Asymptotics in Knuth's parking problem for caravans

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Abstract

We consider a generalized version of Knuth's parking problem, in which caravans consisting of a random number of cars arrive at random on the unit circle. Then each car turns clockwise until it finds a free space to park. Extending a recent work by Chassaing and Louchard [8], we relate the asymptotics for the sizes of blocks formed by occupied spots with the dynamics of the additive coalescent. According to the behavior of the caravan's size tail distribution, several qualitatively different versions of eternal additive coalescent are involved.

Keywords: Parking problem, additive coalescent, bridges with exchangeable increments.

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1 Introduction

The original parking problem of Knuth can be stated as follows. Consider a parking lot with n spaces, identified with the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Initially the parking lot is empty, and $m \leq n$ cars in a queue arrive one by one. Car i tries to park on a uniformly distributed space L_i among the n possible, independently of other cars, but if the space is already occupied, then it tries places labeled $L_i + 1, L_i + 2, \ldots$ until it finally finds a free spot to park. As cars arrive, blocks of consecutive occupied spots are forming. It appears that a *phase transition* occurs at the stage where the parking lot is almost full, more precisely when the number of free spots is of order $\sqrt{n}$. Indeed, while the largest block of occupied spots is of order $\log m$ with high probability as long as $\sqrt{m} = o(n - m)$, a block of size approximately n is present (while the others are of order at most $\log n$) with high probability when $n - m = o(\sqrt{m})$. In the meanwhile, precisely when $n - m$ is of order $\lambda\sqrt{m}$ with $\lambda > 0$, a clustering phenomenon occurs as λ decays. The behavior of this clustering process has been studied precisely by Chassaing and Louchard [8]. It turns out that the process of the relative sizes of occupied blocks is related to the so-called *standard additive coalescent* [10, 1].

The model originates from a problem in Computer Science: spaces in the parking lot should be thought of as elementary memory spaces, each of which can be used to store elementary data (cars). Roughly, our aim in this work is to investigate the more general situation where one wants to store larger files, each requiring several elementary memory spaces. In other words, single cars are replaced by *caravans* of cars, i.e. several cars may arrive simultaneously at the same spot. In this direction, it will be convenient to consider a continuous version of the problem, that goes as follows. Let $p_1, \ldots p_m$ be a sequence of positive real numbers with sum 1, and $s_1, \ldots, s_m$, m distinct locations on the unit circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Imagine that m drops of paint with masses $p_1, \ldots, p_m$, fall successively at locations $s_1, \ldots, s_m$. Each time a drop of paint falls, we brush it clockwise in such a way that the resulting painted portion of $\mathbb{T}$ is covered by a unit density of paint. So at each step the drop of paint is used to cover a new portion of the circle and the total length of the painted part of the circle when $i \leq m$ drops have fallen is $p_1 + \cdots + p_i$. In this setting, drops of paint play the role of caravans, and the painted portion of the circle corresponds to occupied spots in the parking lot.

More precisely, we consider an increasing sequence $(A_0, \ldots, A_m)$ of open subsets of $\mathbb{T}$, starting from $A_0 = \varnothing$ and ending at $A_m = \mathbb{T}$, which can be thought of as the successive painted portions of the circle. Given A_i and the location s_{i+1} from where the $i + 1$-th drop of paint will be brushed, we paint as many space as possible to the right of s_{i+1} with the quantity p_{i+1} of paint, without covering the already painted parts, i.e. the blocks of A_i. Alternatively, we break the $i+1$-th caravan into several pieces, so that to fill as much as possible the holes left by $\mathbb{T} \setminus A_i$ after s_{i+1}, when reading in clockwise order. The last car to park arrives at some location t_{i+1}, and we let A_{i+1} be the union of A_i and the arc between s_{i+1} and t_{i+1}, see Figure 1. More formal definitions will come in Sect. 2.

In particular, A_i is a disjoint union of open intervals and $\mathrm{Leb}(A_i) = p_1 + \ldots + p_i$. Let $\Lambda^{\mathbf{p}}(i)(= \Lambda(p_1, \ldots, p_m, s_1, \ldots, s_m, i))$ be the sequence of the Lebesgue measures of the connected components of A_i, ranked by decreasing order. It will be convenient to view $\Lambda^{\mathbf{p}}(i)$ as an infinite sequence, by completing with an infinite number of zero terms.

Now consider the following random problem. Let $\ell > 0$ be a random variable with

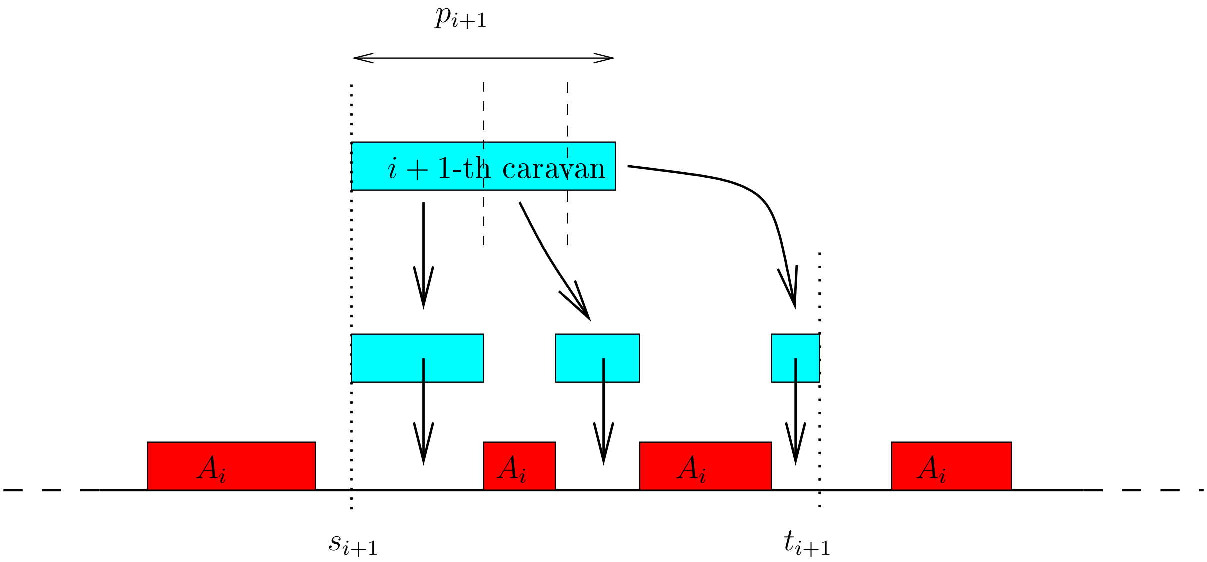

Figure 1: Arrival, splitting and parking of the $i+1$-th caravan in the process

finite expectation $\mu_1 = \mathbb{E}[\ell]$. We say that $\ell \in \mathcal{D}_2$ whenever ℓ has a finite second moment $\mu_2 = \mathbb{E}[\ell^2]$. For $\alpha \in (1,2)$, we say that $\ell \in \mathcal{D}_\alpha$ whenever

$$\mathbb{P}(\ell > x) \underset{x\to\infty}{\sim} cx^{-\alpha} \tag{1}$$

for some $0 < c < \infty$. This implies that ℓ is in the domain of attraction of a spectrally positive stable random variable with index α, and we stress that our results can be extended under this more general hypothesis; (1) is only intended to make things easier. We suppose from now on that $\ell \in \mathcal{D}_\alpha$ for some $\alpha \in (1,2]$, and take a random iid sample $\ell_1, \ell_2, \ldots$ of variables distributed as ℓ, and independently of this sequence, iid uniform random variables on $[0,1)$, $U_1, U_2, \ldots$. For $\varepsilon > 0$, set

$$T_\varepsilon = \inf\{i : \ell_1 + \ldots + \ell_i \geq 1/\varepsilon\},$$

so by the elementary renewal theorem, $T_\varepsilon \sim 1/(\varepsilon\mu_1)$. Then introduce the sequence $(\ell_i^*, 1 \leq i \leq T_\varepsilon)$ defined by

$$\ell_i^* = \ell_i \text{ for } 1 \leq i \leq T_\varepsilon - 1 \quad \text{and} \quad \ell_{T_\varepsilon}^* = \varepsilon^{-1} - (\ell_1 + \ldots + \ell_{T_\varepsilon - 1}),$$

so the terms of ℓ^* sum to $1/\varepsilon$.

