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Abstract

Quantum integrability of classical integrable systems given by quadratic Killing tensors on curved configuration spaces is investigated. It is proven that, using a "minimal" quantization scheme, quantum integrability is insured for a large class of classic examples.

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1 Introduction

One of the main goals of this article is to present a somewhat general framework for the quantization of classical observables on a cotangent bundle which are polynomials at most cubic in momenta. This approach enables us to investigate the quantization of classically Poisson-commuting observables, and hence to tackle the problem of quantum integrability for a reasonably large class of dynamical systems.

What should actually be the definition of quantum integrability is a long standing issue, see, e.g., [37]. The point of view espoused in this paper is the following. Start with a complete set of independent Poisson-commuting classical observables, and use some quantization rule to get a corresponding set of quantum observables; if these operators appear to be still in involution with respect to the commutator, the system will be called integrable at the quantum level.

Our work can be considered as a sequel to earlier and pioneering contributions [11, 4, 5, 22, 34] that provide worked examples of persistence of integrability from the classical to the quantum regime. The general approach we deal with in this paper helps us to highlight the general structure of quantum corrections and to show that the latter actually vanish in most, yet not all, interesting examples.

Returning to the general issue of quantization, let us mention that our choice of quantization procedure, which we might call "minimal", doesn't stem from first principles, e.g., from invariance or equivariance requirements involving some specific symmetry. Although this "minimal" quantization only applies to low degree polynomials on cotangent bundles, it has the virtue of leading automatically to the simplest symmetric operators that guarantee quantum integrability in many cases. In order to provide the explicit form of the quantization scheme, hence of the quantum corrections, we need a symmetric linear connection be given on the base of our cotangent bundle. In most examples where a (pseudo-)Riemannian metric is considered from the outset, this connection will be chosen as the Levi-Civita connection.

To exemplify our construction, we consider a number of examples of classical integrable systems together with their quantization. For instance, our approach for dealing with quantum integrability in somewhat general terms allowed us to deduce the quantum integrability of the Hamiltonian flow for the generalized Kerr-Newman solution of the Einstein-Maxwell equations with a cosmological constant first discovered by Carter [9, 10, 11]. Also does our quantization scheme leads us to an independent proof of the quantum integrability for Stäckel systems originally due to Benenti, Chanu and Rastelli [4, 5].

The paper is organised as follows. In Section 2 we gather the definitions of the Schouten bracket of symmetric contravariant tensor fields on configuration space, M. We make use of Souriau's procedure to present, in a manifestly gauge invariant fashion, the minimal coupling to an external electromagnetic field; this enables us to provide a geometric definition of the so called Schouten-Maxwell bracket. The related definitions of Killing and Killing-Maxwell tensors follow naturally and will be used throughout the rest of the paper. We recall the basics of classical integrable systems, with emphasis on the Stäckel class. The main objective of the present Section is then to revisit some classic examples of integrable systems involving Killing tensors. Naturally starting with the Jacobi system on the ellipsoid, we prove, *en passant*, that it is locally of the Stäckel type, even allowing for an extra harmonic potential. This extends previous work of Benenti [3] related to the geodesic flow of the ellipsoid. Similarly, we show that the Neumann system is also locally Stäckel. A number of additional examples, not of Stäckel type, e.g., the Di Pirro system, and the geodesic flow on various (pseudo-)Riemannian manifolds such as the Kerr-Newman-de Sitter solution and the Multi-Centre solution are also considered.

We introduce, in Section 3, a specific "minimal" quantization scheme for observables at most cubic in momenta on the cotangent bundle T^*M of a smooth manifold M endowed with a symmetric connection ∇, extending a previous proposal [11]. This quantization mapping is shown to be equivariant with respect to the affine group of (M, ∇). The computation of the commutators of quantum observables is then carried out and yields explicit expressions for quantum corrections. We also provide the detailed analysis of quantum integrability for a wide class of examples within the above list.

The concluding section includes a discussion and brings together several remarks about the status of the "minimal" quantization that has been abstracted from the various examples dealt with in this paper. It also opens some prospects for future investigations related to quantum integrability in the spirit of this work.

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2 Classical integrable systems

2.1 Killing tensors

Let us start with the definition of the Schouten bracket of two polynomial functions on the cotangent bundle $(T^*M, \omega = d\xi_i \wedge dx^i)$ of a smooth manifold M. Consider two such homogeneous polynomials $P = P^{i_1\ldots i_k}(x)\xi_{i_1}\ldots\xi_{i_k}$ and $Q = Q^{i_1\ldots i_\ell}(x)\xi_{i_1}\ldots\xi_{i_\ell}$ of degree k and ℓ respectively; we will identify these polynomials with the corresponding smooth symmetric contravariant tensor fields $P^\sharp = P^{i_1\ldots i_k}(x)\partial_{i_1}\otimes\cdots\otimes\partial_{i_k}$ and $Q^\sharp = Q^{i_1\ldots i_\ell}(x)\partial_{i_1}\otimes\cdots\otimes\partial_{i_\ell}$.

The Schouten bracket $[P^\sharp, Q^\sharp]_S$ of the two contravariant symmetric tensors $P^\sharp$ and $Q^\sharp$ (of degree k and ℓ respectively) is the symmetric contravariant $(k+\ell-1)$-tensor corresponding to the Poisson bracket of P and Q, namely

$$[P^\sharp, Q^\sharp]_S = \{P, Q\}^\sharp. \tag{2.1}$$

Using the the Poisson bracket $\{P, Q\} = \partial_{\xi_i} P \partial_i Q - \partial_{\xi_i} Q \partial_i P$, and (2.1), we readily get the local expression of the Schouten bracket of $P^\sharp$ and $Q^\sharp$. If the manifold M is endowed with a symmetric connection ∇, the latter can be written as[1]

$$[P^\sharp, Q^\sharp]_S^{i_1\ldots i_{k+\ell-1}} = k\, P^{i(i_1\ldots i_{k-1}}\nabla_i Q^{i_k\ldots i_{k+\ell-1})} - \ell\, Q^{i(i_1\ldots i_{\ell-1}}\nabla_i P^{i_\ell\ldots i_{k+\ell-1})}. \tag{2.2}$$

If M is, in addition, equipped with a (pseudo-)Riemannian metric, g, we denote by

$$H = \tfrac{1}{2}\, \mathrm{g}^{ij}\xi_i\xi_j \tag{2.3}$$

the Hamiltonian function associated with this structure. The Hamiltonian flow associated with H is nothing but the geodesic flow on T^*M.

A symmetric contravariant tensor field $P^\sharp$ of degree k satisfying $\{H, P\} = 0$ is called a Killing (or Killing-Stäckel) tensor; using now the Levi-Civita connection ∇ in (2.2), this condition reads

$$\nabla^{(i}P^{i_1\ldots i_k)} = 0. \tag{2.4}$$

[1]In this article the round (resp. square) brackets will denote symmetrization (resp. skew-symmetrization) with the appropriate combinatorial factor.

2.2 Killing-Maxwell tensors

2.2.1 Souriau's coupling

In the presence of an electromagnetic field, F, Souriau [33] has proposed to replace the canonical symplectic structure, ω, of T^*M by the twisted symplectic structure $\omega_F = d\xi_i \wedge dx^i + \frac{1}{2}F_{ij}dx^i \wedge dx^j$. The (gauge-invariant) Poisson bivector now reads

$$\pi_F = \partial_{\xi_i} \wedge \partial_i - \tfrac{1}{2}F_{ij}\, \partial_{\xi_i} \wedge \partial_{\xi_j}.$$

The Poisson bracket of two observables P, Q of T^*M is now

$$\{P, Q\}_F = \pi_F(dP, dQ) = \partial_{\xi_i} P \partial_i Q - \partial_{\xi_i} Q \partial_i P - F_{ij}\, \partial_{\xi_i} P \wedge \partial_{\xi_j} Q, \tag{2.5}$$

and the Schouten-Maxwell bracket of two polynomials P and Q is then defined by

$$[P^\sharp, Q^\sharp]_{S,F} = \{P, Q\}^\sharp_F.$$

If the manifold M is endowed with a symmetric connection ∇, the Schouten-Maxwell bracket takes on the following form

$$\begin{aligned}[P^\sharp, Q^\sharp]_{S,F} = \quad & [P^\sharp, Q^\sharp]_S^{i_1 \ldots i_{k+\ell-1}} \partial_{i_1} \otimes \cdots \otimes \partial_{i_{k+\ell-1}} \\ & -k\ell\, F_{ij} P^{i(i_1 \ldots i_{k-1}}\, Q^{i_k \ldots i_{k+\ell-2})j} \partial_{i_1} \otimes \cdots \otimes \partial_{i_{k+\ell-2}}\end{aligned} \tag{2.6}$$

with the expression (2.2) of the Schouten bracket $[\,\cdot\,,\,\cdot\,]_S$.

Suppose now that the manifold M is endowed with a metric g; the Hamiltonian vector field on (T^*M, ω_F) for the Hamiltonian H given by (2.3) yields the the Lorentz equations of motions for a charged test particle moving on (M, g) under the influence of an external electromagnetic field F.

A symmetric contravariant tensor field $P^\sharp$ of degree k on (M, g) is now called a Killing-Maxwell tensor if $\{H, P\}_F = 0$. The Killing-Maxwell equations then read, using (2.6),

$$\nabla^{(i} P^{i_1 \ldots i_k)} = 0 \qquad \& \qquad P^{i(i_1 \ldots i_{k-1}}\, F_i^{i_k)} = 0 \tag{2.7}$$

where $F_i^j = \mathrm{g}^{jm} F_{mi}$, in accordance with previous results [11] obtained with a slightly different standpoint.

The conditions (2.7) are of special importance for proving the classical and quantum integrability of the equations of motion of a charged test particle in the generalized Kerr-Newman background.

