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Local energy statistics in disordered systems: a proof of the local REM conjecture[1]

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Abstract: Recently, Bauke and Mertens conjectured that the local statistics of energies in random spin systems with discrete spin space should in most circumstances be the same as in the random energy model. Here we give necessary conditions for this hypothesis to be true, which we show to hold in wide classes of examples: short range spin glasses and mean field spin glasses of the SK type. We also show that, under certain conditions, the conjecture holds even if energy levels that grow moderately with the volume of the system are considered.

Keywords: universality, level statistics, random energy model, extreme value theory, disordered systems, spin glasses

AMS Subject Classification: 60G70, 82B45

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1. Introduction.

In a recent paper [BaMe], Bauke and Mertens have formulated an interesting conjecture regarding the behaviour of local energy level statistics in disordered systems. Roughly speaking, their conjecture can be formulated as follows. Consider a random Hamiltonian, $H_N(\sigma)$, i.e., a real-valued random function on some product space, $\mathcal{S}^{\Lambda_N}$, where $\mathcal{S}$ is a finite space, typically $\{-1,1\}$, of the form

$$H_N(\sigma) = \sum_{A\subset\Lambda_N} \Phi_A(\sigma), \tag{1.1}$$

where Λ_N are finite subsets of Z^d of cardinality, say, N. The sum runs over subsets, A, of Λ_N and Φ_A are random local functions, typically of the form

$$\Phi_A(\sigma) = J_A \prod_{x\in A} \sigma_x \tag{1.2}$$

where J_A, $A\subset\mathbb{Z}^d$, is a family of (typically independent) random variables, defined on some probability space, $(\Omega,\mathcal{F},\mathbb{P})$, whose distribution is not too singular. In such a situation, for typical σ, $H_N(\sigma)\sim\sqrt{N}$, while $\sup_\sigma H_N(\sigma)\sim N$. Bauke and Mertens then ask the following question: Given a fixed number, E, what is the statistics of the values $N^{-1/2}H_N(\sigma)$ that are closest to this number, and how are configurations, σ, for which these good approximants of E are realised, distributed on $\mathcal{S}^{\Lambda_N}$? Their conjectured answer, which at first glance seems rather surprising, is quite simple: find $\delta_{N,E}$ such that $\mathbb{P}(|N^{-1/2}H_N(\sigma)-E|\le b\delta_{N,E})\sim|\mathcal{S}|^{-N}b$; then, the collection of points, $\delta_{N,E}^{-1}|N^{-1/2}H_N(\sigma)-E|$, over all $\sigma\in\mathcal{S}^{\Lambda_N}$, converges to a Poisson point process on $\mathbb{R}_+$. Furthermore, for any finite k, the k-tuple of configurations, $\sigma^1,\sigma^2,\dots,\sigma^k$, where the k-best approximations are realised, is such that all of its elements have maximal Hamming distance between each other. In other words, the asymptotic behavior of these best approximants of E is the same, as if the random variables $H_N(\sigma)$ were all independent Gaussian random variables with variance N, i.e., as if we were dealing with the random energy model (REM) [Der1]. Bauke and Mertens call this “universal REM like behaviour”.

A comparable result had previously been conjectured by Mertens [Mer1] in the particular case of the *number partitioning problem.* In that case, the function H_N is simply given by

$$H_N(\sigma) = \sum_{i=1}^{N} X_i\sigma_i, \tag{1.3}$$

with X_i i.i.d. random variables uniformly distributed on $[0,1]$, $\sigma_i\in\{-1,1\}$, and one is interested in the distribution of energies near the value zero (which corresponds to an optimal partitioning of the N random variables, X_i, into two groups such that their sum in each group is as similar as possible). This conjecture was later proven by Borgs, Chayes, and Pittel [BCP, BCMP]. It should be noted that in this problem, one needs, of course, take care of the obvious symmetry of the Hamiltonian under the transformation $\sigma \to -\sigma$. An extension of these results in the spirit of the REM conjecture was proven recently in [BCMN], i.e., when the value zero is replaced by an arbitrary value, E.

In [BK2] we generalised this result to the case of the k-partitioning problem, where the random function to be considered is actually vector-valued (consisting of the vector of differences between the sums of the random variables in each of the k subsets of the partition). To be precise, we considered the special case where the subsets of the partition are required to have the same cardinality, N/k (restricted k-partitioning problem). The general approach to the proof we developed in that paper sets the path towards the proof of the conjecture by Bauke and Mertens that we will present here.

The universality conjecture suggests that correlations are irrelevant for the properties of the local energy statistics of disordered systems for energies near "typical energies". On the other hand, we know that correlations must play a rôle for the extremal energies near the maximum of $H_N(\sigma)$. Thus, there are two questions beyond the original conjecture that naturally pose themselves: (i) assume we consider instead of fixed E, N-dependent energy levels, say, $E_N = N^\alpha C$. How fast can we allow E_N to grow for the REM-like behaviour to hold? and (ii) what type of behaviour can we expect once E_N grows faster than this value? We will see that the answer to the first question depends on the properties of H_N, and we will give an answer in models with Gaussian couplings. The answer to question (ii) requires a detailed understanding of $H_N(\sigma)$ as a random process, and we will be able to give a complete answer on only in the case of the GREM, when H_N is a hierarchically correlated Gaussian process. This will be discussed in a separate paper [BK05].

Our paper will be organized as follows. In Chapter 2, we prove an abstract theorem, that implies the REM-like-conjecture under three hypothesis. This will give us some heuristic understanding why and when such a conjecture should be true. In Chapter 3 we then show that the hypothesis of the theorem are fulfilled in two classes of examples: p-spin Sherrington-Kirkpatrick like models and short range Ising models on the lattice. In both cases we establish conditions on how fast E_N can be allowed to grow, in the case when the couplings are Gaussian.

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2. Abstract theorems.

In this section we will formulate a general result that implies the REM property under some concise conditions, that can be verified in concrete examples. This will also allow us to present the broad outline of the structure of the proof without having to bother with technical details. Note that our approach is rather different from that of [BCMN] that involves computations of moments.

Our approach to the proof of the Mertens conjecture is based on the following theorem, which provides a criterion for Poisson convergence in a rather general setting.

Theorem 2.1: *Let $V_{i,M} \geq 0$, $i \in \mathbb{N}$, be a family of non-negative random variables satisfying the following assumptions: for any $\ell \in \mathbb{N}$ and all sets of constants $b_j > 0$, $j = 1, \ldots, \ell$,*

$$\lim_{M\uparrow\infty} \sum_{(i_1,\ldots,i_\ell)\subset\{1,\ldots,M\}} \mathbb{P}(\forall_{j=1}^{\ell} V_{i_j,M} < b_j) \to \prod_{j=1}^{\ell} b_j \tag{2.1}$$

where the sum is taken over all possible sequences of different indices $(i_1, \ldots, i_\ell)$. Then the point process

$$\mathcal{P}_M = \sum_{i=1}^{M} \delta_{V_{i,M}}, \tag{2.2}$$

on $\mathbb{R}_+$, converges weakly in distribution, as $M \uparrow \infty$, to the standard Poisson point process, $\mathcal{P}$ on $\mathbb{R}_+$ (i.e., the Poisson point process whose intensity measure is the Lebesgue measure).

Remark: Theorem 2.1 was proven (in a more general form, involving vector valued random variables) in [BK2]. It is very similar in its spirit to an analogous theorem for the case of exchangeable variables proven in [BM] in an application to the Hopfield model. The rather simple proof in the scalar setting can be found in Chapter 13 of [B].

Naturally, we want to apply this theorem with $V_{i,M}$ given by $|N^{-1/2}H_N(\sigma)-E_N|$, properly normalised.

We will now introduce a setting in which the assumptions of Theorem 2.1 are verified. Consider a product space $\mathcal{S}^N$ where $\mathcal{S}$ is a finite set. We define on $\mathcal{S}^N$ a real-valued random process, $Y_N(\sigma)$. Assume for simplicity that

$$\mathbb{E}Y_N(\sigma) = 0, \quad \mathbb{E}(Y_N(\sigma))^2 = 1. \tag{2.3}$$

Define on $\mathcal{S}^N$

$$b_N(\sigma, \sigma') \equiv \mathrm{cov}(Y_N(\sigma), Y_N(\sigma')). \tag{2.4}$$

Let us also introduce the Gaussian process, Z_N, on $\mathcal{S}^N$, that has the same mean and the same covariance matrix as $Y_N(\sigma)$.

Let G be the group of automorphisms on $\mathcal{S}_N$, such that, for $g \in G$, $Y_N(g\sigma) = Y_N(\sigma)$, and let F be the larger group, such that, for $g \in F$, $|Y_N(g\sigma)| = |Y_N(\sigma)|$. Let

$$E_N = cN^\alpha, \quad c, \alpha \in \mathbb{R}, \;\; 0 \leq \alpha < 1/2, \tag{2.5}$$

be a sequence of real numbers, that is either a constant, $c \in \mathbb{R}$, if $\alpha = 0$, or converges to plus or minus infinity, if $\alpha > 0$. We will define sets Σ_N as follows: If $c \neq 0$, we denote by Σ_N be the set of residual classes of $\mathcal{S}^N$ modulo G; if $c = 0$, we let Σ_N be the set of residual classes modulo F. We will assume throughout that $|\Sigma_N| > \kappa^N$, for some $\kappa > 1$. Define the sequence of numbers

$$\delta_N = \sqrt{\tfrac{\pi}{2}} e^{E_N^2/2} |\Sigma_N|^{-1}. \tag{2.6}$$

Note that δ_N is exponentially small in $N \uparrow \infty$, since $\alpha < 1/2$. This sequence is chosen such that, for any $b \geq 0$,

$$\lim_{N \uparrow \infty} |\Sigma_N| \mathbb{P}(|Z_N(\sigma) - E_N| < b\delta_N) = b. \tag{2.7}$$

For $\ell \in \mathbb{N}$, and any collection, $\sigma^1, \ldots, \sigma^\ell \in \Sigma_N^{\otimes \ell}$, we denote by $B_N(\sigma^1, \ldots, \sigma^\ell)$ the covariance matrix of $Y_N(\sigma)$ with elements

$$b_{i,j}(\sigma^1, \ldots, \sigma^\ell) \equiv b_N(\sigma^i, \sigma^j). \tag{2.8}$$

Assumptions A.

