \; i, \; j, \; k \le 1 + d_c \, ,
$$

since otherwise it would be cheaper to transfer money from the account i to j by passing through k rather than directly. Then, for any "optimal" strategy the effective cost between i and j would be $(1 + \lambda^{ik})(1 + \lambda^{kj}) - 1$.

3.2 The super-replication problem

Following Kabanov (1999), we define the *solvency region* :

$$
K := \left\{ x \in I\!\!R^{1+d_c} \; : \; \exists \, a \in I\!\!M_+^{1+d_c}, \; x^i + \sum_{j=1}^{1+d_c} (a^{ji} - (1 + \lambda^{ij}) a^{ij}) \; \ge \; 0 \;\; \forall \, i \le 1 + d_c \right\} .
$$

The elements of K can be interpreted as the vectors of portfolio holdings such that the no-bankruptcy condition is satisfied, i.e. the liquidation value of the portfolio holdings x, through some convenient transfers, is nonnegative.
Clearly, the set K is a closed convex cone containing the origin. We then introduce the partial ordering $\succeq$ induced by K :

$$\text{for all } x_1, x_2 \in I\!R^{1+d_c}, \quad x_1 \succeq x_2 \text{ if and only if } x_1 - x_2 \in K .$$

A trading strategy (ϕ, L) is said to be *admissible* if there is some some $c,\ \delta \in I\!R$ and δ_f in $I\!R^{d_f}$ such that

$$X_0^{\phi,L}(t) \quad \succeq \quad -(c + \delta_f \cdot S^f(t), \delta S^c(t)) \ , \quad 0 \le t \le T \ . \tag{3.2}$$

Observe that, if $X_0^{\phi,L}$ satisfies the above condition, then, after possibly changing (c, δ, δ_f), it holds for $X_x^{\phi,L}$ too, $x \in I\!R^{1+d_c}$. We denote by $\mathcal{A}$ the set of such trading strategies.

A *contingent claim* is a $(1+d_c)$-dimensional $\mathcal{F}_T$-measurable random variable $g(S(T))$. Here, g maps $I\!R_+^d$ into $I\!R^{1+d_c}$ and satisfies

$$g(s) \succeq -(c + \delta_f \cdot s^f, \delta s^c) \qquad \text{for all } s \text{ in } I\!R_+^d \tag{3.3}$$

for some $c,\ \delta \in I\!R$ and δ_f in $I\!R^{d_f}$.
In the rest of the paper, we shall identify a contingent claim with its pay-off function g. For $i = 2, \ldots, 1 + d_c$, the random variable $g^i(S(T))$ represents a target position in the asset $d_f - 1 + i$, while $g^1(S(T))$ represents a target position in terms of the numéraire.

The *super-replication problem* of the contingent claim g is then defined by

$$p(0, S(0)) \quad := \quad \inf \left\{ w \in I\!R \ : \ \exists\, (\phi, L) \in \mathcal{A},\ X_{w\mathbf{1}_1}^{\phi,L}(T) \succeq g(S(T)) \right\} \ ,$$

i.e. $p(0, S(0))$ is the minimal initial capital which allows to hedge the contingent claim g by means of some admissible trading strategy.

4 The explicit characterization

Before to state our main result, we need to define some additional notations. We first introduce the positive polar of K

$$K^* \ := \ \{\xi \in I\!R^{1+d_c} \ : \ \xi \cdot x \ge 0, \ \ \forall x \in K\} \ = \ \{\xi \in I\!R_+^{1+d_c} \ : \ \xi^j \le \xi^i (1+\lambda^{ij})\} \ , \tag{4.1}$$

together with its (compact) section

$$\Lambda := \{\xi \in K^* \ : \ \xi^1 = 1\} \ \subset \ (0,\infty)^{1+d_c} \ .$$

One easily checks that Λ is not empty since it contains the vector of $I\!R^{1+d_c}$ with all component equal to one. It is a standard result in convex analysis that the partial ordering $\succeq$ can be characterized in terms of Λ by

$$x_1 \ \succeq \ x_2 \quad \text{if and only if} \quad \xi \cdot (x_1 - x_2) \geq 0 \quad \text{for all} \ \ \xi \in \Lambda \ , \tag{4.2}$$

see e.g. Rockafellar (1970).
For $\xi \in I\!R^{1+d_c}$, we denote by $\underline{\xi}$ the vector of $I\!R^{d_c}$ defined by $\underline{\xi}^i = \xi^{i+1}$ for $i \leq d_c$. This amounts to removing the first component. With this notations, we define

$$G(z) = G(z^f, z^c) \ \ := \ \sup_{\xi \in \Lambda} \ \xi \cdot g\left(z^f, \text{diag}[\underline{\xi}]^{-1} z^c\right) \quad \text{for} \ \ z = (z^f, z^c) \ \text{in} \ I\!R_+^{d_f+d_c} \ ,$$

and denote by $\hat{G}$ the concave enveloppe of G with respect to z^c.

4.1 Main result

The main result of this paper requires the additional conditions :
$(\mathbf{H}\lambda)$: $\lambda^{ij} + \lambda^{ji} > 0$ for all $i, j = 1, \ldots, 1 + d_c$, $i \neq j$.
$(\mathbf{H}\sigma)$: For all $1 \leq i \leq d_f$ and $t \leq T$, $\{\sigma(t,z)^{ij}\}_{j \leq d}$ depends only on z^f.
$(\mathbf{Hg})$: g is lower-semicontinuous, $\hat{G}$ is continuous and has linear growth.

The condition $(H\lambda)$ means that there is no way to avoid transaction costs when transacting on the d_c last assets.

Theorem 4.1 *Assume that* $(H\lambda)$-$(H\sigma)$-(Hg) *hold. Then,*

$$p(0, S(0)) \ \ = \ \ \min\left\{w \in I\!R \ : \ \exists\, (\phi, L) \in \mathcal{A}^{BH}, \ X_{w\mathbf{1}_1}^{\phi,L}(T) \succeq g(S(T))\right\} \ ,$$

where

$$\mathcal{A}^{BH} \ \ := \ \ \{(\phi, L) \in \mathcal{A} \ : \ L(t) = L(0) \ \textit{for all} \ 0 \leq t \leq T\} \ .$$

Moreover, there is some $\hat{\Delta} \in I\!R^{d_c}$ *such that*

$$p(0, S(0)) \ \ = \ \ \mathbb{E}\left[C(S^f(T); \hat{\Delta})\right] + \sup_{\xi \in \Lambda} \underline{\xi} \cdot \text{diag}[\hat{\Delta}] S^c(0)$$

where, for $\Delta \in \mathbb{R}^{d_c}$*,*

$$C(S^f(T);\Delta) := \sup_{s^c \in (0,\infty)^{d_c}} \hat{G}\left(S^f(T), s^c\right) - \Delta \cdot s^c \ ,$$

and there is an optimal hedging strategy $(\phi, L) \in \mathcal{A}^{BH}$ *satisfying* $L = \hat{\Delta}$ *on* $[0,T]$*.*

The proof of the last result will be provided in the subsequent sections.

As in the papers quoted in the introduction, we obtain that the cheapest hedging strategy consists in keeping the number of non-freely exchangeable assets, S^c, constant in the portfolio. But here there is a remaining part, namely $g(S(T)) - (0, \text{diag}[\hat{\Delta}]S^c(T))$, which has to be hedged dynamically by investing in the freely exchangeable assets, S^f. It is done by hedging $C(S^f(T);\hat{\Delta})$.

From Theorem 4.1, we can now deduce an explicit formulation for $p(0, S(0))$.

Corollary 4.1 *Let the conditions of Theorem 4.1 hold. Then,*

$$p(0,S(0)) \ \ = \ \ \min_{\Delta \in \mathbb{R}^{d_c}} \left\{ \mathbb{E}\left[C(S^f(T);\Delta)\right] + \sup_{\xi \in \Lambda} \underline{\xi} \cdot \text{diag}[\Delta] S^c(0) \right\} \ .$$

Moreover, if $\hat{\Delta}$ *solves the above optimization problem, then there is an optimal hedging strategy* $(\phi, L) \in \mathcal{A}^{BH}$ *which satisfies* $L = \hat{\Delta}$ *on* $[0,T]$*.*

The proof will be provided in Section 7.

Remark 4.1 Let the conditions of Theorem 4.1 hold.
1. In Touzi (1999), the result of Bouchard and Touzi (2000) is generalized to the case where the initial wealth, before to be increased by the super-replication price, is non-zero, i.e. the following problem is considered :

$$p(0,S(0);x) \ \ := \ \ \inf\left\{ w \in \mathbb{R} \ : \ \exists\, (\phi,L) \in \mathcal{A},\ X^{\phi,L}_{x+w\mathbf{1}_1}(T) \succeq g(S(T)) \right\} \ ,$$

$x \in \mathbb{R}^{1+d_c}$. Our result can be easily extended to this case. Indeed, it suffices to observe from the wealth dynamics that

$$X^{\phi,L}_{x+w\mathbf{1}_1}(T) \succeq g(S(T))$$
$$\Longleftrightarrow$$
$$X^{\phi,L}_{w\mathbf{1}_1}(T) \succeq g(S(T)) - (x^1, \text{diag}[S^c(0)]^{-1}\text{diag}[S^c(T)]\underline{x})$$

where we recall that $\underline{x}$ is obtained from x by dropping the first component. Hence, to characterize $p(0,S(0);x)$, it suffices to replace g by

$$g(s;x) \quad := \quad g(s) - (x^1, \mathrm{diag}[S^c(0)]^{-1}\mathrm{diag}[s^c]\underline{x}) \ .$$

We then deduce from Theorem 4.1 and Corollary 4.1 that, for some $\hat{\Delta}(x) \in I\!R^{d_c}$,

$$\begin{aligned} p(0,S(0);x) &= \min\left\{w \in I\!R \ : \ \exists\,(\phi,L) \in \mathcal{A}^{BH},\ X^{\phi,L}_{x+w\mathbf{1}_1}(T) \succeq g(S(T))\right\} \\ &= \mathbb{E}\left[C(S^f(T);\hat{\Delta}(x),x)\right] + \sup_{\xi\in\Lambda} \underline{\xi}\cdot \mathrm{diag}[\hat{\Delta}(x)]S^c(0) \\ &= \min_{\Delta\in I\!R^{d_c}} \mathbb{E}\left[C(S^f(T);\Delta,x)\right] + \sup_{\xi\in\Lambda} \underline{\xi}\cdot \mathrm{diag}[\Delta]S^c(0) \end{aligned} \tag{4.3}$$

where

$$C(S^f(T);\Delta,x) := \sup_{s^c\in(0,\infty)^{d_c}} \hat{G}\left(S^f(T),s^c\right) - \left(x^1, \mathrm{diag}[S^c(0)]^{-1}\mathrm{diag}[s^c]\underline{x}\right) - \Delta\cdot s^c \ .$$