2.2.2 Standard electromagnetic coupling

A more traditional, though equivalent, means to deal with the coupling to an electromagnetic field, $F = dA$ (locally), is to keep the canonical 1-form, $\alpha = \xi_i dx^i$, on T^*M unchanged, and hence to work with the original Poisson bracket $\{\cdot, \cdot\}$, but to replace the Hamiltonian (2.3) by

$$\widetilde{H} = \tfrac{1}{2}\mathrm{g}^{ij}(\xi_i - A_i)(\xi_j - A_j) \tag{2.8}$$

where the tilde makes it clear that the expressions to consider are actually polynomials in the variables $\xi_i - A_i$, for $i = 1, \ldots, n$; for example, if $P = P^{i_1 \cdots i_k}\xi_{i_1} \ldots \xi_{i_k}$, then

$$\widetilde{P} = P^{i_1 \ldots i_k}(\xi_{i_1} - A_{i_1}) \ldots (\xi_{i_k} - A_{i_k}). \tag{2.9}$$

The equations of motion given by the Hamiltonian vector field for the Hamiltonian (2.8) on $(T^*M, d\alpha)$ are, again, the Lorentz equations of motion.

The Schouten-Maxwell brackets and Schouten brackets for the electromagnetic coupling are related as follows via the corresponding Poisson brackets, viz

$$\{P, Q\}_F = \{\widetilde{P}, \widetilde{Q}\}.$$

In this framework, a Killing-Maxwell tensor, $P^\sharp$, of degree k on (M, g) is defined by the equation $\{\widetilde{H}, \widetilde{P}\} = 0$. The resulting constraints are, again, given by (2.7).

From now on, and in order to simplify the notation, we will omit the $\sharp$-superscript and use the same symbol for symmetric contravariant tensors and the corresponding polynomial functions on T^*M.

2.3 General definition of classical integrability

Let us recall that a dynamical system $(\mathcal{M}, \omega, H)$ is (Liouville) integrable if there exist $n = \frac{1}{2}\dim \mathcal{M}$ independent Poisson-commuting functions $P_1, \ldots, P_n \in C^\infty(\mathcal{M})$ — that is $dP_1 \wedge \cdots \wedge dP_n \neq 0$ and $\{P_k, P_\ell\} = 0$ for all $k, \ell = 1, \ldots, n$ — such that $P_1 = H$.

We will, in the sequel, confine considerations to the case of cotangent bundles, $(\mathcal{M} = T^*M, \omega = d\theta)$ where θ is the canonical 1-form, and of polynomial functions, $P_1, \ldots, P_n$, on T^*M, that is to the case of n Schouten-commuting Killing tensors. Moreover, all examples that we will consider will be given by polynomials of degree two or three.

2.4 The Stäckel systems

These systems on $(T^*M, \omega = d\xi_i \wedge dx^i)$ are governed by the Hamiltonians

$$H = \sum_{i=1}^{n} a^i(x) \left(\tfrac{1}{2}\xi_i^2 + f_i(x^i) \right) \tag{2.10}$$

where the i-th function f_i depends on the coordinate x^i only, and the functions a^i are defined as follows. Let B denote a $\mathrm{GL}(n, \mathbf{R})$-valued function defined on M and such that

$$B(x) = (B_1(x^1) B_2(x^2) \dots B_n(x^n))$$

where the i-th column $B_i(x^i)$ depends on x^i only $(i = 1, \dots, n)$; such a matrix will be called a Stäckel matrix. Then take

$$a(x) = \begin{pmatrix} a^1(x) \\ \vdots \\ a^n(x) \end{pmatrix}$$

to be the first column $A_1(x)$ of the matrix $A(x) = B(x)^{-1}$.

The integrability of such a system follows from the existence of n quadratic polynomials

$$I_\ell = \sum_{i=1}^{n} A^i_\ell(x) \left(\tfrac{1}{2}\xi_i^2 + f_i(x^i) \right), \qquad \ell = 1, \dots, n, \qquad H = I_1. \tag{2.11}$$

We call Stäckel potential every function of the form

$$U_\ell(x) = \sum_{i=1}^{n} A^i_\ell(x) f_i(x^i), \qquad \ell = 1, \dots, n; \tag{2.12}$$

the potential appearing in the Hamiltonian is just U_1.

One can check (see, e.g., [28], p. 101) that the n independent quantities I_ℓ are such that

$$\{I_\ell, I_m\} = \sum_{s,t=1}^{n} (A^s_\ell \, \partial_s A^t_m - A^s_m \, \partial_s A^t_\ell) \xi_s \left(\tfrac{1}{2}\xi_t^2 + f_t \right), \qquad \ell \neq m.$$

The relation $A = B^{-1}$, gives the useful identity[2]

$$\partial_k A^i_j = -C^i_k \, A^k_j, \qquad C^i_k = \sum_{s=1}^{n} A^i_s \frac{dB^s_k}{dx^k}, \tag{2.13}$$

[2] The Einstein summation convention is not used.

which implies

$$A^s_\ell\, \partial_s A^t_m - A^s_m\, \partial_s A^t_\ell = 0, \quad \ell \neq m, \quad s, t = 1, \ldots, n \tag{2.14}$$

and therefore the so defined Stäckel systems are classically integrable.

Remark 2.1. Let us mention an interesting result due to Pars (see [28], p. 102): for a system whose Hamiltonian is of the form (2.10), the Hamilton-Jacobi equation is separable if and only if this system is Stäckel.

Although these systems constitute quite a large class of integrable systems, they do not exhaust the full class. A simple example of a non-Stäckel integrable system was produced by Di Pirro (see Section 2.9).

2.5 The Jacobi integrable system on the ellipsoid

Let $\mathcal{E} \subset \mathbf{R}^{n+1}$ be the n-dimensional ellipsoid defined by the equation $Q_0(y, y) = 1$ where we define, for $y, z \in \mathbf{R}^{n+1}$,

$$Q_\lambda(y, z) = \sum_{\alpha=0}^{n} \frac{y_\alpha z_\alpha}{a_\alpha - \lambda}, \tag{2.15}$$

with $0 < a_0 < a_1 < \ldots < a_n$; the equations $Q_\lambda(y, y) = 1$ define a family of confocal quadrics.

It has been proved by Jacobi (in the case $n = 2$) that the differential equations governing the geodetic motions on the ellipsoid, $\mathcal{E}$, form an integrable system. The same remains true if a quadratic potential is admitted (see [27]). The Hamiltonian of the system, prior to reduction, reads

$$H(p, y) = \frac{1}{2} \sum_{\alpha=0}^{n} p_\alpha^2 + \frac{a}{2} \sum_{\alpha=0}^{n} y_\alpha^2 \tag{2.16}$$

where $p, y \in \mathbf{R}^{n+1}$ and a is some real parameter.

Moser has shown [26] that the following polynomial functions

$$F_\alpha(p, y) = p_\alpha^2 + a y_\alpha^2 + \sum_{\beta \neq \alpha} \frac{(p_\alpha y_\beta - p_\beta y_\alpha)^2}{a_\alpha - a_\beta} \qquad \text{with} \qquad \alpha = 0, 1, \ldots, n, \tag{2.17}$$

are in involution on $(T^*\mathbf{R}^{n+1}, \sum_{\alpha=0}^{n} dp_\alpha \wedge dy_\alpha)$. Those will generate the commuting first integrals of the Jacobi dynamical system on the cotangent bundle $T^*\mathcal{E}$ of the ellipsoid.

Our goal is to deduce from the knowledge of (2.17) the independent quantities in involution $I_1, \ldots, I_n$ on $(T^*\mathcal{E}, d\xi_i \wedge dx^i)$ from the symplectic embedding

$$\iota : T^*\mathcal{E} \hookrightarrow T^*\mathbf{R}^{n+1}$$

given by $Z_1(p,y) = Q_0(y,y) - 1 = 0$ and $Z_2(p,y) = Q_0(p,y) = 0$.

Proposition 2.2. *The restrictions $F_\alpha|_{T^*\mathcal{E}} = F_\alpha \circ \iota$ of the functions (2.17) Poisson-commute on $T^*\mathcal{E}$.*

Proof. We get, using Dirac brackets,

$$\begin{aligned} \{F_\alpha|_{T^*\mathcal{E}}, F_\beta|_{T^*\mathcal{E}}\} &= \{F_\alpha, F_\beta\}|_{T^*\mathcal{E}} \\ &\quad - \frac{1}{\{Z_1, Z_2\}} \left[\{Z_1, F_\alpha\}\{Z_2, F_\beta\} - \{Z_1, F_\beta\}\{Z_2, F_\alpha\}\right]|_{T^*\mathcal{E}} \end{aligned} \tag{2.18}$$

for second-class constraints. Now, the denominator $\{Z_1, Z_2\} = -2\sum_{\alpha=0}^{n} (y_\alpha/a_\alpha)^2$ doesn't vanish while $\{Z_1, F_\alpha\} = 4(p_\alpha y_\alpha/a_\alpha)Z_1 - 4(y_\alpha^2/a_\alpha)Z_2$ is zero on $T^*\mathcal{E}$, for all $\alpha = 0, ..., n$. The fact that $\{F_\alpha, F_\beta\} = 0$ completes the proof. □

The reduced Hamiltonian for the Jacobi system on the ellipsoid $\mathcal{E}$ is plainly

$$H = \frac{1}{2}\sum_{\alpha=0}^{n} \left(p_\alpha^2 + a y_\alpha^2\right)\Big|_{T^*\mathcal{E}} = \frac{1}{2}\sum_{\alpha=0}^{n} F_\alpha\Big|_{T^*\mathcal{E}}. \tag{2.19}$$

In order to provide explicit expressions for the function in involution $I_1, \ldots, I_n$, we resort to Jacobi ellipsoidal coordinates $x^1, \ldots, x^n$ on $\mathcal{E}$. Those are defined by

$$Q_\lambda(y,y) = 1 - \frac{\lambda U_x(\lambda)}{V(\lambda)} \tag{2.20}$$

where

$$U_x(\lambda) = \prod_{i=1}^{n}(\lambda - x^i) \qquad \text{and} \qquad V(\lambda) = \prod_{\alpha=0}^{n}(\lambda - a_\alpha) \tag{2.21}$$

and are such that $a_0 < x^1 < a_1 < x^2 < \ldots < x^n < a_n$. The induced metric, $\mathrm{g} = \sum_{i,j=1}^{n} \mathrm{g}_{ij}(x) dx^i dx^j$, of the ellipsoid $\mathcal{E}$ is given by

$$\mathrm{g}_{ij}(x) = \frac{1}{4}\sum_{\alpha=0}^{n} \frac{y_\alpha^2}{(a_\alpha - x^i)(a_\alpha - x^j)}$$

and retains the form [26]

$$\mathrm{g} = \sum_{i=1}^{n} \mathrm{g}_i(x)(dx^i)^2 \qquad \text{where} \qquad \mathrm{g}_i(x) = -\frac{x^i}{4}\frac{U_x'(x^i)}{V(x^i)} \tag{2.22}$$

which is actually Riemannian because of the previous inequalities. We put for convenience $\mathrm{g}^i(x) = 1/\mathrm{g}_i(x)$.