(i) Let $\mathcal{R}_{N,\ell}^\eta$ *denote the set*

$$\mathcal{R}_{N,\ell}^\eta \equiv \{(\sigma^1, \ldots, \sigma^\ell) \in \Sigma_N^{\otimes \ell} : \forall_{1 \leq i < j \leq \ell} \; |b_N(\sigma^i, \sigma^j)| \leq N^{-\eta}\}. \tag{2.9}$$

Then there exists a continuous decreasing function, $\rho(\eta) > 0$, *on* $]\eta_0, \tilde{\eta}_0[$ *(for some* $\tilde{\eta}_0 \geq \eta_0 > 0$*), and* $\mu > 0$, *such that*

$$|\mathcal{R}_{N,\ell}^\eta| \geq \left(1 - \exp\left(-\mu(\eta) N^{\rho(\eta)}\right)\right) |\Sigma_N|^\ell. \tag{2.10}$$

(ii) Let $\ell \geq 2$, $r = 1, \ldots, \ell - 1$. *Let*

$$\begin{aligned} \mathcal{L}_{N,r}^\ell = \Big\{ (\sigma^1, \ldots, \sigma^\ell) \in \Sigma_N^{\otimes \ell} : \forall_{1 \leq i < j \leq \ell} \, |Y_N(\sigma^i)| \neq |Y_N(\sigma^j)|, \\ \operatorname{rank}(B_N(\sigma^1, \ldots, \sigma^\ell)) = r \Big\} \end{aligned} \tag{2.11}$$

Then there exists $d_{r,\ell} > 0$, such that, for all N large enough,

$$|\mathcal{L}^{\ell}_{N,r}| \leq |\Sigma_N|^r e^{-d_{r,\ell} N}. \tag{2.12}$$

(iii) *For any $\ell \geq 1$, any $r = 1, 2, \ldots, \ell$, and any $b_1, \ldots, b_\ell \geq 0$, there exist constants, $p_{r,\ell} \geq 0$ and $Q < \infty$, such that, for any $\sigma^1, \ldots, \sigma^\ell \in \Sigma_N^{\otimes \ell}$ for which* $\operatorname{rank}(B_N(\sigma^1, \ldots, \sigma^\ell)) = r$,

$$\mathbb{P}\big(\forall_{i=1}^{\ell} : |Y_N(\sigma^i) - E_N| < \delta_N b_i\big) \leq Q \delta_N^r N^{p_{r,\ell}}. \tag{2.13}$$

Theorem 2.2: *Assume the Assumptions A hold. Assume that $\alpha \in [0, 1/2[$ is such that, for some $\eta_1 \leq \eta_2 \in]\eta_0, \tilde{\eta}_0[$, we have:*

$$\alpha < \eta_2/2, \tag{2.14}$$

$$\alpha < \eta/2 + \rho(\eta)/2, \ \forall \eta \in]\eta_1, \eta_2[, \tag{2.15}$$

and

$$\alpha < \rho(\eta_1)/2. \tag{2.16}$$

Furthermore, assume that, for any $\ell \geq 1$, any $b_1, \ldots, b_\ell > 0$, and $(\sigma^1, \ldots, \sigma^\ell) \in \mathcal{R}^{\eta_1}_{N,\ell}$,

$$\mathbb{P}\left(\forall_{i=1}^{\ell} : \ |Y_N(\sigma^i) - E_N| < \delta_N b_i\right) = \mathbb{P}\left(\forall_{i=1}^{\ell} : \ |Z_N(\sigma^i) - E_N| < \delta_N b_i\right) + o(|\Sigma_N|^{-\ell}). \tag{2.17}$$

Then, the point process,

$$\mathcal{P}_N \equiv \sum_{\sigma \in \Sigma_N} \delta_{\{\delta_N^{-1} |Y_N(\sigma) - E_N|\}} \to \mathcal{P} \tag{2.18}$$

converges weakly to the standard Poisson point process $\mathcal{P}$ on $\mathbb{R}_+$.

Moreover, for any $\epsilon > 0$ and any $b \in \mathbb{R}_+$,

$$\mathbb{P}\left(\forall_{N_0} \exists_{N \geq N_0} : \exists_{\sigma, \sigma' : |b_N(\sigma, \sigma')| > \epsilon} : |Y_N(\sigma) - E_N| \leq |Y_N(\sigma') - E_N| \leq \delta_N b\right) = 0. \tag{2.19}$$

Remark: Before giving the proof of the theorem, let us comment on the various assumptions.

(i) Assumption A (i) holds with some η in any reasonable model, but the function $\rho(\eta)$ is model dependent.

(ii) Assumptions A (ii) and (iii) is also apparently valid in most cases, but can be tricky sometimes. An example where (ii) proved difficult is the k-partitioning problem, with $k > 2$.

(iii) Condition (2.19) is essentially a local central limit theorem. In the case $\alpha = 0$ it holds, if the Hamiltonian is a sum over independent random interactions, under mild decay assumptions on the characteristic function of the distributions of the interactions. Note that some such assumptions are obviously necessary, since if the random interactions take on only finitely many values, then also the Hamiltonian will take values on a lattice, whose spacings are not exponentially small, as would be necessary for the theorem to hold. Otherwise, if $\alpha > 0$, this will require further assumptions on the interactions. We will leave this problem open in the present paper. It is of course trivially verified, if the interactions are Gaussian.

Proof: We just have to verify the hypothesis of Theorem 2.1, for $V_{i,M}$ given by $\delta_N^{-1}|Y_N(\sigma) - E_N|$, i.e., we must show that

$$\sum_{(\sigma^1,\dots,\sigma^\ell)\in\Sigma_N^{\otimes l}} \mathbb{P}\left(\forall_{i=1}^{\ell} : |Y_N(\sigma^i) - E_N| < b_i\delta_N\right) \to b_1\cdots b_\ell. \tag{2.20}$$

We split this sum into the sums over the set $\mathcal{R}_{N,\ell}^{\eta_1}$ and its complement. First, by the assumption (2.17)

$$\begin{aligned}&\sum_{(\sigma^1,\dots,\sigma^\ell)\in\mathcal{R}_{N,\ell}^{\eta_1}} \mathbb{P}\left(\forall_{i=1}^{\ell} :\ |Y_N(\sigma^i) - E_N| < b_i\delta_N\right)\\ &= \sum_{(\sigma^1,\dots,\sigma^\ell)\in\mathcal{R}_{N,\ell}^{\eta_1}} \mathbb{P}\left(\forall_{i=1}^{\ell} :\ |Z_N(\sigma^i) - E_N| < b_i\delta_N\right) + o(1).\end{aligned} \tag{2.21}$$

But, with $\mathcal{C}(E_N) = \{\vec{x} = (x_1,\dots,x_\ell) \in \mathbb{R}^\ell : \forall_{i+1}^{\ell} |E_N - x_i| \leq \delta_N b_i\}$,

$$\mathbb{P}\left(\forall_{i=1}^{\ell} : |Z_N(\sigma^i) - E_N| < b_i\delta_N\right) = \int\limits_{\mathcal{C}(E_N)} \frac{e^{-(\vec{z},B_N^{-1}(\sigma^1,\dots,\sigma^\ell)\vec{z})/2}}{(2\pi)^{\ell/2}\sqrt{\det(B_N(\sigma^1,\dots,\sigma^\ell))}} d\vec{z}, \tag{2.22}$$

where $B_N(\sigma^1,\dots,\sigma^\ell)$ is the covariance matrix defined in (2.8). Since δ_N is exponentially small in N, we see that, uniformly for $(\sigma^1,\dots,\sigma^\ell) \in \mathcal{R}_{N,\ell}^{\eta_1}$, the integral (2.22) equals

$$(2\delta_N/\sqrt{2\pi})^\ell (b_1\cdots b_\ell) e^{-(\vec{E}_N,B^{-1}(\sigma^1,\dots,\sigma^\ell)\vec{E}_N)/2}(1+o(1)), \tag{2.23}$$

where we denote by $\vec{E}_N$ the vector $(E_N,\dots,E_N)$.

We treat separately the sum (2.21) taken over the smaller set, $\mathcal{R}_{N,\ell}^{\eta_2} \subset \mathcal{R}_{N,\ell}^{\eta_1}$, and the one over $\mathcal{R}_{N,\ell}^{\eta_1} \setminus \mathcal{R}_{N,\ell}^{\eta_2}$.

Since, by (2.14), η_2 is chosen such that $E_N^2 N^{-\eta_2} \to 0$, by (2.17), (2.22), and (2.23), each term in the sum over $\mathcal{R}_{N,\ell}^{\eta_2}$ equals

$$(2\delta_N/\sqrt{2\pi})^\ell (b_1 \cdots b_\ell) e^{-\frac{1}{2}\|\mathbb{E}_N\|^2(1+O(N^{-\eta_2}))}(1+o(1)) = (b_1 \cdots b_\ell)|\Sigma_N|^{-\ell}(1+o(1)), \quad (2.24)$$

uniformly for $(\sigma^1, \ldots, \sigma^\ell) \in \mathcal{R}_{N,l}^{\eta_2}$. Hence by Assumption A (i)

$$\sum_{(\sigma^1,\ldots,\sigma^\ell)\in\mathcal{R}_{N,\ell}^{\eta_2}} \mathbb{P}\left(\forall_{i=1}^\ell : |Z_N(\sigma^i) - E_N| < b_i\delta_N\right) = |\mathcal{R}_{N,l}^{\eta_2}||\Sigma_N|^{-\ell}(b_1 \cdots b_\ell)(1+o(1)) \to b_1 \cdots b_l. \quad (2.25)$$