Following Chassaing and Louchard [8], we are interested in the formation of macroscopic painted components in the limit when ε tends to 0, at times close to T_ε, i.e. when the circle is almost entirely painted. Specifically, we let

$$\mathbf{X}^{(\varepsilon)}(t) = \Lambda^{\mathrm{P}}(T_\varepsilon - \lfloor t\varepsilon^{-1/\alpha} \rfloor), \qquad t \geq 0,$$

for Λ^{P} defined as above with the data $m = T_\varepsilon, p_i = \varepsilon\ell_i^*, s_i = U_i$. Observe that $T_\varepsilon - [t\varepsilon^{-1/\alpha}]$ decreases when t increases, and therefore, in order to investigate the formation of painted components, we should consider the process $\big(\mathbf{X}^{(\varepsilon)}(t), t \geq 0\big)$ backwards in time. This is what we shall do in Theorem 1, using the exponential time change $t \to e^{-t}$.

Before describing our main result, let us first recall some features of the additive coalescent. The additive coalescent $\mathbf{C}$ is a Markov process with values in the infinite ordered simplex

$$S = \left\{ \mathbf{s} = (s_1, s_2, \dots) : s_1 \geq s_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} s_i \leq 1 \right\}$$

endowed with the uniform distance, whose evolution is described formally by: given that the current state is $\mathbf{s}$, two terms s_i and s_j, $i < j$, of $\mathbf{s}$ are chosen and merge into a single term $s_i + s_j$ (which implies some reordering of the resulting sequence) at a rate equal to $s_i + s_j$. A version $(\mathbf{C}(t), t \in \mathbb{R})$ of this process defined for times describing the whole real axis is called *eternal*. We refer to [1, 3] for background.

As shown in [4], eternal additive coalescents can be encoded by certain bridges. Specifically, let $B = (B(x), 0 \leq x \leq 1))$ be a càdlàg real-valued process with exchangeable increments, such that $B(0) = B(1) = 0$. Suppose further that B has infinite variation and no negative jumps a.s. Then B attains its overall infimum at a unique location V (which is uniformly distributed on $[0, 1]$), and B is continuous at V. Consider the so-called *Vervaat transform* which maps the bridge B into an excursion $\mathcal{E}$ defined by

$$\mathcal{E}(x) = B(V + x) - B(V) \quad \text{for } 0 \leq x \leq 1,$$

where the addition $V + x$ is modulo 1. Finally, we let for $t \geq 0$

$$\mathcal{E}^{(t)}_x = \mathcal{E}(x) - tx \quad \text{for } 0 \leq x \leq 1,$$

and introduce $\mathbf{F}(t)$ as the random element of S defined by the ranked sequence of the lengths of the constancy intervals of the process $\underline{\mathcal{E}}^{(t)} = (\inf_{0 \leq y \leq x} \mathcal{E}^{(t)}(y), 0 \leq x \leq 1)$. Here, a constancy interval means a connected component of the complement of the support of the Stieltjes measure $\mathrm{d}(-\underline{\mathcal{E}})$. Finally, if we define $\mathbf{C}(t) = \mathbf{F}(e^{-t})$, then $\mathbf{C} = (\mathbf{C}(t), -\infty < t < \infty)$ is an eternal additive coalescent (see Section 6.1 for comments and details).

In this work, eternal additive coalescent associated to certain remarkable bridges will play a key role. More precisely, we write $\mathbf{C}^{(2)} = (\mathbf{C}^{(2)}(t), -\infty < t < \infty)$ for the eternal additive coalescent $\mathbf{C}$ constructed above when $B = \mathcal{B}^{(2)}$ is a standard Brownian bridge; so that $\mathbf{C}^{(2)}$ is the so-called standard additive coalescent (cf. [4, 1]). Next, for $1 < \alpha < 2$, we denote by $\sigma^{(\alpha)} = (\sigma^{(\alpha)}(t), t \geq 0)$ a standard spectrally positive stable Lévy process with index α, that is $\sigma^{(\alpha)}$ has independent and stationary increments, no negative jumps, and

$$\mathbb{E}(\exp(-\lambda \sigma^{(\alpha)}(t))) = \exp(t\lambda^{\alpha}), \qquad \text{for all } \lambda \geq 0.$$

We call *standard stable loop* [1] of index α the process $\mathcal{B}^{(\alpha)}$ defined by

$$\mathcal{B}^{(\alpha)}(x) = \sigma^{(\alpha)}(x) - x\sigma^{(\alpha)}(1), \qquad \text{for } 0 \leq x \leq 1. \tag{2}$$

We finally write $\mathbf{C}^{(\alpha)} = (\mathbf{C}^{(\alpha)}(t), -\infty < t < \infty)$ for the eternal additive coalescent $\mathbf{C}$ constructed above when the bridge B is the standard stable loop of index α.

We are now able to state our main result.

[1] We call $\mathcal{B}^{(\alpha)}$ a *loop* and not a *bridge* to avoid a possible confusion: even though $\mathcal{B}^{(\alpha)}$ starts from 0, ends at 0 and has exchangeable increments, it does not have the same law as the stable process $\sigma^{(\alpha)}$ conditioned on $\sigma^{(\alpha)}(1) = 0$!

Theorem 1 *The process* $(\mathbf{X}^{(\varepsilon)}(t), 0 \le t < T_\varepsilon)$ *converges as* $\varepsilon \downarrow 0$ *in the sense of weak convergence of finite-dimensional distributions to some process* $\mathbf{X} = (\mathbf{X}(t), 0 \le t < \infty)$. *The exponential time-changed process* $(\mathbf{X}(e^{-t}), -\infty < t < \infty)$ *is an eternal additive coalescent; more precisely:*
(i) *When* $\alpha = 2$, $(\mathbf{X}(e^{-t}), -\infty < t < \infty)$ *is distributed as*

$$(\mathbf{C}^{(2)}(t + \frac{1}{2}\log(\mu_2/\mu_1) - \log\mu_1), -\infty < t < \infty).$$

(ii) *When* $1 < \alpha < 2$, $(\mathbf{X}(e^{-t}), -\infty < t < \infty)$ *is distributed as*

$$(\mathbf{C}^{(\alpha)}(t + \frac{1}{\alpha}\log\left(\frac{\Gamma(2-\alpha)c}{(\alpha-1)\mu_1}\right) - \log\mu_1), -\infty < t < \infty).$$

It might be interesting to discuss further the role of the parameter α and the interpretation in terms of phase transition. As it was already mentioned, the renewal theorem entails than the number of drops of paint needed for the complete covering is $T_\varepsilon \sim 1/(\varepsilon\mu_1)$, a quantity which is not sensitive to α. It is easy to show that for every $a < 1$, there are no macroscopic painted components when only $[aT_\varepsilon]$ drops of paint have fallen, so the phase transition (i.e. the number of drops which is needed for the appearance of macroscopic components) occurs for numbers close to T_ε. More precisely, the regime for the phase transition is of order $T_\varepsilon - \varepsilon^{-1/\alpha}$; so the phase transition occurs closer to T_ε when α is larger. We would like also to stress that one-dimensional distributions of the limiting additive coalescent process $\mathbf{X}$ depend on α, but not its semigroup which is the same for all $\alpha \in (1, 2]$. A heuristic explanation might be the following: the number of drops needed to complete the covering once the phase transition has occurred is too small (of order $\varepsilon^{-1/\alpha}$) to observe significant differences in the dynamics of aggregation of macroscopic painted components.

Remark. Our model bears some similarity with another parking problem on the circle, where drops of paints fall uniformly on the circle and then are brushed clockwise, but where overlaps are now allowed (some points may be covered this way several times), call it the "random covering of an interval" problem. However, as showed in [6], this last model has very different asymptotics from those of the parking problem, as it turns out that the random covering of an interval is related to Kingman's coalescent rather than the additive coalescent. A shared feature is that the phase transition of the random covering problem appears also when the circle is almost completely covered, but for example the different fragments are ultimately finite in number rather than infinite.

We also mention yet another parking problem, first considered by Rényi (see [15, 9]). In can be formulated as follows: caravans with size ε are placed on $\mathbb{T}$ (the original work rather considers $(0, 1)$) one after another, but the locations s_i where cars park are chosen uniformly among spaces that do not induce overlaps and splitting of caravans, i.e. so that the length of the arc from s_i to t_i is exactly ε. This is done until no uncovered sub-arc of $\mathbb{T}$ with size $\ge \varepsilon$ remains. This process does not involve coalescing blocks of cars, and one is rather interested in the properties of the random number of cars that are able to park.

The method in [8] relies on an encoding *parking* function which is shown to be asymptotically related to a function of standard Brownian bridge, and a representation of the standard additive coalescent due to Bertoin [4]. Our approach to Theorem 1 is close in spirit to that of [8], and uses the representation of eternal additive coalescent that we presented above; we briefly sketch it here. First, we encode the process $\mathbf{X}^{(\varepsilon)}$ by a bridge with exchangeable increments in Sect. 2. In Sect. 3, we show that this bridge converges to some bridge with exchangeable increments that can be represented in terms of the standard Brownian bridge (for $\alpha = 2$) or the standard stable loop (for $1 < \alpha < 2$). Theorem 1 then follows readily.