2. The set of initial wealth which allow to hedge g,

$$\Gamma(g) \quad := \quad \left\{x \in I\!R^{1+d_c} \ : \ \exists\,(\phi,L)\in\mathcal{A},\ X^{\phi,L}_x(T) \succeq g(S(T))\right\} \ ,$$

can be written in

$$\Gamma(g) \quad = \quad \left\{x \in I\!R^{1+d_c} \ : \ p(0,S(0);x) \le 0\right\} \ .$$

3. In the limit case where $d_f = 0$, we recover the result of Bouchard and Touzi (2000) and Touzi (1999).

4.2 Example

We conclude this section with a simple example. We consider a two dimensional Black and Scholes model, i.e. $d_f = d_c = 1$, $\sigma(t,s) = \sigma \in I\!M^2$ with σ invertible. In this case, we have

$$\Lambda \quad = \quad \left\{(1,y) \in I\!R^2 \ : \ \frac{1}{1+\lambda^{21}} \le y \le 1+\lambda^{12}\right\} \quad , \quad \lambda^{21}+\lambda^{12} \ > \ 0 \ .$$

We take g of the form

$$g(s) \quad = \quad \left([s^1-K^1]^+\mathbf{1}_{\{s^2>K^2\}}\right)\mathbf{1}_1 \ = \ \left([s^f-K^1]^+\mathbf{1}_{\{s^c>K^2\}}\right)\mathbf{1}_1$$

with K^1, $K^2 > 0$. Then,

$$G(s) \;=\; \Big([s^f - K^1]^+ \mathbf{1}_{\{s^c > \tilde K^2\}}\Big)\,\mathbf{1}_1 \quad \text{and} \quad \hat G(s) \;=\; [s^f - K^1]^+ \Big((s^c/\tilde K^2) \wedge 1\Big)\ ,$$

where $\tilde K^2 = K^2/(1+\lambda^{21})$. For $\Delta \in I\!R$, we have

$$\begin{aligned} C(s^f;\Delta) \;&=\; \sup_{s^c \in (0,\infty)^{d_c}} [s^f - K^1]^+ \Big((s^c/\tilde K^2) \wedge 1\Big) - \Delta s^c \\ &=\; \begin{cases} \Big([s^f - K^1]^+ - \Delta \tilde K^2\Big)\,\mathbf{1}_{\{0 \le \Delta \tilde K^2 \le [s^f - K^1]^+\}}\ , & \text{if } \Delta \ge 0\ , \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Then, by Corollary 4.1,

$$\begin{aligned} p(0,S(0)) \;&=\; \min_{\Delta \ge 0} \mathbb{E}\left[C(S^f(T);\Delta)\right] + (1+\lambda^{12})\Delta S^c(0) \\ &=\; \min_{\Delta \ge 0}\ \mathbb{E}\left[\Big([S^f(T) - K^1]^+ - \Delta \tilde K^2\Big)\,\mathbf{1}_{\{0 \le \Delta \tilde K^2 \le [S^f(T) - K^1]^+\}}\right] \\ &+\; (1+\lambda^{12})\Delta S^c(0) \\ &=\; \min_{\Delta \ge 0}\ \mathbb{E}\left[\Big([S^f(T) - K^1 - \Delta \tilde K^2]^+\Big)\right] + (1+\lambda^{12})\Delta S^c(0)\ , \end{aligned} \tag{4.4}$$

where the expectation is convex in Δ. Then, if the optimal Δ is different from 0, it must satisfy the first order condition

$$-\,\tilde K^2\,\mathbb{E}\left[\mathbf{1}_{\{S^f(T) - K^1 \ge \Delta \tilde K^2\}}\right] + (1+\lambda^{12})S^c(0) \;=\; 0\ . \tag{4.5}$$

We consider two different cases.

1. If $\mathbb{P}\left[S^f(T) - K^1 \ge 0\right] \;\le\; (1+\lambda^{12})S^c(0)/\tilde K^2$, then, either the only solution of (4.5) is 0 or (4.5) has no solution. It follows that the optimum in (4.4) is achieved by $\hat\Delta = 0$. Therefore

$$p(0,S(0)) \;=\; \mathbb{E}\left[[S^f(T) - K^1]^+\right]\ ,$$

and, by the Clark-Ocone's formula, the optimal hedging strategy (ϕ, L) is defined by $L = 0$ and $\phi(t) = \mathbb{E}\left[S^f(T)\mathbf{1}_{\{S^f(T) \ge K^1\}} \mid \mathcal{F}_t\right]/S^f(t)$.

2. If $\mathbb{P}\left[S^f(T) - K^1 \ge 0\right] \;>\; (1+\lambda^{12})S^c(0)/\tilde K^2$, then (4.5) has a unique solution $\hat\Delta > 0$ which satisfies

$$p \;:=\; p(0,S(0)) \;=\; \mathbb{E}\left[[S^f(T) - K^1 - \hat\Delta \tilde K^2]^+\right] + (1+\lambda^{12})\hat\Delta S^c(0)\ .$$

Observe that, in this model, $\hat{\Delta}$ can be computed explicitly in terms of the inverse of the cumulated distribution of the gaussian distribution. Let (ϕ, L) be defined by

$$L(t) \;=\; \hat{\Delta} \quad \text{and} \quad \phi(t) \;=\; \mathbb{E}\left[S^f(T)\mathbf{1}_{\{S^f(T)-K^1\geq\hat{\Delta}\tilde{K}^2\}} \mid \mathcal{F}_t\right]/S^f(t) \quad \text{on } t \leq T\;.$$

By the Clark-Ocone's formula, we have

$$X^{\phi,L}_{p\mathbf{1}_1}(T) \;=\; \left([S^f(T)-K^1-\hat{\Delta}\tilde{K}^2]^+\;,\;\hat{\Delta}S^c(T)\right)\;.$$

For ease of notations, let us define

$$\Psi := [S^f(T)-K^1]^+\mathbf{1}_{\{S^c(T)\geq K^2\}}\;.$$

Fix $\omega \in \{\Psi \geq 0\}$. Then, $S^f(T) \geq K^1$ and $S^c(T) \geq K^2$. If $S^f(T) - K^1 \leq \hat{\Delta}\tilde{K}^2$ then $X^{\phi,L}_{p\mathbf{1}_1}(T) = (0, \hat{\Delta}S^c(T))$. Recalling the definition of $\tilde{K}^2$, we then obtain

$$X^{\phi,L}_{p\mathbf{1}_1}(T) \;=\; (0,\hat{\Delta}S^c(T)) \;\succeq\; (\hat{\Delta}\tilde{K}^2,0) \;\succeq\; (\Psi,0)\;.$$

If $S^f(T) - K^1 > \hat{\Delta}\tilde{K}^2$, then

$$X^{\phi,L}_{p\mathbf{1}_1}(T) \;=\; \left(S^f(T)-K^1-\hat{\Delta}\tilde{K}^2,\hat{\Delta}S^c(T)\right) \;\succeq\; (S^f(T)-K^1,0) \;=\; (\Psi,0)\;.$$

On $\{\Psi = 0\}$, we have $X^{\phi,L}_{p\mathbf{1}_1}(T) \succeq 0 = (\Psi, 0)$ since $\hat{\Delta} > 0$.

5 Fictitious markets

In this section, we follow the arguments of Bouchard and Touzi (2000), i.e. we introduce an auxiliary control problem which can be interpreted as a super-replication problem in a fictitious market without transaction costs but were S^c is replaced by a controlled process evolving in the "bid-ask" spreads associated to the transaction costs λ. This is obtained by introducing a controlled process $f(Y^{(a,b)})$, see below, which evolves in Λ. Then, the fictitious market is constructed by replacing S by $(S^f, \mathrm{diag}[\underline{f}(Y^{(a,b)})]S^c)$ and $g(S(T))$ by $f(Y^{(a,b)}(T)) \cdot g(S(T))$. In this paper, we shall not enter into the detailed construction as it follows line by line the arguments of Bouchard and Touzi (2000) up to obvious modifications (at the level of notations). We only state the most important results and refer to Bouchard and Touzi (2000) for the proofs, see also the survey paper Touzi (1999).

5.1 Parameterization of the fictitious markets

We first parameterize the compact set Λ. Since K^* is a polyhedral closed convex cone, we can find a family $e = (e_i)_{i \le n}$ in $(0,\infty)^{1+d_c}$, for some $n \ge 1$, such that, for all $\alpha \in I\!R^n_+$, $\sum_{i=1}^n \alpha^i e_i = 0$ implies $\alpha = 0$, and $K^* = \{\sum_{i=1}^n \alpha^i e_i \ , \ \alpha \in I\!R^n_+\}$. Then, we define the map f from $(0,\infty)^n$ into Λ by

$$f(y) \ := \ \left(\sum_{i=1}^n y^i e_i\right) / \left(\sum_{i=1}^n y^i e_i^1\right) \quad , \ y \in (0,\infty)^n \ .$$

Before to go on with the definition of the fictitious markets, we list some useful properties of f and Λ.

Lemma 5.1 *Let $(H\lambda)$ hold. Then,*
(i) There is some $\delta > 0$ such that $0 < \xi^i + \frac{1}{\xi^i} \le \delta$ for each $\xi \in \Lambda$.
(ii) On $(0,\infty)^n$, the rank of the Jacobian matrix Df of f is d_c.