Using (2.20) and (2.21), we find the local expressions $y_\alpha(x)$ via the formula

$$y_\alpha^2 = a_\alpha \frac{\prod_{i=1}^{n}(a_\alpha - x^i)}{\prod_{\beta\neq\alpha}(a_\alpha - a_\beta)} \tag{2.23}$$

and then obtain the constrained coordinate functions

$$p_\alpha(\xi, x) = -\tfrac{1}{2} y_\alpha \sum_{i=1}^{n} \frac{\mathrm{g}^i(x)\xi_i}{(a_\alpha - x^i)} \tag{2.24}$$

given by the induced canonical 1-form $\sum_{i=1}^n \xi_i\, dx^i = \iota^*\left(\sum_{\alpha=0}^n p_\alpha dy_\alpha\right)$.

The Hamiltonian (2.19) on $(T^*\mathcal{E}, d\xi_i \wedge dx^i)$ is then found to be

$$H = \frac{1}{2}\sum_{i=1}^{n} \mathrm{g}^i(x)\xi_i^2 + \frac{a}{2}\left[\sum_{\alpha=0}^{n} a_\alpha - \sum_{i=1}^{n} x^i\right]. \tag{2.25}$$

Note that the potential term is obtained from the large λ behaviour

$$Q_\lambda(y, y) \sim \frac{1}{\lambda}\sum_{\alpha=0}^{n} y_\alpha^2 + \frac{1}{\lambda^2}\sum_{\alpha=0}^{n} a_\alpha y_\alpha^2 + \cdots$$

which can be computed using relation (2.20). One gets

$$Q_\lambda(y, y) \sim \frac{1}{\lambda}\left[\sum_{\alpha=0}^{n} a_\alpha - \sum_{i=1}^{n} x^i\right] + \cdots$$

One relates the conserved quantities (2.17) to their reduced expressions on $T^*\mathcal{E}$ by computing, using (2.24) and (2.23), the expression of $F_\alpha|_{T^*\mathcal{E}}$. One gets the

Proposition 2.3. *The Moser conserved quantities* $(F_\alpha|_{T^*\mathcal{E}})_{\alpha=0,\ldots,n}$ *retain the form*

$$F_\alpha|_{T^*\mathcal{E}} = \frac{a_\alpha\, G_{a_\alpha}(\xi, x)}{\prod\limits_{\beta\neq\alpha}(a_\alpha - a_\beta)}$$

where

$$G_\lambda(\xi, x) = \sum_{i=1}^{n} g^i(x) \prod_{j\neq i}(\lambda - x^j)\xi_i^2 + a\prod_{i=1}^{n}(\lambda - x^i). \tag{2.26}$$

It is useful to introduce the notation $\sigma_k^i(x)$ for the symmetric functions of order $k = 0, 1, \ldots, n-1$ of the variables $(x^1, \ldots, x^n)$, with the exclusion of index i, namely

$$\prod_{j\neq i}(\lambda - x^j) = \sum_{k=1}^{n}(-1)^{k-1}\lambda^{n-k}\sigma_{k-1}^i(x). \tag{2.27}$$

We note that, from the above definition, $\sigma_0^i(x) = 1$.

It is also worthwhile to introduce other symmetric functions, $\sigma_k(x)$, via

$$\prod_{j=1}^{n}(\lambda - x^j) = \sum_{k=0}^{n}(-1)^k\lambda^{n-k}\sigma_k(x). \tag{2.28}$$

We thus have

$$G_\lambda(\xi, x) = \sum_{i=1}^{n}(-1)^{i-1}\lambda^{n-i}I_i(\xi, x) + a(-\lambda)^n \tag{2.29}$$

where the independent functions I_i $(i = 1, \ldots, n)$ are in involution and can be written as

$$I_i(\xi, x) = \sum_{j=1}^{n} A_i^j(x)\xi_j^2 - a\sigma_i(x) \qquad \text{with} \qquad A_i^j(x) = g^j(x)\sigma_{i-1}^j(x). \tag{2.30}$$

In the case $i = 1$, we recover the Hamiltonian (2.25), i.e.,

$$H = \frac{1}{2}I_1 + \frac{a}{2}\sum_{\alpha=0}^{n} a_\alpha.$$

Proposition 2.4. *The Jacobi system on $T^\star\mathcal{E}$ defines a Stäckel system, with Stäckel matrix*

$$B^i_k(x^k) = (-1)^i\,\frac{(x^k)^{n+1-i}}{4V(x^k)} \tag{2.31}$$

and potential functions

$$f_k(x^k) = a\frac{(x^k)^{n+1}}{4V(x^k)} \tag{2.32}$$

for $i, k = 1, \dots, n$.

Proof. It is obvious from its definition that B is a Stäckel matrix. We just need to prove that $A = B^{-1}$. To this aim we first prove a useful identity. Let us consider the integral in the complex plane

$$\frac{1}{2i\pi}\int_{|z|=R}\frac{z^{n-i}}{(z-\lambda)}\frac{U_x(\lambda)}{U_x(z)}\,dz.$$

When $R \to \infty$ the previous integral vanishes because the integrand vanishes as $1/R^2$ for large R. We then compute this integral using the theorem of residues and we get the identity

$$\sum_{k=1}^{n}\frac{(x^k)^{n-i}}{U_x'(x^k)}\prod_{j\neq k}(\lambda - x^j) = \lambda^{n-i}. \tag{2.33}$$

Equipped with this identity let us now prove that

$$\sum_{k=1}^{n} B^i_k\, A^k_j = \delta^i_j.$$

Multiplying this relation by $(-1)^{j-1}\lambda^{n-j}$ and summing over j from 1 to n, we get the equivalent relation

$$\sum_{k=1}^{n} B^i_k \sum_{j=1}^{n}(-1)^{j-1}\lambda^{n-j}A^k_j = (-1)^{i-1}\lambda^{n-i},$$

which becomes, using (2.30) and (2.27):

$$\sum_{k=1}^{n} B^i_k\, g^k(x)\prod_{j\neq k}(\lambda - x^j) = (-1)^{i-1}\lambda^{n-i}.$$

Using the explicit form of $g^k(x)$ given in (2.22) and of the matrix B, this relation reduces to the identity (2.33) and this completes the derivation of (2.31).

In order to get the functions $f_i(x^i)$ as in (2.10), let us resort to (2.30) and solve, for the unknown f_i, the following equation

$$-a\sigma_i(x) = \sum_{j=1}^{n} A_i^j(x) f_j.$$

Multiplying both sides by B_k^i, summing over i from 1 to n, and using (2.31) we get

$$\begin{aligned} f_k &= -a\sum_{i=1}^{n} B_k^i \sigma_i(x) = -\frac{a}{4V(x^k)} \sum_{i=1}^{n} (-1)^i (x^k)^{n+1-i} \sigma_i(x) \\ &= -\frac{a}{4V(x^k)} \left[\sum_{i=0}^{n} (-1)^i (x^k)^{n+1-i} \sigma_i(x) - (x^k)^{n+1} \right]. \end{aligned}$$

In view of (2.28), we have $\sum_{i=0}^{n} (-1)^i (x^k)^{n-i} \sigma_i(x) = \prod_{j=1}^{n} (x^k - x^j) = 0$, which completes the proof. $\square$

Remark 2.5. 1. The fact that the geodesic flow on $T^\star\mathcal{E}$ is a Stäckel system was first proved by Benenti in [3]. We have given here a new derivation, which makes the link between Moser's conserved quantities on $T^*\mathbf{R}^{n+1}$ and the Stäckel conserved quantities on $T^*\mathcal{E}$. We have extended this link to the case where Jacobi's potential is admitted.

2. Checking that the unconstrained observables I_i are in involution is most conveniently done using their generating function (2.26). Indeed it is easy to verify the relation

$$\{G_\lambda(\xi, x), G_\mu(\xi, x)\} = 0, \qquad \lambda,\ \mu \in \mathbf{R},$$

which implies, via (2.29), and upon expansion in powers of λ and μ, the relations $\{I_i, I_j\} = 0$ for any $i, j = 1, \ldots, n$.

3. Some authors [2, 22] have quantized the full set of commuting observables for the geodesic flow of the ellipsoid $\mathcal{E} \subset \mathbf{R}^{n+1}$ in its unconstrained form, namely on $T^*\mathbf{R}^{n+1}$. Notice though that in the reduction process from $T^*\mathbf{R}^{n+1}$ to $T^*\mathcal{E}$ quantum corrections may prove necessary in order to insure self-adjointness of the quantized observables. Our point of view will be to perform the classical reduction in the first place and then to quantize the observables directly on $T^*\mathcal{E}$ via a specific procedure that will be described in Section 3.

