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EXACT MINIMAX RISK FOR DENSITY ESTIMATORS IN NON-INTEGER SOBOLEV CLASSES

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Abstract

The L_2-minimax risk in Sobolev classes of densities with non-integer smoothness index is shown to have an analog form to that in integer Sobolev classes. To this end, the notion of Sobolev classes is generalized to fractional derivatives of order $\beta \in \mathbb{R}^+$. A minimax kernel density estimator for such a classes is found. Although there exists no corresponding proof in the literature so far, the result of this article was used implicitly in numerous papers. A certain necessity that this gap had to be filled, can thus not be denied.

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1 Introduction

When trying to describe the goodness of an estimator, minimax performance is one optimality criterium possible to be consulted. The minimax risk of density estimators can be regarded in various settings, e.g. we differentiate between the local risk in a single point and the integrated risk over the whole curve. Several loss functions have been under consideration, such as absolute, quadratic and supremum norm, Hellinger and Kullback-Leibler distance. But exact asymptotics is up to now limited to a few special cases: to supremum risk in Hölder classes, and to mean integrated square error (MISE) in analytical and in Sobolev classes.
The latter has been examined for quite a while since in 1983, Efroimovich and Pinsker completed the asymptotic minimax rate of the lower bound of MISE (Samarow [8]) by the still lacking asymptotically exact constant, using tools that are common in information theory. The results were enhanced and new methods of proof found by Golubev [4] and [5], Golubev, Levit [6] and Schipper [10]. Sobolev classes are classes of L_2-integrable functions, in the present problem densities, for which smoothness is measured through the L_2-norm of their β^{th} derivative, $\beta \in \mathbb{N}$.

$$\mathcal{S}_\beta(L) \quad = \quad \left\{ f \in L_2 \middle| \int \left(f^{(\beta)}(x) \right)^2 dx \leq L \right\}, \quad L < \infty \tag{1}$$

Nowadays it is well known that

$$\inf_{\widetilde{f}_n} \sup_{f \in \mathcal{S}_\beta(L)} n^{\frac{2\beta}{2\beta+1}} E_f \|\widetilde{f}_n - f\|_2^2 \quad = \quad \gamma(\beta, L)\Big(1 + o(1)\Big) \tag{2}$$

where $$\gamma(\beta, L) \;=\; (2\beta+1)\left(\frac{\pi(2\beta+1)(\beta+1)}{\beta}\right)^{-\frac{2\beta}{2\beta+1}} L^{\frac{1}{2\beta+1}}$$

is Pinsker's constant. Estimators attaining minimax rates of convergernce have been studied in abundance, e.g. kernel estimators, but also wavelet estimators and a wide range of others. More care has to be taken when envisaging asymptotically exact minimax estimators.
However, the characterization of the smoothness of a given density function is incomplete when just assigning it to some $\mathcal{S}_\beta(L)$, $\beta \in \mathbb{N}$. Recalling the Sobolev criterion,

$$L \;\geq\; \int \left(f^{(\beta)}(x)\right)^2 dx \;=\; \frac{1}{2\pi}\int |\omega^\beta \widehat{f}(\omega)|^2 d\omega,$$

we immediately observe that $\mathcal{S}_\beta(L)$ contains densities which do not lie in $\mathcal{S}_{\beta+1}$, although for suitably chosen $L' < \infty$ and $\varepsilon < 1$, they certainly do satisfy $\frac{1}{2\pi}\int |\omega^{\beta+\varepsilon}\widehat{f}(\omega)|^2 d\omega \leq L'$.
The present article is interested in the question of whether the minimax risk can also be calculated for such generalized Sobolev classes. Corresponding claims are implicit in a number of recent papers, yet their proofs cover but the entire case. For our purpose we will employ the concept of the so-called fractional derivative after Riemann and Liouville, thoroughly discussed in Samko [9]:

$$f^{(\beta)}(x) \;=\; \frac{d^\beta}{dx^\beta} f(x) \;=\; \frac{1}{\Gamma(\lceil\beta\rceil - \beta)}\,\frac{d^{\lceil\beta\rceil}}{dx^{\lceil\beta\rceil}} \int t^{\beta-\lceil\beta\rceil} f(x+t)dt$$

with $\lceil x \rceil$ the smallest integer greater than the positive real number x. For $\beta \in \mathbb{N}$, $f^{(\beta)}$ is the β^{th} derivative of f, for $\beta \in \mathbb{R}^+\backslash\mathbb{N}$ it is the β^{th} fractional derivative of f (Samko [9], p. 137). In case $f^{(\beta)}$ is continuous and L_1-integrable, then $\widehat{f^{(\beta)}}(\omega) = (-i\omega)^\beta \widehat{f}(\omega)$. The other way around, if $(-i\cdot\mathrm{id})^\beta \widehat{f}$ is L_1- or L_2-integrable, the inverse transform from the Fourier into the time domain exists and for our purpose we define:

$$f^{(\beta)}(x) \;:=\; \frac{1}{2\pi}\int (-i\omega)^\beta \widehat{f}(\omega) e^{-ix\omega} dx \tag{3}$$

Existence and uniqueness of the β^{th} fractional derivative of f follow thus from $\frac{1}{2\pi}\int |\omega^\beta \widehat{f}(\omega)|^2 d\omega \leq L$, and Parseval's equality gives $\int (f^{(\beta)}(x))^2 dx = \frac{1}{2\pi}\int |\omega^\beta \widehat{f}(\omega)|^2 d\omega$.
Adopting the idea of Schipper [10], we find upper and lower bounds for the asymptotic minimax risk in $\mathcal{S}_\beta(L)$, $\beta > 1/2$, which are then shown to converge towards each other. Thereby it will be verified that the minimax risk is determined by $n^{2\beta/(2\beta+1)}\gamma(\beta, L)$, where $\gamma(\beta, L)$ is an analogue of Pinsker's constant. A minimax kernel function for kernel density estimation is obtained as a byproduct from the calculation. On benefit of our a statement, it is for instance possible to show the asymptotically exact minimax-adaptivity of non-parametric estimation procedures such as the recently proposed Stein's blockwise estimator for densities (Rigollet [7]) and the cross-validation kernel choice for density estimation (Dalelane [1]).
The calculation of the upper bound in Schipper [10] actually holds for both entire and non-entire smoothness indeces, so Schipper's Theorem 3 (*Math. Meth. of Statistics* (1996), Vol. 5 No. 3, page 258-260) applies directly. To show the lower bound in Section 2, we replace original problem of estimating a curve by the problem of estimating a finite-dimensional parameter θ (of increasing dimension). A lower bound for the risk of such an estimator may be found by means of the van Trees inequality. The Bayesian risk over a least favorable parametric family of densities $\mathcal{F}_\Theta$, and a least favorable prior distribution Λ on the space of finite-dimensional parameters Θ, such that $f_\theta \in \mathcal{S}_\beta(L)$ with a high probability, provides us

with a lower bound for the minimax risk on $\mathcal{S}_\beta(L)$. It is exactly this gap in the literature: f_θ asymptotically in $\mathcal{S}_\beta(L)$ for $\beta \notin \mathbb{N}$, which we have been able to close in the present paper. Although the demonstrations follow in general the same lines as Schipper [10], the least favorable family of densities had to be constructed in a different way. The proof of the essential property (Theorem 2) applies Riemann-Liouville calculus along with approximations in the Fourier domain and is not similar to Schipper [10].
The result for the lower bound can be considered as a special case of the theorem in Golubev [5], who yields lower bounds for the quadratic risk of non-parametric estimation problems in a variety of elliptic density classes via Local Asymptotic Normality. Unfortunately the proof in Golubev [5] is heavily abbreviated (the proof of a claim corresponding to our Theorem 2 is actually omitted) and not easy to retrace. We hope that by our detailed proof, we are able to somehow enlighten the complicated matters.