Proof. See Bouchard and Touzi (2000) for the easy proof. □

In order to alleviate the notations, we define $\bar f$, $\underline{F}$ and $\bar F$ as

$$\bar f^i \ = \ \underline{f}^{i-d_f} \mathbf{1}_{\{i > d_f\}} + \mathbf{1}_{\{i \le d_f\}} \ , \ \underline{F} \ = \ \mathrm{diag}[\underline{f}] \quad \text{and} \quad \bar F \ = \ \mathrm{diag}[\bar f] \ .$$

The map $\bar f$ coincides with f on its last d_c components, while the first d_f ones are set to one. Here, $\bar f$, $\bar F$ and $\underline{F}$ take values in $I\!R^d$, $I\!M^d$ and $I\!M^{d_c}$ respectively.

Given some arbitrary parameter $\mu > 0$, we then define for all $(y_0, z_0) \in (0,\infty)^n \times I\!R^d_+$ the continuous function α^{y_0,s_0} on $[0,T] \times I\!R^d_+ \times (0,\infty)^n \times I\!M^{n,d} \times I\!R^n$ as

$$\alpha^{y_0,s_0}(t,s,y,a,b) := \begin{cases} A(t,s,y,a,b) & \text{if } \sum_{i=1}^d \sum_{j=1}^n \left(|s^i - z_0^i| + |\ln \frac{y^j}{y_0^j}| \right) < \mu \\ \text{constant} & \text{otherwise,} \end{cases} \tag{5.1}$$

where

$$\begin{aligned} A(t,s,y,a,b) \ = \ \sigma(t,s)^{-1} \bar F(y)^{-1} \ & \left\{ D\bar f(y) \mathrm{diag}[y] b \right. \\ & + \frac{1}{2} \mathrm{Vect} \left[\mathrm{Tr} \left(D^2 \bar f^i(y) \mathrm{diag}[y] a a' \mathrm{diag}[y] \right) \right]_{i \le d} \\ & \left. + \mathrm{Vect} \left[\left(D\bar f(y) \mathrm{diag}[y] a \sigma(t,s)' \right)_{ii} \right]_{i \le d} \right\} \ . \end{aligned}$$

Let $\mathcal{D}$ be the set of all bounded progressively measurable processes $(a,b) = \{(a(t),b(t)), 0 \le t \le T\}$ where a and b are valued respectively in $I\!M^{n,d}$ and $I\!R^n$. For all (t,y,z) in $[0,T]\times(0,\infty)^{n+d}$ and (a,b) in $\mathcal{D}$, we introduce the controlled process $Y^{(a,b)}_{t,y,z}$ defined on $[t,T]$ as the solution of the stochastic differential equation

$$\begin{aligned} dY(r) &= \operatorname{diag}[Y(r)]\left[(b(r)+a(r)\alpha^{y,s}(r,S_{t,s}(r),Y(r),a(r),b(r)))\,dt + a(r)dW(r)\right] \\ Y(t) &= y\ , \end{aligned} \qquad (5.2)$$

where $S_{t,s}$ is the solution of (3.1) with the condition $S_{t,s}(t) = s$ and $s = \bar{F}(y)^{-1}z$. It follows from our assumption on σ that $\alpha^{y_0,s_0}(t,s,y,a,b)$ is a random Lipschitz function of y, so that the process $Y^{(a,b)}_{t,y,z}$ is well defined on $[t,T]$. For each (a,b) in $\mathcal{D}$, we define the process $Z^{(a,b)}_{t,y,z}$ by

$$Z^{(a,b)}_{t,y,z} = \bar{F}(Y^{(a,b)}_{t,y,z})S_{t,s} \quad \text{with} \quad s = \bar{F}(y)^{-1}z\ . \qquad (5.3)$$

Observe that $(Z^{(a,b)}_{t,y,z})^f = S^f_{t,s}$.

5.2 Super-replication in the fictitious markets

Let ϕ be a progressively measurable process valued in $I\!R^d$ satisfying

$$\sum_{i=1}^{d}\int_0^T |\phi^i(t)|^2 d\langle Z^{(a,b),i}_{t,y,z}(t)\rangle < \infty\ . \qquad (5.4)$$

Then, given $x \ge 0$, we introduce the process $X^{(a,b)^\phi}_{t,x,y,z}$ defined by

$$X^{(a,b)^\phi}_{t,x,y,z}(r) = x + \int_t^r \phi(s)\cdot dZ^{(a,b)}_{t,y,z}(s) \qquad (5.5)$$

and we denote by $\mathcal{B}^{(a,b)}(t,x,y,z)$ the set of all such processes ϕ satisfying the additional condition

$$X^{(a,b)^\phi}_{t,x,y,z}(r) \ \ge\ -c - \delta\cdot Z^{(a,b)}_{t,y,z}(r)\ ,\ \ t \le r \le T\ ,\ \ \text{for some } (c,\delta)\in I\!R^{1+d}\ . \qquad (5.6)$$

We finally define the auxiliary stochastic control problems

$$\begin{aligned} u^{(a,b)}(t,y,z) \ :=\ \inf\Big\{ & x \in I\!R\ :\ \exists\phi\in\mathcal{B}^{(a,b)}(t,x,y,z)\ , \\ & X^{(a,b)^\phi}_{t,x,y,z}(T) \ge f\left(Y^{(a,b)}_{t,y,z}(T)\right)\cdot g\left(S_{t,s}(T)\right)\Big\}\ , \end{aligned} \qquad (5.7)$$

with $s=\bar{F}(y)^{-1}z$, and

$$u(t,y,z) := \sup_{(a,b)\in\mathcal{D}} u^{(a,b)}(t,y,z)\ . \tag{5.8}$$

The value function $u^{(a,b)}(t,y,z)$ coincides with the super-replication price of the modified claim $f\left(Y^{(a,b)}_{t,y,z}(T)\right)\cdot g\left(S_{t,s}(T)\right)$ in the market formed by the assets $Z^{(a,b)}_{t,y,z}$ without transaction costs. The function $u(t,y,z)$ is the upper-bound of these prices over all the "controlled" fictitious markets. We refer to Bouchard and Touzi (2000) for a more detailed discussion.

5.3 Viscosity properties of u_*

We can now provide a first lower bound for $p(0,S(0))$ which is similar to the one provided by Bouchard and Touzi (2000) in the case $d_f=0$.
For $(t,y,z)\in[0,T]\times(0,\infty)^n\times I\!R^d_+$, we define the lower semicontinuous function u_* by

$$u_*(t,y,z) \ \ := \ \ \liminf_{\substack{(t',y',z')\to(t,y,z)\\ (t',y',z')\in[0,T)\times(0,\infty)^{n+d}}} u(t',y',z')\ .$$

Contrary to Bouchard and Touzi (2000), we need to extend the definition of u_* to the whole subspace $[0,T]\times(0,\infty)^n\times I\!R^d_+$ (in opposition to $[0,T]\times(0,\infty)^{n+d}$). Although, we are only interested by u_* on $[0,T)\times(0,\infty)^{n+d}$, since $S(0)\in(0,\infty)^d$, this extension will be useful to apply the comparison theorem of Proposition 6.2 below.

Theorem 5.1 *Let $(H\lambda)$ and (Hg) hold. Then u_* satisfies :*
(i) For all $y\in(0,\infty)^n$, we have $p(0,S(0))\ge u_*(0,y,\bar{F}(y)S(0))$.
(ii) u_* is independent of its variable y.
(iii) u_* is a viscosity supersolution on $[0,T)\times(0,\infty)^n\times I\!R^d_+$ of

$$\inf_{a\in I\!M^{n,d}} -\mathcal{H}^a\varphi \ge 0\ ,$$

where, for a smooth function φ,

$$\mathcal{H}^a\varphi \ \ := \ \ \frac{\partial\varphi}{\partial t}+\frac{1}{2}\mathrm{Tr}\left[\Gamma^{a'}D^2_{zz}\varphi\Gamma^a\right]$$

and

$$\Gamma^a(t,y,z) \ \ := \ \ \mathrm{diag}[z]\left(\sigma(t,\bar{F}(y)^{-1}z)+\bar{F}(y)^{-1}D\bar{f}(y)\mathrm{diag}[y]a\right)\ .$$

(iv) For all $(y,z) \in (0,\infty)^n \times I\!R^d_+$

$$u_*(T,y,z) \;\; \geq \;\; G(z) \; .$$

This result is obtained by following line by line the arguments of Sections 6, 7 and 8 in Bouchard and Touzi (2000), see also Touzi (1999). Since its proof is rather long, we omit it.

In Bouchard and Touzi (2000), the above characterization was sufficient to solve the super-replication problem. Indeed, in the case where $d_f = 0$, one can show that u_* is concave with respect to z and non-increasing in t. This turns out to be sufficient to show that it corresponds to the price of the cheapest buy-and-hold super-hedging strategy in the original market. In our context, where $d_f \geq 1$, we can only show that u_* is concave with respect to z^c and there is no reason why it should be concave in z (in particular if $g(s^f, s^c)$ depends only on s^f). We therefore have to work a little more.
As a first step, we rewrite the above PDE in a more tractable way. For all $(t,z) \in [0,T] \times I\!R^d_+$ and $\mu \in I\!M^{d_c,d}$, we define

$$\sigma^\mu(t,z) \;\; := \;\; \mathrm{diag}[z] \left[\sigma(t,z)^{ij} \mathbf{1}_{i \leq d_f} + \mu^{ij} 1_{i > d_f}\right]_{1 \leq i,j \leq d}$$

where, for real numbers (a^{ij}), $[a^{ij}]_{1 \leq i,j \leq d}$ denotes the square d-dimensional matrix M defined by $M^{ij} = a^{ij}$.
Since u_* does not depend on its y variable, from now on, we shall omit it if not required by the context.

Corollary 5.1 *Let* $(H\lambda)$*,* $(H\sigma)$ *and* (Hg) *hold. Then,*
(i) u_* *is a viscosity supersolution on* $[0,T) \times I\!R^d_+$ *of*

$$\inf_{\mu \in I\!M^{d_c,d}} -\mathcal{G}^\mu \varphi \geq 0 \; , \tag{5.9}$$

where, for a smooth function φ *and* $\mu \in I\!M^{d_c,d}$*,*

$$\mathcal{G}^\mu \varphi(t,z) \;\; = \;\; \frac{\partial \varphi}{\partial t} + \frac{1}{2} Tr\left[\sigma^\mu(t,z)' D^2_{zz}\varphi(t,z) \sigma^\mu(t,z)\right] \; .$$

(ii) For each $(t,z^f) \in [0,T) \times (0,\infty)^{d_f}$*, the map* $z^c \in (0,\infty)^{d_c} \mapsto u_*(t,z^f,z^c)$ *is concave.*

(iii) For all $z \in \mathbb{R}^d_+$

$$u_*(T,z) \ \ \geq \ \ \hat{G}(z)\ , \tag{5.10}$$

where we recall that $\hat{G}$ is the concave envelope of G with respect to its last d_c variables.