2.6 The Neumann system

The Neumann Hamiltonian on $(T^*\mathbf{R}^{n+1}, \sum_{\alpha=0}^n dp_\alpha \wedge dy_\alpha)$ is

$$H = \tfrac{1}{2}\sum_{\alpha=0}^{n}\left(p_\alpha^2 + a_\alpha y_\alpha^2\right) \tag{2.34}$$

with the real parameters $0 < a_0 < a_1 < \ldots < a_n$. Under the symplectic reduction, with the second class constraints

$$Z_1(p,y) = \sum_{\alpha=0}^{n} y_\alpha^2 - 1 = 0, \qquad\qquad Z_2(p,y) = \sum_{\alpha=0}^{n} p_\alpha y^\alpha = 0, \tag{2.35}$$

it becomes a dynamical system on $(T^*S^n, d\xi_i \wedge dx^i)$.

This system is classically integrable, with the following commuting first integrals of the Hamiltonian flow in $T^*\mathbf{R}^{n+1}$:

$$F_\alpha(p,y) = y_\alpha^2 + \sum_{\beta\neq\alpha}\frac{\left(p_\alpha y_\beta - p_\beta y_\alpha\right)^2}{a_\alpha - a_\beta} \qquad \text{with} \qquad \alpha = 0, 1, \ldots, n. \tag{2.36}$$

The symplectic embedding

$$\iota : T^*S^n \hookrightarrow T^*\mathbf{R}^{n+1}$$

given by $Z_1(p,y) = 0$ and $Z_2(p,y) = 0$ preserves the previous conservation laws. Indeed the Poisson brackets of the restrictions $F_\alpha|_{T^*\mathcal{E}} = F_\alpha \circ \iota$ of the functions F_α are still given by the Dirac brackets (2.18) of the second class constraints (2.35). This time we have

$$\{Z_1, Z_2\} = -2\sum_{\alpha=0}^{n} y_\alpha^2 \neq 0, \qquad \{Z_1, F_\alpha\} = 0,$$

which gives again

$$\{F_\alpha|_{T^*\mathcal{E}}, F_\beta|_{T^*\mathcal{E}}\} = 0.$$

Let us introduce an adapted coordinate system on $(T^*S^n, d\xi_i \wedge dx^i)$ much in the same manner as for the ellipsoid.

We start with the following definition [26] of a coordinate system $(x^1, \ldots, x^n)$ on S^n:

$$Q_\lambda(y,y) = \sum_{\alpha=0}^{n}\frac{y_\alpha^2}{a_\alpha - \lambda} = -\frac{\prod_{i=1}^n(\lambda - x^i)}{\prod_{\alpha=0}^n(\lambda - a_\alpha)}.$$

The following inequalities hold: $0 < a_0 < x^1 < a_1 < \ldots < x^n < a_n$. We get, in the same way as before,

$$y_\alpha^2 = \frac{\prod_{i=1}^n (a_\alpha - x^i)}{\prod_{\beta \neq \alpha}(a_\alpha - a_\beta)} \tag{2.37}$$

together with the following expression of the round metric $\mathrm{g} = \sum_{\alpha=0}^n dy_\alpha^2|_{S^n}$ in terms of the newly introduced coordinates, namely

$$\mathrm{g} = \sum_{i=1}^n \mathrm{g}_i(x)(dx^i)^2 \qquad \text{with} \qquad \mathrm{g}_i(x) = -\frac{U_x'(x^i)}{4V(x^i)} \tag{2.38}$$

with the notation (2.21). Again, we put for convenience $\mathrm{g}^i(x) = 1/\mathrm{g}_i(x)$.

Our goal is to deduce from the knowledge of (2.36) the independent quantities in involution $I_1, \ldots, I_n$ on $(T^*S^n, d\xi_i \wedge dx^i)$. The formula (2.24) relating unconstrained and constrained momenta still holds and yields the

Proposition 2.6. *The Neumann system $(F_\alpha|_{T^*S^n})_{\alpha=0,\ldots,n}$ retains the following form*

$$F_\alpha|_{T^*S^n} = -\frac{G_{a_\alpha}(\xi, x)}{\prod\limits_{\beta \neq \alpha}(a_\alpha - a_\beta)}$$

where

$$G_\lambda(\xi, x) = \sum_{i=1}^n g^i(x) \prod_{j \neq i}(\lambda - x^j)\xi_i^2 + \prod_{j=1}^n (\lambda - x^j).$$

Let us, again, posit

$$G_\lambda(\xi, x) = \sum_{i=1}^n (-1)^{i-1} \lambda^{n-i} I_i(\xi, x) + \lambda^n$$

where the independent functions I_i $(i = 1, \ldots, n)$ are in involution and can be written as

$$I_i(\xi, x) = \sum_{j=1}^n A_i^j(x)\xi_j^2 - \sigma_i(x) \qquad \text{with} \qquad A_i^j(x) = g^j(x)\sigma_{i-1}^j(x), \tag{2.39}$$

where the symmetric functions $\sigma_i(x)$ are as in (2.28).

Using the relations

$$\sigma_1(x) = \sum_{i=1}^n x^i, \qquad \text{and} \qquad \sum_{\alpha=0}^n a_\alpha y_\alpha^2 = \sum_{\alpha=0}^n a_\alpha - \sum_{i=1}^n x^i,$$

one can check that the Hamiltonian (2.34) is $H = \frac{1}{2} I_1$.

Proposition 2.7. *The Neumann flow on* (T^*S^n, H) *defines a Stäckel system, with Stäckel matrix*

$$B^i_k(x^k) = (-1)^i \, \frac{(x^k)^{n-i}}{4V(x^k)}$$

and potential functions

$$f_k(x^k) = \frac{(x^k)^n}{4V(x^k)} \tag{2.40}$$

for $i, k = 1, \ldots, n$.

Proof. To check that $A = B^{-1}$, it is enough to use the identity (2.33). The computation of the potential functions f_k proceeds along the same lines as in the proof of Proposition 2.4. □

Remark 2.8. The involution property $\{I_i, I_j\} = 0$ for $i, j = 1, \ldots, n$, similarly to the case of the ellipsoid, is seen to follow from the relation $\{G_\lambda(\xi, x), G_\mu(\xi, x)\} = 0$.

2.7 Test particles in generalized Kerr-Newman background

Plebanski and Demianski have constructed in [29, 30] a class of metrics generalizing the Kerr-Newman solution in 4-dimensional spacetime. The former are also known as the Kerr-Newman-Taub-NUT-de Sitter solutions of the Einstein-Maxwell equations. The metric, in the coordinate system $(x^1, x^2, x^3, x^4) = (p, q, \sigma, \tau)$, retains the form

$$\mathrm{g} = \frac{X}{p^2+q^2}(d\tau + q^2 d\sigma)^2 - \frac{Y}{p^2+q^2}(d\tau - p^2 d\sigma)^2 + \frac{p^2+q^2}{X}\,dp^2 + \frac{p^2+q^2}{Y}\,dq^2 \tag{2.41}$$

with

$$X = \gamma - g^2 + 2np - \epsilon p^2 - \frac{\Lambda}{3}\,p^4, \qquad \& \qquad Y = \gamma + e^2 - 2mq + \epsilon q^2 - \frac{\Lambda}{3}\,q^4, \tag{2.42}$$

where (m, γ) are related to the mass and angular momentum of the Kerr black hole, (e, g) to the electric and magnetic charge; n is the NUT charge, and Λ the cosmological constant. The remaining parameter ϵ can be scaled out to ± 1 or 0.

This metric, g, together with the electromagnetic field, locally given by $F = dA$ where

$$A = \frac{1}{p^2+q^2}\Big[(eq + gp)d\tau + pq(gq - ep)d\sigma\Big], \tag{2.43}$$

provide an exact solution of the Einstein-Maxwell equations with cosmological constant Λ. Let us notice for further use that

$$\nabla_i A^i = 0. \tag{2.44}$$

Upon defining the 1-forms

$$\begin{aligned}
K &= \sqrt{\frac{Y}{2(p^2+q^2)}}\,(d\tau - p^2\,d\sigma) + \sqrt{\frac{p^2+q^2}{2Y}}\,dq,\\
L &= \sqrt{\frac{Y}{2(p^2+q^2)}}\,(d\tau - p^2\,d\sigma) - \sqrt{\frac{p^2+q^2}{2Y}}\,dq,\\
M_1 &= \sqrt{\frac{p^2+q^2}{X}}\,dp,\\
M_2 &= \sqrt{\frac{X}{p^2+q^2}}\,(d\tau + q^2 d\sigma),
\end{aligned}$$

one constructs the 2-form

$$\mathcal{Y} = pK \wedge L - qM_1 \wedge M_2. \tag{2.45}$$

One can check that the twice-symmetric tensor $P = -\mathcal{Y}^2$, namely $P_{ij} = -\mathcal{Y}_{ik}\mathcal{Y}_{\ell j}\mathrm{g}^{k\ell}$, is a Killing-Maxwell tensor (see (2.7)), given by

$$P = p^2(K \otimes L + L \otimes K) + q^2(M_1 \otimes M_1 + M_2 \otimes M_2). \tag{2.46}$$

We thus recover Carter's result [11] about the integrability of the Hamiltonian flow for a charged test particle in the generalized Kerr-Newman background in a different manner.

Remark 2.9. The 2-form $\mathcal{Y}$ in (2.45) defines what is usually called a Killing-Yano tensor [21, 8].

The four conserved quantities in involution for the generalized Kerr-Newman system are, respectively,

$$\widetilde{H} = \tfrac{1}{2}\,\mathrm{g}^{ij}(\xi_i - A_i)(\xi_j - A_j), \qquad \widetilde{P} = P^{ij}(\xi_i - A_i)(\xi_j - A_j) \tag{2.47}$$

where P is as in (2.46), and

$$\widetilde{S} = \xi_3 - A_3, \qquad \widetilde{T} = \xi_4 - A_4. \tag{2.48}$$

2.8 The Multi-Centre geodesic flow

The class of Multi-Centre Euclidean metrics in 4 dimensions retain, in a local coordinate system $(x^i) = (t, (y^a)) \in \mathbf{R} \times \mathbf{R}^3$, the form

$$\mathrm{g} = \frac{1}{V(y)}(dt + A_a(y)dy^a)^2 + V(y)\gamma \tag{2.49}$$

with $\gamma = \delta_{ab}\, dy^a dy^b$ the flat Euclidean metric in 3-space, and $dV = \pm\star(dA)$ where $\star$ is the Hodge star for γ. These conditions insure that the metric (2.49) is Ricci-flat.