2 Minimax bounds

Let $X_1, \ldots, X_n$ be i.i.d. random variables with common density function f and let $\widetilde{f}_n$ be an arbitrary estimator for f depending but on the sample.

Theorem 1 *(see Schipper [10] Theorem 3)* *Let $\mathcal{S}_\beta(L)$ be the Sobolev class of those L_2-integrable densities, which satisfy $\frac{1}{2\pi}\int |\omega^\beta \widehat{f}(\omega)|^2 d\omega \leq L$ for some constants $\beta > 0$ and $L < \infty$. Then it holds, that*

$$\inf_{\widetilde{f}_n} \sup_{\mathcal{S}_\beta(L)} n^{\frac{2\beta}{2\beta+1}} \; E_f \|\widetilde{f}_n - f\|_2^2 \;\; \leq \;\; \gamma(\beta, L)$$

The bound is maintained by a kernel estimator with the minimax kernel K_β, that is the inverse Fourier transform of $\widehat{K}_\beta(\omega) = \left(1 - c_{\min}(L) \cdot |\omega|^\beta\right)_+$ with $c_{\min} = \left(\frac{nL\pi(2\beta+1)(\beta+1)}{\beta}\right)^{-\beta/(2\beta+1)}$.

Generally speaking, the derivation of the lower bound proceeds similarly to Schipper [10] (Subsection 4.1, page 262-268). However it is not the same, and so we give a little more detail. The following steps lead to the desired result, which can partly be effected analogously to Schipper [10], partly new proofs had to found:
1) Construction of a least favorable parametric family of densities $\mathcal{F}_\Theta$, proof that the elements of $\mathcal{F}_\Theta$ are contained in an ε-neighborhood of the considered Sobolev class $\mathcal{S}_\beta(L)$. Both our center function of $\mathcal{F}_\Theta$ and our perturbation functions had to be constructed in a distinct way to Schipper [10], whereas the parameter set Θ is the same. The proof of our Theorem 2 is different to that of Schipper's corresponding lemmata (Lemma 1 through 4).
2) Definition of a least favorable prior distribution Λ on the parameter set Θ, proof that under Λ the elements of $\mathcal{F}_\Theta$ are contained in $\mathcal{S}_\beta(L)$ itself with high probability. This time, Schipper's distribution Λ and his proof (Lemma 5, p. 266-267) are possible to transfer to our context.
3) Main approximation of the lower bound via the Bayes risk over $\mathcal{F}_\Theta$ with respect to Λ by means of the van Trees inequality. Again the proof of Schipper's Proposition 2 (p. 267-268) resembles our demonstration.

The problem of searching a lower bound for the minimax risk over the Sobolev class $\mathcal{S}_\beta(L)$, can be reverted to a parametric subset of $\mathcal{S}_\beta$. Whether the minimax risk over the subclass coincides with the minimax risk over $\mathcal{S}_\beta(L)$, obviously depends on the difficulty of the estimation problem within the subclass. We achieve our aim using the adjacent construction: Let us assume $\beta > 1/2$ and let f_0 be the following density from $\mathcal{S} := \bigcap_{\beta\in\mathbb{R}^+} \mathcal{S}_\beta$

$$f_0(x) \;:=\; \begin{cases} \frac{1}{c_a}\exp\Big\{-\frac{a}{(x+1/2)(1/2-x)}\Big\}, & -1/2 \le x \le 1/2 \\ 0, \text{ otherwise} \end{cases} \tag{4}$$

with c_a defined such that f_0 is a density, and for technical reason, the constant a satisfying $\int |\widehat{f}_0(\omega)\frac{\sin\omega/2}{\omega/2}|d\omega = 2\pi$. Since $\int |\omega^\beta \widehat{f}_0(\omega)|^2 d\omega < \infty$ for all $\beta < \infty$, also $\int |\omega^\beta \widehat{f}_0(\omega)| d\omega$ exists for all $\beta < \infty$. Let g_A be the indicator function on $[-A+1/2, A-1/2]$ times the factor $\frac{1}{2A-1}$, i.e.

$$g_A(x) \;:=\; \frac{1}{2A-1}\, I_{[-A+\frac{1}{2}\,,\,A-\frac{1}{2}]}(x). \tag{5}$$

Then $f_0 * g_A$ is a symmetric density within $\mathcal{S}$, that takes the constant value $\frac{1}{2A-1}$ on $[-A+1, A-1]$, and decreases smoothly towards 0 on $[A-1, A]$ and $[-A, -A+1]$. In order to constitute a sufficiently difficult estimation problem departing from this very smooth density, let us add some perturbation functions to $f_0 * g_A$:

$$\varphi_k(x) \;:=\; \begin{cases} \frac{1}{\sqrt{A}}\cos\frac{k\pi x}{A}\; I_{[-A,A]}(x), & k>0 \\ \frac{1}{\sqrt{A}}\sin\frac{k\pi x}{A}\; I_{[-A,A]}(x), & k<0 \end{cases} \tag{6}$$

These perturbations will be weighted by factors θ_k, where $\theta = (\ldots, \theta_{-2}, \theta_{-1}, \theta_1, \theta_2, \ldots)$ is (asymptotically) in the set:

$$\Theta_A(L) \;:=\; \left\{\theta \in \mathbb{R}^\infty \Big| \sum_{k\neq 0} |\theta_k| \le A^{-2\beta+1} \text{ and } \sum_{k\neq 0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\beta} \le 4A^2 L\right\} \tag{7}$$

The set $\{f_\theta |\, \theta \in \Theta_A(L)\}$ will from now on be the family of densities under consideration.

$$f_\theta(x) \;:=\; \frac{1}{b(\theta)}\; f_0 * g_A(x)\Big(1 + \sum_{k\neq 0}\theta_k\varphi_k(x)\Big), \tag{8}$$

where $b(\theta)$ is the normalizing constant. We cannot prove that $\{f_\theta |\, \theta\in\Theta_A(L)\} \subseteq \mathcal{S}_\beta(L)$, but instead that for all $\varepsilon > 0$ there exists an $A_\varepsilon < \infty$, so that for every $A \ge A_\varepsilon$ the following holds: $\sup_{\theta\in\Theta_A(L)} \|f_\theta^{(\beta)}\|_2^2 \le L + \varepsilon$.