Proof. (i). Recall from Lemma 5.1 that the rank of $Df(y)$ is d_c whenever $y \in (0,\infty)^n$. Since $\bar{f}^i = 1$ for $i \leq d_f$, we deduce from $(H\sigma)$ that, for each $\mu \in \mathbb{M}^{d_c,d}$, we can find some $a \in \mathbb{M}^{n,d}$ such that $\Gamma^a(t,y,z) = \sigma^\mu(t,z)$. Then, the first result follows from Theorem 5.1.
(ii). For φ satisfying (5.9) we must have, on $[0,T)\times(0,\infty)^d$, $-Tr[\mu' D^2_{z^c z^c}\varphi\mu] \geq 0$ for all $\mu \in \mathbb{M}^{d_c,d}$ since otherwise we would get a contradiction of (5.9) by considering $\delta\mu$ and sending δ to infinity. Then, the concavity property follows from the same argument as in Lemma 8.1 of Bouchard and Touzi (2000).
(iii). In view of the boundary condition of Theorem 5.1, is suffices to show that $u_*(T,z)$ is concave with respect to z^c. To see this, fix $(z^f, z^c_1, z^c_2) \in \mathbb{R}^{d_f+2d_c}_+$ and observe that, by (ii) and definition of u_*,

$$\begin{aligned}
u_*(T,z^f,(z^c_1+z^c_2)/2) \ &= \liminf_{\substack{(\tilde{t},\tilde{z}^f,\tilde{z}^c_1,\tilde{z}^c_2)\to(T,z^f,z^c_1,z^c_2)\\ (\tilde{t},\tilde{z}^f,\tilde{z}^c_1,\tilde{z}^c_2)\in[0,T)\times(0,\infty)^{d_f+2d_c}}} u_*(\tilde{t},\tilde{z}^f,(\tilde{z}^c_1+\tilde{z}^c_2)/2) \\
&\geq \liminf_{\substack{(\tilde{t},\tilde{z}^f,\tilde{z}^c_1,\tilde{z}^c_2)\to(T,z^f,z^c_1,z^c_2)\\ (\tilde{t},\tilde{z}^f,\tilde{z}^c_1,\tilde{z}^c_2)\in[0,T)\times(0,\infty)^{d_f+2d_c}}} \frac{1}{2}\left(u_*(\tilde{t},\tilde{z}^f,\tilde{z}^c_1)+u_*(\tilde{t},\tilde{z}^f,\tilde{z}^c_2)\right) \\
&\geq \frac{1}{2}\left(u_*(T,z^f,z^c_1)+u_*(T,z^f,z^c_2)\right)\ .
\end{aligned}$$

□

6 A tractable lower bound for u_*

Let us introduce some additional notations. Given $\kappa \geq 0$, we define U_κ as the set of all elements M of $\mathbb{M}^{d_c,d}$ such that $|M| \leq \kappa$. We then denote by $\mathcal{U}_\kappa$ the collection of all U_κ-valued predictable processes. To each $\mu \in \bigcup_{\kappa\geq 0}\mathcal{U}_\kappa$, we associate the controlled process $Z^\mu_{t,z}$ defined as the solution of

$$Z(s) \ \ = \ \ z + \int_t^s \sigma^{\mu(r)}(r,Z(r))dW(r) \quad t \leq s \leq T\ . \tag{6.1}$$

Observe that, under $(H\sigma)$,

$$(Z^{\mu}_{t,z})^f \;=\; S^f_{t,z}\;.$$

The aim of this Section is to prove the following result.

Proposition 6.1 *Let $(H\lambda)$, $(H\sigma)$ and (Hg) hold. Then, for all $z \in (0,\infty)^d$,*

$$u_*(0,z) \;\geq\; \sup_{\mu\in\mathcal{U}} \mathbb{E}\left[\hat{G}(Z^{\mu}_{0,z}(T))\right]\;,$$

where $\mathcal{U}$ denotes the set of all $I\!M^{d_c,d}$-valued square integrable predictable processes μ such that $Z^{\mu}_{0,z}$ is a martingale for all $z \in (0,\infty)^d$.

Using an approximation argument combined with the martingale property of the Z^{μ}'s, the concavity of u_* with respect to z^c and assumption $(H\sigma)$, this will allow us to show that, for all $z = (z^f, z^c) \in (0,\infty)^{d_f} \times (0,\infty)^{d_c}$ and $\Delta \in \partial_{z^c} u_*(0, z^f, z^c)$, we have

$$u_*(0,z^f,z^c) \;\geq\; \mathbb{E}\left[\sup_{\tilde{z}^c\in(0,\infty)^{d_c}} \left\{\hat{G}(S^f_{0,z}(T),\tilde{z}^c) - \Delta\cdot\tilde{z}^c\right\}\right] + \Delta\cdot z^c\;,$$

where, $\partial_{z^c} u_*(0, z^f, z^c)$ is the subgradient of the mapping $z^c \mapsto u_*(0, z^f, z^c)$, see Corollary 7.1 below. This last lower bound for u_* will turn out to be enough to conclude the proof, see Section 7.

To this purpose, we shall first consider the auxiliary control problems

$$v_\kappa(t,z) \;:=\; \sup_{\mu\in\mathcal{U}_\kappa} \mathbb{E}\left[\hat{G}\left(Z^{\mu}_{t,z}(T)\right)\right] \qquad (t,z,\kappa) \in [0,T]\times(0,\infty)^d\times(0,\infty)\;, \quad (6.2)$$

and show that $u_* \geq \sup_{\kappa>0} v^*_\kappa$, where v^*_κ is the upper-semicontinuous function defined on $[0,T] \times I\!R^d_+$ by

$$v^*_\kappa(t,z) \;:=\; \limsup_{\substack{(t',z')\to(t,z)\\ (t',z')\in[0,T)\times(0,\infty)^d}} v_\kappa(t',z')\;.$$

This will be done by means of a comparison argument on the PDE defined by (5.9)-(5.10) with U_κ substituted to $I\!M^{d_c,d}$.
In the next subsection, we show that v^*_κ is a viscosity subsolution of (5.9)-(5.10) with U_κ substituted to $I\!M^{d_c,d}$. Then, we provide the comparison theorem. We conclude the proof of Proposition 6.1 in the last subsection.

6.1 Viscosity properties of v^*_κ

We start with the subsolution property in the interior of the domain. The proof is rather standard now but, as it is short, we provide it for completeness.

Lemma 6.1 *For each $\kappa > 0$, v^*_κ is a viscosity subsolution on $[0,T) \times I\!R^d_+$ of*

$$\inf_{\mu \in U_\kappa} -\mathcal{G}^\mu \varphi \;\; \leq \;\; 0 \, .$$

Proof. Let $\varphi \in C^2([0,T] \times I\!R^d)$ and (t_0, z_0) be a strict global maximizer of $v^*_\kappa - \varphi$ on $[0,T) \times I\!R^d_+$ such that $(v^*_\kappa - \varphi)(t_0, z_0) = 0$. We assume that

$$\inf_{\mu \in U_\kappa} -\mathcal{G}^\mu \varphi(t_0, z_0) \;\; > \;\; 0 \, , \tag{6.3}$$

and work towards a contradiction. If (6.3) holds, then it follows from our continuity assumptions on σ that there exists some $t_0 < \eta < T - t_0$ such that

$$\inf_{\mu \in U_\kappa} -\mathcal{G}^\mu \varphi(t,z) \;\; \geq \;\; 0 \quad \text{ for all } (t,z) \in B_0 := B((t_0,z_0),\eta) \, . \tag{6.4}$$

Recall that $B((t_0, z_0), \eta)$ is the open ball of radius η centered on (t_0, z_0), see the notations section. Let $(t_n, z_n)_{n \geq 0}$ be a sequence in $B_0 \cap ([0,T) \times (0,\infty)^d)$ such that

$$(t_n, z_n) \;\longrightarrow\; (t_0, z_0) \quad \text{and} \quad v_\kappa(t_n, z_n) \;\longrightarrow\; v^*_\kappa(t_0, z_0)$$

and notice that

$$v_\kappa(t_n, z_n) - \varphi(t_n, z_n) \;\longrightarrow\; 0 \, . \tag{6.5}$$

Next, define the stopping times

$$\theta^\mu_n \;\; := \;\; T \wedge \inf \{ s > t_n \; : \; (s, Z^\mu_n(s)) \notin B_0 \}$$

where μ is any element of $\mathcal{U}_\kappa$ and $Z^\mu_n := Z^\mu_{t_n,z_n}$. Let $\partial_p B_0 = [t_0, t_0 + \eta] \times \partial B(z_0, \eta) \cup \{t_0 + \eta\} \times B(z_0, \eta)$ denote the parabolic boundary of B_0 and observe that

$$0 \; > \; -\zeta \;\; := \;\; \sup_{(t,z) \in \partial_p B_0 \cap ([0,T] \times I\!R^d_+)} (v^*_\kappa - \varphi)(t,z)$$

since (t_0, z_0) is a strict maximizer of $v^*_\kappa - \varphi$. Then, for a fixed $\mu \in \mathcal{U}_\kappa$, we deduce from Itô's Lemma and (6.4) that

$$\varphi(t_n, z_n) \;\; \geq \;\; \mathbb{E}\left[\varphi(\theta^\mu_n, Z^\mu_n(\theta^\mu_n))\right] \geq \mathbb{E}\left[v^*_\kappa(\theta^\mu_n, Z^\mu_n(\theta^\mu_n)) + \zeta\right] \geq \zeta + \mathbb{E}\left[\hat{G}(Z^\mu_n(T))\right] \, ,$$

where we used the fact that $\varphi \geq v_\kappa^* \geq v_\kappa$ and

$$v_\kappa\left(\theta_n^\mu, Z_n^\mu(\theta_n^\mu)\right) \;\geq\; \mathbb{E}\left[\hat{G}(Z_n^\mu(T)) \mid \mathcal{F}_{\theta_n^\mu}\right] .$$

By arbitrariness of $\mu \in \mathcal{U}_\kappa$, it follows from the previous inequality that

$$\varphi(t_n, z_n) \;\geq\; \zeta + v_\kappa(t_n, z_n) .$$

In view of (6.5), this leads to a contradiction since $\zeta > 0$. □

We now turn to the boundary condition.