Now let us consider the remaining set $\mathcal{R}_{N,\ell}^{\eta_1} \setminus \mathcal{R}_{N,\ell}^{\eta_2}$ (if it is non-empty, i.e., if strictly $\eta_1 < \eta_2$), and let us find $\eta_1 = \eta^0 < \eta^1 < \cdots < \eta^n = \eta_2$, such that

$$\alpha < \eta^i/2 + \rho(\eta^{i+1})/2 \quad \forall i = 0, 1, \ldots, n-1, \quad (2.26)$$

which is possible due to the assumption (2.15). Then let us split the sum over $\mathcal{R}_{N,l}^{\eta_1} \setminus \mathcal{R}_{N,\ell}^{\eta_2}$ into n sums, each over $\mathcal{R}_{N,\ell}^{\eta_i} \setminus \mathcal{R}_{N,\ell}^{\eta_{i+1}}$, $i = 0, 1, \ldots, n-1$. By (2.17), (2.22), and (2.23), we have, uniformly for $(\sigma^1, \ldots, \sigma^\ell) \in \mathcal{R}_{N,\ell}^{\eta^i}$,

$$\begin{aligned}\mathbb{P}\left(\forall_{i=1}^\ell : |Z_N(\sigma^i) - E_N| < b_i\delta_N\right) &= (2\delta_N/\sqrt{2\pi})^\ell (b_1 \cdots b_\ell) e^{-\frac{1}{2}\|\mathbb{E}_N\|^2(1+O(N^{-\eta^i}))}(1+o(1)) \\ &\le C|\Sigma_N|^{-\ell} e^{N^{2\alpha-\eta^i}},\end{aligned} \quad (2.27)$$

for some constant $C < \infty$. Thus by Assumption A (i),

$$\begin{aligned}\sum_{\mathcal{R}_{N,l}^{\eta_i}\setminus\mathcal{R}_{N,l}^{\eta_{i+1}}} \mathbb{P}(\forall_{i=1}^\ell : |Z_N(\sigma^i) - E_N| < b_i\delta_N) &\le C|\Sigma_N^{\otimes l} \setminus \mathcal{R}_{N,l}^{\eta_{i+1}}||\Sigma_N|^{-\ell} e^{N^{2\alpha-\eta^i}} \\ &\le C\exp\left(-\mu(\eta^{i+1})N^{\rho(\eta^{i+1})} + N^{2\alpha-\eta^i}\right),\end{aligned} \quad (2.28)$$

that, by (2.26), converges to zero, as $N \to \infty$, for any $i = 0, 1, \ldots, n-1$. So the sum (2.21) over $\mathcal{R}_{N,l}^{\eta_1} \setminus \mathcal{R}_{N,l}^{\eta_2}$ vanishes.

Now we turn to the sum over collections, $(\sigma^1, \ldots, \sigma^\ell) \notin \mathcal{R}_{N,l}^{\eta_1}$. We distinguish the cases when $\det(B_N(\sigma^1, \ldots, \sigma^\ell)) = 0$ and $\det(B_N(\sigma^1, \ldots, \sigma^\ell)) \neq 0$. For the contributions from the latter case, using Assumptions A (i) and (iii), we get readily that,

$$\begin{aligned}\sum_{\substack{(\sigma^1,\ldots,\sigma^\ell)\notin\mathcal{R}_{N\ell}^{\eta_1} \\ \operatorname{rank}(B_N(\sigma^1,\ldots,\sigma^\ell))=\ell}} \mathbb{P}\left(\forall_{i=1}^\ell |Y_N(\sigma^i) - E_N| < \delta_N b_i\right) &\le |\Sigma_N|^\ell e^{-\mu(\eta_1)N^{\rho(\eta_1)}} Q|\delta_N|^\ell N^{p_\ell} \\ &\le CN^{p_\ell}\exp\left(-\mu(\eta_1)N^{\rho(\eta_1)} + \ell E_N^2/2\right).\end{aligned} \quad (2.29)$$

The right-hand side of (2.29) tends to zero exponentially fast, if condition (2.16) is verified.

Finally, we must deal with the contributions from the cases when the covariance matrix is degenerate, namely

$$\sum_{\substack{(\sigma^1,\dots,\sigma^\ell)\in\Sigma_N^{\otimes l}\\ \mathrm{rank}(B_N(\sigma^1,\dots,\sigma^\ell))=r}} \mathbb{P}(\forall_{i=1}^{\ell} : |Y_N(\sigma^i)-E_N|<b_i\delta_N), \tag{2.30}$$

for $r=1,\dots,\ell-1$. In the case $c=0$, this sum is taken over the set $\mathcal{L}^r_{N,\ell}$, since σ and σ' in Σ_N are different, iff $|Y_N(\sigma)|\neq|Y_N(\sigma')|$, by definition of Σ_N. In the case $c\neq 0$, this sum is taken over ℓ-tuples $(\sigma^1,\dots,\sigma^\ell)$ of different elements of Σ_N, i.e., such that $Y_N(\sigma^i)\neq Y_N(\sigma^j)$, for any $1\le i<j\le\ell$. But for all N large enough, all terms in this sum over ℓ-tuples, $(\sigma^1,\dots,\sigma^\ell)$, such that $Y_N(\sigma^i)=-Y_N(\sigma^j)$, for some $1\le i<j\le\ell$, equal zero, since the events $\{|Y_N(\sigma^i)-E_N|<b_i\delta_N\}$ and $\{|-Y_N(\sigma^i)-E_N|<b_j\delta_N\}$, with $E_N=cN^\alpha$, $c\neq 0$, are disjoint. Therefore (2.30) is reduced to the sum over $\mathcal{L}^r_{N,\ell}$ in the case $c\neq 0$ as well. Then, by Assumptions A (ii) and (iii), it is bounded from above by

$$|\mathcal{L}^r_{N,\ell}|Q(\delta_N)^r N^{p_{r,\ell}} \le |\Sigma_N|^r e^{-d_{r,\ell}N} Q(\delta_N)^r N^{p_{r,\ell}} \le C e^{-d_{r,\ell}N} e^{\ell E_N^2/2} N^{p_{r,\ell}}. \tag{2.31}$$

This bound converges to zero exponentially fast, since $E_N^2=c^2N^{2\alpha}$, with $\alpha<1/2$. This concludes the proof of the first part of the theorem.

The second assertion (2.19) is elementary: by (2.29) and (2.31), the sum of terms $\mathbb{P}(\forall_{i=1}^2 : |Y_N(\sigma^i)-E_N|<\delta_N b)$ over all pairs, $(\sigma^1,\sigma^2)\in\Sigma_N^{\otimes 2}\setminus\mathcal{R}^{\eta_1}_{N,2}$, such that $\sigma^1\neq\sigma^2$, converges to zero exponentially fast. Thus (2.19) follows from the Borel-Cantelli lemma. $\diamondsuit$

Finally, we remark that the results of Theorem 2.2 can be extended to the case when $\mathbb{E}Y_N(\sigma)\neq 0$, if $\alpha=0$, i.e., $E_N=c$. Note that, e.g. the unrestricted number partitioning problem falls into this class. Let now $Z_N(\sigma)$ be the Gaussian process with the same mean and covariances as $Y_N(\sigma)$. Let us consider both the covariance matrix, B_N, and the mean of Y_N, $\mathbb{E}Y_N(\sigma)$, as random variables on the probability space $(\Sigma_N,\mathcal{B}_N,\mathbb{E}_\sigma)$, where $\mathbb{E}_\sigma$ is the uniform law on Σ_N. Assume that, for any $\ell\ge 1$,

$$B_N(\sigma^1,\dots,\sigma^\ell)\stackrel{\mathcal{D}}{\to} I_d, \quad N\uparrow\infty, \tag{2.32}$$

where I_d denotes the identity matrix, and

$$\mathbb{E}Y_N(\sigma)\stackrel{\mathcal{D}}{\to} D, \quad N\uparrow\infty, \tag{2.33}$$

where D is some random variable D. Let

$$\widetilde{\delta}_N = \sqrt{\tfrac{\pi}{2}} K^{-1} |\Sigma_N|^{-1}. \tag{2.34}$$

where

$$K \equiv \mathbb{E} e^{-(c-D)^2/2}. \tag{2.35}$$

Theorem 2.3: *Assume that, for some $R>0$, $|\mathbb{E}Y_N(\sigma)| \le N^R$, for all $\sigma\in\Sigma_N$. Assume that (2.10) holds for some $\eta>0$ and that (ii) and (iii) of Assumptions A are valid. Assume that there exists a set, $\mathcal{Q}_N \subset \mathcal{R}^\eta_{N,\ell}$, such that (2.17) is valid for any $(\sigma^1,\dots,\sigma^\ell)\in\mathcal{Q}_N$, and that $|\mathcal{R}^\eta_{N,\ell}\setminus\mathcal{Q}_N| \le |\Sigma_N|^\ell e^{-N^\gamma}$, with some $\gamma>0$. Then, the point process*

$$\mathcal{P}_N \equiv \sum_{\sigma\in\Sigma_N} \delta_{\widetilde{\delta}_N^{-1}|Y_N(\sigma)-E_N|} \to \mathcal{P} \tag{2.36}$$

converges weakly to the standard Poisson point process $\mathcal{P}$ on $\mathbb{R}_+$.

Proof. We must prove again the convergence of the sum (2.20), that we split into three sums: the first over $\mathcal{Q}_N$, the second over $\mathcal{R}^\eta_{N,\ell}\setminus\mathcal{Q}_N$, and the third over the complement of the set $\mathcal{R}^\eta_{N,\ell}$. By assumption, (2.17) is valid on $\mathcal{Q}_N$, and thus the terms of the first sum are reduced to

$$\begin{aligned}&\int_{\forall i=1,\dots,\ell:|z_i-c|<\widetilde{\delta}_N b_i} \frac{e^{-((\vec z-\mathbb{E}\vec Y_N(\sigma))B_N^{-1}(\sigma^1,\dots,\sigma^\ell)(\vec z-\mathbb{E}\vec Y_N(\sigma)))/2}}{(2\pi)^{\ell/2}\sqrt{\det(B_N(\sigma^1,\dots,\sigma^\ell))}} d\vec z\\ &\qquad = (2\tilde\delta_N/\sqrt{2\pi})^\ell (b_1\cdots b_\ell) e^{-(\vec c-\vec{\mathbb{E}}Y_N(\sigma))B^{-1}(\sigma^1,\dots,\sigma^\ell)(\vec c-\mathbb{E}\vec Y_N(\sigma))/2}(1+o(1)),\end{aligned} \tag{2.37}$$

with $\vec c\equiv(c,\dots,c)$, and $E\vec Y_N(\sigma)\equiv(\mathbb{E}Y_N(\sigma^1),\dots,\mathbb{E}Y_N(\sigma^\ell))$, since δ_N is exponentially small and $|\mathbb{E}Y_N(\sigma)|\le N^R$. By definition of $\tilde\delta_N$, the quantities (2.37) are at most $O(|\Sigma_N|^{-\ell})$, while, by the estimate (2.10) and by the assumption on the cardinality of $\mathcal{R}^\eta_{N,\ell}\setminus\mathcal{Q}_N$, the number of ℓ-tuples of configurations in $\Sigma_N^{\otimes l}\setminus\mathcal{R}^\eta_{N,\ell}$ and in $\mathcal{R}^\eta_{N,\ell}\setminus\mathcal{Q}_N$ is exponentially smaller than $|\Sigma_N|^\ell$. Hence