Theorem 2 *Let f_θ and $\Theta_A(L)$ be defined as above. Then, as $A \longrightarrow \infty$:*

$$\sup_{\Theta_A(L)} \|f_\theta^{(\beta)}\|_2^2 \;=\; L + o(1)$$

This theorem is the main assertion of our paper. Filling the gap in the hitherto existing literature, it enables us to go on proving the minimax bound for non-integer Sobolev classes. Its cumbersome and unpleasantly lengthy proof is to be found in Section 5.

The next step leading to the lower bound requires the definition of a prior distribution Λ, which is done accordingly to [10], so as to yield a parameter θ of finite dimension: Let $\varepsilon > 0$, $W > 0$ and $\sigma_k^2 > 0$

$$\lambda(\theta) \;=\; \prod_{0<|k|<W}\lambda_k(\theta_k)\prod_{|k|\ge W}\delta_0(\theta_k), \tag{9}$$

where $\delta_0(.)$ is the Dirac function on 0, and for $|k| < W$: $\lambda_k(\theta_k)$ are absolutely continuous densities with $E\theta_k^2 = \sigma_k^2$, $\theta_k^2 \leq G^2\sigma_k^2$ (Λ-f.s.) for some $G < \infty$, and the Fisher information $I_k := \int \frac{\lambda_k'^2(\theta_k)}{\lambda_k(\theta_k)}\, d\theta_k \leq (1+\varepsilon)\sigma_k^{-2}$ (with respect to the translation group $\{\lambda_k(.-u)|u \in \mathbb{R}\}$). (These conditions are satisfied, for example, by independent bounded, zero mean random variables $\sigma_k\xi_k$, $|k| < W$, with $|\xi_k| < G$, $E\xi_k^2 = 1$ and the Fisher-information of the density of ξ_k smaller than $1+\varepsilon$.) Let us set

$$\begin{aligned} W &= \frac{A}{\pi}\Big(\frac{L(1-\varepsilon)n(2\beta+1)(\beta+1)\pi}{\beta}\Big)^{\frac{1}{2\beta+1}} \\ \sigma_k^2 &= \frac{4A}{n}\Big(\Big|\frac{W}{k}\Big|^{\beta} - 1\Big)_+ \end{aligned} \tag{10}$$

As W grows with $n \longrightarrow \infty$, the dimension of the parameter θ will tend to infinity, allowing for more and more perturbation functions φ_k in the definition of f_θ. At the end of Section 3 it will be shown that σ_k^2 and W of this form approximately maximize the lower bound of the minimax risk for the prior distribution Λ.
Since Λ is not supported on $\Theta_A(L)$, we will have to show that at least the probability of $\theta \in \Theta_A(L)$ grows with $n \longrightarrow \infty$:
First consider that λ has a bounded support, $|\theta_k| \leq G\sigma_k$ for $|k| < W$, and else $\theta_k = 0$. With the above construction of σ_k^2 and W, letting $A \sim \ln n$, condition $\sum |\theta_k| \leq A^{-2\beta+1}$ is fulfilled for n sufficiently large. Lemma 1 takes care of $\sum \theta_k^2 \left(\frac{k\pi}{A}\right)^{2\beta} \leq 4A^2L$.

Lemma 1 *For the prior distribution Λ defined above, with W and σ_k^2 as in (10), it holds that for $n \longrightarrow \infty$:*

$$P_\lambda\Big(\theta \notin \Theta_A(L)\Big) = o(n^{-1})$$

This lemma corresponds to Lemma 5 in Schipper [10], p. 266. Its proof is exactly the same (p. 266-267) and we abstain from quoting it here (also see Dalelane [1] for more details).

Theorem 3 *For $L < \infty$, $\beta > 1/2$ and $\gamma(\beta, L)$ equal to Pinsker's constant we have:*

$$\liminf_{n\to\infty} \inf_{\widetilde{f}_n} \sup_{f \in \mathcal{S}_\beta(L)} n^{\frac{2\beta}{2\beta+1}}\, E_f\|\widetilde{f}_n - f\|_2^2 \;\geq\; \gamma(\beta, L)$$

Proof Let us at first reduce the supremum of the risk by restricting the set of density functions. According to Theorem 2 we know that for $A \sim \ln n$, $\lim_{A\to\infty}\{f_\theta|\theta \in \Theta_A(L)\} \subseteq \mathcal{S}_\beta(L)$.

$$\liminf_{n\to\infty} \inf_{\widetilde{f}_n} \sup_{S_\beta(L)} E_f\|\widetilde{f}_n - f\|_2^2 \;\geq\; \liminf_{n\to\infty} \inf_{\widetilde{f}_n} \sup_{\Theta_A(L)} E_{f_\theta}\|\widetilde{f}_n - f_\theta\|_2^2 \tag{11}$$

For any fixed A, we find a lower bound for the supremum over $\Theta_A(L)$ through the Bayesian risk with respect to Λ.

$$\inf_{\widetilde{f}_n} \sup_{\Theta_A(L)} E_{f_\theta}\|\widetilde{f}_n - f_\theta\|_2^2 \;\geq\; \inf_{\widetilde{f}_n} \int_{\Theta_A(L)} E_{f_\theta}\|\widetilde{f}_n - f_\theta\|_2^2\, d\Lambda(\theta) \tag{12}$$

In inequality (20) of the proof of Theorem 2 it will be shown that $1-A^{-3/2} \le b(\theta) \le 1+A^{-3/2}$. Furthermore, because of orthonormality $\| \sum \theta_k \varphi_k \|_2^2 = \sum \theta_k^2 \le \sum |\theta_k| \le A^{-2\beta+1}$, (28). So we can derive, for all $\theta \in \Theta_A(L)$:

$$\begin{aligned}
\|f_\theta\|_2 &= \frac{1}{b(\theta)} \Big\| f_0 * g_A \Big(1 - \sum_{k\neq 0} \theta_k \varphi_k\Big) \Big\|_2 \\
&\le \frac{1}{b(\theta)} \ \max f_0 * g_A \ \Big\| I_{[-A,A]} + \sum_{k\neq 0} \theta_k \varphi_k \Big\|_2 \\
&\le \frac{\text{const.}}{\sqrt{A}} \ =: \ \frac{1}{\sqrt{A_0}}
\end{aligned}$$

Because the set of all densities with $\|f\|_2 \le 1/A_0$ is convex, we may in (13) also restrict the set estimators to $\|\widetilde{f}_n\|_2^2 \le 1/A_0$ without increasing the supremum.