For some special potentials $V(y)$, the geodesic flow is integrable as shown in [20, 13, 35]. The four conserved quantities in involution are given by

$$H = \tfrac{1}{2}\mathrm{g}^{ij}\xi_i\xi_j, \qquad K = K^i\xi_i, \qquad L = L^i\xi_i, \qquad P = P^{ij}\xi_i\xi_j, \tag{2.50}$$

where K and L are two commuting Killing vectors and P a Killing 2-tensor whose expressions can be found in the previous References.

2.9 The Di Pirro system

Di Pirro has proved (see, e.g., [28], p. 113) that the Hamiltonian on $T^*\mathbf{R}^3$

$$H = \frac{1}{2(\gamma(x^1,x^2) + c(x^3))}\left[a(x^1,x^2)\xi_1^2 + b(x^1,x^2)\xi_2^2 + \xi_3^2\right] \tag{2.51}$$

admits one and only one additional first integral given by

$$P = \frac{1}{(\gamma(x^1,x^2) + c(x^3))}\left[c(x^3)\left(a(x^1,x^2)\xi_1^2 + b(x^1,x^2)\xi_2^2\right) - \gamma(x^1,x^2)\xi_3^2\right]. \tag{2.52}$$

In the case where the metric defined by H in (2.51) possesses a Killing vector, the system becomes integrable though not of Stäckel type. This happens, e.g., if (i) $c(x^3) = \mathrm{const.}$, or (ii) $a = b$ and γ depend on $r = \sqrt{(x^1)^2 + (x^2)^2}$ only.

3 A quantization scheme for integrable systems

We wish to deal now with the quantum version of the preceding examples. Let us start with some preliminary considerations:

1. There is no universally accepted procedure of quantization, i.e., of a linear identification, $\mathcal{Q}$, of a space of classical observables with some space of linear symmetric operators on a Hilbert space. One — among many — of the pathways to construct such a quantization mapping has been to demand that the mapping $\mathcal{Q}$ be equivariant with respect to some Lie group of symplectomorphisms of classical phase space.

2. Similarly, there is no universally accepted notion of quantum integrability. However, given a classical integrable system $P_1, \ldots, P_n$ on a symplectic manifold $(\mathcal{M}, \omega)$, and a quantization mapping $\mathcal{Q} : P_i \mapsto \widehat{P}_i$, we will say that such a system is integrable in the quantum sense if $[\widehat{P}_i, \widehat{P}_j] = 0$ for all $i, j = 1, \ldots, n$.

3. A large number of integrable systems involve *quadratic* observables. We will thus choose to concentrate on this important — yet very special — case, both from the classical and quantum viewpoint.

4. Among all possible quantization procedures, the search for integrability-preserving ones (if any) should be of fundamental importance. The quantization of quadratic observables we will present below might serve as a starting point for such a programme.

3.1 Quantizing quadratic and cubic observables

Let us recall that the space $\mathcal{F}_\lambda(M)$ of λ-densities on M is defined as the space of sections of the complex line bundle $|\Lambda^n T^*M|^\lambda \otimes \mathbb{C}$. In the case where the configuration manifold is orientable, (M, vol), such a λ-density can be, locally, cast into the form $\phi = f|\mathrm{vol}|^\lambda$ with $f \in C^\infty(M)$ which means that ϕ transforms under the action of $a \in \mathrm{Diff}(M)$ according to $f \mapsto a_*f|(a_*\mathrm{vol})/\mathrm{vol}|^\lambda$.

The completion $\mathcal{H}(M)$ of the space of compactly supported half-densities, $\lambda = \frac{1}{2}$, is a Hilbert space canonically attached to M that will be used throughout this article. The scalar product of two half-densities reads

$$\langle \phi, \psi \rangle = \int_M \overline{\phi}\, \psi$$

where the bar stands for complex conjugation.

We will assume that configuration space is endowed with a (pseudo-)Riemannian structure, (M, g); and denote by $|\mathrm{vol}_\mathrm{g}|$ the corresponding density and by Γ_{ij}^k the associated Christoffel symbols.

The quantization now introduced is a linear invertible mapping from the space of quadratic observables $P = P_2^{jk}(x)\xi_j\xi_k + P_1^j(x)\xi_j + P_0(x)$ to the space of second-order differential operators on $\mathcal{H}(M)$, viz $A = \widehat{P} = A_2^{jk}(x)\nabla_j\nabla_k + A_1^j(x)\nabla_j + A_0(x)\mathbf{1}$ where the covariant derivative of half-densities $\nabla_j\phi = \partial_j\phi - \frac{1}{2}\Gamma^k_{jk}\phi$ (or, locally, $\nabla_j\phi = (\partial_j f)|\mathrm{vol_g}|^{\frac{1}{2}}$) has been used. We furthermore require that the principal symbol be preserved (see below (3.1), (3.2) and (3.3)), and that $\widehat{P}$ be formally self-adjoint, i.e., $\langle\phi, \widehat{P}\psi\rangle = \langle\widehat{P}\phi, \psi\rangle$ for all compactly supported $\phi, \psi \in \mathcal{F}_{\frac{1}{2}}(M)$.

The quantization reads

$$A_2^{jk} = -P_2^{jk} \tag{3.1}$$

$$A_1^j = iP_1^j - \nabla_k P_2^{jk} \tag{3.2}$$

$$A_0 = P_0 + \frac{i}{2}\nabla_j P_1^j \tag{3.3}$$

and admits the alternative form

$$\widehat{P} = -\nabla_j \circ P_2^{jk} \circ \nabla_k + \frac{i}{2}\left(P_1^j \circ \nabla_j + \nabla_j \circ P_1^j\right) + P_0\mathbf{1} \tag{3.4}$$

which makes clear the symmetry of the quantum operators.

Remark 3.1. The formula (3.4) was originally used by Carter [11] for proving the quantum integrability of the equations of motion of charged test particles in the Kerr-Newman solution.

Remark 3.2. It is worth mentioning that formula (3.4) actually corresponds at the same time to the projectively equivariant quantization [24, 16] and to the conformally equivariant quantization [17, 15] $\mathcal{Q}_{0,1}(P) : \mathcal{F}_0(M) \to \mathcal{F}_1(M)$ restricted to quadratic polynomials.

One can check the relations:

$$[\widehat{P}_0, \widehat{Q}_1] = i[P_0, Q_1]_S = i\{\widehat{P_0, Q_1}\}, \tag{3.5}$$

$$[\widehat{P}_0, \widehat{Q}_2] = -\tfrac{1}{2}\left(\nabla_j \circ [P_0, Q_2]_S^j + [P_0, Q_2]_S^j \circ \nabla_j\right) = i\{\widehat{P_0, Q_2}\}, \tag{3.6}$$

$$[\widehat{P}_1, \widehat{Q}_1] = -\tfrac{1}{2}\left(\nabla_j \circ [P_1, Q_1]_S^j + [P_1, Q_1]_S^j \circ \nabla_j\right) = i\{\widehat{P_1, Q_1}\}. \tag{3.7}$$

Quantum corrections appear explicitly whenever $k+\ell>2$, as can be seen from the next commutators:

$$[\widehat{P}_1,\widehat{Q}_2]=i\widehat{\{P_1,Q_2\}}+i\widehat{A}_{P_1,Q_2} \tag{3.8}$$

where

$$A_{P_1,Q_2}=\frac{1}{2}\nabla_j{\circ}Q_2^{jk}{\circ}\nabla_k(\nabla_\ell P_1^\ell) \tag{3.9}$$

is a scalar quantum correction that may vanish in some special instances, e.g., if the vector-field P_1 is divergence-free (in particular if it is a Killing vector-field).

The previous formulæ can be found, in a different guise, in[11]. Here, we will go one step further and compute the commutators $[\widehat{P}_2,\widehat{Q}_2]$ which involve third-order differential operators. To that end, we propose to quantize homogeneous cubic polynomials according to

$$\widehat{P}_3=-\frac{i}{2}\left(\nabla_j{\circ}P_3^{jk\ell}{\circ}\nabla_k{\circ}\nabla_\ell+\nabla_j{\circ}\nabla_k{\circ}P_3^{jk\ell}{\circ}\nabla_\ell\right) \tag{3.10}$$

as a “minimal” choice to insure the symmetry of the resulting operator.

Remark 3.3. The formula (3.10) precisely coincides with the projectively equivariant quantization [7] $\mathcal{Q}_{0,1}(P):\mathcal{F}_0(M)\to\mathcal{F}_1(M)$ restricted to cubic polynomials.

The previously mentioned commutator is actually given by

$$\begin{aligned}
[\widehat{P}_2,\widehat{Q}_2] \;&=\; [P_2,Q_2]_S^{jk\ell}\nabla_j{\circ}\nabla_k{\circ}\nabla_\ell \\
&\quad +\frac{3}{2}\left(\nabla_j[P_2,Q_2]_S^{jk\ell}\right)\nabla_k{\circ}\nabla_\ell \\
&\quad +\left[\frac{1}{2}\left(\nabla_j\nabla_k[P_2,Q_2]_S^{jk\ell}\right)+\frac{2}{3}\left(\nabla_k B^{k\ell}_{P_2,Q_2}\right)\right]\nabla_\ell
\end{aligned} \tag{3.11}$$

where the skew-symmetric tensor

$$\begin{aligned}
B^{jk}_{P,Q} \;&=\; P^{\ell[j}\nabla_\ell\nabla_m Q^{k]m}+P^{\ell[j}R^{k]}{}_{m,n\ell}Q^{mn}-(P\leftrightarrow Q) \\
&\quad -\nabla_\ell P^{m[j}\nabla_m Q^{k]\ell}-P^{\ell[j}R_{\ell m}Q^{k]m}
\end{aligned} \tag{3.12}$$

satisfies, in addition, $B_{P,Q}=-B_{Q,P}$. We have used the following convention for the Riemann and Ricci tensors, viz $R^\ell{}_{i,jk}=\partial_j\Gamma^\ell_{ik}-(j\leftrightarrow k)+\ldots$, and $R_{ij}=R^k{}_{i,kj}$.