Lemma 6.2 *Under (Hg), for each $\kappa > 0$ and $z \in I\!R_+^d$, $v_\kappa^*(T, z) \leq \hat{G}(z)$.*

Proof. For ease of notations, we write $\bar{v}_\kappa(z)$ for $v_\kappa^*(T, z)$. Let f be in $C^2(I\!R^d)$ and $z_0 \in I\!R_+^d$ be such that

$$0 \;=\; (\bar{v}_\kappa - f)(z_0) = \max_{I\!R_+^d}(\bar{v}_\kappa - f) .$$

We assume that

$$\bar{v}_\kappa(z_0) - \hat{G}(z_0) \;=\; f(z_0) - \hat{G}(z_0) \;>\; 0 , \tag{6.6}$$

and work towards a contradiction to the definition of v_κ.

1. Define on $[0, T] \times I\!R^d$

$$\varphi(t, z) \;:=\; f(z) + c|z - z_0|^2 + (T - t)^{\frac{1}{2}}$$

and notice that for all $z \in I\!R^d$

$$\frac{\partial \varphi}{\partial t}(t, z) \;\longrightarrow\; -\infty \quad \text{as} \quad t \longrightarrow T . \tag{6.7}$$

Since U_κ is compact, there is some $\eta,\ \tilde{\eta} > 0$ such that

$$\inf_{\mu \in U_\kappa} -\mathcal{G}^\mu \varphi(t, z) \;>\; 0 \quad \text{for all } (t, z) \in B_0 := [T - \tilde{\eta}, T) \times \bar{B}(z_0, \eta) . \tag{6.8}$$

Since $\varphi(T, z_0) = f(z_0)$ and $\hat{G}$ is continuous, see (Hg), it follows from (6.6) that we can choose η, $\tilde{\eta}$ such that, for some $\varepsilon > 0$, we also have

$$\varphi(T, z) - \hat{G}(z) \;>\; \varepsilon \quad \text{for all } z \in \bar{B}(z_0, \eta) \cap I\!R_+^d , \tag{6.9}$$

and, by upper-semicontinuity of $v^*_\kappa - f$,

$$v^*_\kappa(t,z) \;\le\; f(z)+\alpha \quad \text{for some } \alpha>0 \text{ on } B_0\cap([0,T]\times I\!R^d_+)\ . \tag{6.10}$$

By definition of φ, we also have

$$\varphi(t,z) \;\;\ge\;\; f(z)+c\eta^2 \quad \text{on } [T-\tilde\eta,T]\times\partial\bar B(z_0,\eta)\ , \tag{6.11}$$

where, by possibly taking a smaller $\tilde\eta$, we can choose c large enough so that

$$c\eta^2 \;\;\ge\;\; \alpha+\varepsilon \tag{6.12}$$

and (6.8)-(6.9)-(6.10) still holds, see (6.7).
2. Let (s_n,ξ_n) be a sequence in $[T-\tilde\eta/2,T)\times$ $(\bar B(z_0,\eta)\cap(0,\infty)^d)\subset B_0$ satisfying

$$(s_n,\xi_n)\longrightarrow(T,z_0)\ ,\ \ s_n<T \quad\text{and}\quad v^*_\kappa(s_n,\xi_n)\longrightarrow\bar v_\kappa(z_0)\ .$$

Let (t_n,z_n) be a maximizer of $(v^*_\kappa-\varphi)$ on $[s_n,T]\times$ $(\bar B(z_0,\eta)\cap[0,\infty)^d)\subset B_0$. For all n, let $(t^k_n,z^k_n)_k$ be a subsequence in $[s_n,T)\times(\bar B(z_0,\eta)\cap(0,\infty)^d)$ satisfying

$$(t^k_n,z^k_n)\longrightarrow(t_n,z_n)\quad\text{and}\quad v_\kappa(t^k_n,z^k_n)\longrightarrow v^*_\kappa(t_n,z_n)\ .$$

We shall prove later that

$$(t_n,z_n)\longrightarrow(T,z_0)\quad\text{and}\quad v^*_\kappa(t_n,z_n)\longrightarrow\bar v_\kappa(z_0) \tag{6.13}$$

and that there exists a subsequence of $(t^k_n,z^k_n)_{k,n}$, relabelled (t'_n,z'_n), satisfying

$$(t'_n,z'_n)\to(T,z_0)\quad\text{and}\quad v_\kappa(t'_n,z'_n)\to\bar v_\kappa(z_0)\ \ ,\text{ where } t'_n<T \text{ for all } n\ \ . \tag{6.14}$$

3. For all n, we define the stopping times

$$\theta^\mu_n \;:=\; T\wedge\inf\{s>t'_n\ :\ (s,Z^\mu_n(s))\in\partial B_0\}\ \ ,\ \ \mu\in U_\kappa\ ,$$

where $Z^\mu_n:=Z^\mu_{t'_n,z'_n}$, together with $\mathcal{J}^\mu_n:=\{\theta^\mu_n<T\}$.

4. We can now prove the required contradiction. For fixed $\mu\in\mathcal{U}_\kappa$, we deduce from Itô's Lemma and (6.8) that

$$\varphi(t'_n,z'_n) \;\;\ge\;\; \mathbb{E}\left[\varphi(\theta^\mu_n,Z^\mu_n(\theta^\mu_n))\right]\ .$$

Recalling (6.9), (6.11), (6.10), we obtain

$$\varphi(t'_n, z'_n) \;\geq\; \mathbb{E}\left[\left(\hat{G}(Z^\mu_n(T)) + \varepsilon\right) \mathbf{1}_{(\mathcal{J}^\mu_n)^c} + \left(c\eta^2 - \alpha + v^*_\kappa(\theta^\mu_n, Z^\mu_n(\theta^\mu_n))\right) \mathbf{1}_{\mathcal{J}^\mu_n}\right] .$$

Since $v_\kappa(T,\cdot) = \hat{G}(\cdot)$, we deduce from the previous inequality and (6.12) that

$$\varphi(t'_n, z'_n) \;\geq\; \mathbb{E}\left[\varepsilon + v_\kappa(\theta^\mu_n, Z^\mu_n(\theta^\mu_n))\right] \;\geq\; \varepsilon + \mathbb{E}\left[\hat{G}(Z^\mu_n(T))\right] .$$

By (6.14) and definition of φ, we can also choose n such that that $v_\kappa(t'_n, z'_n) \geq \varphi(t'_n, z'_n) - \varepsilon/2$, so that

$$v_\kappa(t'_n, z'_n) \;\geq\; \varepsilon/2 + \mathbb{E}\left[\hat{G}(Z^\mu_n(T))\right]$$

which, by arbitrariness of $\mu \in \mathcal{U}_\kappa$, contradicts the definition of $v_\kappa(t'_n, z'_n)$.

5. It remains to prove (6.13) and (6.14). Clearly, $t_n \to T$. Let $\hat{z} \in \bar{B}(z_0, \eta) \cap [0,\infty)^d$ be such that $z_n \to \hat{z}$, along some subsequence. Then, by definition of f and z_0, we have

$$\begin{aligned} 0 &\geq (\bar{v}_\kappa - f)(\hat{z}) - (\bar{v}_\kappa - f)(z_0) \\ &\geq \limsup_{n\to\infty} (v^*_\kappa - \varphi)(t_n, z_n) + c|\hat{z} - z_0|^2 - (v^*_\kappa - \varphi)(s_n, \xi_n) \\ &\geq c|\hat{z} - z_0|^2 \geq 0 , \end{aligned}$$

where the third inequality is obtained by definition of (t_n, z_n). Then, $\hat{z} = z_0$ and, by continuity of φ and definition of (s_n, ξ_n), $v^*_\kappa(t_n, z_n) \to \bar{v}_\kappa(z_0)$. This also proves that

$$\lim_n \lim_k (t^k_n, z^k_n) = (T, z_0) \quad \text{and} \quad \lim_n \lim_k v_\kappa(t^k_n, z^k_n) = \bar{v}_\kappa(z_0) . \tag{6.15}$$

Now assume that $\text{card}\{(n,k) \in I\!N \times I\!N \;:\; t^k_n = T\} = \infty$. Since $v_\kappa(T,\cdot) = \hat{G}(\cdot)$ and $\hat{G}$ is continuous, there exists a subsequence, relabelled (t^k_n, z^k_n), such that

$$\limsup_n \limsup_k v_\kappa(t^k_n, z^k_n) \leq \hat{G}(z_0) .$$

Since by assumption $\hat{G}(z_0) < f(z_0) = \bar{v}_\kappa(z_0)$, this leads to a contradiction with (6.15). Hence, $\text{card}\{(n,k) \in I\!N \times I\!N \;:\; t^k_n = T\} < \infty$, and, using a diagonalization argument, we can construct a subsequence $(t'_n, z'_n)_n$ of $(t^k_n, z^k_n)_{n,k}$ satisfying (6.14).
□

6.2 The comparison theorem

Proposition 6.2 *Let V be an upper semicontinuous viscosity subsolution and U be a lower semicontinuous viscosity supersolution on $[0,T)\times I\!R^d_+$ of*

$$\inf_{\mu\in U_\kappa} -\mathcal{G}^\mu\varphi \;=\; 0\;.$$

Assume that V and U satisfy the linear growth condition

$$|V(t,z)|+|U(t,z)|\le K\ (1+|z|)\quad,\ (t,z)\in[0,T)\times I\!R^d_+\ ,\quad K>0\;.$$

Then,

$$V(T,.)\ \le\ U(T,.)\quad \textit{implies}\quad V\ \le\ U\quad \textit{on}\ [0,T]\times I\!R^d_+\;.$$

Proof. 1. Let λ be some positive parameter and consider the functions

$$u(t,z)\ :=\ e^{\lambda t}U(t,z)\quad \text{and}\quad v(t,z)\ :=\ e^{\lambda t}V(t,z)\;.$$

It is easy to verify that the functions u and v are, respectively, a lower semicontinuous viscosity supersolution and an upper semicontinuous viscosity subsolution on $[0,T)\times I\!R^d_+$ of

$$\lambda\varphi-\frac{\partial\varphi}{\partial t}-\sup_{\mu\in U_\kappa} Tr\left[\sigma^{\mu\prime}D^2_{zz}\varphi\sigma^\mu\right]\ =\ 0\;. \tag{6.16}$$

Moreover u and v satisfy

$$u(T,z)\ \ge\ v(T,z)\quad \text{for all}\quad z\ \in\ I\!R^d_+\ ,$$

as well as the linear growth condition

$$|v(t,z)|+|u(t,z)|\le A\ (1+|z|)\quad,\ (t,z)\in[0,T)\times I\!R^d_+\ ,\quad A>0\;. \tag{6.17}$$

Through the following steps of the proof we are going to show that $u\ge v$ on the entire domain $[0,T]\times I\!R^d_+$, which is equivalent to $U\ge V$ on $[0,T]\times I\!R^d_+$.