$$\begin{aligned}&\sum_{(\sigma^1,\dots,\sigma^\ell)\in\mathcal{Q}_N} \mathbb{P}(\forall_{i=1}^\ell : |Y_N(\sigma^i)-E_N|<b_i\delta_N)\\ &=\sum_{(\sigma^1,\dots,\sigma^\ell)\in\mathcal{Q}_N} (2\tilde\delta_N/\sqrt{2\pi})^\ell (b_1\cdots b_\ell) e^{-(\vec c-\vec{\mathbb{E}}Y_N(\sigma))B^{-1}(\sigma^1,\dots,\sigma^\ell)(\vec c-\mathbb{E}\vec Y_N(\sigma))/2}(1+o(1))+o(1)\\ &=\sum_{(\sigma^1,\dots,\sigma^\ell)\in\Sigma_N^{\otimes\ell}} (2\tilde\delta_N/\sqrt{2\pi})^\ell (b_1\cdots b_\ell) e^{-(\vec c-\vec{\mathbb{E}}Y_N(\sigma))B^{-1}(\sigma^1,\dots,\sigma^\ell)(\vec c-\mathbb{E}\vec Y_N(\sigma))/2}(1+o(1))+o(1)\\ &=\frac{b_1\cdots b_\ell}{|\Sigma_N|^\ell K^\ell}\sum_{(\sigma^1,\dots,\sigma^\ell)\in\Sigma_N^{\otimes\ell}} e^{-(\vec c-\vec{\mathbb{E}}Y_N(\sigma))B^{-1}(\sigma^1,\dots,\sigma^\ell)(\vec c-\mathbb{E}\vec Y_N(\sigma))/2}(1+o(1))+o(1).\end{aligned} \tag{2.38}$$

The last quantity converges to $b_1 \cdots b_\ell$, by the assumptions (2.32), (2.33) and (2.35).

The sum of the probabilities, $\mathbb{P}(\forall_{i=1}^{\ell} : |Y_N(\sigma) - E_N| < \delta_N b_i)$, over all ℓ-tuples of $\mathcal{R}_{N,\ell}^{\eta} \setminus \mathcal{Q}_N$, contains at most $|\Sigma_N|^{\ell} e^{-N^{-\gamma}}$ terms, while, by Assumption A (iii), (and since, for any $(\sigma^1, \dots, \sigma^\ell) \in \mathcal{R}_{N,\ell}^{\eta}$, the rank of $B_N(\sigma^1, \dots, \sigma^\ell)$ equals ℓ) each term is at most of order $|\Sigma_N|^{-\ell}$, up to a polynomial factor. Thus this sum converges to zero.

Finally, the sum of the same probabilities over the collections $(\sigma^1, \dots, \sigma^\ell) \in \Sigma_N^{\otimes l} \setminus \mathcal{R}_{N,\ell}^{\eta}$ converges to zero, exponentially fast, by the same arguments as those leading to (2.29) and (2.31), with $\eta_1 = \eta$. $\diamondsuit$

3. Examples

We will now show that the assumptions of our theorem are verified in a wide class of physically relevant models: 1) the Gaussian p-spin SK models, 2) SK-models with non-Gaussian couplings, and 3) short-range spin-glasses. In the last two examples we consider only the case $\alpha = 0$.

3.1 p-spin Sherrington-Kirkpatrick models, $0 \leq \alpha < 1/2$.

In this subsection we illustrate our general theorem in the class of Sherrington-Kirkpatrick models. Consider $\mathcal{S} = \{-1, 1\}$.

$$H_N(\sigma) = \frac{\sqrt{N}}{\sqrt{\binom{N}{p}}} \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq N} J_{i_1,\dots,i_p} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_p} \tag{3.1}$$

is the Hamiltonian of the p-spin Sherrington-Kirkpatrick model, where $J_{i_1,\dots,i_p}$ are independent standard Gaussian random variables.

The following elementary proposition concerns the symmetries to the Hamiltonian.

Proposition 3.1: *Assume that, for any $0 < i_1 < \cdots < i_p \leq N$, $\sigma_{i_1} \cdots \sigma_{i_p} = \sigma'_{i_1} \cdots \sigma'_{i_p}$. Then, if p is pair, either $\sigma_i = \sigma'_i$, for all $i = 1, \dots, N$, or $\sigma_i = -\sigma'_i$, for all $i = 1, \dots, N$, and, if p is odd, then $\sigma_i = \sigma'_i$, for all $i = 1, \dots, N$. Assume that, for any $0 < i_1 < \cdots < i_p \leq N$, $\sigma_{i_1} \cdots \sigma_{i_p} = -\sigma'_{i_1} \cdots \sigma'_{i_p}$. This is impossible, if p is pair and $\sigma_i = -\sigma'_i$, for all $i = 1, \dots, N$, if p is odd.*

This proposition allows us to construct the space Σ_N: If p is odd and $c \neq 0$, $\Sigma_N = \mathcal{S}^N$, thus $|\Sigma_N| = 2^N$. If p is even, or $c = 0$, Σ_N consists of equivalence classes where configurations σ and $-\sigma$ are identified, thus $|\Sigma_N| = 2^{N-1}$.

Theorem 3.2: *Let $p \geq 1$ be odd. Let $\Sigma_N = \mathcal{S}^N$. If $p = 1$ and $\alpha \in [0, 1/4[$, and, if $p = 3, 5, \dots,$, and $\alpha \in [0, 1/2[$, for any constant $c \in \mathbb{R} \setminus \{0\}$ the point process*

$$\mathcal{P}_N \equiv \sum_{\sigma \in \Sigma_N} \delta_{\{\delta_N^{-1} |N^{-1/2} H_N(\sigma) - cN^\alpha|\}} \tag{3.2}$$

where $\delta_N = 2^{-N} e^{+c^2 N^{2\alpha}/2} \sqrt{\frac{\pi}{2}}$, converges weakly to the standard Poisson point process, $\mathcal{P}$, on $\mathbb{R}_+$.

Let p be even. Let Σ_N be the space of equivalence classes of $\mathcal{S}^N$ where σ and $-\sigma$ are identified. For any $\alpha \in [0, 1/2[$ and any constant, $c \in \mathbb{R}$, the point process

$$\mathcal{P}_N \equiv \sum_{\sigma \in \Sigma_N} \delta_{\{(2\delta_N)^{-1} |N^{-1/2} H_N(\sigma) - cN^\alpha|\}} \tag{3.3}$$

converges weakly to the standard Poisson point process, $\mathcal{P}$, on $\mathbb{R}_+$. The result (3.3) holds true as well in the case of $c = 0$, for p odd.

Proof of Theorem 3.2. We have to verify the assumptions of Theorem 2.2 for the process $N^{-1/2} H_N(\sigma) = Y_N(\sigma)$. The elements of the covariance matrix (2.8) are:

$$b_{j,j}(\sigma^1, \dots, \sigma^\ell) = 1, \quad \forall_{j=1}^\ell; \tag{3.4}$$

$$b_{j,m}(\sigma^1, \dots, \sigma^\ell) = \binom{N}{p}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq N} \sigma_{i_1}^j \dots \sigma_{i_p}^j \sigma_{i_1}^m \dots \sigma_{i_p}^m, \ \forall_{1 \leq j < m \leq \ell}. \tag{3.5}$$

It has been observed in [BKL] that its non-diagonal elements can be written as

$$b_{j,m}(\sigma^1, \dots, \sigma^\ell) = \sum_{k=0}^{[p/2]} (-N)^{-k} \binom{2k}{p} (k-1)!! \Big(\tfrac{1}{N} \sum_{q=1}^N \sigma_q^j \sigma_q^m \Big)^{p-2k} (1 + O(1/N)). \tag{3.6}$$

Now let us verify the Assumption A (i). Let

$$\mathcal{Q}_{N,\ell,q}^\zeta = \Big\{ (\sigma^1, \dots, \sigma^l) \in \Sigma_N^{\otimes l} : \forall_{1 \leq i < j \leq \ell} \ \Big| N^{-1} \sum_{q=1}^N \sigma_q^i \sigma_q^j \Big| < qN^{-\zeta} \Big\}. \tag{3.7}$$

The ℓ-tuples of this set satisfy the following property: for any $\delta_2, \dots, \delta_\ell \in \{-1, 1\}^{\ell-1}$, the sets of sites $A_{\delta_2, \dots, \delta_\ell} = \{ i : \sigma_i^2 = \delta_2 \sigma_i^1, \sigma_i^3 = \delta_3 \sigma_i^1, \dots, \sigma_i^\ell = \delta_\ell \sigma_i^1 \}$ has the cardinality $N 2^{-(\ell-1)} + O(N^{1-\zeta})$. Then it is an easy combinatorial computation to check that there exists $h > 0$, such that, for any $q \in \mathbb{R}_+$, and any $\zeta \in]0, 1/2[$,

$$|\mathcal{Q}_{N,\ell,q}^\zeta| \geq |\Sigma_N|^\ell (1 - \exp(-hN^{1-2\zeta})), \tag{3.8}$$

for all N large enough. By the representation (3.6), we have $\bigcap_{k=0}^{p} \mathcal{Q}_{N,\ell,q}^{-(k-\eta)/(p-2k)} \subset \mathcal{R}_{N,\ell}^{\eta}$, with $q = (p \max_{k=0,\dots,[p/2]} \binom{2k}{p}(k-1)!!)^{-1}$. But, for any $\eta \in]0, p/2[$, and any $k = 0, 1, \dots, [p/2]$, $\mathcal{Q}_{N,\ell,q}^{-(k-\eta)/(p-2k)} \subset \mathcal{Q}_{N,\ell,q}^{-(k+1-\eta)/(p-2(k+1))}$. Therefore,

$$\mathcal{Q}_{N,\ell,q}^{\eta/p} \subset \mathcal{R}_{N,\ell}^{\eta}. \tag{3.9}$$

Thus, due to (3.8), Assumption A (i) is verified with $\rho(\eta) = 1 - 2\eta/p$, for $\eta \in]0, p/2[$.