We can rewrite the commutator (3.11) with the help of the quantization prescription (3.4) and (3.10) as

$$[\widehat{P}_2,\widehat{Q}_2]=i\widehat{\{P_2,Q_2\}}+i\widehat{A}_{P_2,Q_2} \tag{3.13}$$

where

$$A_{P_2,Q_2} = -\frac{2}{3}\left(\nabla_k B^{k\ell}_{P_2,Q_2}\right)\xi_\ell \tag{3.14}$$

is a divergence-free vector-field associated with the tensor (3.12) and providing the potential quantum correction for quadratic polynomials; recall that, according to (3.4), one has $\widehat{A}_{P_2,Q_2} = (i/2)(A^\ell_{P_2,Q_2}\circ\nabla_\ell + \nabla_\ell\circ A^\ell_{P_2,Q_2})$.

We thus have the

Proposition 3.4. *The commutator of the quantum operators $\widehat{P}$ and $\widehat{Q}$ associated with two general quadratic polynomials $P = P_2 + P_1 + P_0$ and $Q = Q_2 + Q_1 + Q_0$ reads*

$$\frac{1}{i}[\widehat{P},\widehat{Q}] = \widehat{\{P,Q\}} + \widehat{A}_{P_2,Q_2} + \widehat{A}_{P_1,Q_2} - \widehat{A}_{Q_1,P_2} \tag{3.15}$$

where the third-order differential operator $\widehat{\{P,Q\}}$ is given by (3.10).

Proof. The formula (3.15) results trivially from the previously computed commutators and from collecting the anomalous terms appearing in (3.8) and (3.13) only. □

Remark 3.5. In the special case where $Q_2 = H$ as given by (2.3), the anomalous tensor (3.12) takes the form

$$B^{jk}_{P,H} = -\tfrac{1}{2}\nabla^{[j}\nabla_\ell P^{k]\ell} - P^{\ell[j}R^{k]}_\ell$$

and reduces to

$$B^{jk}_{P,H} = -P^{\ell[j}R^{k]}_\ell \tag{3.16}$$

if P is a Killing tensor [11].

Remark 3.6. In the particular case where $H = \frac{1}{2}\mathrm{g}^{jk}(\xi_j - eA_j)(\xi_k - eA_k)$ is the Hamiltonian of the electromagnetic coupling, our quantum commutator (3.15) reduces to Carter's formula (6.16) in [11].

The purpose of our article is, indeed, to study, using explicit examples, how classical integrability behaves under the "minimal" quantization rules proposed in [11] and somewhat extended here. The next section will be devoted to the computation of the quantum corrections in (3.8) and (3.13) for all the examples that have been previously introduced.

3.2 The equivariance Lie algebra

So far, the transformation property of the quantization rules (3.4) and (3.10) under a change of coordinates has been put aside. It is mandatory to investigate if these rules are consistent with the map $\mathcal{Q} : P \mapsto \widehat{P}$ (which has been defined for cubic polynomials, $P = \sum_{k=0}^{3} P^{i_1 \cdots i_k} \xi_{i_1} \ldots \xi_{i_k}$, only) being equivariant with respect to some Lie subgroup of the group of diffeomorphisms of configuration space, M.

Restricting considerations to the infinitesimal version of the sought equivariance, we will therefore look for the set $\mathfrak{g}$ of all vector fields X with respect to which our quantization is equivariant, namely $L_X \mathcal{Q} = 0$. From its very definition, $\mathfrak{g}$ is a Lie subalgebra of the Lie algebra, $\mathrm{Vect}(M)$, of vector fields of M. The previous condition means that, for each polynomial P, the following holds:

$$L_X(\mathcal{Q}(P)\phi) - \mathcal{Q}(L_X P)\phi - \mathcal{Q}(P) L_X \phi = 0 \tag{3.17}$$

where $L_X \phi$ denotes the Lie derivative of the half-density ϕ of M with respect to the vector field $X \in \mathfrak{g}$ and $L_X P = \{X, P\}$ is the Poisson bracket of $X = X^i \xi_i$ and P.

Let us recall that, putting locally $\phi = f|\mathrm{vol}|^{\frac{1}{2}} \in \mathcal{F}_{\frac{1}{2}}$ with $f \in C^\infty(M)$, we get the following expression for the Lie derivative: $L_X \phi = (Xf + \frac{1}{2}\mathrm{div}(X) f)|\mathrm{vol}|^{\frac{1}{2}}$, or with a slight abuse of notation, $L_X \phi = X^j \nabla_j \phi + \frac{1}{2}(\nabla_j X^j)\phi = \frac{1}{2}(X^j \circ \nabla_j + \nabla_j \circ X^j)\phi$, that is

$$L_X \phi = \frac{1}{i} \widehat{X} \phi \tag{3.18}$$

for any $X \in \mathrm{Vect}(M)$.

The equivariance condition (3.17) must hold for any $\phi \in \mathcal{F}_{\frac{1}{2}}$ and thus translates into

$$[\widehat{X}, \widehat{P}] = i \widehat{\{X, P\}} \tag{3.19}$$

for any $X \in \mathfrak{g}$ and any cubic polynomial P. The Condition (3.19) characterizes the Lie algebra $\mathfrak{g}$ we are looking for. We will consider successively the case of polynomials of increasing degree:

(i) Returning to the previous relations (3.5), (3.7) together with $X = P_1$ and $P = Q_0 + Q_1$, we readily find that the Lie algebra $\mathfrak{g}_1$ spanned by the solutions of (3.19) restricted to polynomials P of degree one is $\mathfrak{g}_1 = \mathrm{Vect}(M)$.

(ii) Let us now proceed to the case of quadratic polynomials $P = P^{jk} \xi_j \xi_k$. The relations (3.7) and (3.9) give, in that case, the following equivariance defect

$$[\widehat{X}, \widehat{P}] - i \widehat{\{X, P\}} = \frac{i}{2} \nabla_j \circ P^{jk} \circ \nabla_k (\nabla_\ell X^\ell) \mathbf{1}. \tag{3.20}$$

This defect vanishes for any such P iff $\nabla_k(\nabla_\ell X^\ell) = 0$, i.e.,

$$d(\mathrm{div}(X)) = 0. \tag{3.21}$$

The vector fields X with constant divergence span now a subspace $\mathfrak{g}_2 \subset \mathfrak{g}_1$ which is, indeed, an infinite dimensional Lie subalgebra of $\mathrm{Vect}(M)$. The "minimal" quantization restricted to quadratic polynomials is therefore equivariant with respect to the group of all diffeomorphisms which preserve the volume up to a multiplicative nonzero constant.

(iii) Let us finally consider homogeneous cubic polynomials $P = P^{jk\ell}\xi_j\xi_k\xi_\ell$ and compute the equivariance defect in this case. A tedious calculation leads to

$$[\widehat{X}, \widehat{P}] - i\widehat{\{X, P\}} = i\widehat{Z}, \qquad Z = Z^j\xi_j, \tag{3.22}$$

with

$$Z^j = \nabla_k\left[P^{jk\ell}\nabla_\ell \mathrm{div}(X) - P^{\ell m[j}L_X\Gamma^{k]}_{\ell m}\right] \tag{3.23}$$

where

$$L_X\Gamma^k_{\ell m} = \nabla_\ell\nabla_m X^k - R^k{}_{m,n\ell}X^n \tag{3.24}$$

is the Lie derivative of the symmetric linear connection ∇ with respect to the vector field X.

Proposition 3.7. *The Lie algebra $\mathfrak{g} \subset \mathrm{Vect}(M)$ with respect to which the "minimal" quantization (3.4) and (3.10) is equivariant is $\mathrm{aff}(M, \nabla)$, the Lie algebra of affine vector fields of (M, ∇).*

Proof. The equivariance condition (3.19), defining the Lie algebra $\mathfrak{g}_3$ we are looking for, is equivalent to $Z = 0$ in (3.22) for all symmetric tensor fields $P^{jk\ell}$, i.e., thanks to (3.23) to

$$T^{jk\ell}_k\nabla_\ell \mathrm{div}(X) - T^{\ell m[j}_k L_X\Gamma^{k]}_{\ell m} = 0$$

for all tensor fields $T^{\ell m j}_k = T^{(\ell m j)}_k$. This readily implies that

$$2\delta^j_{(i}\delta^k_\ell\nabla_{m)}\mathrm{div}(X) + \delta^j_{(i}L_X\Gamma^k_{\ell m)} - \delta^k_{(i}L_X\Gamma^j_{\ell m)} = 0.$$

Summing over $i = j$, one gets

$$2n\delta^k_m\nabla_\ell \mathrm{div}(X) + 4\delta^k_\ell\nabla_m \mathrm{div}(X) + (n+1)L_X\Gamma^k_{\ell m} - \delta^k_m L_X\Gamma^i_{\ell i} - \delta^k_\ell L_X\Gamma^i_{mi} = 0,$$

where $n = \dim(M)$, hence $\nabla_i \mathrm{div}(X) = 0$ and $L_X\Gamma^k_{ij} = \delta^k_i\varphi_j + \delta^k_j\varphi_i$ for some 1-form φ depending upon the (projective) vector field X. The expression (3.24) of the Lie derivative of the symmetric connection ∇ then yields $L_X\Gamma^j_{ij} = (n+1)\varphi_i = 0$ since we have found that $\nabla_i\nabla_j X^j = 0$. This entails $L_X\Gamma^k_{ij} = 0$, proving that $\mathfrak{g} = \mathfrak{g}_3$ is nothing but the Lie algebra $\mathrm{aff}(M,\nabla)$ of affine vector fields. $\square$

We thus obtain the nested equivariance Lie algebras

$$\mathfrak{g} = \mathrm{aff}(M,\nabla) \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 = \mathrm{Vect}(M)$$

where $\mathfrak{g}_2$ is the Lie algebra of vector fields with constant divergence. (Note that if M is compact without boundary, $\mathfrak{g}_2$ reduces to the Lie algebra of divergence-free vector fields.)