$$\begin{aligned}
\inf_{\widetilde{f}_n} \int_{\Theta_A(L)} E_{f_\theta} \|\widetilde{f}_n - f_\theta\|_2^2 \, d\Lambda(\theta) &= \inf_{\|\widetilde{f}_n\|_2^2 \le A_0^{-1}} \int_{\Theta_A(L)} E_{f_\theta} \|\widetilde{f}_n - f_\theta\|_2^2 \, d\Lambda(\theta) && (13) \\
&\ge \inf_{\|\widetilde{f}_n\|_2^2 \le A_0^{-1}} \int E_{f_\theta} \|\widetilde{f}_n - f_\theta\|_2^2 \, d\Lambda(\theta) - \frac{4}{A_0} P_\lambda\Big(\theta \notin \Theta_A(L)\Big) && (14) \\
&= \inf_{\|\widetilde{f}_n\|_2^2 \le A_0^{-1}} \int E_{f_\theta} \|\widetilde{f}_n - f_\theta\|_2^2 \, d\Lambda(\theta) \ + \ o(n^{-1}) \\
&\ge \inf_{\widetilde{f}_n} E_\lambda E_{f_\theta} \|\widetilde{f}_n - f_\theta\|_2^2 \ + \ o(n^{-1}) && (15)
\end{aligned}$$

Due to $\|f_\theta\|_2^2 \le A_0^{-1}$ and $\|\widetilde{f}_n\|_2^2 \le A_0^{-1}$ it holds in (14) that $\|\widetilde{f}_n - f_\theta\|_2^2 \le 4A_0^{-1}$. In (15) we return to the complete set of estimators.

Since f_θ has bounded support, i.e. $[-A, A]$, it is equivalent, as regards the quadratic risk, either to estimate the function f_θ in the time domain or its Fourier coefficients. ($\widehat{f}_\theta(0) = 1$ is known)

$$\begin{aligned}
& E_\lambda E_{f_\theta} \|\widetilde{f}_n - f_\theta\|_2^2 \\
= \ & E_\lambda E_{f_\theta} \ \frac{1}{2A} \sum_{\kappa \neq 0} \Big| \widehat{\widetilde{f}_n}\left(\frac{\kappa\pi}{A}\right) - \widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right) \Big|^2 \\
= \ & E_\lambda E_{f_\theta} \ \frac{1}{2A} \sum_{\kappa \neq 0} \ \mathrm{Re}^2 \left(\widehat{\widetilde{f}_n}\left(\frac{\kappa\pi}{A}\right) - \widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right) \right) + \ \mathrm{Im}^2 \left(\widehat{\widetilde{f}_n}\left(\frac{\kappa\pi}{A}\right) - \widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right) \right) \\
= \ & E_\lambda E_{f_\theta} \ \frac{1}{2A} \sum_{\kappa \neq 0} \left(\mathrm{Re} \widehat{\widetilde{f}_n}\left(\frac{\kappa\pi}{A}\right) - \mathrm{Re} \widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right) \right)^2 + \left(\mathrm{Im} \widehat{\widetilde{f}_n}\left(\frac{\kappa\pi}{A}\right) - \mathrm{Im} \widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right) \right)^2 \qquad (16)
\end{aligned}$$

The van Trees inequality (Gill, Levit [3]) may now be applied on every single summand. For technical reason, the real parts are derived with respect to $\theta_{|\kappa|}$, while the imaginary ones are derived with respect to $\theta_{-|\kappa|}$.

$$\begin{aligned}
E_\lambda E_{f_\theta} \left[\mathrm{Re} \widehat{\widetilde{f}_n}\left(\frac{\kappa\pi}{A}\right) - \mathrm{Re} \widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right) \right]^2 &\ge \frac{E_\lambda^2 \left[\partial \, \mathrm{Re} \widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right) / \partial \theta_{|\kappa|} \right]}{n E_\lambda I_{f_\theta}(\theta_{|\kappa|}) + I_{|\kappa|}} \\
& \qquad\qquad\qquad\qquad\qquad\qquad (17) \\
E_\lambda E_{f_\theta} \left[\mathrm{Im} \widehat{\widetilde{f}_n}\left(\frac{\kappa\pi}{A}\right) - \mathrm{Im} \widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right) \right]^2 &\ge \frac{E_\lambda^2 \left[\partial \, \mathrm{Im} \widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right) / \partial \theta_{-|\kappa|} \right]}{n E_\lambda I_{f_\theta}(\theta_{-|\kappa|}) + I_{-|\kappa|}}
\end{aligned}$$

where we denote $I_{f_\theta}(\theta_\kappa) = \int \frac{(\partial f_\theta(x)/\partial\theta_\kappa)^2}{f_\theta(x)}\,dx$. I_κ is the "Fisher information" of λ_κ and by construction $\leq (1+\varepsilon)\sigma_\kappa^{-2}$ for $|\kappa| < W$ and $= \infty$ for $|\kappa| \geq W$, respectively. Hence all summands with $|\kappa| \geq W$ vanish from the sum. Approximations for $I_{f_\theta}(\theta_\kappa)$, $\partial\,\mathrm{Re}\,\widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right)/\partial\theta_{|\kappa|}$ und $\partial\,\mathrm{Im}\,\widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right)/\partial\theta_{-|\kappa|}$ are available from

Lemma 2 *For $A \longrightarrow \infty$:*

$$I_{f_\theta}(\theta_\kappa) = \frac{1+o(1)}{2A} \qquad \frac{\partial\,\mathrm{Re}\widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right)}{\partial\theta_{|\kappa|}} = \frac{1+o(1)}{2\sqrt{A}} \qquad \frac{\partial\,\mathrm{Im}\widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right)}{\partial\theta_{-|\kappa|}} = \frac{1+o(1)}{2\sqrt{A}}$$

with $o(1)$ independent of κ and θ_κ. The proof is postponed to Section 5. From (15) completed by (16), the van Trees approximation (17) and Lemma 2 we thus have:

$$\begin{aligned}
\inf_{\widetilde{f}_n} E_\lambda E_{f_\theta}\|\widetilde{f}_n - f_\theta\|_2^2 &= \inf_{\widetilde{f}_n} E_\lambda E_{f_\theta}\frac{1}{2A}\sum_{\kappa\neq 0}\left|\widetilde{\widehat{f}_n}\left(\frac{\kappa\pi}{A}\right) - \widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right)\right|^2 \\
&= E_\lambda E_{f_\theta}\frac{1}{2A}\sum_{\kappa\neq 0}\left(\mathrm{Re}\widetilde{\widehat{f}_n}\left(\frac{\kappa\pi}{A}\right) - \mathrm{Re}\widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right)\right)^2 + \left(\mathrm{Im}\widetilde{\widehat{f}_n}\left(\frac{\kappa\pi}{A}\right) - \mathrm{Im}\widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right)\right)^2 \\
&\geq \frac{1}{2A}\sum_{\kappa\neq 0}\frac{E_\lambda^2\left[\partial\,\mathrm{Re}\widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right)/\partial\theta_{|\kappa|}\right]}{nE_\lambda I_{f_\theta}(\theta_{|\kappa|}) + I_{|\kappa|}} + \frac{E_\lambda^2\left[\partial\,\mathrm{Im}\widehat{f}_\theta\left(\frac{\kappa\pi}{A}\right)/\partial\theta_{-|\kappa|}\right]}{nE_\lambda I_{f_\theta}(\theta_{-|\kappa|}) + I_{-|\kappa|}} \\
&= \frac{1}{2A}\sum_{0<|\kappa|<W}\frac{\left(\frac{1+o(1)}{2\sqrt{A}}\right)^2}{n\,\frac{1+o(1)}{2A} + (1+\varepsilon)\sigma_{|\kappa|}^{-2}} + \frac{\left(\frac{1+o(1)}{2\sqrt{A}}\right)^2}{n\,\frac{1+o(1)}{2A} + (1+\varepsilon)\sigma_{-|\kappa|}^{-2}} \\
&= \frac{1+o(1)}{2A(1+\varepsilon)}\sum_{0<|\kappa|<W}\frac{1}{n + 2A\sigma_\kappa^{-2}} \qquad (18)
\end{aligned}$$