Let us now check the Assumption A (ii). To estimate the cardinality of $\mathcal{L}_{N,\ell}^{r}$, we need to introduce an ℓ by $\binom{N}{p}$ matrix, $C_p(\sigma^1, \dots, \sigma^\ell)$, as follows. For any given $\sigma^1, \dots, \sigma^\ell$, the jth column is composed of all $\binom{N}{p}$ products, $\sigma_{i_1}^{j} \sigma_{i_2}^{j} \cdots \sigma_{i_p}^{j}$, over all subsets $1 \le i_1 < i_2 < \dots < i_p \le N$. Then we have

$$C_p^T(\sigma^1, \dots, \sigma^\ell) C_p(\sigma^1, \dots, \sigma^\ell) = \binom{N}{p}^{-1} B_N(\sigma^1, \dots, \sigma^\ell). \tag{3.10}$$

Let $\sigma^1, \dots, \sigma^\ell$ be such that $\operatorname{rank}(B_N(\sigma^1, \dots, \sigma^\ell)) = r < \ell$. Then, r columns of the matrix $C_p(\sigma^1, \dots, \sigma^\ell)$ form a basis of its ℓ columns. Assume that these are, e.g., the first r columns. The matrix $C_p(\sigma^1, \dots, \sigma^r)$ can contain at most 2^r different rows. We will show that, for any $(\sigma^1, \dots, \sigma^\ell) \in \mathcal{L}_{N,\ell}^{r}$, it can in fact not contain all 2^r rows, due to the following proposition.

Proposition 3.3: *Assume that an $2^r \times r$ matrix, A, with elements, 1 or -1, consists of all 2^r different rows. Assume that a column of length 2^r with elements 1 or -1 is a linear combination of the columns of A. Then this column is a multiple (with coefficient $+1$ or -1) of one of the columns of the matrix A.*

Proof. The proof can be carried out by induction over r. A generalisation of this fact is proven in [BK-npp].

Now, if the matrix $C_p(\sigma^1, \dots, \sigma^r)$ contained all 2^r rows, then, by Proposition 3.3, for any $j = r+1, \dots, \ell$, there would exist $m = 1, \dots, r$, such that, either, for any $0 < i_1 < \dots < i_p \le N$, $\sigma_{i_1}^{j} \cdots \sigma_{i_p}^{j} = \sigma_{i_1}^{m} \cdots \sigma_{i_p}^{m}$, or, for any $0 < i_1 < \dots < i_p \le N$, $\sigma_{i_1}^{j} \cdots \sigma_{i_p}^{j} = -\sigma_{i_1}^{m} \cdots \sigma_{i_p}^{m}$, which would imply $|Y_N(\sigma^j)| = |Y_N(\sigma^m)|$. But this is excluded by the definition of $\mathcal{L}_{N,\ell}^{r}$.

Thus, for any $(\sigma^1, \dots, \sigma^\ell) \in \mathcal{L}_{N,\ell}^{r}$, the matrix $C_p(\sigma^1, \dots, \sigma^r)$ contains at most $2^r - 1$ different rows. There are $O((2^r-1)^N)$ possibilities to construct such a matrix. Furthermore, there is only an N-independent number of possibilities to complete it by adding linear combinations of its columns to $C_p(\sigma^1, \dots, \sigma^\ell)$. To see this, consider the restriction of $C_p(\sigma^1, \dots, \sigma^r)$ to any r linearly independent rows. There are not more than $2^{r(\ell-r)}$ ways to complete it by

$(\ell - r)$ columns of ± 1 of length r, that are linear combinations of its r columns. But each such choice determines uniquely linear coefficients in these linear combinations and hence the completion of the whole $C_p(\sigma^1, \ldots, \sigma^r)$ up to $C_p(\sigma^1, \ldots, \sigma^\ell)$. Thus $|\mathcal{L}^r_{N,\ell}| = O((2^r - 1)^N)$.

It remains to verify the Assumption A (iii). This is easy: if $\operatorname{rank}(B_N(\sigma^1, \ldots, \sigma^\ell)) = r$, then r of the random variables $Y_N(\sigma^1), \ldots, Y_N(\sigma^\ell)$ are linearly independent. Assume that these are, e.g., $Y_N(\sigma^{i_1}), \ldots, Y_N(\sigma^{i_r})$. Then the covariance matrix $B_N(\sigma^{i_1}, \ldots, \sigma^{i_r})$ is non-degenerate, and the corresponding probability is bounded from above by

$$\mathbb{P}(\forall_{j=1}^r |Y_N(\sigma^{i_j}) - E_N| < \delta_N b_{i_j}) \leq \frac{(2\delta_N)^r (b_{i_1} \cdots b_{i_r})}{\sqrt{(2\pi)^r \det B_N(\sigma^{i_1}, \ldots, \sigma^{i_r})}}. \tag{3.11}$$

¿From the representation of the matrix elements of $, B_N(\sigma^{i_1}, \ldots, \sigma^{i_r}))$, (3.5), one sees that the determinant, $(\det B_N(\sigma^{i_1}, \ldots, \sigma^{i_r}))$, is a finite polynomial in the variables N^{-1}, and thus its inverse can grow at most polynomially.

Thus, we have established that Assumption A is verified. We now turn to conditions (2.14), (2.15), and (2.16) on α. Since $\rho(\eta) = 1 - 2\eta/p$, for $\eta \in]0, p/2[$, we should find $\eta_1, \eta_2 \in]0, p/2[$ such that $\alpha < \eta_2/2$, $\alpha < \eta/2 + 1/2 - \eta/p$ for $\eta \in]\eta_1, \eta_2[$, and $\alpha < 1/2 - \eta_1/p$. We see that, for any $p \geq 2$ and $\alpha \in]0, 1/2[$, it is possible to fix $\eta_1 > 0$ small enough, and $\eta_2 \in]0, p/2[$ close enough to 1, such that these assumptions are satisfied. If $p = 1$, then such a choice is possible only for $\alpha \in]0, 1/4[$. The assumption (2.17) need not be verified here as $Y_N(\sigma)$ is a Gaussian process. $\diamondsuit$

Remark: Values $p = 1, \alpha = 1/4$. The value $\alpha = 1/4$ is likely to be the true critical value in the case $p = 1$. In this case, one can check that the principle part of our sum gives a contribution of the form

$$\frac{\text{const}(1 + o(1))}{\sqrt{(2\pi N)^{k(k-1)/2}}} \sum_{\substack{m_{1,2}, \ldots, m_{k-1,k} \\ \forall i \neq j: |m_{i,j}| < N^{\eta - 1/2}}} \exp\left(c^2 N^{2\alpha} \sum_{1 \leq i < j \leq N} m_{i,j}(1+o(1)) - \frac{N}{2} \sum_{1 \leq i < j \leq k} m_{i,j}^2 (1+o(1))\right), \tag{3.12}$$

which in turn is easily seem to be of order $(e^{c^2/2})^{k(k-1)/2}$, that it it does not behave like a constant to the power k. Note that the term proportional to $\sqrt{N}$ in the exponents arises from the off-diagonal part of the covariance matrix B_N.

If $\alpha > 1/4$, the contribution from the (3.12) is already of order $(e^{N^{4\alpha-1} c^2/2})^{k(k-1)/2}$, which cannot be compensated by any normalisation of the form δ_N^k. Thus at least the conditions of Theorem 2.1 cannot hold in this case.

3.2. Generalized p-spin SK models at level $\alpha = 0$.

In this subsection we generalize Theorem 3.2 to the case of non-Gaussian process in the case of non-zero mean and $\alpha = 0$. Let $p \geq 1$, $U_{i_1,i_2,\dots,i_p}$ be any $\binom{N}{p}$ i.i.d. random variables with $\mathbb{E}U = a$ and $\operatorname{Var} U = 1$. Let

$$H_N(\sigma) = \frac{\sqrt{N}}{\sqrt{\binom{N}{p}}} \sum_{1\le i_1<i_2<\dots<i_p\le N} U_{i_1,\dots,i_p}\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_p}. \tag{3.13}$$

Let $\phi(s) = \mathbb{E}e^{is(U-a)}$ be the generating function of $(U - a)$.

Assumption B. *We will assume in this section that $E|U|^3 < \infty$ and $|\phi(s)| = O(|s|^{-1})$, as $|s| \to \infty$.*

Remark: The decay assumption on the Fourier transform is not optimal, but some condition of this type is needed, as the result cannot be expected for discrete distributions, where the number of possible values the Hamiltonian takes on would be finite.

We consider $Y_N(\sigma) = N^{-1/2}H_N(\sigma)$. The state space Σ_N is defined as in the previous example. The covariance matrix, given by (3.6), converges in law to the identity matrix by the law of large numbers. Furthermore, analogously to (3.6), we see that $\mathbb{E}Y_N(\sigma) = Q_p(N^{-1/2}\sum_{i=1}^N \sigma_i)$, where

$$Q_p(x) = \sum_{k=0}^{[p/2]} (-1)^k \binom{2k}{p} (k-1)!!x^{p-2k}. \tag{3.14}$$

By the central limit theorem, $\mathbb{E}Y_N(\sigma) \xrightarrow{\mathcal{D}} Q_p(J)$ where J is a standard Gaussian random variable. Hence, (2.32) and (2.33) are verified and we may define the constant

$$K_p \equiv \mathbb{E}\exp\Big(- (c - aQ_p(J))^2/2\Big) \tag{3.15}$$

Then, $\widetilde{\delta}_N = K_p^{-1}|\Sigma_N|^{-1}(\sqrt{2\pi}/2)$, with $|\Sigma_N| = 2^N$ for p odd and $|\Sigma_N| = 2^{N-1}$ for p even.

Theorem 3.4:

(i) Let p be odd. Let $\Sigma_N = \mathcal{S}^N$. For any $c \neq 0$, the point process

$$\mathcal{P}_N \equiv \sum_{\sigma\in\Sigma_N} \delta\{2^N K_p(2/\sqrt{2\pi})|Y_N(\sigma) - c|\} \tag{3.16}$$

converges weakly to the standard Poisson point process on $\mathbb{R}_+$.

(ii) Let p be odd and $c=0$, or let p be even and $c\neq 0$. Denote by Σ_N the space of the 2^{N-1} equivalence classes in $\mathcal{S}^N$ where σ and $-\sigma$ are identified. Then the point process

$$\mathcal{P}_N \equiv \sum_{\sigma\in\Sigma_N} \delta\{2^{N-1}K_p(2/\sqrt{2\pi})|Y_N(\sigma)-c|\}, \tag{3.17}$$

converges weakly to the standard Poisson point process on $\mathbb{R}_+$.