Conspicuously, our quantization scheme turns out to be equivariant with respect to a rather small Lie subgroup of $\mathrm{Diff}(M)$, namely of the affine group of (M,∇). It would be interesting to investigate to what extent the equivariance under the sole affine group, $\mathrm{GL}(n,\mathbf{R}) \ltimes \mathbf{R}^n$, of a flat affine structure (M,∇) allows one to uniquely extend to the whole algebra of polynomials the quantization scheme we have devised for cubic polynomials.

3.3 The quantum Stäckel system

The quantization of the general Stäckel system (see Section 2.4) has first been undertaken by Benenti, Chanu and Rastelli in [4, 5]. We will derive, here, the covariant expression of the quantum correction associated to the "minimal" quantization, with the help of the results obtained in Section 3.1.

Denote by $I_i = I_{2,i} + I_{0,i}$ the i-th Stäckel conserved quantity, $i = 1,\ldots,n$, in (2.11) where the indices 0 and 2 refer to the degree of homogeneity with respect to the coordinates ξ. Applying (3.15) with $P_1 = Q_1 = 0$, $P_2 = I_{2,i}$ and $Q_2 = I_{2,j}$ one gets

$$[\widehat{I}_i, \widehat{I}_j] = [\widehat{I}_{2,i}, \widehat{I}_{2,j}] = i\widehat{A}_{I_{2,i},I_{2,j}} = \frac{2}{3}\left(\nabla_k B^{k\ell}_{I_{2,i},I_{2,j}}\right)\nabla_\ell.$$

Remark 3.8. This result shows that there are no quantum corrections produced by the potential term. More generally, start with a system defined by independent, homogeneous, quadratic observables $H_1,\ldots,H_n$ which is integrable at the classical and quantum levels. Consider a new set of observables $H_1+U_1,\ldots,H_n+U_n$ obtained by adding potential terms $U_1,\ldots,U_n$; if the new system is classically integrable, it will remain integrable at the quantum level.

We are now in position to prove the following

Proposition 3.9. *The quantum correction (3.12) of a general Stäckel system, with commuting conserved quantities $I_1, \ldots, I_n$ defined by (2.11), retains the form*

$$B^{k\ell}_{I_{2,i},I_{2,j}} = -2 I_{2,i}^{s[k} R_{st} I_{2,j}^{\ell]t} \tag{3.25}$$

for $i, j = 1, \ldots, n$, where R_{st} denotes the components of the Ricci tensor of the metric associated with the Hamiltonian I_1.

Proof. As a preliminary remark, let us observe that the Stäckel metric, given by (2.10), needs not be Riemannian. So we will write it

$$\mathrm{g} = \sum_{i=1}^{n} \frac{(dx^i)^2}{A_1^i(x)} = \sum_{a=1}^{n} \eta_a (\theta^a)^2 \tag{3.26}$$

where $(\theta^a = dx^a/\sqrt{|A_1^a|})_{a=1,\ldots,n}$ is the orthonormal moving coframe and the signature of g is given by $\eta_a = \mathrm{sign}(A_1^a)$. We will denote by $(e_a = \sqrt{|A_1^a|}\partial_a)_{a=1,\ldots,n}$ the associated orthonormal frame with respect to the metric $\eta_{ab} = \eta_a \delta_{ab}$ used to raise and lower frame indices.

Let us recall, in order to fix the notation, that the connection form ω satisfies the structure equation $d\theta^a + \omega^a{}_b \wedge \theta^b = 0$ and the associated curvature form, Ω, given by $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$, is expressed in terms of the Riemann tensor by $\Omega^a{}_b = \frac{1}{2} R^a{}_{b,cd}\, \theta^c \wedge \theta^d$. The indices $a, \ldots, d$ run from 1 to n and the Einstein summation convention is used when no ambiguity arises. Denoting by $R^\ell{}_{i,jk}$ the local components of the Riemann tensor, we have $R^a{}_{b,cd} = \theta^a_\ell\, R^\ell{}_{i,jk}\, e^i_b e^j_c e^k_d$.

We start off with the calculation of the connection form, ω, and of some components of the curvature form, Ω. Straightforward computation, using relation (2.13), then yields for the non-vanishing components of the connection

$$\omega_{ab,a} = \tfrac{1}{2} \eta_b\, C^a_b \frac{|A_1^b|^{3/2}}{|A_1^a|}, \quad a \neq b, \qquad \omega_{ab,c} = \omega_{ab}(e_c),$$

the other nontrivial components $\omega_{ab,b}$ are obtained accordingly. For the curvature, a lengthy computation gives the special components

$$R_{ac,cb} = 3\left(-\eta_a \omega_{ca,c}\, \omega_{ab,a} - \eta_b \omega_{cb,c}\, \omega_{ba,b} + \eta_c \omega_{ca,c}\, \omega_{cb,c}\right), \qquad a \neq b, \tag{3.27}$$

which will be needed in the sequel.

Two last ingredients are the introduction of the frame components of various objects. We will denote the Killing tensor $I_{2,i}$ (resp. $I_{2,j}$) as P (resp. Q). Their frame components $P = P^{bc}\, e_b \otimes e_c$, and similarly for Q, will be

$$P^{bc} = p_b \delta_{bc}, \qquad p_b = \frac{A_i^b}{2|A_1^b|}, \qquad Q^{bc} = q_b \delta_{bc}, \qquad q_b = \frac{A_j^b}{2|A_1^b|}. \tag{3.28}$$

The covariant derivative will have the frame components

$$\mathcal{D}_c P_{ab} = e_c(P_{ab}) - \omega^s{}_{a,c} P_{sb} - \omega^s{}_{b,c} P_{as}.$$

The equations which express that P^{ab} is a Killing tensor are now

$$\begin{aligned} e_b(p_a) &= 2\omega_{ab,a}(\eta_a p_a - \eta_b p_b), \quad a \neq b, \\ e_a(p_a) &= 0, \end{aligned} \tag{3.29}$$

where the repeated indices are not summed over. One can check that they hold true using the explicit form of p_a given in (3.28) and the identity (2.13).

Using all of the previous information one can compute the frame components of the various pieces appearing in the tensor $B^{ij}_{P,Q}$. We have successively

$$P^{s[i}\nabla_s \nabla_t Q^{j]t} - (P \leftrightarrow Q) =$$

$$\sum_{l\neq i,j} (4\omega_{li,l}\omega_{lj,l} - 3\eta_l\eta_i\omega_{li,l}\omega_{ij,i} - 3\eta_l\eta_j\omega_{lj,l}\omega_{ji,j}) \left[p_i q_j - \eta_l p_l \eta_i q_j + \eta_l q_l \eta_i p_j - (i \leftrightarrow j)\right]$$

and

$$\nabla_s P^{t[i} \nabla_t Q^{j]s} = \tfrac{1}{2} \sum_l \omega_{li,l}\omega_{lj,l} \left[p_i q_j - \eta_l p_l \eta_i q_j + \eta_l q_l \eta_i p_j - (i \leftrightarrow j)\right].$$

Combining these relations, and using (3.27), we get

$$P^{s[i}\nabla_s \nabla_t Q^{j]t} - (P \leftrightarrow Q) - \nabla_s P^{t[i}\nabla_t Q^{j]s} =$$

$$\tfrac{1}{2} \sum_l \eta_l R_{il,lj} \left[p_i q_j - \eta_l p_l \eta_i q_j + \eta_l q_l \eta_i p_j - (i \leftrightarrow j)\right].$$

Let us then compute

$$P^{s[i} R^{j]}{}_{u,vs} Q^{uv} - (P \leftrightarrow Q) = \tfrac{1}{2} \sum_l \eta_l R_{il,lj} \left[\eta_l p_l \eta_i q_j - \eta_l q_l \eta_i p_j - (i \leftrightarrow j)\right].$$

Collecting all the pieces leaves us with

$$\begin{aligned} P^{s[i}\nabla_s \nabla_t Q^{j]t} + P^{s[i} R^{j]}{}_{u,vs} Q^{uv} - (P \leftrightarrow Q) - \nabla_s P^{t[i}\nabla_t Q^{j]s} &= \\ \tfrac{1}{2} \textstyle\sum_l \eta_l R_{il,lj}(p_i q_j - p_j q_i). \end{aligned} \tag{3.30}$$

The last sum is nothing but the frame components of the tensor $-P^{s[i}R_{st}Q^{j]t}$, so that we have obtained the tensorial relation

$$P^{s[i}\nabla_s\nabla_t Q^{j]t} + P^{s[i}R^{j]}{}_{u,vs}Q^{uv} - (P \leftrightarrow Q) - \nabla_s P^{t[i}\nabla_t Q^{j]s} = -P^{s[i}R_{st}Q^{j]t}, \quad (3.31)$$

which implies

$$B^{ij}_{P,Q} = -2P^{s[i}R_{st}Q^{j]t}, \qquad (3.32)$$

in agreement with [5]. This ends the proof of Proposition 3.9. □

Now we can come to the central point of our analysis: is a Stäckel system integrable at the quantum level? The answer is given by the following

Corollary 3.10. *([4, 5]) A Stäckel system is integrable at the quantum level iff*

$$R_{ij} = 0 \qquad \textit{for } i \neq j, \text{ where } i, j = 1, \ldots, n, \qquad (3.33)$$

in the special coordinates which are constituent to this system.