All sums obtained from W and σ_κ^2 through (18), i.e. from a prior distribution Λ satisfying Lemma 1, are thus lower bounds of the minimax risk.
What we are searching for is a bound as large as possible, we hence maximize (18) subject to the constraint $\sum \sigma_\kappa^2\left(\frac{\kappa\pi}{A}\right)^{2\beta} \leq (1-\varepsilon)4A^2L$, such that $P(\theta \notin \Theta_A(L)) = o(n^{-1})$ remains valid. The solution to this problem is W and σ_κ^2 from (10). The maximum in (18) can be approximated as follows:

$$\begin{aligned}
&\frac{1}{2A(1+\varepsilon)n}\sum_{0<|\kappa|<W}\left(\left|\frac{W}{\kappa}\right|^\beta - 1\right)\left|\frac{\kappa}{W}\right|^\beta \\
=\;& \frac{1}{A(1+\varepsilon)n}\sum_{0<\kappa<W}\left(1 - \left(\frac{\kappa}{W}\right)^\beta\right) \\
=\;& \frac{1}{A(1+\varepsilon)n}\,\frac{\beta}{\beta+1}\,W\Big(1+o(1)\Big) \\
=\;& (2\beta+1)\left(\frac{(2\beta+1)(\beta+1)\pi}{\beta n}\right)^{-\frac{2\beta}{2\beta+1}} L^{\frac{1}{2\beta+1}}\,\frac{(1-\varepsilon)^{\frac{1}{2\beta+1}}}{1+\varepsilon}\Big(1+o(1)\Big) \\
=\;& n^{-\frac{2\beta}{2\beta+1}}\,\gamma(\beta,L)\,\Big(1+o(1)\Big) \qquad (19)
\end{aligned}$$

Combining (11) with (12), (15), (18) and (19), we obtain the required result:

$$\liminf_{n\to\infty}\;\inf_{\widetilde{f}_n}\;\sup_{\mathcal{S}_\beta(L)}\; n^{\frac{2\beta}{2\beta+1}}\,E_f\|\widetilde{f}_n - f\|_2^2 \;\geq\; \gamma(\beta,L) \qquad \square$$

3 Remaining Proofs

Proof of **Theorem 2** For f_θ defined in equation (8), it holds that

$$\begin{aligned}\|f_\theta^{(\beta)}\|_2 &= \frac{1}{b(\theta)}\Big\|(f_0 * g_A)^{(\beta)} + \Big(f_0 * g_A \sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta)}\Big\|_2 \\ &\leq \frac{1}{b(\theta)}\Big\|(f_0 * g_A)^{(\beta)}\Big\|_2 + \frac{1}{b(\theta)}\Big\|\Big(f_0 * g_A \sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta)}\Big\|_2\end{aligned}$$

$b(\theta)$, $\|(f_0 * g_A)^{(\beta)}\|_2^2$ and $\|(f_0 * g_A \sum \theta_k\varphi_k)^{(\beta)}\|_2^2$ are then considered one by one. Remember definition (6): $\varphi_k(x) = A^{-1/2}\cos(\pi k/A)I(|x| \leq A)$ for $k > 0$, and the same with sine for $k < 0$. Take first the normalizing constant $b(\theta)$:

$$\begin{aligned}b(\theta) &= \int_{-A}^{A} f_0 * g_A(x)\left(1 + \sum_{k\neq 0}\theta_k\varphi_k(x)\right)dx \\ &= 1 + \sum_{k\neq 0}\theta_k\int_{-A}^{A} f_0 * g_A(x)\varphi_k(x)dx \\ &= 1 + \sum_{k>0}\theta_k\int_{-A}^{A} f_0 * g_A(x)\frac{1}{\sqrt{A}}\cos\frac{k\pi x}{A}\,dx + \sum_{k<0}\theta_k\int_{-A}^{A} f_0 * g_A(x)\frac{1}{\sqrt{A}}\sin\frac{k\pi x}{A}\,dx \\ &= 1 + \sum_{k>0}\theta_k\int_{-A}^{A} f_0 * g_A(x)\frac{1}{\sqrt{A}}\cos\frac{k\pi x}{A}\,dx + 0 \\ &= 1 + \frac{1}{\sqrt{A}}\sum_{k>0}\theta_k\left[\int_{-A}^{A}\frac{1}{2A-1}\cos\frac{k\pi x}{A}\,dx - 2\int_{A-1}^{A}\left(\frac{1}{2A-1} - f_0 * g_A(x)\right)\cos\frac{k\pi x}{A}\,dx\right] \\ &= 1 + \frac{1}{\sqrt{A}}\sum_{k>0}\theta_k\left[0 - 2\int_{A-1}^{A}\left(\frac{1}{2A-1} - f_0 * g_A(x)\right)\cos\frac{k\pi x}{A}\,dx\right]\end{aligned}$$

For the second term on the right-hand side we have:

$$\begin{aligned}\frac{2}{\sqrt{A}}\Big|\sum_{k>0}\theta_k\int_{A-1}^{A}\left(\frac{1}{2A-1} - f_0 * g_A(x)\right)\cos\frac{k\pi x}{A}\,dx\Big| &\leq \frac{2}{\sqrt{A}}\sum_{k>0}|\theta_k|\int_{A-1}^{A}\frac{1}{2A-1}\Big|\cos\frac{k\pi x}{A}\Big|\,dx \\ &\leq \frac{2}{\sqrt{A}(2A-1)}\sum_{k>0}|\theta_k| \\ &\leq \frac{1}{\sqrt{A}(A-1/2)}\,A^{-2\beta+1},\end{aligned}$$

so that for $\beta > 1/2$ and A sufficiently large, it follows that

$$1 - A^{-3/2} \;\leq\; b(\theta) \;\leq\; 1 + A^{-3/2} \tag{20}$$

$(f_0 * g_A)^{(\beta)}$ is integrable in L_2. So instead of the L_2-norm of $f_0 * g_A$ in the time domain, by Parseval's equality we may as well study the L_2-norm of its Fourier transform.