We argue by contradiction, and assume that for some (t_0,z_0) in $[0,T]\times I\!R^d_+$

$$0\ <\ \delta\ :=v(t_0,z_0)-u(t_0,z_0)\;.$$

2. Following Barles et al. (2003), we introduce the following functions. For some positive parameter α, we set

$$\phi_\alpha(z,z') \;=\; \left[1+|z|^2\right]\left[\varepsilon+\alpha|z'|^2\right] \quad \text{and} \quad \Phi_\alpha(t,z,z') \;=\; e^{L(T-t)}\phi_\alpha(z+z',z-z')\;.$$

Here, L and ε are positive constants to be chosen later and we don't write the dependence of ϕ_α, Φ_α and Ψ_α with respect to them.
By the linear growth condition (6.17), the upper semicontinuous function Ψ_α defined by

$$\Psi_\alpha(t,z,z') \;:=\; v(t,z)-u(t,z')-\Phi_\alpha(t,z,z')$$

is such that for all (t,z,z') in $[0,T]\times I\!R^{2d}_+$

$$\begin{aligned}\Psi_\alpha(t,z,z') \;&\le\; A\left(1+|z|+|z'|\right)-\min\{\varepsilon,\alpha\}\left(|z-z'|^2+|z+z'|^2+1\right)\\ &\le\; A\left(1+|z|+|z'|\right)-\min\{\varepsilon,\alpha\}\left(|z|^2+|z'|^2\right)\;.\end{aligned}$$

We deduce that Ψ_α attains its maximum at some $(t_\alpha,z_\alpha,z'_\alpha)$ in $[0,T]\times I\!R^{2d}_+$. The inequality $\Psi_\alpha(t_0,z_0,z_0)\le\Psi_\alpha(t_\alpha,z_\alpha,z'_\alpha)$ reads

$$\Psi_\alpha(t_\alpha,z_\alpha,z'_\alpha) \;\ge\; \delta-\varepsilon\left(1+4|z_0|^2\right)e^{LT}\;.$$

Hence, ε can be chosen sufficiently small (depending on L and $|z_0|$) so that

$$v(t_\alpha,z_\alpha)-u(t_\alpha,z'_\alpha) \;\ge\; \Psi_\alpha(t_\alpha,z_\alpha,z'_\alpha) \;\ge\; \delta-\varepsilon\left(1+4|z_0|^2\right)e^{LT} \;>\; 0\;. \qquad (6.18)$$

From (6.18) and (6.17), we get

$$\begin{aligned}0 \;\le\; \frac{\alpha}{2}|z_\alpha-z'_\alpha|^2+\frac{\varepsilon}{2}|z_\alpha+z'_\alpha|^2 \;&\le\; v(t_\alpha,z_\alpha)-u(t_\alpha,z'_\alpha)-\frac{\varepsilon}{2}|z_\alpha+z'_\alpha|^2-\frac{\alpha}{2}|z_\alpha-z'_\alpha|^2\\ &\le\; A\left(1+|z_\alpha|+|z'_\alpha|\right)-\min\left\{\frac{\varepsilon}{2},\frac{\alpha}{2}\right\}\left(|z_\alpha|^2+|z'_\alpha|^2\right)\;.\end{aligned}$$

We deduce that $\{\alpha|z_\alpha-z'_\alpha|\}_{\alpha>0}$ as well as $\{(z_\alpha,z'_\alpha)\}_{\alpha>0}$ are bounded. Therefore, after possibly passing to a subsequence, we can find $(\bar t,\bar z)\in[0,T]\times I\!R^d_+$ such that

$$(t_\alpha,z_\alpha,z'_\alpha)\to(\bar t,\bar z,\bar z) \quad \text{as} \quad \alpha\to\infty\;.$$

Since $v\,-\,u$ is upper semicontinuous, it follows from (6.18) that

$$v(\bar t,\bar z)-u(\bar t,\bar z) \;\ge\; \limsup_{\alpha\to\infty} v(t_\alpha,z_\alpha)-u(t_\alpha,z'_\alpha) \;\ge\; \delta-\varepsilon\left(1+4|z_0|^2\right)e^{LT} \;>\; 0\;.$$

Since $u(T,.) \geq v(T,.)$ on $\mathbb{R}^d_+$, $\bar{t}$ is in $[0,T)$, hence for α sufficiently large t_α is in $[0,T)$.

3. Let α be sufficiently large so that

$$|z_\alpha - z'_\alpha| \ < \ 1 \quad \text{and} \quad t_\alpha \ \in \ [0,T) \ .$$

Since $(t_\alpha, z_\alpha, z'_\alpha)$ is a maximum point of Ψ_α, by the fundamental result in the User's Guide to Viscosity Solutions (Theorem 8.3 in Crandall et al. 1993), for each $\eta > 0$, there are numbers a_1^η , a_2^η in $\mathbb{R}$, and symmetric matrices X^η and Y^η in $\mathbb{M}^d$ such that

$$\begin{aligned}
(a_1^\eta, D_z\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha), X^\eta) \ &\in \ \bar{\mathcal{P}}^{2,+}(v)(t_\alpha, z_\alpha) \ , \\
(a_2^\eta, -D_{z'}\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha), Y^\eta) \ &\in \ \bar{\mathcal{P}}^{2,+}(u)(t_\alpha, z'_\alpha) \ ,
\end{aligned}$$

with

$$a_1^\eta - a_2^\eta \ = \ \frac{\partial \Phi_\alpha}{\partial t}(t_\alpha, z_\alpha, z'_\alpha) \ = \ -L\,\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha) \ ,$$

and

$$\begin{pmatrix} X^\eta & 0 \\ 0 & -Y^\eta \end{pmatrix} \ \leq \ M + \eta M^2 \ , \quad \text{where} \quad M \ := \ D^2_{(z,z')}\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha) \ . \tag{6.19}$$

Since v is a viscosity subsolution and u is a viscosity supersolution of (6.16) on $[0,T) \times \mathbb{R}^d_+$, we must have

$$\begin{aligned}
\lambda v(t_\alpha, z_\alpha) - a_1^\eta - \frac{1}{2}\sup_{\mu \in U_\kappa} Tr\,[\sigma^\mu(t_\alpha, z_\alpha)' X^\eta \sigma^\mu(t_\alpha, z_\alpha)] \ &\leq \ 0 \ , \\
\lambda u(t_\alpha, z_\alpha) - a_2^\eta - \frac{1}{2}\sup_{\mu \in U_\kappa} Tr\,[\sigma^\mu(t_\alpha, z'_\alpha)' Y^\eta \sigma^\mu(t_\alpha, z'_\alpha)] \ &\geq \ 0 \ .
\end{aligned}$$

Taking the difference we get

$$\begin{aligned}
& \lambda\,(v(t_\alpha, z_\alpha) - u(t_\alpha, z'_\alpha)) + L\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha) \\
\leq \ & \frac{1}{2}\sup_{\mu \in U_\kappa} Tr\,[\sigma^\mu(t_\alpha, z_\alpha)' X^\eta \sigma^\mu(t_\alpha, z_\alpha)] - \frac{1}{2}\sup_{\mu \in U_\kappa} Tr\,[\sigma^\mu(t_\alpha, z'_\alpha)' Y^\eta \sigma^\mu(t_\alpha, z'_\alpha)] \\
\leq \ & \frac{1}{2}\sup_{\mu \in U_\kappa} \{Tr\,[\sigma^\mu(t_\alpha, z_\alpha)' X^\eta \sigma^\mu(t_\alpha, z_\alpha)] - Tr\,[\sigma^\mu(t_\alpha, z'_\alpha)' Y^\eta \sigma^\mu(t_\alpha, z'_\alpha)]\} \ .
\end{aligned} \tag{6.20}$$

Let $(e_i, i = 1, ..., d)$ be an orthonormal basis of $\mathbb{R}^d$, and for each μ in U_κ set

$$\xi_i^\mu \ := \ \sigma^\mu(t_\alpha, z_\alpha)e_i \quad \text{and} \quad \zeta_i^\mu \ := \ \sigma^\mu(t_\alpha, z'_\alpha)e_i$$

so that

$$
\begin{aligned}
& Tr\left[\sigma^\mu(t_\alpha,z_\alpha)' X^\eta \sigma^\mu(t_\alpha,z_\alpha)\right] - Tr\left[\sigma^\mu(t_\alpha,z'_\alpha)' Y^\eta \sigma^\mu(t_\alpha,z'_\alpha)\right] \\
= & \sum_{i=1}^d X^\eta \xi_i^\mu \cdot \xi_i^\mu - Y^\eta \zeta_i^\mu \cdot \zeta_i^\mu
\end{aligned}
$$

and, by (6.19),

$$
Tr\left[\sigma^\mu(t_\alpha,z_\alpha)' X^\eta \sigma^\mu(t_\alpha,z_\alpha)\right] - Tr\left[\sigma^\mu(t_\alpha,z'_\alpha)' Y^\eta \sigma^\mu(t_\alpha,z'_\alpha)\right] \;\le\; \sum_{i=1}^d (M+\eta M^2)\beta_i^\mu \cdot \beta_i^\mu \;,
$$

where β_i^μ is the $2d$-dimensional column vector defined by : $\beta_i^\mu := ({\xi_i^\mu}', {\zeta_i^\mu}')'$. Letting η go to zero, and using (6.20), we get

$$
\lambda\left(v(t_\alpha,z_\alpha) - u(t_\alpha,z'_\alpha)\right) + L\Phi_\alpha(t_\alpha,z_\alpha,z'_\alpha) \;\le\; \frac{1}{2}\sup_{\mu\in U_\kappa}\left\{\sum_{i=1}^d M\beta_i^\mu\cdot\beta_i^\mu\right\} . \tag{6.21}
$$