The rest of this work is organized as follows. In Section 2 we provide a representation of the painted components in terms of a bridge and its Vervaat's transform. The convergence of these bridges when ε tends to 0 is established in Section 3, and that of the sequence of the sizes of the painted components in Section 4. Section 5 is devoted to a brief discussion of the analogous discrete setting (i.e. Knuth's parking for caravans), and finally some complements are presented in Section 6.

2 Bridge representation

We develop a representation of the parking process with the help of bridges with exchangeable increments, which is crucial to our study.

Let us first give the proper definition the of sequence ($\varnothing = A_0, \ldots, A_m$) of the Introduction. We identify the circle $\mathbb{T}$ with $[0, 1)$ and write $p_{\mathbb{T}} : \mathbb{R} \to \mathbb{T}$ for the canonical projection. If A is a measurable subset of $\mathbb{T}$ (identified with $[0, 1)$), let F_A be its repartition function defined by $F_A(x) = \mathrm{Leb}([0, x] \cap A)$ for $0 \leq x < 1$, where Leb is Lebesgue measure. Also, extend F_A on the whole real line with the formula $F_A(x + 1) = F_A(x) + F_A(1-)$. Given A_i for some $0 \leq i \leq m - 1$, let

$$t_{i+1} = \inf\{x \geq s_{i+1} : F_{A_i}(x) + p_{i+1} - (x - s_{i+1}) \leq F_{A_i}(s_{i+1})\}.$$

Notice that the arc $p_{\mathbb{T}}((s_{i+1}, t_{i+1}))$ oriented clockwise from s_{i+1} to $p_{\mathbb{T}}(t_{i+1})$ has length $t_{i+1} - s_{i+1} \geq p_{i+1}$. Then let A_{i+1} be the interior of the closure of $p_{\mathbb{T}}((s_{i+1}, t_{i+1})) \cup A_i$. The point in taking the closure and then the interior is that we consider that two painted connected components of $\mathbb{T}$ that are at distance 0 constitute in fact a single painted connected component.

Define

$$h^{\mathbf{p}}_{i+1}(x) = F_{A_i}(x) - F_{A_i}(s_{i+1}) + p_{i+1} - (x - s_{i+1}) \qquad s_{i+1} \leq x \leq t_{i+1},$$

and $h^{\mathbf{p}}_{i+1}(x) = 0$ in $[t_{i+1}, s_{i+1} + 1)$, so $h^{\mathbf{p}}_i$ is a càdlàg function (right-continuous with left-limits) on $[s_{i+1}, s_{i+1} + 1)$. Consider it as a function on $\mathbb{T}$ by letting $h^{\mathbf{p}}_{i+1}(x) = h^{\mathbf{p}}_{i+1}(y)$ where y is the element of $[s_{i+1}, s_{i+1} + 1) \cap p_{\mathbb{T}}^{-1}(x)$. The quantity $h^{\mathbf{p}}_{i+1}(x)$ can be thought of as the quantity of cars of the $i + 1$-th caravan that try to park at x. See Figure 2.

We consider the *profile*

$$H^{\mathbf{p}}_i = \sum_{j=1}^{i} h^{\mathbf{p}}_j \tag{3}$$

of the parking at step $0 \leq i \leq m$, so $H^{\mathbf{p}}_i(x)$ is the total quantity of cars that have tried (successfully or not) to park at x (with the convention that $H^{\mathbf{p}}_i(1) = H^{\mathbf{p}}_i(0)$) before the $i + 1$-th caravan has arrived.

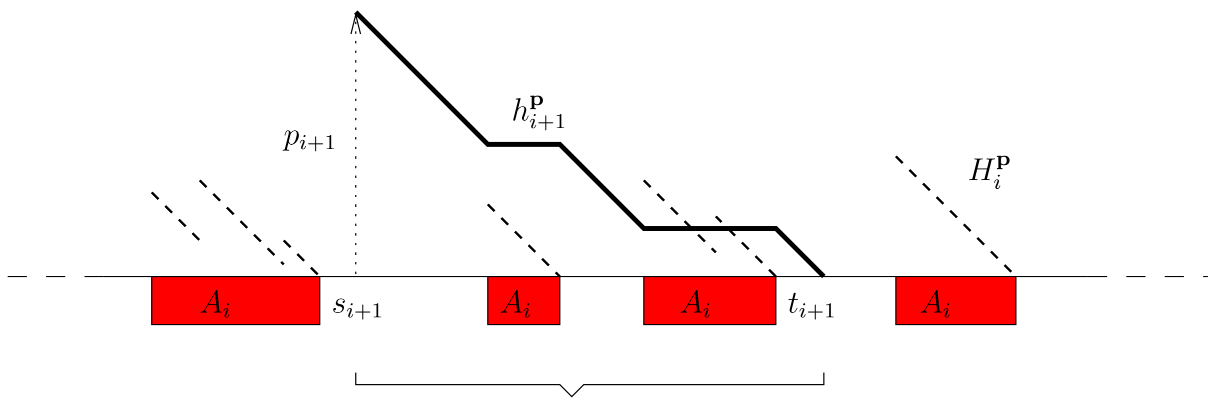

Figure 2: The function $h^{\mathbf{P}}_{i+1}$ (thick line) corresponding to the $i+1$-th caravan of Figure 1. The blocks of A_i are represented under the axis, and the dashed lines represent the profile $H^{\mathbf{P}}_i$ (it gives more information than A_i alone). The bracket under the figure indicates how A_{i+1} is obtained by formation of a new block comprising the blocks of A_i between s_{i+1} and t_{i+1}

Lemma 1 *For* $1 \le i \le m$,
(i) *the set* A_i *is the interior of the support of* $H^{\mathbf{P}}_i$.
(ii) $H^{\mathbf{P}}_i(t_i-) = 0$.
(iii) $H^{\mathbf{P}}_i$ *jumps at times* $s_1, \ldots, s_i$ *with respective jump magnitudes* $p_1, \ldots, p_i$, *and has a drift with slope* -1 *on its support. That is, if* $[v, v'] \subseteq \operatorname{supp}(H^{\mathbf{P}}_i)$,

$$H^{\mathbf{P}}_i(x+v) = H^{\mathbf{P}}_i(v-) - x + \sum_{j=1}^{i} p_j \mathbb{1}_{\{v \le s_j \le v+x\}} \qquad 0 \le x \le v' - v.$$

Proof. Properties (i) and (iii) are easily shown using a recursion on i and splitting the behavior of $h^{\mathbf{P}}_i$ on A_{i-1} and $A_i \setminus A_{i-1}$. We give some details for (ii). For $i \ge 1$, notice that by definition t_i cannot be a point of increase of $F_{A_{i-1}}$, i.e. a point such that $F_{A_{i-1}}(t_i - \varepsilon) < F_{A_{i-1}}(t_i) < F_{A_{i-1}}(t_i + \varepsilon)$ for every $\varepsilon > 0$. Therefore, $t_i \notin A_{i-1}$ and $h^{\mathbf{P}}_j(t_i) = h^{\mathbf{P}}_j(t_i-) = 0$ for $j < i$. Since it follows by continuity of $F_{A_{i-1}}$ that $h^{\mathbf{P}}_i(t_i-) = 0$, (ii) is proved. □

Consider the *bridge function*:

$$b^{\mathbf{P}}_i(x) = -x + \sum_{j=1}^{i} p_j \mathbb{1}_{\{x \ge s_j\}} \qquad 0 \le x < 1,$$

which starts from $b^{\mathbf{P}}_i(0) = 0$ and ends at $b^{\mathbf{P}}_i(1-) = p_1 + \ldots + p_i - 1$. We extend $b^{\mathbf{P}}_i$ to a function on $\mathbb{R}$ by setting $b^{\mathbf{P}}_i(x+1) = b^{\mathbf{P}}_i(x) + b^{\mathbf{P}}_i(1-)$. For any $v \in [0,1)$, it is easily seen using (iii) in Lemma 1 that

$$H^{\mathbf{P}}_i(x+v) = H^{\mathbf{P}}_i(v-) + b^{\mathbf{P}}_i(x+v) - \left(0 \wedge \inf_{u \in [v, v+x]} (H^{\mathbf{P}}_i(v-) + b^{\mathbf{P}}_i(u))\right).$$