$$\Big\|(f_0 * g_A)^{(\beta)}\Big\|_2^2 = \int_{-A}^{A}\Big|(f_0 * g_A)^{(\beta)}(x)\Big|^2 dx$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int \left|\omega^{\beta}\widehat{f}_0(\omega)\widehat{g}_A(\omega)\right|^2 d\omega \\
&= \frac{1}{2\pi}\int \left|\omega^{\beta}\widehat{f}_0(\omega)\frac{2\sin(A-1/2)\omega}{(2A-1)\omega}\right|^2 d\omega \\
&= \left(\frac{1}{A-1/2}\right)^2 \frac{1}{2\pi}\int \left|\omega^{\beta-1}\widehat{f}_0(\omega)\sin(A-1/2)\omega\right|^2 d\omega \\
&\leq \left(\frac{1}{A-1/2}\right)^2 \frac{1}{2\pi}\int \left|\omega^{\beta-1}\widehat{f}_0(\omega)\right|^2 d\omega \\
&\leq \left(\frac{1}{A-1/2}\right)^2 \|f_0^{(\beta-1)}\|_2^2 \qquad (21)
\end{aligned}
$$

For $\beta \geq 1$, clearly $\|f_0^{(\beta-1)}\|_2^2 < \infty$ because f_0 lies in $\mathcal{S}$. For $1/2 < \beta < 1$ we can calculate $\|f_0^{(\beta-1)}\|_2^2 = \frac{1}{2\pi}\int |\omega^{\beta-1}\widehat{f}_0(\omega)|^2 d\omega \leq \|f_0\|_2^2 + \frac{1}{\pi}|2\beta-1|^{-1}$, which is also less than infinity.
The consideration of the last and most important term $\|(f_0 * g_A \cdot \sum \theta_k\varphi_k)^{(\beta)}\|_2^2$ requires a little knowledge about fractional derivatives. For two sufficiently regular functions f and g, the Leibnitz formula takes the following form:

$$
(f\cdot g)^{(\beta)} = \sum_{i=0}^{\infty}\binom{\beta}{i} f^{(i)}\cdot g^{(\beta-i)}
$$

where $\binom{\beta}{i}$ an analogue to the binomial coefficient with natural numbers:

$$
\binom{\beta}{i} = \frac{\beta!}{i!(\beta-i)!} = \frac{\beta(\beta-1)(\beta-2)\cdots}{i!\ (\beta-i)(\beta-i-1)\cdots} = \frac{\beta\cdots(\beta-i+1)}{i!}
$$

As usual, $\binom{\beta}{0} = 1$. Now we apply this expansion to $(f_0 * g_A \cdot \sum \theta_k\varphi_k)^{(\beta)}$. Recall the definition: $\lfloor x\rfloor$ is the integer part of a real number x, and for x positive (as in our case) $\lceil x\rceil := \lfloor x\rfloor + 1$.

$$
\begin{aligned}
&\Big\|\Big(f_0 * g_A \cdot \sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta)}\Big\|_2 \\
&= \Big\|\sum_{i=0}^{\infty}\binom{\beta}{i}(f_0 * g_A)^{(i)}\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta-i)}\Big\|_2 \\
&\leq \sum_{i=0}^{\lfloor\beta\rfloor}\binom{\beta}{i}\Big\|(f_0 * g_A)^{(i)}\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta-i)}\Big\|_2 + \Big\|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}(f_0 * g_A)^{(i)}\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta-i)}\Big\|_2 \\
&\leq \sum_{i=0}^{\lfloor\beta\rfloor}\binom{\beta}{i}\max|(f_0 * g_A)^{(i)}|\,\Big\|\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta-i)}\Big\|_2 + \Big\|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}(f_0 * g_A)^{(i)}\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta-i)}\Big\|_2 \\
&\leq \sum_{i=0}^{\lfloor\beta\rfloor}\binom{\beta}{i}\max|g_A|\,\|f_0^{(i)}\|_1\Big\|\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta-i)}\Big\|_2 + \Big\|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}(f_0 * g_A)^{(i)}\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta-i)}\Big\|_2 \\
&= \sum_{i=0}^{\lfloor\beta\rfloor}\binom{\beta}{i}\frac{\|f_0^{(i)}\|_1}{2A-1}\Big\|\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta-i)}\Big\|_2 + \Big\|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}(f_0 * g_A)^{(i)}\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta-i)}\Big\|_2 \qquad (22)
\end{aligned}
$$

where $\|f_0^{(i)}\|_1$ is of course equal to 1 for $i = 0$ and finite for $i = 1,\ldots,\lfloor\beta\rfloor$. When $\beta\in\mathbb{N}$, then

$\binom{\beta}{i} = 0$ for all $i \geq \lceil\beta\rceil$, so there is no residual. In the next step we employ:

$$\Big\|\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\gamma)}\Big\|_2^2 \;=\; \sum_{k\neq 0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\gamma} \tag{23}$$

for all γ, proven in (28). Furthermore for $\Theta_A(L)$, $\sum|\theta_k| \leq A^{-2\beta+1}$ and $\sum\theta_k^2\left(\frac{k\pi}{A}\right)^{2\beta} \leq 4A^2L$ had been determined in (7). Therefrom we can show in (29) that

$$\sum_{k\neq 0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\beta-l} \;\leq\; (1+4L)A^{2-l} \qquad \text{for } \; 0 < l < 2\beta \tag{24}$$

Hence continuing at inequality number (22):

$$\begin{aligned}
&\Big\|\Big(f_0 * g_A\cdot\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta)}\Big\|_2 \\
&= \sum_{i=0}^{\lfloor\beta\rfloor}\binom{\beta}{i}\frac{\|f_0^{(i)}\|_1}{2A-1}\sqrt{\sum_{k\neq 0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2(\beta-i)}} + \Big\|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\,(f_0*g_A)^{(i)}\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta-i)}\Big\|_2 \\
&\leq \frac{1}{2A-1}\sqrt{4LA^2} + \sum_{i=1}^{\lfloor\beta\rfloor}\binom{\beta}{i}\frac{\|f_0^{(i)}\|_1}{2A-1}\sqrt{(1+4L)A^{2-i}} \\
&\quad + \Big\|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\,(f_0*g_A)^{(i)}\Big(\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta-i)}\Big\|_2
\end{aligned} \tag{25}$$

For the residual we apply Lemma 3 to our functions. It states that for functions with support in $[-A,A]$:

$$\Big\|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\,f^{(i)}\cdot g^{(\beta-i)}\Big\|_2^2 \;=\; o(A^2)\;\|\widehat{f^{(\lceil\beta\rceil)}}\|_1^2\;\|g\|_2^2$$

Setting $f := f_0 * g_A$ and $g := \sum\theta_k\varphi_k$, we proceed at inequality number (25):

$$\begin{aligned}
\Big\|\Big(f_0 * g_A\cdot\sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta)}\Big\|_2 \;&\leq\; \frac{1}{2A-1}\sqrt{4LA^2} + \sum_{i=1}^{\lfloor\beta\rfloor}\binom{\beta}{i}\frac{\|f_0^{(i)}\|_1}{2A-1}\sqrt{(1+4L)A^{2-i}} \\
&\quad + o(A)\;\Big\|\widehat{(f_0*g_A)}^{(\lceil\beta\rceil)}\Big\|_1\;\Big\|\sum_{k\neq 0}\theta_k\varphi_k\Big\|_2 \\
&= \frac{1}{2A-1}\sqrt{4LA^2} + \sum_{i=1}^{\lfloor\beta\rfloor}\binom{\beta}{i}\frac{\|f_0^{(i)}\|_1}{2A-1}\sqrt{(1+4L)A^{2-i}} \\
&\quad + o(A)\;\Big\|\widehat{(f_0*g_A)}^{(\lceil\beta\rceil)}\Big\|_1\;\sqrt{\sum_{k\neq 0}\theta_k^2}
\end{aligned} \tag{26}$$