Proof of Theorem 3.4. We should check the assumptions of Theorem 2.3. The Assumptions A (i), for any $\eta\in]0,p/2[$, and (ii) have been already verified in the proof of Theorem 3.2. We must check (iii) and also the assertion (2.17) on an appropriate subset $\mathcal{Q}_N$.

We will use the construction of the matrix $C_p(\sigma^1,\dots,\sigma^\ell)$ explained in the proof of Theorem 3.2, see (3.10). Let us introduce the Fourier transform

$$f^{\sigma^1,\dots,\sigma^\ell}(t_1,\dots,t_\ell) = \mathbb{E}\exp\Big(i[t_1(Y_N(\sigma^1)-\mathbb{E}Y_N(\sigma^1))+\cdots+t_k(Y_N(\sigma^k)-\mathbb{E}Y_N(\sigma^\ell))]\Big). \tag{3.18}$$

A simple computation shows that

$$f^{\sigma^1,\dots,\sigma^\ell}(t_1,\dots,t_\ell) = \prod_{m=1}^{\binom{N}{p}} \phi\Big(\binom{N}{p}^{-1/2}\{C_p(\sigma^1,\dots,\sigma^\ell)\vec{t}\}_m\Big), \tag{3.19}$$

where $\{C_p(\sigma^1,\dots,\sigma^\ell)\vec{t}\}_m$ is the mth coordinate of the product of the matrix $C_p(\sigma^1,\dots,\sigma^\ell)$ with the vector $\vec{t}=(t_1,\dots,t_\ell)$.

Assumption A (iii) is valid due to the following proposition.

Proposition 3.5: *There exists a constant, $Q=Q(r,\ell,b_1,\dots,b_\ell)>0$, such that, for any $(\sigma^1,\dots,\sigma^\ell)\in\Sigma_N^{\otimes\ell}$, any $r\leq\ell$, if $\operatorname{rank} B_N(\sigma^1,\dots\sigma^\ell)=r$,*

$$\mathbb{P}\left(\forall_{i=1}^{\ell}: |Y_N(\sigma^i)-c|\leq\widetilde{\delta}_N b_i\right) \leq [\widetilde{\delta}_N]^r Q N^{pr/2+1}. \tag{3.20}$$

Proof. Recall that it follows from the hypothesis that the rank of the matrix $C_p(\sigma^1,\dots,\sigma^\ell)$ equals r. Let us remove from this matrix $\ell-r$ columns such that the remaining r columns are linearly independent. They correspond to a certain subset of r configurations. Without loss of generality, we may assume that they are $\sigma^1,\dots,\sigma^r$, i.e., we obtain the matrix $C_p(\sigma^1,\dots,\sigma^r)$. Obviously,

$$\mathbb{P}\left(\forall_{i=1}^{\ell}: |Y_N(\sigma^i)-c|\leq\widetilde{\delta}_N b_i\right) \leq \mathbb{P}\left(\forall_{j=1}^{r}: |Y_N(\sigma^j)-c|\leq\widetilde{\delta}_N b_i\right) \tag{3.21}$$

Then

$$\mathbb{P}\left(\forall_{i=1}^{\ell} : |Y_N(\sigma^i) - c| \leq \widetilde{\delta}_N b_i\right)$$
$$\leq \frac{1}{(2\pi)^r} \lim_{D\to\infty} \int_{[-D,D]^r} \left|f_N^{\sigma^1,\dots,\sigma^r}(t_1,\dots,t_r)\right| \prod_{j=1}^{r} \frac{e^{it_jb_j\widetilde{\delta}_N} - e^{-it_jb_j\widetilde{\delta}_N}}{2it_j} dt_j. \tag{3.22}$$

As $\widetilde{\delta}_N = O(2^{-N})$, the integrand in (3.22) is bounded by

$$\left|\frac{e^{it_jb_j\widetilde{\delta}_N} - e^{-it_jb_j\widetilde{\delta}_N}}{2it_j}\right| \leq \min\left(Q_0 2^{-N}, 2|t_j|^{-1}\right), \tag{3.23}$$

with a constant, $Q_0 = Q_0(b_j)$. Next, let us choose in the matrix $C_p(\sigma^1,\dots,\sigma^r)$ any r linearly independent rows and construct from them an $r\times r$ matrix, $\bar{C}^{r\times r}$. Then, by (3.19) and by Assumption B on $\phi(s)$

$$|f_N^{\sigma^1,\dots,\sigma^r}(\vec{t})| \leq \prod_{j=1}^{r} \left|\phi\left(\binom{N}{p}^{-1/2}\{\bar{C}^{r\times r}\vec{t}\}_j\right)\right| \leq \prod_{j=1}^{r} \min\left(1, \tilde{Q}_0 N^{p/2}\left|\{\vec{t}\bar{C}^{r\times r}\}_j\right|^{-1}\right), \tag{3.24}$$

with $\tilde{Q}_0 > 0$. Hence, the absolute value of the integral (3.22) is bounded by the sum of two terms,

$$Q_0(b_1)\cdots Q_0(b_r)2^{-Nr} \int_{\|\vec{t}\|<2^{Nr}} \prod_{j=1}^{r} \min\left(1, \tilde{Q}_0 N^{p/2}\left|\{\bar{C}^{r\times r}\vec{t}\}_j\right|^{-1}\right) dt_j$$
$$+ \int_{\|\vec{t}\|>2^{Nr}} \prod_{j=1}^{r}(2t_m^{-1}) \prod_{j=1}^{r} \min\left(1, \tilde{Q}_0 N^{p/2}\left|\{\bar{C}^{r\times r}\vec{t}\}_j\right|^{-1}\right) dt_j. \tag{3.25}$$

Recall that the matrix $\bar{C}^{r\times r}$ has matrix elements ± 1 and rank r. Since the total number of such matrices is at most 2^{r^2}, the smallest absolute value of the determinant of all such matrices is some positive number that does not depend on N, but only on r. Therefore, the change of variables, $\vec{\eta} = \bar{C}^{r\times r}\vec{t}$, in the first term shows that the integral over $\|\vec{t}\| < 2^{rN}$ is of order at most $N^{pr/2}\ln 2^{rN} \sim N^{pr/2+1}$. Thus the first term of (3.25) is bounded by $Q_1 2^{-Nr}N^{pr/2+1}$, with some constant $Q_1 < \infty$. Using the change of variables $\vec{\eta} = 2^{-rN}\vec{t}$ in the second term of (3.25), one can see that the integral over $\|\vec{t}\| > 2^{Nr}$ is bounded by $Q_2 2^{-Nr}N^{pr/2}$, with some constant $Q_2 < \infty$. This concludes the proof. $\diamondsuit$

Finally, let us fix any $\eta \in]0,1/2[$ and introduce $\mathcal{Q}_N = \mathcal{Q}_{N,\ell,q}^{\eta/p}$ (defined in (3.7)) with $q = (p\max_{k=0,\dots,[p/2]}\binom{2k}{p}(k-1)!!)^{-1}$. By (3.9) and (3.8), it is a subset of $\mathcal{R}_{N,\ell}^{\eta}$, and $|\Sigma_N^{\otimes\ell}\setminus\mathcal{Q}_N|$ is smaller than $2^{N\ell}e^{-hN^{1-2\eta}}$, with some $h > 0$. We need to verify (2.17) for $\mathcal{Q}_N$. We abbreviate

$$\vec{W}_N \equiv v^{-1}\left((c - \mathbb{E}Y_N(\sigma^1)),\dots,(c - \mathbb{E}Y_N(\sigma^\ell))\right). \tag{3.26}$$

For any $\sigma^1,\dots,\sigma^\ell \in \mathcal{Q}_N$, we split

$$\mathbb{P}(\forall_{i=1}^\ell |Y_N(\sigma^i)-c| < b_i\widetilde{\delta}_N) = \sum_{m=1}^4 I_N^m(\sigma^1,\dots,\sigma^\ell), \tag{3.27}$$

where

$$\begin{aligned} I_N^1(\sigma^1,\dots,\sigma^\ell) = &\int_{\mathbb{R}^\ell} \prod_{j=1}^\ell \frac{e^{it_jb_j\widetilde{\delta}_N} - e^{it_jb_j\widetilde{\delta}_N}}{2it_j} e^{i\vec{t}\cdot\vec{W}_N} e^{-\vec{t}B_N(\sigma^1,\dots,\sigma^\ell)\vec{t}/2} d\vec{t} \\ &- \int_{\|\vec{t}\|\geq \epsilon N^{p/6}} \prod_{j=1}^\ell \frac{e^{it_jb_j\widetilde{\delta}_N} - e^{-it_jb_j\widetilde{\delta}_N}}{2it_j} e^{i\vec{t}\cdot\vec{W}_N} e^{-\vec{t}B_N(\sigma^1,\dots,\sigma^\ell)\vec{t}/2} d\vec{t}, \end{aligned} \tag{3.28}$$

$$I_N^2(\sigma^1,\dots,\sigma^\ell) = \int_{\|\vec{t}\|<\epsilon N^{p/6}} \prod_{j=1}^\ell \frac{e^{it_jb_j\widetilde{\delta}_N} - e^{-it_jb_j\widetilde{\delta}_N}}{2it_j} e^{i\vec{t}\cdot\vec{W}_N} \left(f_N^{\sigma^1,\dots,\sigma^\ell}(\vec{t}) - e^{-\vec{t}B_N(\sigma^1,\dots,\sigma^\ell)\vec{t}/2}\right) d\vec{t}, \tag{3.29}$$

$$I_N^3(\sigma^1,\dots,\sigma^\ell) = \int_{\epsilon N^{p/6}<\|\vec{t}\|<\delta\sqrt{N^p}} \prod_{j=1}^\ell \frac{e^{it_jb_j\widetilde{\delta}_N} - e^{-it_jb_j\widetilde{\delta}_N}}{2it_j} e^{i\vec{t}\cdot\vec{W}_N} f_N^{\sigma^1,\dots,\sigma^\ell}(\vec{t}) d\vec{t}, \tag{3.30}$$

and

$$I_N^4(\sigma^1,\dots,\sigma^\ell) = (2\pi)^{-\ell} \lim_{D\to\infty} \int_{[-D,D]^\ell \cap \|\vec{t}\|>\delta\sqrt{N^p}} \prod_{j=1}^\ell \frac{e^{it_jb_j\widetilde{\delta}_N} - e^{-it_jb_j\widetilde{\delta}_N}}{2it_j} e^{i\vec{t}\cdot\vec{W}_N} f_N^{\sigma^1,\dots,\sigma^\ell}(\vec{t}) d\vec{t}, \tag{3.31}$$

with some $\epsilon,\delta>0$ to be chosen later.