Suppose v is such that $H^{\mathbf{P}}_m(v-) = 0$ (here $H^{\mathbf{P}}_m(0-) = H^{\mathbf{P}}_m(1-)$), call such a number a *last empty spot*. By (ii), Lemma 1, the set of last empty spots is not empty since it contains

t_m. On the other hand, by (i) in the same lemma, the support of $H_m^{\mathbf{p}}$ is the closure of A_m which has measure 1, hence it is $\mathbb{T}$. By (iii), we conclude by letting $v = 0, v' \uparrow 1$ that $H_m^{\mathbf{p}}(x) = H_m^{\mathbf{p}}(0) + b_m^{\mathbf{p}}(x)$ for $0 \le x < 1$, so for $x = t_m-$, $H_m^{\mathbf{p}}(0) = -b_m^{\mathbf{p}}(t_m-) = -\inf b_m^{\mathbf{p}}$ necessarily since $H_m^{\mathbf{p}}$ is non-negative. This implies that the last empty spots are those v's such that $b_m^{\mathbf{p}}(v-) = \inf b_m^{\mathbf{p}}$. We choose one of them by letting

$$V = \inf\{x \in [0,1] : b_m^{\mathbf{p}}(x-) = \inf_{u\in[0,1]} b_m^{\mathbf{p}}(u)\},$$

the first location when the infimum of $b_m^{\mathbf{p}}$ is reached. We have proved

Lemma 2 *For any $0 \le x < 1, 1 \le i \le m$,*

$$H_i^{\mathbf{p}}(x+V) = b_i^{\mathbf{p}}(x+V) - \inf_{u\in[V,V+x]} b_i^{\mathbf{p}}(u).$$

Recall that we are interested in $\Lambda^{\mathbf{p}}(i)$, the ranked sequence of the lengths of the interval components of A_i, where A_i can be viewed as the painted portion of the circle after i drops of paint have fallen, or the set of occupied spots after the i-th caravan has arrived. Lemma 1(i) enables us to identify A_i as the interior of support of the function $H_i^{\mathbf{p}}$, and since the Lebesgue measure of the interval components of the interior of the support of $H_i^{\mathbf{p}}$ is not affected by a cyclic shift, we record the following simple identification

Lemma 3 *For every $i = 1, \dots, m$, $\Lambda^{\mathbf{p}}(i)$ coincides with the ranked lengths of the intervals of constancy of the function*

$$x \longmapsto \inf_{u\in[V,V+x]} b_i^{\mathbf{p}}(u)\,, \qquad x \in [0,1].$$

3 Convergence of bridges

We now consider a rescaled randomized version of the bridges introduced above. Let $B^{(\varepsilon)} = \varepsilon^{-1+1/\alpha} b_m^{\mathbf{p}}$, where $b_m^{\mathbf{p}}$ is obtained as above with data $m = T_\varepsilon, p_i = \varepsilon \ell_i^*, s_i = U_i$, and these quantities are introduced in the Introduction. So for $0 \le x \le 1$

$$B^{(\varepsilon)}(x) = -\varepsilon^{-1+1/\alpha} x + \sum_{i=1}^{T_\varepsilon} \varepsilon^{1/\alpha} \ell_i^* \mathbb{1}_{\{x \ge U_i\}} = \varepsilon^{1/\alpha} \sum_{i=1}^{T_\varepsilon} \ell_i^* (\mathbb{1}_{\{x\ge U_i\}} - x),$$

because $\ell_1^* + \ldots + \ell_{T_\varepsilon}^* = 1/\varepsilon$. Recall that $\mathcal{B}^{(2)}$ denotes the standard Brownian bridge, and $\mathcal{B}^{(\alpha)}$ the standard stable loop with index α as defined in (2).

Lemma 4 *As $\varepsilon \downarrow 0$, the bridge $B^{(\varepsilon)}$ converges weakly on the space $\mathbb{D}$ of càdlàg paths endowed with Skorokhod's topology, to a bridge with exchangeable increments $B = (B(x), 0 \le x \le 1)$. More precisely:*

(i) *If $\alpha = 2$ then B is distributed as $\sqrt{\mu_2/\mu_1}\,\mathcal{B}^{(2)}$.*

(ii) *If $\alpha \in (1,2)$, then B is distributed as*

$$\left(\frac{\Gamma(2-\alpha)c}{(\alpha-1)\mu_1}\right)^{\frac{1}{\alpha}} \mathcal{B}^{(\alpha)}\,.$$

The proof of Lemma 4(ii) will use the following well-known representation:

$$\left(\frac{\Gamma(2-\alpha)c}{(\alpha-1)\mu_1}\right)^{\frac{1}{\alpha}} \mathcal{B}^{(\alpha)}(x) = \sum_{i=1}^{\infty} \Delta_i \left(\mathbb{1}_{\{x \geq U_i\}} - x\right) , \qquad 0 \leq x \leq 1,$$

where $(U_i, i \geq 1)$ is a sequence of i.i.d. uniform$(0,1)$ r.v.'s, $(\Delta_i, i \geq 1)$ is the ranked sequence of the atoms of a Poisson measure on $(0,\infty)$ with intensity $\alpha c \mu_1^{-1} x^{-1-\alpha} \mathrm{d}x$, and these two sequences are independent. More precisely, the series in the right-hand side does not converge absolutely, but is taken in the sense

$$\sum_{i=1}^{\infty} \Delta_i \left(\mathbb{1}_{\{x \geq U_i\}} - x\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_i \left(\mathbb{1}_{\{x \geq U_i\}} - x\right) ,$$

where the limit is uniform in the variable x, a.s. This representation follows immediately from the celebrated Lévy-Itô decomposition, specified for the stable process $\sigma^{(\alpha)}$, as the process of the jumps of the latter is a Poisson point process on $\mathbb{R}_+$ with intensity $\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} x^{-1-\alpha} \mathrm{d}x$. See also Kallenberg [11].

Proof. Following Kallenberg [11], we represent the jump sizes of the bridge $B^{(\varepsilon)}$ by the random point measure

$$\psi_\varepsilon = \sum_{i=1}^{T_\varepsilon} (\varepsilon^{1/\alpha} \ell_i^*)^2 \delta_{\varepsilon^{1/\alpha} \ell_i^*}.$$

By Theorem 2.3 in [11], we have to show:

$$\text{if } \alpha = 2, \text{ then } \psi_\varepsilon \to (\mu_2/\mu_1)\delta_0, \tag{4}$$

and

$$\text{if } \alpha < 2, \text{ then } \psi_\varepsilon \to \psi := \sum_{i=1}^{\infty} \Delta_i^2 \delta_{\Delta_i}, \tag{5}$$

where the convergence is in law with respect to the weak topology on measures on $[0,\infty)$, and in (5), $(\Delta_i, i \geq 1)$ is the ranked sequence of the atoms of a Poisson measure on $(0,\infty)$ with intensity $\alpha c \mu_1^{-1} x^{-1-\alpha} \mathrm{d}x$.

Case (i) is easier to treat. Indeed, notice that the total mass of ψ_ε is

$$\psi_\varepsilon(\mathbb{R}_+) = \varepsilon \sum_{i=1}^{T_\varepsilon} (\ell_i^*)^2 = \varepsilon T_\varepsilon \frac{(\ell_{T_\varepsilon}^*)^2 + \sum_{i=1}^{T_\varepsilon - 1} \ell_i^2}{T_\varepsilon}.$$

Since $\ell_i^* \leq \ell_i$, the law of large numbers gives $\psi_\varepsilon(\mathbb{R}_+) \to \mu_2/\mu_1$.

Now let

$$m_\varepsilon^* := \sqrt{\varepsilon} \max_{1 \leq i \leq T_\varepsilon} \ell_i^* \qquad \text{and} \qquad M_n := \sqrt{\varepsilon} \max_{1 \leq i \leq n} \ell_i,$$

so to prove (4), it suffices to show that $m_\varepsilon^* \to 0$ in probability. Notice that $m_\varepsilon^* \leq M_{T_\varepsilon}$.

Let $\eta > 0$ and $K > \mu_1^{-1}$. Then

$$\begin{aligned} \mathbb{P}(m_\varepsilon^* > \eta) &= \mathbb{P}(m_\varepsilon^* > \eta, T_\varepsilon \leq K\varepsilon^{-1}) + \mathbb{P}(m_\varepsilon^* > \eta, T_\varepsilon > K\varepsilon^{-1}) \\ &\leq \mathbb{P}(M_{\lfloor K\varepsilon^{-1} \rfloor} > \eta) + \mathbb{P}(T_\varepsilon > K\varepsilon^{-1}). \end{aligned}$$

The second term converges to 0 since $\varepsilon\mu_1 T_\varepsilon \to 1$ a.s. For the first term, notice that

$$\mathbb{P}(M_{\lfloor K\varepsilon^{-1}\rfloor} \le \eta) = (1 - \mathbb{P}(\ell > \eta/\sqrt{\varepsilon}))^{\lfloor K\varepsilon^{-1}\rfloor}.$$

Taking logarithms and checking that $\varepsilon^{-1}\mathbb{P}(\ell^2 > \eta^2/\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ (which holds since $\mathbb{E}[\ell^2] < \infty$), we finally obtain that $\mathbb{P}(M_{\lfloor K\varepsilon^{-1}\rfloor} \le \eta) \to 1$. This completes the proof of (4).