After having derived the claim of Theorem 2, we will show in (30) that

$$\|\widehat{(f_0*g_A)}^{(\lceil\beta\rceil)}\|_1 \;\leq\; \frac{\|\widehat{f_0^{(\lfloor\beta\rfloor)}}\|_1}{A-1/2}, \qquad \text{where} \quad \|\widehat{f_0^{(\lfloor\beta\rfloor)}}\|_1 \;<\; \infty \tag{27}$$

Furthermore $\sum \theta_k^2 \le \sum |\theta_k| \le A^{-2\beta+1}$, but $-2\beta+1<0$, such that (26) can be continued as

$$\begin{aligned}
\Big\| \Big(f_0 * g_A \cdot \sum_{k\neq 0} \theta_k \varphi_k \Big)^{(\beta)} \Big\|_2 &< \frac{1}{2A-1}\sqrt{4LA^2} + \sum_{i=1}^{\lfloor\beta\rfloor} \binom{\beta}{i} \frac{\|f_0^{(i)}\|_1}{2A-1}\sqrt{(1+4L)A^{2-i}} \\
&\quad + o(A)\, \frac{\|\widehat{f_0^{(\lfloor\beta\rfloor)}}\|_1}{A-1/2}\sqrt{A^{-2\beta+1}} \\
&= \frac{1}{2A-1}\sqrt{4LA^2} + \sum_{i=1}^{\lfloor\beta\rfloor} \binom{\beta}{i} \frac{\|f_0^{(i)}\|_1}{2A-1}\sqrt{(1+4L)A^{2-i}} \\
&\quad + o(A)\, \frac{\frac{1}{2\pi}\|\widehat{f_0^{(\lfloor\beta\rfloor)}}\|_1}{A-1/2}\, o(1) \\
&= \sqrt{L}\Big(1+o(1)\Big) + O\left(A^{-1/2}\right) + o(1)
\end{aligned}$$

This result in connection with (20) and (21) completes Theorem 2:

$$\begin{aligned}
\|f_\theta^{(\beta)}\|_2 &\le \frac{1}{b(\theta)}\Big\|(f_0 * g_A)^{(\beta)}\Big\|_2 + \frac{1}{b(\theta)}\Big\|\Big(f_0 * g_A \sum_{k\neq 0}\theta_k\varphi_k\Big)^{(\beta)}\Big\|_2 \\
&= O\left(A^{-1}\right)\|f_0^{(\beta-1)}\|_2 + \sqrt{L}\Big(1+o(1)\Big) \\
&= \sqrt{L} + o(1)
\end{aligned}$$

Still we are left to prove the intermediate assertions (23), (24) and (27).
As an exception to the ordinary case, sine and cosine enjoy an easy to calculate fractional derivative: $\sin^{(\gamma)}(ax) = a^\gamma \sin\,(ax+\gamma\pi/2)$ and the like for cosine (Samko [9], p. 174). Obviously, the orthogonality between our functions φ_k is preserved through derivation.

$$\begin{aligned}
&\int \Big(\sum_{k\neq 0}\theta_k \varphi_k^{(\gamma)}(x)\Big)^2 dx \\
=& \int \sum_{k\neq 0}\theta_k^2\, \varphi_k^{(\gamma)}(x)^2 dx \\
=& \int_{-A}^{A} \sum_{k>0}\theta_k^2 \,\frac{1}{A}\left(\frac{k\pi}{A}\right)^{2\gamma}\cos^2\left(\frac{k\pi x}{A}+\frac{\gamma\pi}{2}\right) + \sum_{k<0}\theta_k^2\,\frac{1}{A}\left(\frac{k\pi}{A}\right)^{2\gamma}\sin^2\left(\frac{k\pi x}{A}+\frac{\gamma\pi}{2}\right)\,dx \\
=& \sum_{k>0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\gamma}\frac{1}{A}\int_{-A}^{A}\cos^2\left(\frac{k\pi x}{A}+\frac{\gamma\pi}{2}\right) + \sum_{k<0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\gamma}\frac{1}{A}\int_{-A}^{A}\sin^2\left(\frac{k\pi x}{A}+\frac{\gamma\pi}{2}\right)\,dx \\
=& \sum_{k>0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\gamma}\frac{1}{A}\int_{-A}^{A}\cos^2\frac{k\pi x}{A}\,dx + \sum_{k<0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\gamma}\frac{1}{A}\int_{-A}^{A}\sin^2\frac{k\pi x}{A}\,dx \\
=& \sum_{k>0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\gamma}\frac{1}{A}\,A + \sum_{k<0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\gamma}\frac{1}{A}\,A \\
=& \sum_{k\neq 0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\gamma} \qquad (28)
\end{aligned}$$

Referring to step (24), $0 < l < 2\beta$:

$$\begin{aligned}
\sum_{k\neq 0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\beta-l} &= \sum_{0\neq|k|\leq A^2/\pi}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\beta-l} + \sum_{|k|>A^2/\pi}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\beta}\left(\frac{k\pi}{A}\right)^{-l}\\
&\leq A^{2\beta-l}\sum_{0\neq|k|\leq A^2/\pi}\theta_k^2 + A^{-l}\sum_{|k|>A^2/\pi}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\beta}\\
&\leq A^{2\beta-l}\sum_{k\neq 0}|\theta_k| + A^{-l}\sum_{k\neq 0}\theta_k^2\left(\frac{k\pi}{A}\right)^{2\beta}\\
&\leq A^{2\beta-l}\cdot A^{-2\beta+1} + A^{-l}\cdot 4LA^2\\
&= A^{1-l} + 4LA^{2-l}\\
&\leq (1+4L)A^{2-l} \qquad (29)
\end{aligned}$$

Proof of (27):

$$\begin{aligned}
\|\widehat{(f_0 * g_A)^{(\lceil\beta\rceil)}}\|_1 &= \int\left|\omega^{\lceil\beta\rceil}\widehat{f}_0(\omega)\widehat{g}_A(\omega)\right|d\omega\\
&= \int\left|\omega^{\lceil\beta\rceil}\widehat{f}_0(\omega)\,\frac{2\sin(A-1/2)\omega}{(2A-1)\omega}\right|d\omega\\
&= \frac{1}{A-1/2}\int\left|\omega^{\lfloor\beta\rfloor}\widehat{f}_0(\omega)\,\sin(A-1/2)\omega\right|d\omega\\
&\leq \frac{\|\widehat{f_0^{(\lfloor\beta\rfloor)}}\|_1}{A-1/2} \qquad (30)
\end{aligned}$$