4. In this last step, we are going to see that, for a convenient choice of the positive constant L, inequality (6.21) leads to a contradiction to (6.18).
Notice that

$$
M \;=\; e^{L(T-t_\alpha)}\begin{pmatrix} D_{zz}\phi_\alpha + D_{z'z'}\phi_\alpha + 2D_{zz'}\phi_\alpha & D_{zz}\phi_\alpha - D_{z'z'}\phi_\alpha \\ D_{zz}\phi_\alpha - D_{z'z'}\phi_\alpha & D_{zz}\phi_\alpha + D_{z'z'}\phi_\alpha - 2D_{zz'}\phi_\alpha \end{pmatrix}
$$

the (partial) Hessian matrices of ϕ_α being taken at the point $(z_\alpha + z'_\alpha, z_\alpha - z'_\alpha)$. Then, for μ in U_κ

$$
\begin{aligned}
\sum_{i=1}^d M\beta_i^\mu\cdot\beta_i^\mu \;=\; & e^{L(T-t_\alpha)}\sum_{i=1}^d D_{zz}\phi_\alpha\,(\xi_i^\mu + \zeta_i^\mu)\cdot(\xi_i^\mu + \zeta_i^\mu) \\
& + 2D_{zz'}\phi_\alpha\,(\xi_i^\mu + \zeta_i^\mu)\cdot(\xi_i^\mu - \zeta_i^\mu) \\
& + D_{z'z'}\phi_\alpha\,(\xi_i^\mu - \zeta_i^\mu)\cdot(\xi_i^\mu - \zeta_i^\mu) \;.
\end{aligned} \tag{6.22}
$$

Since for each μ in U_κ, $|\mu|$ is bounded by κ, we verify using the Lipschitz property of the function $z \mapsto diag[z]\sigma(t,z)$ that for some positive constant C, for each z, z' in $I\!R_+^d$ and t in $[0,T]$

$$
|\sigma^\mu(t,z) - \sigma^\mu(t,z')| \;\le\; C|z-z'| \quad\text{and}\quad |\sigma^\mu(t,z)| \;\le\; C\left(1+|z|\right) .
$$

In the following C denotes a positive constant (independent of α, ε and L) which value may change from line to line.
4.1. Since α satisfies $|z_\alpha - z'_\alpha| \le 1$, for $i = 1, .., d$,

$$D_{zz}\phi_\alpha\,(\xi_i^\mu + \zeta_i^\mu)\cdot(\xi_i^\mu + \zeta_i^\mu) \;\le\; C|D_{zz}\phi_\alpha|\left[(1+|z_\alpha|)^2 + (1+|z'_\alpha|)^2\right] \;.$$

From $|D_{zz}\phi_\alpha| \le 2(\varepsilon + \alpha|z_\alpha - z'_\alpha|^2)$ and the previous estimate, we deduce that

$$D_{zz}\phi_\alpha\,(\xi_i^\mu + \zeta_i^\mu)\cdot(\xi_i^\mu + \zeta_i^\mu) \;\;\le\;\; C(\varepsilon + \alpha|z_\alpha - z'_\alpha|^2)\left(1 + |z_\alpha + z'_\alpha|^2\right) \;. \quad (6.23)$$

4.2. For $i = 1, .., d$

$$D_{zz'}\phi_\alpha\,(\xi_i^\mu + \zeta_i^\mu)\cdot(\xi_i^\mu - \zeta_i^\mu) \;\le\; C|D_{zz'}\phi_\alpha|\left[(1+|z_\alpha|) + (1+|z'_\alpha|)\right]|z_\alpha - z'_\alpha| \;.$$

Since $|D_{zz'}\phi_\alpha| \le 4\alpha|z_\alpha - z'_\alpha||z_\alpha + z'_\alpha|$ and $|z_\alpha - z'_\alpha| \le 1$, we deduce that

$$D_{zz'}\phi_\alpha\,(\xi_i^\mu + \zeta_i^\mu)\cdot(\xi_i^\mu - \zeta_i^\mu) \;\;\le\;\; C(\varepsilon + \alpha|z_\alpha - z'_\alpha|^2)\left(1 + |z_\alpha + z'_\alpha|^2\right) \;. \quad (6.24)$$

4.3. For $i = 1, .., d$

$$D_{z'z'}\phi_\alpha\,(\xi_i^\mu - \zeta_i^\mu)\cdot(\xi_i^\mu - \zeta_i^\mu) \;\le\; C|D_{z'z'}\phi_\alpha||z_\alpha - z'_\alpha|^2$$

and since $|D_{z'z'}\phi_\alpha| \le 2\alpha\,(1 + |z_\alpha + z'_\alpha|^2)$, we get

$$D_{z'z'}\phi_\alpha\,(\xi_i^\mu - \zeta_i^\mu)\cdot(\xi_i^\mu - \zeta_i^\mu) \;\;\le\;\; C(\varepsilon + \alpha|z_\alpha - z'_\alpha|^2)\left(1 + |z_\alpha + z'_\alpha|^2\right) \;. \quad (6.25)$$

Finally, collecting the estimates (6.23), (6.24) and (6.25), we deduce from (6.22) that for some positive constant $\tilde{C}$ (independent of L, ε and α)

$$\sum_{i=1}^{d} M\beta_i^\mu\cdot\beta_i^\mu \;\;\le\;\; \tilde{C}e^{L(t-t_\alpha)}(\varepsilon + \alpha|z_\alpha - z'_\alpha|^2)\left(1 + |z_\alpha + z'_\alpha|^2\right) \;\;=\;\; \tilde{C}\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha) \;.$$

Hence, if we take $L \ge \frac{\tilde{C}}{2}$, then (6.21) reads

$$\lambda(v(t_\alpha, z_\alpha) - u(t_\alpha, z'_\alpha)) \;\le\; (\frac{\tilde{C}}{2} - L)\Phi_\alpha(t_\alpha, z_\alpha, z'_\alpha) \;\le\; 0$$

which is in contradiction with (6.18).

□

6.3 Proof of Proposition 6.1

We first make use of Proposition 6.2 to obtain the intermediary inequality $u_* \geq \sup_{\kappa>0} v_\kappa^*$.

Corollary 6.1 *Under* $(H\lambda)$, $(H\sigma)$ *and* (Hg), *for each* $\kappa > 0$, *we have* $u_* \geq v_\kappa^*$ *on* $[0,T] \times I\!R_+^d$.

Proof. In view of Lemmas 6.1, 6.2, Corollary 5.1 and Proposition 6.2, it suffices to show that u_* and v_κ^* have linear growth. To check this condition for v_κ^*, it suffices to recall that Z^μ is a martingale and use assumption (Hg). We now consider u_*. First, recall from (Hg) that $\hat{G}$ has linear growth. Using Lemma 5.1, we deduce that, for each $(a,b) \in \mathcal{D}$ and $(t,y,z) \in [0,T] \times (0,\infty)^{n+d}$, we have

$$f\left(Y_{t,y,z}^{(a,b)}(T)\right) \cdot g\left(S_{t,s}(T)\right) \;\leq\; \hat{G}\left(Z_{t,y,z}^{(a,b)}(T)\right) \;\leq\; \delta\left(1 + \sum_{i=1}^{d} Z_{t,y,z}^{(a,b)^i}(T)\right)$$

where $s = \bar{F}(y)^{-1}z$ and δ is some positive constant. It follows from the definition of $u^{(a,b)}$ that

$$u(t,y,z) \;=\; \sup_{(a,b)\in\mathcal{D}} u^{(a,b)}(t,y,z) \leq \delta\left(1 + \sum_{i=1}^{d} z^i\right) . \tag{6.26}$$

Now, observe that, for $(a,b) = (0,0)$, $Y^{(0,0)}$ is constant so that $Z^{(0,0)^i}$ coincides with S^i up to a multiplicative constant, $i \leq d$. Hence, $Z^{(0,0)}$ is a $\mathbb{P}$-martingale and it follows from the definition of $u^{(0,0)}$ that

$$u(t,y,z) \;\geq\; u^{(0,0)}(t,y,z) \;\geq\; \mathbb{E}\left[f\left(y\right) \cdot g\left(S_{t,s}(T)\right)\right] \;,\; s = \bar{F}(y)^{-1}z \;.$$

Using Lemma 5.1 and (3.3), we then deduce as above that

$$u(t,y,z) \;\geq\; u^{(0,0)}(t,y,z) \;\geq\; -\hat{\delta}\left(1 + \sum_{i=1}^{d} z^i\right)$$

for some positive constant $\hat{\delta}$. Combining the last inequality with (6.26) shows that u_* has linear growth. □

Proof of Proposition 6.1. 1. We first show that $\{Z_{0,z}^\mu \;,\; \mu \in \cup_{\kappa\geq 0}\mathcal{U}_\kappa\}$ is dense in probability in $\{Z_{0,z}^\mu \;,\; \mu \in \mathcal{U}\}$. To see this, take $\mu \in \mathcal{U}$ and consider the sequence defined by $\mu_\kappa := \mu \mathbf{1}_{|\mu|\le\kappa} \in \mathcal{U}_\kappa$, $\kappa \in I\!N$. Recalling that $Z^{\mu,i}_{0,z} = S^i_{0,z}$ for $i \le d_f$, see assumption $(H\sigma)$, we deduce from the Itô's isometry that

$$\mathbb{E}\left[\sum_{i=1}^{d} |\ln(Z^{\mu,i}_{0,z}(T)) - \ln(Z^{\mu_\kappa,i}_{0,z}(T))|^2\right]$$
$$\le \delta\mathbb{E}\left[\int_0^T |\mu(t)-\mu_\kappa(t)|^2 + ||\mu(t)|^2 - |\mu_\kappa(t)|^2|\,dt\right]$$

for some $\delta > 0$. Since $\mu_\kappa \to \mu$ dt $\times$ d$\mathbb{P}$-a.e. and, by definition of $\mathcal{U}$, μ is square integrable, we deduce from the dominated convergence theorem that $\ln(Z^{\mu_\kappa,i}_{0,z}(T))$ goes to $\ln(Z^{\mu,i}_{0,z}(T))$ in L^2, $i \le d$. It follows that the convergence holds $\mathbb{P}-$a.s. along some subsequence.