Now we turn our attention to (5). It suffices to show that for every function $f : [0,\infty) \to [0,\infty)$, say of class $\mathcal{C}^1$ with bounded derivative

$$\lim_{\varepsilon\to 0} \mathbb{E}\left(\exp(-\langle \psi_\varepsilon, f\rangle)\right) = \mathbb{E}\left(\exp(-\langle \psi, f\rangle)\right); \tag{6}$$

see for instance Section II.3 in Le Gall [12]. In this direction, recall from the classical formula for Poisson random measures that

$$\mathbb{E}\left(\exp(-\langle \psi, f\rangle)\right) = \frac{\alpha c}{\mu_1}\int_0^\infty (1 - \exp(-y^2 f(y)))y^{-1-\alpha}dy.$$

To start with, we observe from the renewal theorem that $\varepsilon^{1/\alpha}\ell^*_{T_\varepsilon}$ converges to 0 in probability as $\varepsilon \to 0$, so in (6), we may replace ψ_ε by

$$\psi'_\varepsilon = \sum_{i=1}^{T_\varepsilon - 1} (\varepsilon^{1/\alpha}\ell_i)^2 \delta_{\varepsilon^{1/\alpha}\ell_i}.$$

Next, for every $a \ge 0$, we consider the random measure

$$\psi_{\varepsilon,a} = \sum_{i=1}^{a/\varepsilon} (\varepsilon^{1/\alpha}\ell_i)^2 \delta_{\varepsilon^{1/\alpha}\ell_i}.$$

Again, by the (elementary) renewal theorem, $\varepsilon T_\varepsilon \to \mu_1^{-1}$ in probability, so for every $\eta > 0$, the event

$$\langle \psi_{\varepsilon,\mu_1^{-1}-\eta}, f\rangle \;\le\; \langle \psi'_\varepsilon, f\rangle \;\le\; \langle \psi_{\varepsilon,\mu_1^{-1}+\eta}, f\rangle \tag{7}$$

has a probability which tends to 1 as $\varepsilon \to 0$.

Now

$$\mathbb{E}\left(\exp(-\langle \psi_{\varepsilon,a}, f\rangle)\right) = \mathbb{E}\left(\exp(-f(\varepsilon^{1/\alpha}\ell)(\varepsilon^{1/\alpha}\ell)^2)\right)^{a/\varepsilon}.$$

Taking logarithms, we have to estimate

$$\begin{aligned}
& \frac{a}{\varepsilon}\mathbb{E}\left(1 - \exp(-f(\varepsilon^{1/\alpha}\ell)(\varepsilon^{1/\alpha}\ell)^2)\right) \\
= \;& \frac{a}{\varepsilon}\int_0^\infty \varepsilon^{2/\alpha}(2xf(\varepsilon^{1/\alpha}x) + \varepsilon^{1/\alpha}x^2 f'(\varepsilon^{1/\alpha}x))\exp(-(\varepsilon^{1/\alpha}x)^2 f(\varepsilon^{1/\alpha}x))\mathbb{P}(\ell > x)dx \\
= \;& \frac{a}{\varepsilon}\int_0^\infty (2yf(y) + y^2 f'(y))\exp(-y^2 f(y))\mathbb{P}(\ell > y/\varepsilon^{1/\alpha})dx.
\end{aligned}$$

By (1) and dominated convergence, we see that the preceding quantity converges as $\varepsilon \to 0$ towards

$$ac\int_0^\infty (2yf(y) + y^2 f'(y))\exp(-y^2 f(y))y^{-\alpha}dx = \alpha ac\int_0^\infty (1 - \exp(-y^2 f(y)))y^{-1-\alpha}dx.$$

Taking $a = \mu_1^{-1} \pm \eta$, using (7) and letting η tend to 0, we see that (6) holds, which completes the proof of the statement. □

4 Convergence of $\mathbf{X}^{(\varepsilon)}$

In this section, we deduce Theorem 1 from Lemmas 3,4. Recall the definition of the bridge $b_i^{\mathbf{p}}$ in Section 2. For $i \leq T_\varepsilon$, let $B_i^{(\varepsilon)}$ be the bridge $\varepsilon^{-1+1/\alpha} b_i^{\mathbf{p}}$ with data $p_j = \varepsilon \ell_j^*, s_j = U_j$, so $B_{T_\varepsilon}^{(\varepsilon)} = B^{(\varepsilon)}$. Let also V_ε be the left-most location of the infimum of $B^{(\varepsilon)}$, and

$$\mathbf{V}B^{(\varepsilon)}(x) = B^{(\varepsilon)}(x + V_\varepsilon) - \inf B^{(\varepsilon)}, \qquad 0 \leq x \leq 1$$

the Vervaat transform of $B^{(\varepsilon)}$. By Lemma 3, $\mathbf{X}^{(\varepsilon)}(t) = \Lambda^{\mathbf{p}}(T_\varepsilon - \lfloor t\varepsilon^{-1/\alpha} \rfloor)$ coincides with the ranked sequence of lengths of constancy intervals of the infimum process of

$$B^{(\varepsilon)}_{T_\varepsilon - \lfloor t\varepsilon^{-1/\alpha}\rfloor}(x + V_\varepsilon) - \inf B^{(\varepsilon)}, \qquad 0 \leq x \leq 1,$$

where the constant $-\inf B^{(\varepsilon)}$ has no effect and is added for future considerations.

Lemma 5 *For every $t \geq 0$, the difference*

$$B^{(\varepsilon)}(x) - B^{(\varepsilon)}_{T_\varepsilon - \lfloor t\varepsilon^{-1/\alpha}\rfloor}(x) = \varepsilon^{1/\alpha} \sum_{j=0}^{\lfloor t\varepsilon^{-1/\alpha} - 1\rfloor} \ell^*_{T_\varepsilon - i} \mathbb{1}_{\{x \geq U_{T_\varepsilon - i}\}} \qquad 0 \leq x \leq 1$$

converges in probability for the uniform norm to the pure drift $x \mapsto t\mu_1 x$ as $\varepsilon \downarrow 0$.

Proof. Recall from the renewal theorem that $\varepsilon^{1/\alpha}\ell_{T_\varepsilon} \to 0$ in probability as $\varepsilon \downarrow 0$. Therefore, we might start the sum appearing in the statement from $j = 1$. Now, the sequences $(\ell_1, \ldots, \ell_{T_\varepsilon - 1})$ and $(\ell_{T_\varepsilon - 1}, \ldots, \ell_1)$ have the same distribution. Up to doing the substitution, Lemma 5 for fixed s is therefore a simple application of the strong law of large numbers. The conclusion is obtained by standard monotonicity arguments. $\square$

As a consequence of Lemmas 4, 5, and the fact that $s \mapsto t\mu_1 s$ is continuous, the process

$$B^{(\varepsilon)}_{T_\varepsilon - \lfloor t\varepsilon^{-1/\alpha}\rfloor}(x + V_\varepsilon) - \inf B^{(\varepsilon)} = \mathbf{V}B^{(\varepsilon)}(x) - \left(B^{(\varepsilon)}(x + V_\varepsilon) - B^{(\varepsilon)}_{T_\varepsilon - \lfloor t\varepsilon^{-1/\alpha}\rfloor}(x + V_\varepsilon)\right)$$

converges in the Skorokhod space to

$$\mathcal{E}^{(t\mu_1)} = (\mathcal{E}(x) - t\mu_1 x, 0 \leq x \leq 1),$$

where

$$\mathcal{E}(x) = B(x + V) - \inf B, \qquad 0 \leq x \leq 1$$

is the Vervaat transform of the limiting bridge B which appears in Lemma 4, V being the a.s. unique location of its infimum. Now letting $\underline{\mathcal{E}}^{(t)}$ be the infimum process of $\mathcal{E}^{(t)}$ and $\mathbf{F}(t)$ be the decreasing sequence of lengths of constancy intervals of $\underline{\mathcal{E}}^{(t)}$, we have

Proposition 1 *The process $(\mathbf{X}^{(\varepsilon)}(t), t \geq 0)$ converges to $(\mathbf{F}(\mu_1 t), t \geq 0)$ in the sense of weak convergence of finite-dimensional marginals.*