The first part of $I_N^1(\sigma^1,\dots,\sigma^\ell)$ is exactly the quantity $\mathbb{P}(\forall_{i=1}^\ell : |Z_N(\sigma^i)-c| < b_i\widetilde{\delta}_N)$. Note that

$$\Big|\prod_{j=1}^\ell \frac{e^{it_jb_j\widetilde{\delta}_N} - e^{-t_jb_j\widetilde{\delta}_N}}{2it_j}\Big| \leq Q2^{-N\ell}, \tag{3.32}$$

with some $Q<\infty$. Then the second part of I_N^1 is exponentially smaller than $2^{-\ell N}$, for all $(\sigma^1,\dots,\sigma^\ell)\in\mathcal{Q}_N$. We must show that I_N^2, I_N^3, I_N^4 are $o(2^{-N\ell})$, for all $(\sigma^1,\dots,\sigma^\ell)\in\mathcal{Q}_N$. This is easy due to the following proposition.

Proposition 3.6: *There exist constants, $C<\infty$, $\epsilon,\theta,\delta>0$, such that, for all $(\sigma^1,\dots,\sigma^\ell)\in\mathcal{Q}_N$, the following estimates hold:*

(i) For all $\|\vec{t}\|<\epsilon N^{p/6}$,

$$\left|f_N^{\sigma^1,\dots,\sigma^\ell}(\vec{t}) - e^{-\vec{t}B_N(\sigma^1,\dots,\sigma^\ell)\vec{t}/2}\right| \leq \frac{C\|\vec{t}\|^3}{\sqrt{N^p}} e^{-\vec{t}B_N(\sigma^1,\dots,\sigma^\ell)\vec{t}/2}. \tag{3.33}$$

(ii) For all $\|\vec{t}\| < \delta\sqrt{N^p}$,

$$\left|f_N^{\sigma^1,\dots,\sigma^\ell}(\vec{t})\right| \leq e^{-\theta\|\vec{t}\|^2}. \tag{3.34}$$

Proof. The proof is elementary and completely analogous to the corresponding estimate in the proof of the Berry-Essen inequality. All details are completely analogous to those in the proof of Lemma 3.5 in [BK2] and therefore are omitted. ♢

Using (3.33) and (3.32), we see that $I_N^2(\sigma^1,\dots,\sigma^l) = O(N^{-p/2})2^{-N\ell}$. The third term, $I_N^3(\sigma^1,\dots,\sigma^l)$, is exponentially smaller than $2^{-N\ell}$ by (3.34).

Finally, by (3.32) we may estimate $I_N^4(\sigma^1,\dots\sigma^\ell)$ roughly as

$$|I_N^4(\sigma^1,\dots\sigma^\ell)| \leq Q2^{-\ell N} \int\limits_{\|\vec{t}\|>\delta\sqrt{N^p}} |f_N^{\sigma^1,\dots,\sigma^k}(\vec{t})|d\vec{t}, \tag{3.35}$$

with some constant $Q < \infty$. By the construction of the set $\mathcal{Q}_N$ (3.7), for any $(\sigma^1,\dots,\sigma^\ell) \in \mathcal{Q}_N$, the matrix $C_1(\sigma^1,\dots,\sigma^\ell)$, (i.e., the matrix with N rows, the kth row being $\sigma_k^1, \sigma_k^2, \dots, \sigma_k^\ell$), contains at least $2^{\ell-1}$ possible different rows, each row being present at least $2^{-\ell}N(1+o(1))$ times. Consequently, each of these rows is present in the matrix $C_p(\sigma^1,\dots,\sigma^\ell)$ at least $2^{-\ell}N^p(1+o(1))$ times, for any $p \geq 2$. Then, by (3.19), $f_N^{\sigma^1,\dots,\sigma^\ell}(\vec{t})$ is the product of at least $2^{\ell-1}$ different characteristic functions, each is taken to the power at least $2^{-\ell}N^p(1+o(1))$. Let us fix from a set of different rows of $C_p(\sigma^1,\dots,\sigma^\ell)$ ℓ linearly independent ones, and denote by $\bar{C}$ the square matrix composed of them. Then there exists $\zeta(\delta) > 0$, such that $\sqrt{\vec{t}\bar{C}^T\bar{C}\vec{t}/v^2} \geq \zeta$, for all $\vec{t}$, with $\|\vec{t}\| > \delta$. Changing variables $\vec{s} = \binom{N}{p}^{-1/2}\bar{C}\vec{t}$ in (3.35), one gets the bound

$$|I_N^4(\sigma^1,\dots,\sigma^\ell)| \leq Q2^{-\ell N}N^{p\ell/2} \int\limits_{\|\vec{s}\|>\zeta} \prod_{m=1}^{\ell} \left|\phi(s_m)\right|^{2^{-(\ell-1)}N^p(1+o(1))} ds_m. \tag{3.36}$$

Assumption B made on $\phi(s)$ implies that $\phi(s)$ is aperiodic, and thus $|\phi(s)| < 1$, for any $s \neq 0$. Moreover, for any $\zeta > 0$, there exists $h(\zeta) > 0$, such that $|\phi(s)| < 1 - h(\zeta)$, for all s with $|s| > \zeta/\ell$. Therefore, the right-hand side of (3.36) does not exceed

$$Q2^{-N\ell}N^{p\ell/2}(1-h(\zeta))^{2^{-(\ell-1)}N^p(1+o(1))-2} \int\limits_{\|\vec{s}\|>\eta} \prod_{m=1}^{\ell} \left|\phi(s_m)\right|^2 ds_m, \tag{3.37}$$

where the integral is finite again due to Assumption B. Therefore, $I_N^4(\sigma^1,\dots,\sigma^\ell)$ is exponentially smaller than $2^{-N\ell}$. This concludes the proof of (2.17) on $\mathcal{Q}_N$ and of the theorem. ♢

3.3. Short range spin glasses.

As a final example, we consider short-range spin glass models. To avoid unnecessary complications, we will look at models on the d-dimensional torus, Λ_N, of length N. We consider Hamiltonians of the form

$$H_N(\sigma) \equiv -N^{-d/2} \sum_{A \subset \Lambda_N} r_A J_A \sigma_A \tag{3.38}$$

where e $\sigma_A \equiv \prod_{x \in A} \sigma_x$, r_A are given constants, and J_A are random variables. We will introduce some notation:

(a) Let $\mathcal{A}_N$ denote the set of all $A \subset \Lambda_N$, such that $r_A \neq 0$.

(b) For any two subsets, $A, B \subset \Lambda_N$, we say that $A \sim B$, iff there exists $x \in \Lambda_N$ such that $B = A + x$. We denote by $\mathcal{A}$ the set of equivalence classes of $\mathcal{A}_N$ under this relation.

We will assume that the constants, r_A, and the random variables, J_A, satisfy the following conditions:

(i) $r_A = r_{A+x}$, for any $x \in \Lambda_N$;

(ii) there exists $k \in \mathbb{N}$, such that any equivalence class in $\mathcal{A}$ has a representative $A \subset \Lambda_k$; we will identify the set $\mathcal{A}$ with a uniquely chosen set of representatives contained in Λ_k.

(iii) $\sum_{A \subset \Lambda_N:} r_A^2 = N^d$.

(iv) J_A, $A \in \mathbb{Z}^d$, are a family of independent random variables, such that

(v) J_A and J_{A+x} are identically distributed for any $x \in \mathbb{Z}^d$;

(vi) $\mathbb{E}J_A = 0$ and $\mathbb{E}J_A^2 = 1$, and $\mathbb{E}J_A^3 < \infty$;

(vii) For any $A \in \mathcal{A}$, the Fourier transform $\phi_A(s) \equiv \mathbb{E}\exp(isJ_A)$, of J_A satisfies $|\phi_A(s)| = O(|s|^{-1})$ as $|s| \to \infty$.

Observe that $\mathbb{E}H_N(\sigma) = 0$,

$$b(\sigma, \sigma') \equiv N^{-d}\mathbb{E}H_N(\sigma)H_N(\sigma') = N^{-d} \sum_{A \subset \Lambda_N} r_A^2 \sigma_A \sigma'_A \leq 1 \tag{3.39}$$

where equality holds, if $\sigma = \sigma'$.

Note that $Y_N(\sigma) = Y_N(\sigma')$ (resp. $Y_N(\sigma) = -Y_N(\sigma')$), if and only if, for all $A \in \mathcal{A}_N$, $\sigma_A = \sigma'_A$ (resp. $\sigma_A = -\sigma'_A$). E.g., in the standard Edwards-Anderson model, with nearest neighbor pair interaction, if σ_x differs from σ'_x on every second site, x, then $Y_N(\sigma) = -Y_N(\sigma')$, and if $\sigma' = -\sigma$, $Y_N(\sigma) = Y_N(\sigma')$. In general, we will consider two configurations, $\sigma, \sigma' \in S^{\Lambda_N}$, as equivalent, iff for all $A \in \mathcal{A}_N$, $\sigma_A = \sigma'_A$. We denote the set of these equivalence classes by Σ_N. We will assume in the sequel that there is a finite constant, $\Gamma \geq 1$, such that $|\Sigma_N| \geq 2^{N^d}\Gamma^{-1}$. In the special case of $c = 0$, the equivalence relation will be extended to include the case $\sigma_A = -\sigma'_A$, for all $A \in \mathcal{A}_N$. In most reasonable examples (e.g. whenever nearest neighbor pair interactions are included in the set $\mathcal{A}$), the constant $\Gamma \leq 2$ (resp. $\Gamma \leq 4$, if $c = 0$)).

Theorem 3.7: *Let $c \in \mathbb{R}$, and Σ_N be the space of equivalence classes defined before. Let $\delta_N \equiv |\Sigma_N|^{-1} e^{c^2/2}\sqrt{\frac{\pi}{2}}$. Then the point process*

$$\mathcal{P}_N \equiv \sum_{\sigma \in \Sigma_N} \delta_{\{\delta_N^{-1}|H_N(\sigma)-c|\}}, \tag{3.40}$$

converges weakly to the standard Poisson point process on $\mathbb{R}_+$.