Proof. The Killing tensors $I_{2,i}$ are diagonal, for $i = 1, \ldots, n$, in the Stäckel coordinate system, and the proof follows from (3.25). □

The conditions (3.33) are known as the Robertson conditions [31], as interpreted by Eisenhart [18]. Quite recently, Benenti et al [4] have refined the definition of the separability of the Schrödinger equation and shown that, for Stäckel systems, the Robertson conditions are necessary and sufficient for the separability of the Schrödinger equation. As mentioned in Remark 2.1, the classical integrability is equivalent to the separability of the Hamilton-Jacobi equation; the situation for these systems can be therefore summarized by the following diagram:

$$\begin{array}{ccc} \text{Classical integrability} & \Longleftrightarrow & \text{separable Hamilton-Jacobi} \\ & & \\ \Downarrow \quad \text{provided} \quad R_{ij} = 0 \quad (i \neq j) & & \\ & & \\ \text{Quantum integrability} & \Longleftrightarrow & \text{separable Schrödinger} \end{array}$$

3.4 The quantum ellipsoid and Neumann systems

It is now easy to prove that the ellipsoid geodesic flow (see section 2.5), including the potential given in (2.16), is integrable at the quantum level. Using the coordinates (x^i) and the (Riemannian) metric given by (2.22), one can check that the Ricci tensor has components

$$R_{ij} = \frac{\mathcal{N}}{x^i} \sum_{s \neq i} \frac{1}{x^s} \, \mathrm{g}_{ij}, \qquad \mathcal{N} = \frac{a_0 a_1 \cdots a_n}{x^1 \cdots x^n},$$

and therefore satisfies the Robertson conditions. As already emphasized, the occurrence of an additional potential is irrelevant for the quantum analysis since the potential terms do not generate quantum corrections (see Remark 3.8).

Similarly we get the quantum integrability for the Neumann system (see Section 2.6) using the metric on S^n given by (2.38). The Ricci tensor being given by

$$R_{ij} = (n-1)\mathrm{g}_{ij},$$

the Robertson conditions are again satisfied.

3.5 The quantum generalized Kerr-Newman system

The quantization of the four commuting observables (2.47) and (2.48) is straightforward.

In view of the relations given in Section 3 all quantum commutators vanish except for $[\widehat{\widetilde{H}}, \widehat{\widetilde{P}}]$; this is due to the fact that the conserved quantities $\widetilde{S}$ and $\widetilde{T}$ (see (2.48)) are Killing-Maxwell vector fields.

The anomalous terms in the previous commutator are A_{P_2,H_2}, A_{P_1,H_2} and A_{P_2,H_1} where $P_2 = P^{ij}\xi_i\xi_j$, $H_2 = \frac{1}{2}\mathrm{g}^{ij}\xi_i\xi_j$, $P_1 = -2P^{ij}\xi_i A_j$ and $H_1 = -\mathrm{g}^{ij}\xi_i A_j$.

The vector field A_{P_2,H_2} given by (3.14) actually vanishes because, cf. (3.16), $B^{jk}_{P_2,H_2} = -P^{\ell[j}R^{k]}_{\ell} = 0$ as a consequence of (2.7); indeed the tensor P anti-commutes with the electromagnetic field strength F, implying that it commutes with the stress-energy electromagnetic tensor, hence with the Ricci tensor in view of the Einstein-Maxwell equations [11].

The two other anomalous terms (3.9) also vanish as it turns out that $\nabla_j A^j = 0$ (see (2.44)) and $\nabla_j(P^{jk}A_k) = 0$.

This derivation reproduces and extends Carter's results to the generalized Kerr-Newman solution, in a somewhat shorter manner.

Remark 3.11. Our analysis of quantum integrability for the generalized Kerr-Newman solution in 4 dimensions can be carried over into recent work [19, 23, 32] dealing with 5-dimensional black holes. In these cases, classical integrability follows from the existence of 3 Killing vectors and 1 quadratic Killing tensor, besides the Hamiltonian. These metrics being Einstein, the above arguments given for the generalized Kerr-Newman case apply just as well, insuring quantum integrability. This fact is in agreement with the separability of the Laplace operator.

3.6 The quantum Multi-Centre system

For this example too, the quantization is straightforward. The single point to be checked for quantum integrability is just the commutator $[\widehat{H}, \widehat{P}]$, with the possible quantum correction (3.16) given by $-P^{\ell[j}R_{\ell}^{k]}$. Here it vanishes trivially since these metrics are Ricci-flat.

3.7 The quantum Di Pirro system

As seen in Section 2.9, the classical integrability of this system is provided by three commuting observables: on the one hand H, P respectively given by (2.51) and (2.52), and $T = \xi_3$ if $c(x^3) = \text{const.}$, and on the other hand H, P and $J = \xi_1 x^2 - \xi_2 x^1$ if $a = b, \gamma$ depend on r only.

At the quantum level, the Killing vectors $\widehat{T}$ and $\widehat{J}$ do commute with $\widehat{H}$ according to (3.8) and (3.9). As for the commutator $[\widehat{P}, \widehat{H}]$ of the quantized Killing tensors, it is given by (3.16), namely $B_{P,H} = -\frac{1}{2}P^{\ell[j}R_{\ell}^{k]}\,\partial_j \wedge \partial_k$, and one finds

$$\begin{aligned} B_{P,H} \quad = \quad & -\frac{3}{16}\,\frac{c'(x^3)}{(\gamma(x^1,x^2)+c(x^3))^3}\big(a(x^1,x^2)\partial_1\gamma(x^1,x^2)\,\partial_1 \wedge \partial_3 \\ & + b(x^1,x^2)\partial_2\gamma(x^1,x^2)\,\partial_2 \wedge \partial_3\big). \end{aligned}$$

For the system (H, P, T), this quantum correction vanishes since $c'(x^3) = 0$, implying quantum integrability. However, for the system (H, P, J), in the generic case $\gamma \neq \text{const.}$, we get $B_{P,H} \neq 0$, showing that the minimal quantization rules may produce quantum corrections.

4 Discussion and outlook

It would be worthwhile to get insight into the status of our "minimal" quantization rules and to their relationship with other bona fide quantization procedures. Among the latter, let us mention those obtained by geometric means, and more specifically by imposing equivariance of the quantization mapping, $\mathcal{Q}$, with respect to some symmetry group, G, e.g., a group of automorphisms of a certain geometric structure on configuration space, M. We refer to the articles [24, 15, 16, 17, 6] for a detailed account on equivariant quantization. The two main examples are respectively the projectively, $G = \mathrm{SL}(n+1, \mathbf{R})$, and conformally, $G = \mathrm{O}(p+1, q+1)$, equivariant quantizations which have been shown to be uniquely determined [24, 17, 15, 16]. For instance, the conformally equivariant quantization $\mathcal{Q}_{\frac{1}{2}} : \mathcal{F}_{\frac{1}{2}}(M) \to \mathcal{F}_{\frac{1}{2}}(M)$ has been explicitly computed for quadratic [15] and cubic [25] observables; for example, if $P = P^{ij}\xi_i\xi_j$ we then have

$$\mathcal{Q}_{\frac{1}{2}}(P) = \widehat{P} + \beta_3\,\nabla_i\nabla_j(P^{ij}) + \beta_4\,\mathrm{g}^{ij}\mathrm{g}_{k\ell}\nabla_i\nabla_j(P^{k\ell}) + \beta_5\,R_{ij}P^{ij} + \beta_6\,R\mathrm{g}_{ij}P^{ij} \tag{4.1}$$

where the "minimal" quantum operator

$$\widehat{P} = -\nabla_i{\circ}P^{ij}{\circ}\nabla_j \tag{4.2}$$

is given by (3.4), together with $\beta_3 = -n/(4(n+1))$, $\beta_4 = -n/(4(n+1)(n+2))$, $\beta_5 = n^2/(4(n-2)(n+1))$, $\beta_6 = -n^2/(2(n^2-4)(n^2-1))$, assuming $n = \dim(M) > 2$. In (4.1) we denote by R_{ij} the components of the Ricci tensor and by R the scalar curvature. The formula (4.1) provides a justification of the term "minimal" for the mapping $P \mapsto \widehat{P}$ given by (3.4) and (3.10).

We have checked that, in the special instance of the geodesic flow of the ellipsoid discussed in Section 2.5, the quantum commutators of the observables I_i defined in (2.30), namely $[\mathcal{Q}_{\frac{1}{2}}(I_i), \mathcal{Q}_{\frac{1}{2}}(I_j)]$, fail to vanish for $i \neq j = 1, \ldots, n$. Had we started from the expression (4.1) with adjustable coefficients $\beta_3, \ldots, \beta_6$, the requirement that the latter commutator be vanishing imposes $\beta_3 = \ldots = \beta_6 = 0$, leading us back to the minimal quantization rule (4.2).

Despite their nice property of preserving, to a large extent, integrability (from classical to quantum), the "minimal" quantization rules still remain an ad hoc procedure, defined for observables at most cubic in momenta, and do not follow from any sound constructive principle, be it of a geometric or an algebraic nature. The quest for a construct leading unambiguously to a genuine "minimal" quantization procedure remains an interesting challenge. As discussed in Section 3.2, the equivariance assumption with respect to the affine group might be helpful for determining the sought "minimal" quantization of polynomials of higher degree. This analysis is required for the quantization of, e.g., the newly discovered integrable systems [14] which involve cubic Killing tensors.

Another field of applications of the present work could be the search for quantum integrability of the geodesic flow on the higher dimensional generalizations of the Kerr metric which have been lately under intense study [19, 12, 36].

Still another perspective for future work would be to generalise the previous computation of quantum corrections to the case of classical integrability in the presence of an electromagnetic field in a purely gauge invariant manner. In particular the approach presented in Section 2.2 should be further extended at the quantum level via the quantization of the Schouten-Maxwell brackets.

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