Proof. The technical point is that Skorokhod convergence of $B^{(\varepsilon)}_{T_\varepsilon - \lfloor t\varepsilon^{-1/\alpha}\rfloor}(x+V_\varepsilon) - \inf B^{(\varepsilon)}$ to $\mathcal{E}^{(t\mu_1)}$, though it does imply convergence of respective infimum processes, does not *a priori* imply that of the ranked sequence of lengths of constancy intervals of these processes. However, this convergence does hold because for every $t \geq 0$, if (a,b) is such a constancy interval, then $\mathcal{E}^{(t\mu_1)}(x) > \mathcal{E}^{(t\mu_1)}(a)$ for $x \in (a,b)$, a.s. See e.g. Lemmas 4 and 7 in [5]. □

This proposition proves Theorem 1. Indeed, recall from Lemma 4 that $B = c_\alpha \mathcal{B}_\alpha$, where $c_2 = \sqrt{\mu_2/\mu_1}$ and for $1 < \alpha < 2$

$$c_\alpha = \left(\frac{\Gamma(2-\alpha)c}{(\alpha-1)\mu_1}\right)^{\frac{1}{\alpha}}.$$

Then plainly, $\mathbf{F}(e^{-t}) = \mathbf{C}^{(\alpha)}(t + \log c_\alpha)$, and hence the limiting process $\mathbf{X}(e^{-t})$ is distributed as $\mathbf{F}(\mu_1 e^{-t}) = \mathbf{C}^{(\alpha)}(t + \log c_\alpha - \log \mu_1)$.

5 Related results for a discrete problem

In situations involving parking problems, it may be more natural to consider discrete parking lots, i.e. $\mathbb{Z}/n\mathbb{Z}$ instead of the unit circle, and caravans with integer sizes, e.g. as in Knuth's original parking problem. Each caravan chooses a random spot, uniform on $\mathbb{Z}/n\mathbb{Z}$, and tries to park at that spot. Studying the frequencies of blocks of cars fits with our general framework by taking ℓ with integer values, $\varepsilon = 1/n$ and $s_i = \lfloor nU_i \rfloor/n$. Rename by T_n the former quantity T_ε (the number of caravans). Let

$$\widetilde{B}^{(n)}(x) = n^{1/\alpha}\sum_{i=1}^{T_n}\left(\frac{\ell_i^*}{n}\mathbb{1}_{\{x \geq \lfloor nU_i \rfloor/n\}} - x\right) \qquad 0 \leq x \leq 1$$

$$B^{(n)}(x) = n^{1/\alpha}\sum_{i=1}^{T_n}\left(\frac{\ell_i^*}{n}\mathbb{1}_{\{x \geq U_i\}} - x\right) \qquad 0 \leq x \leq 1,$$

so $B^{(n)}$ would be $B^{(1/n)}$ in the notation above. The analog of Lemma 5 is still true when replacing $B^{(n)}$ by $\widetilde{B}^{(n)}$, without essential change in the proof. Thus to obtain the very same conclusions as in the preceding sections, it suffices to check a result similar to Lemma 4. Namely, we must prove that $\widetilde{B}^{(n)} \to B$ in the Skorokhod space as $n \to \infty$. Now it is easy to check that a.s., $|\widetilde{B}^{(n)}(x) - B^{(n)}(\lceil nx - \rceil/n)| \leq n^{1/\alpha}/n$ for every $n \geq 1, x \in [0,1]$, because no U_i is rational a.s. Therefore, it suffices to check that $B^{(n)}(\lceil n \cdot + \rceil/n)$ converges to B in distribution for the Skorokhod topology on $\mathbb{D}$. Up to using Skorokhod's representation theorem, this is done by taking $f_n = B^{(n)}$ and $\kappa_n(x) = \lceil nx + \rceil/n$ in the next lemma.

Lemma 6 *Let $(f_n, n \geq 1)$ be a sequence of functions converging in $\mathbb{D}$ to f. For $n \in \mathbb{N}$ let also κ_n be a right-continuous non-decreasing function (not necessarily bijective) from $[0,1]$ to $[0,1]$, such that the sequence (κ_n) converges to the identity function uniformly on $[0,1]$. Then $f_n \circ \kappa_n \to f$ in $\mathbb{D}$.*

Proof. First consider the case $f_n = f$ for every n. Fix $\varepsilon > 0$. Let κ_n^{-1} be the right-continuous inverse of κ_n defined by

$$\kappa_n^{-1}(x) = \inf\{y \in [0,1] : \kappa_n(y) > x\}.$$

It is easy to prove that $\kappa_n(\kappa_n^{-1}(x)-) \leq x \leq \kappa_n(\kappa_n^{-1}(x))$ for every x. Since f is càdlàg, one may find $0 = x_0 < x_1 < \ldots < x_k = 1$ such that the oscillation $\omega(f, [x_i, x_{i+1})) < \varepsilon$ for $0 \leq i \leq k-1$, where

$$\omega(f, A) = \sup_{x,y \in A} |f(x) - f(y)|.$$

Since κ_n approaches the identity, for n large we may assume $\kappa_n(\kappa_n^{-1}(x_i)) < \kappa_n(\kappa_n^{-1}(x_{i+1})-)$ for $0 \leq i \leq k-1$. Define a time-change λ_n (i.e. an increasing bijection between $[0,1]$ and $[0,1]$) by interpolating linearly between the points $(0,0), (\kappa_n^{-1}(x_i), x_i), 1 \leq i \leq k-1, (1,1)$.

Now let $x \in [0,1]$. Suppose $\kappa_n^{-1}(x_i) \leq x < \kappa_n^{-1}(x_{i+1})$ for some $0 \leq i \leq k-1$, and notice that $x_i \leq \kappa_n(\kappa_n^{-1}(x_i)) \leq \kappa_n(x) < \kappa_n(\kappa_n^{-1}(x_{i+1})-) \leq x_{i+1}$. Therefore, $\kappa_n(x)$ belongs to $[x_i, x_{i+1})$ as well as $\lambda_n(x)$ by definition of λ_n, and

$$|f(\kappa_n(x)) - f(\lambda_n(x))| \leq \omega(f, [x_i, x_{i+1})) \leq \varepsilon.$$

Else, one must have $x < \kappa_n^{-1}(0)$ or $x \geq \kappa_n^{-1}(1)$, and the result is similar. Finally, doing the same reasoning for $\varepsilon = \varepsilon_n$ converging to 0 slowly enough gives the existence of some time-changes λ_n converging to the identity uniformly such that $\sup_{x\in[0,1]} |f(\kappa_n(x)) - f(\lambda_n(x))| \leq 2\varepsilon_n$, hence giving convergence of $f \circ \kappa_n$ to f in the Skorokhod space.

In the general case, for every $n \geq 0$ let λ_n be a time-change such that λ_n converges to the identity as $n \to \infty$ and $f_n \circ \lambda_n$ converges to f uniformly. Take $\kappa'_n = \lambda_n^{-1} \circ \kappa_n$. Then $f_n \circ \kappa_n - f \circ \kappa'_n \to 0$ uniformly, so it suffices to show that $f \circ \kappa'_n \to f$ in $\mathbb{D}$, which is done by the former discussion. □

In particular, we recover and extend a certain number of results from [8].

6 Complements

In this section, we would like to provide some information on the eternal additive coalescents $\mathbf{C}^{(\alpha)}$ for $1 < \alpha < 2$, which appear in Theorem 1.

6.1 Mixture of extremes

To start with, we should like to specify the representation of $\mathbf{C}^{(\alpha)}$ as a mixture of so-called *extreme* eternal additive coalescents ([3], [5]). In this direction, let us first consider a sequence $\boldsymbol{\theta} = (\theta_0, \theta_1, \theta_2, \ldots)$ of non-negative numbers satisfying $\sum_{i \geq 0} \theta_i^2 = 1$ and

$$\text{either } \theta_0 > 0 \qquad \text{or} \qquad \sum_{i \geq 0} \theta_i = \infty. \tag{8}$$

Following Kallenberg [11] we associate to $\boldsymbol{\theta}$ a *bridge with exchangeable increments*

$$B^{\theta}(x) = \theta_0 \beta(x) + \sum_{i \geq 1} \theta_i (\mathbb{1}_{\{x \geq U_i\}} - x) \qquad 0 \leq x \leq 1 \tag{9}$$

where $(U_i, i \geq 1)$ denotes a sequence of iid uniform variables and β is an independent standard Brownian bridge. We write $\mathbf{C}^{\theta}$ for the eternal additive coalescent associated to the bridge $B = B^{\theta}$ as explained in the Introduction and call such $\mathbf{C}^{\theta}$ extreme.