If, moreover, the random variables J_A are Gaussian, then, for any $c \in \mathbb{R}$, and $0 \leq \alpha < 1/4$, with $\delta_N \equiv |\Sigma_N|^{-1} e^{N^{2\alpha}c^2/2}\sqrt{\frac{\pi}{2}}$, the point process

$$\mathcal{P}_N \equiv \sum_{\sigma \in \Sigma_N} \delta_{\{\delta_N^{-1}|H_N(\sigma)-cN^\alpha|\}}, \tag{3.41}$$

converges weakly to the standard Poisson point process on $\mathbb{R}_+$.

Proof. We will now show that the assumptions A of Theorem 2.3 hold. First, the point (i) of Assumption A is verified due to the following proposition.

Proposition 3.8:_Let $\mathcal{R}^\eta_{N,\ell}$ be defined as in (2.9). Then, in the setting above, for all $0 \leq \eta < \frac{1}{2}$,_

$$|\mathcal{R}^\eta_{N,\ell}| \geq |\Sigma_N|^\ell \left(1 - e^{-hN^{d(1-2\eta)}}\right), \tag{3.42}$$

with some constant $h > 0$.

Proof. Let $\mathbb{E}_\sigma$ denote the expectation under the uniform probability measure on $\{-1,1\}^{\Lambda_N}$. We will show that there exists a constant, $K > 0$, such that, for any σ', and any $0 \leq \delta_N \leq 1$

$$\mathbb{P}_\sigma(\sigma : b(\sigma,\sigma') > \delta_N) \leq \exp(-K\delta_N^2 N^d). \tag{3.43}$$

Note that without loss, we can take $\sigma' \equiv 1$. We want to use the exponential Chebyshev inequality and thus need to estimate the Laplace transform

$$\mathbb{E}_\sigma \exp\left(tN^{-d}\sum_{A \in \Lambda_N} r_A^2 \sigma_A\right). \tag{3.44}$$

Let us assume for simplicity that $N = nk$ is a multiple of k, and introduce the sub-lattice, $\Lambda_{N,k} \equiv \{0, k, \dots, (n-1)k, nk\}^d$. Write

$$\sum_{A \in \Lambda_N} r_A^2 \sigma_A = \sum_{A \in \mathcal{A}} \sum_{y \in \Lambda_{N,k}} \sum_{x \in \Lambda_k} r_{A+y+x}^2 \sigma_{A+y+x} \equiv \sum_{x \in \Lambda_k} Z_x(\sigma) \tag{3.45}$$

where

$$Z_x(\sigma) = \sum_{y \in \Lambda_{N,k}} Y_{y,x}(\sigma) \tag{3.46}$$

has the nice feature that, for fixed x, the summands

$$Y_{x,y}(\sigma) \equiv \sum_{A \in \mathcal{A}} r_{A+y+x}^2 \sigma_{A+y+x}$$

are independent for different $y, y' \in \Lambda_{n,k}$ (since the sets $A+y+x$ and $A'+y'+x$ are disjoint for any $A, A' \in \Lambda_k$). Using the Hölder inequality repeatedly,

$$\begin{aligned}
\mathbb{E}_\sigma \exp\left(t \sum_{x \in \Lambda_k} Z_x(\sigma) \right) &\leq \prod_{x \in \Lambda_k} \left[\mathbb{E}_\sigma e^{tk^d Z_x(\sigma)} \right]^{k^{-d}} \\
&= \prod_{x \in \Lambda_k} \prod_{y \in \Lambda_{N,k}} \left[\mathbb{E}_\sigma e^{tk^d Y_{x,y}(\sigma)} \right]^{k^{-d}} \\
&= \left[\mathbb{E}_\sigma e^{tk^d Y_{0,0}(\sigma)} \right]^{N^d k^{-d}}
\end{aligned} \tag{3.47}$$

It remains to estimate the Laplace transform of $Y_{0,0}(\sigma)$,

$$\mathbb{E}_\sigma \exp\left(tk^d Y_{0,0}(\sigma) \right) = \mathbb{E}_\sigma \left(tk^d \sum_{A \in \Lambda_k} r_A^2 \sigma_A \right), \tag{3.48}$$

and, since $\mathbb{E}_\sigma \sigma_A = 0$, using that $e^x \leq 1 + x + \frac{x^2}{2} e^{|x|}$,

$$\mathbb{E}_\sigma \exp\left(tk^d \sum_{A \in \Lambda_k} r_A^2 \sigma_A \right) \leq \mathbb{E}_\sigma \exp\left(\frac{t^2}{2} k^{2d} \left(\sum_{A \in \Lambda_k} r_A^2 \right) e^{tk^d \sum_{A \in \Lambda_k} r_A^2} \right) \equiv \mathbb{E}_\sigma \exp\left(\frac{t^2}{2} C e^{tD} \right), \tag{3.49}$$

so that

$$\mathbb{E}_\sigma \exp\left(tN^{-d} \sum_{x \in \Lambda_k} Z_x(\sigma) \right) \leq \exp\left(N^{-d} \frac{t^2}{2} C' e^{N^{-d} tD} \right), \tag{3.50}$$

with constants, C, C', D, that do not depend on N. To conclude the proof of the lemma, the exponential Chebyshev inequality gives,

$$\mathbb{P}_\sigma \left[b(\sigma, \sigma') > \delta_N \right] \leq \exp\left(-\delta_N t + N^{-d} \frac{t^2}{2} C' e^{tN^{-d} D} \right). \tag{3.51}$$

Choosing $t = \epsilon N^d \delta_N$, this gives

$$P_\sigma \left[b(\sigma, \sigma') > \delta_N \right] \leq \exp\left(-\epsilon \delta_N^2 N^d \left(1 - \epsilon C' e^{\epsilon \delta_N D}/2 \right) \right) \tag{3.52}$$

Choosing ϵ small enough, but independent of N, we obtain the assertion of the lemma. $\diamondsuit$

To verify Assumptions A (ii) and (iii), we need to introduce the matrix $C = C(\sigma^1, \dots, \sigma^\ell)$ with ℓ columns and $|\mathcal{A}_N|$ rows, indexed by the subsets $A \in \mathcal{A}_N$: the elements of each of its column are $r_A \sigma_A^1, r_A \sigma_A^2, \dots, r_A \sigma_A^\ell$, so that $C^T C$ is the covariance matrix, $B_N(\sigma^1, \dots, \sigma^\ell)$, up to a multiplicative factor N^d.

The assumption (ii) is verified due to Proposition 3.3. In fact, let us reduce C to the matrix $\tilde{C} = \tilde{C}(\sigma^1, \dots, \sigma^\ell)$ with columns $\sigma_A^1, \sigma_A^2, \dots, \sigma_A^\ell$, without the constants r_A. Then, exactly as in the case of p-spin SK models, by Proposition 3.3, for any $(\sigma^1, \dots, \sigma^\ell) \in \mathcal{L}_{N^d, r}^\ell$ the matrix $\tilde{C}(\sigma^1, \dots, \sigma^\ell)$ can contain at most $2^r - 1$ different columns. Hence, $|\mathcal{L}_{N^d, r}^\ell| = O((2^r - 1)^{N^d})$ while $|\Sigma_N|^r \geq (2^{N^d}/\Gamma)^r$.

The assumption (iii) is verified as well, and its proof is completely analogous to that of Proposition 3.5. The key observation is that, again, the number of possible non-degenerate matrices $\bar{C}^{r \times r}$ that can be obtained from $C_p(\sigma^1, \dots, \sigma^\ell)$ is independent of N. But this is true since, by assumption, the number different constants r_A is N-independent.

Finally, we define $\mathcal{Q}_N$ as follows. For any $A \in \mathcal{A}$, let

$$\mathcal{Q}_{N,\ell}^{\eta; A} = \Big\{ (\sigma^1, \dots, \sigma^\ell) : \forall_{1 \leq i < j \leq \ell} \; r_A^2 \sum_{x \in \mathbb{Z}^d : x + A \subset \Lambda_N} \sigma_A^i \sigma_A^j < |\mathcal{A}|^{-1} N^{-\eta} \Big\}. \tag{3.53}$$

Let us define $\mathcal{Q}_N = \bigcap_{A \in \mathcal{A}} \mathcal{Q}_{N,\ell}^{\eta; A} \subset \mathcal{R}_{N,\ell}^\eta$. By Proposition 3.8, applied to a model where $|\mathcal{A}| = 1$, for any $A \in \mathcal{A}$, we have $|\mathcal{S}_N^{\otimes \ell} \setminus \mathcal{Q}_{N,\ell}^{\eta; A}| \leq 2^{N^d} \exp(-h_A N^{d(1-2\eta)})$, with some $h_A > 0$. Hence, $|\mathcal{R}_{N,\ell}^\eta \setminus \mathcal{Q}_N|$ has cardinality smaller than $|\Sigma_N|^\ell \exp(-h N^{d(1-2\eta)})$, with some $h > 0$. The verification of (2.17) on $\mathcal{Q}_N$ is analogous to the one in Theorem 3.4, using the analogue of Proposition 3.6. We only note a small difference in the analysis of the term I_N^4 where we use the explicit construction of $\mathcal{Q}_N$. We represent the corresponding generating function as the product of $|\mathcal{A}|$ terms over different equivalence classes of $\mathcal{A}$, with representatives $A \subset \Lambda_k$, each term being $\prod_{x \in \mathbb{Z}^d : x + A \in \Lambda_N} \phi(N^{-d/2} r_A (t_1 \sigma_{x+A}^1 + \dots + t_\ell \sigma_{x+A}^\ell))$. Next, we use the fact that for any $(\sigma^1, \dots, \sigma^\ell) \in \mathcal{Q}_N$ each of these $|\mathcal{A}|$ terms is a product of at least $2^\ell - 1$ (and of coarse at most 2^ℓ) different terms, each is taken to the power $|\mathcal{A}|^{-1} N^d 2^{-\ell} (1 + o(1))$. This proves the first assertion of the theorem.

The proof of the second assertion, i.e., the case $\alpha > 0$ with Gaussian variables J_A is immediate from the estimates above and the abstract Theorem 2.2, in view of the fact that the condition (2.17) is trivially verified. $\diamondsuit$

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