2. By Corollary 6.1, we have

$$u_*(0,z) \;\ge\; \sup_{\kappa>0}\sup_{\mu\in\mathcal{U}_\kappa} \mathbb{E}\left[\hat G(Z^{\mu}_{0,z}(T))\right] .$$

Since $\hat G$ has linear growth, see (Hg), there is some $(c,\Delta) \in I\!R \times I\!R^d$ such that $\hat G(Z^\mu_{0,z}(T)) + \Delta\cdot Z^\mu_{0,z}(T) \ge -c$. Since $Z^\mu_{0,z}$ is a martingale, it follows that

$$u_*(0,z) \;\ge\; \sup_{\kappa>0}\sup_{\mu\in\mathcal{U}_\kappa} \mathbb{E}\left[\hat G(Z^{\mu}_{0,z}(T)) + \Delta\cdot Z^\mu_{0,z}(T)\right] - \Delta\cdot z .$$

Using 1. and Fatou's Lemma, we then deduce that

$$u_*(0,z) \;\ge\; \sup_{\mu\in\mathcal{U}} \mathbb{E}\left[\hat G(Z^{\mu}_{0,z}(T)) + \Delta\cdot Z^\mu_{0,z}(T)\right] - \Delta\cdot z .$$

Since for $\mu \in \mathcal{U}$, $Z^\mu_{0,z}$ is also a martingale, the result follows. □

7 Proof of Theorem 4.1 and Corollary 4.1

In order to conclude the proof, we shall now exploit the concavity property of Corollary 5.1.

Remark 7.1 Fix $\mu \in \mathcal{U}$, $t \le T$ and $(s^f, z^c) \in (0,\infty)^{d_f} \times (0,\infty)^{d_c}$. Then, it follows from assumption $(H\sigma)$ that $Z^{\mu,f}_{t,(s^f,z^c)} = S^f_{t,(s^f,z^c)}$ which depends only on (t, s^f). On the other hand $Z^{\mu,c}_{t,(s^f,z^c)}$ depends only on (t, z^c, μ). In the following, we shall then slightly abuse notations and simply write S^f_{t,s^f} for $S^f_{t,s}$ and $Z^{\mu,c}_{t,z^c}$ for $Z^{\mu,c}_{t,(s^f,z^c)}$.

Corollary 7.1 *Let the conditions* $(H\lambda)$, $(H\sigma)$ *and* (Hg) *hold. Then, for all* $(s^f, z^c) \in (0,\infty)^{d_f} \times (0,\infty)^{d_c}$ *and* $\Delta \in \partial_{z^c} u_*(0, s^f, z^c)$, *we have*

$$u_*(0, s^f, z^c) \;\geq\; \mathbb{E}\left[\sup_{\tilde{z}^c \in (0,\infty)^{d_c}} \left\{\hat{G}(S^f_{0,s^f}(T), \tilde{z}^c) - \Delta \cdot \tilde{z}^c\right\}\right] + \Delta \cdot z^c \,.$$

Proof. By definition of Δ and Corollary 5.1, we have

$$u_*(0, s^f, z^c) \;=\; \sup_{\tilde{z}^c \in (0,\infty)^{d_c}} \left\{u_*(0, s^f, \tilde{z}^c) - \Delta \cdot (\tilde{z}^c - z^c)\right\} \,.$$

Since, for each $\tilde{z}^c \in (0,\infty)^{d_c}$ and $\mu \in \mathcal{U}$, $\mathbb{E}\left[Z^{\mu,c}_{0,\tilde{z}^c}\right] = \tilde{z}^c$, it follows from Proposition 6.1 that

$$u_*(0, s^f, z^c) \;\geq\; \sup_{\tilde{z}^c \in (0,\infty)^{d_c}} \sup_{\mu \in \mathcal{U}} \mathbb{E}\left[\hat{G}\left(S^f_{0,s^f}(T), Z^{\mu,c}_{0,\tilde{z}^c}(T)\right) - \Delta \cdot Z^{\mu,c}_{0,\tilde{z}^c}(T)\right] + \Delta \cdot z^c \,.$$

Since $\hat{G}$ is continuous, we deduce from the representation theorem that

$$\begin{aligned} u_*(0, s^f, z^c) \;&\geq\; \sup_{\xi \in L^\infty(\mathcal{B}_\kappa;\mathcal{F}_T)} \mathbb{E}\left[\hat{G}(S^f_{0,s^f}(T), \xi) - \Delta \cdot \xi\right] + \Delta \cdot z^c \\ &\geq\; \mathbb{E}\left[\max_{\tilde{z}^c \in \mathcal{B}_\kappa} \hat{G}(S^f_{0,s^f}(T), \tilde{z}^c) - \Delta \cdot \tilde{z}^c\right] + \Delta \cdot z^c \,, \end{aligned}$$

where $\mathcal{B}_\kappa := \{\alpha \in (0,\infty)^{d_c} \;:\; |\ln(\alpha^i)| \leq \kappa \;,\; i \leq d_c\}$. The result then follows from monotone convergence. □

We can now conclude the proof of our main result.

Proof of Theorem 4.1. For ease of notations, we only write S for $S_{0,S(0)}$ and $s = (s^f, s^c)$ for $S(0)$. In view of Theorem 5.1, we have

$$p(0, S(0)) \;\geq\; \sup_{\xi \in \Lambda} u_*(0, s^f, \mathrm{diag}[\underline{\xi}] s^c) \,. \tag{7.1}$$

1. Recalling that Λ is compact and u_* is concave in its last d_c variables, there is some $\hat{\xi} \in \Lambda$ which attains the optimum in the above inequality. Moreover, by standards arguments of calculus of variations, we can find some $\hat{\Delta} \in \partial_{z^c} u_*(0, s^f, \mathrm{diag}[\underline{\hat{\xi}}] s^c)$ such that

$$(\mathrm{diag}[\hat{\Delta}] s^c) \cdot (\underline{\hat{\xi}} - \underline{\xi}) \;\geq\; 0 \text{ for all } \xi \in \Lambda \,. \tag{7.2}$$

From (4.2), we deduce that $(\mathrm{diag}[\hat{\Delta}]s^c \cdot \underline{\hat{\xi}}, 0) \succeq (0, \mathrm{diag}[\hat{\Delta}]s^c)$.
2. Set

$$\delta \ := \ \mathrm{diag}[\hat{\Delta}]s^c \cdot \underline{\hat{\xi}} \quad \text{and} \quad \hat{C}(S^f(T)) \ := \ \sup_{\tilde{z}\in(0,\infty)^{d_c}} \hat{G}(S^f(T), \tilde{z}) - \hat{\Delta}\cdot\tilde{z} \ ,$$

so that, by (7.1) and Corollary 7.1,

$$p(0, S(0)) \ \geq \ p \ := \ \mathbb{E}\left[\hat{C}(S^f(T))\right] + \delta \ . \tag{7.3}$$

Since by $(H\sigma)$ the dynamics of S^f depends only on S^f, it follows that there is some $\mathbb{R}^{d_f}$-valued predictable process ϕ satisfying $\int_0^T |\phi(t)|^2 dt < \infty$ such that

$$X^f(\cdot) \ := \ p - \delta + \int_0^{\cdot} \phi(t)\cdot dS^f(t) \text{ is a martingale and } X^f(T) \ = \ \hat{C}(S^f(T)) \ . \tag{7.4}$$

3. By combining 1. and 2., we deduce that there is some strategy (ϕ, L) such that $L(t) = L(0)$, $X^{\phi,L}_{p\mathbf{1}_1}(0) = (p-\delta, \mathrm{diag}[\hat{\Delta}]s^c)$ and

$$X^{\phi,L}_{p\mathbf{1}_1}(T) \ = \ \Big(X^f(T), \mathrm{diag}[\hat{\Delta}]S^c(T)\Big) \ \succeq \ \Big(\hat{C}(S^f(T)), \mathrm{diag}[\hat{\Delta}]S^c(T)\Big) \ .$$

Using (7.4), (4.2) and the definition of $\hat{C}$ this implies that

$$\tilde{\xi}\cdot X^{\phi,L}_{p\mathbf{1}_1}(T) - \xi\cdot g\left(S^f(T), \mathrm{diag}[\underline{\xi}]^{-1}\mathrm{diag}[\underline{\tilde{\xi}}]S^c(T)\right) \ \geq \ 0 \quad \text{for all } \ \tilde{\xi}, \ \xi\in\Lambda \ .$$

Considering the case where $\xi = \tilde{\xi}$ and using (4.2) leads to

$$X^{\phi,L}_{p\mathbf{1}_1}(T) \ \succeq \ g(S(T)) \ .$$

In view of (7.3) it remains to check that $(\phi, L) \in \mathcal{A}$, but this readily follows from (7.4) and assumption (3.3). □

Proof of Corollary 4.1. In view of Theorem 4.1, we only have to show that

$$p(0, S(0)) \ \leq \ \inf_{\Delta\in\mathbb{R}^{d_c}} \mathbb{E}\left[C(S^f(T);\Delta)\right] + \sup_{\xi\in\Lambda}\underline{\xi}\cdot\mathrm{diag}[\Delta]S^c(0) \ .$$

To see this, fix some $\Delta \in \mathbb{R}^{d_c}$ such that

$$\tilde{p} \ := \ \mathbb{E}\left[C(S^f(T);\Delta)\right] + \sup_{\xi\in\Lambda}\underline{\xi}\cdot\mathrm{diag}[\Delta]S^c(0) \ < \ \infty \ ,$$

which is possible by Theorem 4.1. Then, by the same argument as in the proof of Theorem 4.1, see above, we obtain that there exists some $(\phi, L) \in \mathcal{A}^{BH}$, with $L(t) = \Delta$ on $t \leq T$, such that $X^{\phi,L}_{\tilde{p}\mathbf{1}_1} \succeq g(S(T))$. This proves the required inequality as well as the last statement of the Corollary. □

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