According to [3, Theorem 15], every eternal version of the additive coalescent $\mathbf{C}$ can be obtained as a mixing of shifted versions of extreme eternal additive coalescents $\mathbf{C}^{\theta}$, i.e.

$\mathbf{C}$ can be expressed in the form $(\mathbf{C}^{\theta^*}(t-t^*), t \in \mathbb{R})$ with $\boldsymbol{\theta}^*, t^*$ random. Equivalently, $\mathbf{C}$ can be viewed as the eternal additive coalescent constructed in the Introduction from the bridge with exchangeable increments $B = e^{t^*} B^{\theta^*}$. As observed by Aldous and Pitman [3], the mixing variables $\boldsymbol{\theta}^*, t^*$ can be recovered from the initial behavior of $\mathbf{C}$:

$$e^{t^*}\theta_i^* = \lim_{t\to-\infty} e^{-t}\mathbf{C}_i(t) \quad \text{and} \quad e^{2t^*} = \lim_{t\to-\infty} e^{-2t}\sum_{i=1}^{\infty}\mathbf{C}_i^2(t)\,.$$

In the case of the standard stable loop $\mathcal{B}^{(\alpha)}$ with $1<\alpha<2$, recall from the Lévy-Itô decomposition that $\theta_0^* = 0$ and $(e^{t^*}\theta_1^*, e^{t^*}\theta_2^*, \ldots) = (\Delta_1, \Delta_2, \ldots)$ is the ranked sequence of the atoms of a Poisson random measure on $(0,\infty)$ with intensity $\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}x^{-1-\alpha}dx$. In particular,

$$e^{2t^*} = \sum_{i=1}^{\infty}\Delta_i^2 \tag{10}$$

has the law of a (positive) stable variable with index $\alpha/2$ and

$$\theta_i^* = \Delta_i/e^{t^*}\,, \qquad i = 1,2,\ldots \tag{11}$$

is such that the sequence of squares $((\theta_1^*)^2, (\theta_2^*)^2, \ldots)$ is distributed according to the Poisson-Dirichlet law $\mathrm{PD}(\alpha/2, 0)$; see Pitman and Yor [14].

We also stress that every coalescent $\mathbf{C}^\theta$ can be obtained as a limit of appropriate caravan parking problems, which are quite natural given the results of [3, 5]. Precisely, suppose that a sequence of probabilities $\mathbf{p}^n = (p_1^n, \ldots, p_{m_n}^n)$ satisfying $p_1^n \geq \ldots \geq p_{m_n}^n > 0$ is given, and satisfies

$$\max_{1\leq i\leq m_n} p_i^n \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad \sigma(\mathbf{p}^n)^{-1}p_i^n \underset{n\to\infty}{\longrightarrow} \theta_i \qquad i\geq 1 \tag{12}$$

for a sequence $\boldsymbol{\theta}$ as described above, and where $\sigma(\mathbf{p}) = \sqrt{\sum_{i=1}^m p_i^2}$ when $\mathbf{p} = (p_1, \ldots, p_m)$. For every n, let τ_n be a uniform permutation on $\{1, 2, \ldots, m_n\}$. Consider the parking problem where the caravans which try to park successively have magnitudes $p_{\tau_n(1)}^n, p_{\tau_n(2)}^n, \ldots$. Let $U_1, U_2, \ldots$ be independent uniform$(0,1)$ random variables independent of τ_n, so we may consider the bridge with exchangeable increments

$$B^{(n)}(s) = \sigma(\mathbf{p}^n)^{-1}\left(-x + \sum_{i=1}^{m_n} p_{\tau_n(i)}^n \mathbb{1}_{\{s\geq U_i\}}\right), \qquad 0\leq s\leq 1.$$

Kallenberg's theorem shows that under the asymptotic assumptions on $\mathbf{p}^n$, $B^{(n)}$ converges in distribution to the bridge B^θ defined above.

Now for $t\geq 0$, let $I_t^n = \inf\{i\geq 1 : \sum_{j=i+1}^{m_n} p_{\tau_n(j)}^n \leq t\}$. The following analogue of Lemma 5 holds.

Lemma 7 *For every $t\geq 0$, the process*

$$\sigma(\mathbf{p}^n)^{-1}\sum_{i=I_t^n+1}^{m_n} p_{\tau_n(i)}^n \mathbb{1}_{\{s\geq U_i\}}\,, \qquad 0\leq s\leq 1$$

converges in probability for the uniform norm to the pure drift $s\mapsto ts$ as $n\to\infty$.

Proof. The key to this lemma is to show that

$$\max_{i\geq I_t^n} \sigma(\mathbf{p}^n)^{-1} p^n_{\tau_n(i)} \to 0 \tag{13}$$

in probability as $n \to \infty$. The result is then obtained via the so-called "weak law of large numbers for sampling without replacement": if $x_i^n, 1 \leq i \leq n$ is a sequence with sum t satisfying $\max_{1\leq i\leq n} x_i^n \to 0$ as $n \to \infty$, and if τ_n is a uniform permutation on $\{1,\ldots,n\}$, then for every rational $r \in [0,1]$, $\sum_{i=1}^n x^n_{\tau_n(i)} \mathbb{1}_{\{r\geq U_i\}} \to tr$ in probability (in fact in L^2). The result in probability remains true if $x_i^n, 1 \leq i \leq n$ is random with sum t, and $\max_{1\leq i\leq n} x_i^n \to 0$ in probability. One concludes that the process $(\sum_{i=1}^n x^n_{\tau_n(i)} \mathbb{1}_{\{s\geq U_i\}}, 0 \leq s \leq 1)$ converges in probability to $(ts, 0 \leq s \leq 1)$ for the uniform norm by a monotonicity argument. The lemma is then proved by letting $x_1 = \sigma(\mathbf{p}^n)^{-1} p^n_{\tau_n(I_t^n+1)}, x_2 = \sigma(\mathbf{p}^n)^{-1} p^n_{\tau_n(I_t+2)}, \ldots, x_{m_n - I_t^n} = \sigma(\mathbf{p}^n)^{-1} p^n_{\tau_n(m_n)}, x_{m_n - I_t^n + 1} = t - \sum_{i=I_t^n+1}^{m_n} \sigma(\mathbf{p}^n)^{-1} p^n_{\tau_n(I_t^n+1)}$ (note that this last term is $\leq \sigma(\mathbf{p}^n)^{-1} p^n_{\tau_n(I_t^n)}$, which goes to 0).

So let us show (13). To this end, let $0 < \rho < 1$, then $X_n^\rho := \sum_{i=\lfloor \rho m_n \rfloor}^{m_n} \sigma(\mathbf{p}^n)^{-1} p^n_{\tau_n(i)} \to \infty$ in probability, since $E[X_n^\rho] \sim \sigma(\mathbf{p}^n)^{-1}(1-\rho)$ goes to infinity (notice $\sigma(\mathbf{p}) \leq p_1$) while $E[(X_n^\rho)^2] \sim E[X_n^\rho]^2$, as a simple computation shows. Therefore, $I_t^n \sim m_n$ in probability. Consequently, for any $K \in \mathbb{N}$, the quantity $P(\tau_n^{-1}(1) < I_t^n, \ldots, \tau_n^{-1}(K) < I_t^n)$ goes to 1, so $\min_{i\geq I_t^n} \tau_n(i) \to \infty$ in probability. But then, for any $\varepsilon > 0$, if K is such that $\theta_K < \varepsilon/2$, then $\sigma(\mathbf{p}^n)^{-1} p_K^n \leq \varepsilon$ for n large. Up to taking n even larger, with probability close to 1, $\tau_n(i) \geq K$ for $i \geq I_t^n$ and therefore $\max_{i\geq I_t^n} \sigma(\mathbf{p}^n)^{-1} p^n_{\tau_n(i)} \leq \varepsilon$, hence (13). □

One deduces, as around the proof of Proposition 1, the following claim. Let $\mathbf{X}^{(n)}(t) = \Lambda^{\mathbf{p}^n\circ\tau_n}(I_t^n)$ be as above with data $m = m_n, p^n_{\tau_n(i)}, 1 \leq i \leq m_n, s_i = U_i$. Then

Proposition 2 *As $n \to \infty$, under the asymptotic regime (12), the process $(\mathbf{X}^{(n)}(t), t \geq 0)$ converges in the sense of weak convergence of finite-dimensional marginals to the time-reversed eternal additive coalescent $(\mathbf{C}^\theta(-\log(t)), t \geq 0)$.*

6.2 On the marginal distributions

It would also be interesting to determine the marginal laws of the fragmentation $\mathbf{F}^{(\alpha)}(t) := \mathbf{C}^{(\alpha)}(-\log t)$. The task seems quite difficult if started from the description of $\mathbf{F}^{(\alpha)}(t)$ in terms of lengths of constancy intervals of Vervaat transform of bridges, because excursion theory seems powerless here, unlike in [13]. In particular, the fact that the fragmentation is based on stable loops and not stable bridges impedes the application of results of Miermont [13] on additive coalescents based on bridges of certain Lévy processes.

Another way to start the exploration is to use the representation of fragmentation processes $\mathbf{F}^\theta(t) := \mathbf{C}^\theta(-\log t)$ described in the preceding section with the help of Inhomogeneous Continuum Random Trees (ICRT) discussed in [3]. In particular, it is easy to obtain the first moment of a size-biased pick[2] $\mathbf{F}_\dagger(t)$ from the sequence $\mathbf{F}(t)$ for any fixed t, as follows.

Let us recall the basic facts on the ICRT($\boldsymbol{\theta}$) construction of $\mathbf{F}^\theta$. The ICRT can be viewed via a *stick-breaking construction* as the metric completion of the positive real line

[2]Recall that a size-biased pick $X_\dagger$ from a (random) positive sequence $(X_i, i \geq 1)$ with sum $0 < S < \infty$ a.s. is a random variable of the form X_{i^*}, where $P(i^* = i | X_j, j \geq 1) = X_i/S$.

$\mathbb{R}_+$ endowed with a non standard metric. Precisely, suppose we are given the following independent random elements:

- A Poisson process $\{(U_i, V_i), i \geq 1\}$ on the octant $\mathbb{O} = \{(x, y) : 0 < y < x\} \subset \mathbb{R}_+^2$, with intensity $\theta_0 \mathrm{d}x\mathrm{d}y \mathbb{1}_{\mathbb{O}}$, so in particular $\{U_i, i \geq 1\}$ is a Poisson process with intensity $\theta_0 x \mathrm{d}x \mathbb{1}_{x \geq 0}$,

- A sequence of independent Poisson processes $\{\xi_{i,j}, j \geq 1\}, i = 1, 2, \dots$ with respective intensities $\theta_i \mathrm{d}x \mathbb{1}_{x \geq 0}, i = 1, 2, \dots$.

We distinguish the points $(V_i, i \geq 1), (\xi_{i,1}, i \geq 1)$ as *joinpoints*, while $(U_i, i \geq 1), (\xi_{i,j}, i \geq 1, j \geq 2)$ are called *cutpoints*. If η is a cutpoint, let η^* be its associated joinpoint, i.e. $U_i^* = V_i, \xi_{i,j}^* = \xi_{i,1}$. By the assumption on $\boldsymbol{\theta}$, it is a.s. possible to arrange the cutpoints by increasing order $0 < \eta_1 < \eta_2 < \dots$. We then construct a family $\mathcal{R}(k), k \geq 1$ of "reduced trees" as follows. Cut the set $(0, \infty)$ into "branches" $(\eta_i, \eta_{i+1}]$, where by convention $\eta_0 = 0$. Let $\mathcal{R}(1)$ be the segment $(0, \eta_1]$, endowed with the usual distance $d_1(x, y) = |x - y|$. Then given $\mathcal{R}(k), d_k$, we obtain $\mathcal{R}(k+1)$ by adding the branch $(\eta_k, \eta_{k+1}]$ somewhere on $\mathcal{R}(k)$, and we plant the left-end η_k on the joinpoint η_k^* (since a.s. $\eta^* < \eta$, the point η_k^* is indeed an element of $\mathcal{R}(k)$). Precisely, $\mathcal{R}(k+1) = (0, \eta_{k+1}]$ and $d_{k+1}(x, y) = d_k(x, y)$ if $x, y \in \mathcal{R}(k)$, $d_{k+1}(x, y) = |x - y|$ if $x, y \in (\eta_k, \eta_{k+1}]$, and $d_{k+1}(x, y) = x - \eta_k + d_k(y, \eta_k^*)$ if $x \in (\eta_k, \eta_{k+1}], y \in \mathcal{R}(k)$. As the distances d_k are compatible by definition, this defines a random metric space $(0, \infty), d$ such that the restriction of d to $\mathcal{R}(k)$ is d_k, we call its metric completion $\mathcal{T}^\theta$ the ICRT($\boldsymbol{\theta}$), its elements are called *vertices*. The point $\varnothing = \lim_{n \to \infty} 1/n$ is distinguished and called the *root*.

One can see that $\mathcal{T}^\theta$ is an $\mathbb{R}$-tree, i.e. a complete metric space such that for any $x, y \in \mathcal{T}^\theta$ there is a unique simple path $[[x, y]]$ from x to y, which is isometric to the segment $[0, d(x, y)]$, i.e. is a geodesic. Moreover, it can be endowed with a natural measure μ^θ which is the weak limit as $n \to \infty$ of the empirical measures $n^{-1} \sum_{i=1}^n \delta_{\eta_i}$. This measure is non-atomic and supported on *leaves*, i.e. vertices $x \in \mathcal{T}^\theta$ such that $x \notin [[\varnothing, y]] \setminus \{y\}$ for any $y \in \mathcal{T}^\theta$. Non-leaf vertices form a set called the *skeleton*. A second natural measure is the Lebesgue measure λ on $\mathcal{T}^\theta$, i.e. the unique measure such that $\lambda([[x, y]]) = d(x, y)$ for any x, y, and this measure is supported on the skeleton.

Now for each t consider a Poisson measure on $\mathcal{T}^\theta$ with atoms $\{x_i^t, i \geq 1\}$, with intensity $t\lambda(\mathrm{d}x)$, so the different processes are coupled in the natural way as t varies, i.e. $\{x_i^t, i \geq 1\}$ increases with t. These points disconnect the tree into a forest of disjoint connected tree components, order them as $\mathcal{F}_i^\theta(t), i \geq 1$ by decreasing order of μ^θ-mass. Then the process $((\mu^\theta(\mathcal{F}_i^\theta(t)), i \geq 1), t \geq 0)$ of these μ^θ-masses has same law as $\mathbf{F}^\theta$. A size-biased pick from this sequence of masses is then obtained as the μ^θ-mass of the tree component at time t that contains an independent μ^θ-sample, conditionally on $(\mathcal{T}^\theta, \mu^\theta)$. Therefore, if $\mathbf{F}_\dagger^\theta(t)$ denotes such a size-biased pick, $E[\mathbf{F}_\dagger^\theta(t)]$ is the probability that two independent μ^θ-samples X_1, X_2 belong to the same tree component of the cut tree, i.e. that no atom of the Poisson measure at time t falls in the path $[[X_1, X_2]]$, and hence it equals $E[e^{-td(X_1, X_2)}]$.

It turns out [3] that $d(X_1, X_2)$ has same law as the length η_1 of the first branch (i.e. the length of $\mathcal{R}(1)$). It is easy to see (see also [7]) that this branch's length has law

$$P(\eta_1 > r) = e^{-\theta_0^2 r 2/2} \prod_{i=1}^{\infty} (1 + \theta_i r) e^{-\theta_i r}.$$

In our setting, recall that the random sequence θ^* is related to that of the atoms (Δ_i) of a Poisson measure on $(0,\infty)$ with intensity $\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}x^{-1-\alpha}dx$ by (10) and (11). Observe that $\mathbf{F}^{(\alpha)}(t) = \mathbf{F}^{\theta^*}(t/e^{t^*})$, and since we must take a Poisson process with intensity t/e^{t^*} on the skeleton of the ICRT(θ^*), terms e^{t^*} cancel out and $E[\mathbf{F}_\dagger^{(\alpha)}(t)] = E[e^{-t\eta}]$ where

$$\begin{aligned} P(\eta \geq r) &= E\left[\prod_{i=1}^{\infty}(1+r\Delta_i)e^{-r\Delta_i}\right] \\ &= \exp\left(-\int_0^\infty \frac{\alpha(\alpha-1)\mathrm{d}x}{\Gamma(2-\alpha)x^{1+\alpha}}(1-\exp(-rx+\log(1+rx)))\right) \\ &= \exp(-(\alpha-1)r^\alpha), \end{aligned}$$

which is a Weibull distribution. This gives (at least in principle) the first moment

$$E[\mathbf{F}_\dagger^{(\alpha)}(t)] = \int_0^\infty \alpha(\alpha-1)r^{\alpha-1}\exp\left(-tr-(\alpha-1)r^\alpha\right)\mathrm{d}r.$$

In principle, this method could be used for the computation of moments of higher order, where the length η_1 of the first branch would simply be replaced by the total length η_k of $\mathcal{R}(k)$. Unfortunately, the distribution of η_k is complicated for $k=3$, and seems intractable for higher k's ([2]).

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