$\|\widehat{f_0^{(\lfloor\beta\rfloor)}}\|_1$ exists, because we chose $f_0 \in \mathcal{S}$. This concludes the proof of Theorem 2. □

Lemma 3 *For functions f and g, which are both L_2-integrable, sufficiently regular and have support in $[-A, A]$, it holds that*

$$\left\|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\,f^{(i)}\cdot g^{(\beta-i)}\right\|_2^2 = o(A^2)\,\left\|\widehat{f^{\lceil\beta\rceil}}\right\|_1^2\cdot\|g\|_2^2$$

Proof This proof takes a detour via Fourier coefficients. Begin with the following discussion: The power function is an analytical function. We may thus for instance expand $(\frac{\kappa\pi}{A})^\beta$ into an infinite Taylor series at point $\frac{(\kappa-\lambda)\pi}{A}$.

$$\left(\frac{\kappa\pi}{A}\right)^\beta = \sum_{i=0}^{\infty}\binom{\beta}{i}\left(\frac{\lambda\pi}{A}\right)^i\left(\frac{(\kappa-\lambda)\pi}{A}\right)^{\beta-i}$$

We cut the Tailor expansion of $\left(\frac{\kappa\pi}{A}\right)^\beta$ after $\lfloor\beta\rfloor$ and bound the residual.

$$\begin{aligned}
&\left|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\left(\frac{\lambda\pi}{A}\right)^i\left(\frac{(\kappa-\lambda)\pi}{A}\right)^{\beta-i}\right|\\
=\ &\left|\sum_{i=0}^{\infty}\binom{\beta}{\lceil\beta\rceil+i}\left(\frac{\lambda\pi}{A}\right)^{\lceil\beta\rceil+i}\left(\frac{(\kappa-\lambda)\pi}{A}\right)^{\beta-\lceil\beta\rceil-i}\right|
\end{aligned}$$

$$
\begin{aligned}
&= \Big|\sum_{i=0}^{\infty} \frac{\beta\cdots(\beta-\lceil\beta\rceil-i+1)}{(\lceil\beta\rceil+i)!}\Big(\frac{\lambda\pi}{A}\Big)^{\lceil\beta\rceil+i}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)^{\beta-\lceil\beta\rceil-i}\Big| \\
&= \Big|\frac{\beta\cdots(\beta-\lceil\beta\rceil+1)}{\lceil\beta\rceil!}\ \Big(\frac{\lambda\pi}{A}\Big)^{\lceil\beta\rceil}\sum_{i=0}^{\infty}\frac{(\beta-\lceil\beta\rceil)\cdots(\beta-\lceil\beta\rceil-i+1)}{(\lceil\beta\rceil+i)\cdots(\lceil\beta\rceil+1)}\ \Big(\frac{\lambda\pi}{A}\Big)^{i}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)^{\beta-\lceil\beta\rceil-i}\Big| \\
&\le \binom{\beta}{\lceil\beta\rceil}\ \Big|\frac{\lambda\pi}{A}\Big|^{\lceil\beta\rceil}\sum_{i=0}^{\infty}\Big|\frac{(\beta-\lceil\beta\rceil)\cdots(\beta-\lceil\beta\rceil-i+1)}{i!}\ \Big(\frac{\lambda\pi}{A}\Big)^{i}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)^{\beta-\lceil\beta\rceil-i}\Big|
\end{aligned}
$$

The product $(\beta-\lceil\beta\rceil)\cdots(\beta-\lceil\beta\rceil-i+1)$ consists of i factors, which are all negative. We can write $|(\beta-\lceil\beta\rceil)\cdots(\beta-\lceil\beta\rceil-i+1)| = (-1)^i(\beta-\lceil\beta\rceil)\cdots(\beta-\lceil\beta\rceil-i+1)$, such that

$$
\begin{aligned}
&\Big|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\ \Big(\frac{\lambda\pi}{A}\Big)^{i}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)^{\beta-i}\Big| \\
&\le \binom{\beta}{\lceil\beta\rceil}\ \Big|\frac{\lambda\pi}{A}\Big|^{\lceil\beta\rceil}\sum_{i=0}^{\infty}(-1)^i\frac{(\beta-\lceil\beta\rceil)\cdots(\beta-\lceil\beta\rceil-i+1)}{i!}\ \Big|\frac{\lambda\pi}{A}\Big|^{i}\Big|\frac{(\kappa-\lambda)\pi}{A}\Big|^{\beta-\lceil\beta\rceil-i} \\
&= \binom{\beta}{\lceil\beta\rceil}\ \Big|\frac{\lambda\pi}{A}\Big|^{\lceil\beta\rceil}\sum_{i=0}^{\infty}\binom{\beta-\lceil\beta\rceil}{i}\ \Big(\frac{-|\lambda|\pi}{A}\Big)^{i}\Big(\frac{|\kappa-\lambda|\pi}{A}\Big)^{\beta-\lceil\beta\rceil-i} \\
&= \binom{\beta}{\lceil\beta\rceil}\ \Big|\frac{\lambda\pi}{A}\Big|^{\lceil\beta\rceil}\Big(\frac{(|\kappa-\lambda|-|\lambda|)\pi}{A}\Big)^{\beta-\lceil\beta\rceil}
\end{aligned}
$$

Since we know that $-1<\beta-\lceil\beta\rceil<0$, we can approximte $(\frac{(|\kappa-\lambda|-|\lambda|)\pi}{A})^{\beta-\lceil\beta\rceil} = O(A^{\beta-\lceil\beta\rceil}) = o(A)$. Now we expand the tail of our Leibnitz formula into a Fourier series and plug in the bound of the Taylor series:

$$
\begin{aligned}
&\Big\|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\ f^{(i)}\cdot g^{(\beta-i)}\Big\|_2^2 \\
&= \frac{1}{2A}\sum_{\kappa\in\mathbb{Z}}\left(\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\ \widehat{f^{(i)}}*\widehat{g^{(\beta-i)}}\Big(\frac{\kappa\pi}{A}\Big)\right)^2 \\
&= \frac{1}{2A}\sum_{\kappa\in\mathbb{Z}}\left(\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\ \frac{1}{2A}\sum_{\lambda\in\mathbb{Z}}\widehat{f^{(i)}}\Big(\frac{\lambda\pi}{A}\Big)\widehat{g^{(\beta-i)}}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)\right)^2 \\
&= \frac{1}{2A}\sum_{\kappa\in\mathbb{Z}}\left(\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\ \frac{1}{2A}\sum_{\lambda\in\mathbb{Z}}\Big(\frac{\lambda\pi}{A}\Big)^{i}\widehat{f}\Big(\frac{\lambda\pi}{A}\Big)\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)^{\beta-i}\widehat{g}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)\right)^2 \\
&= \frac{1}{2A}\sum_{\kappa\in\mathbb{Z}}\left(\frac{1}{2A}\sum_{\lambda\in\mathbb{Z}}\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\ \Big(\frac{\lambda\pi}{A}\Big)^{i}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)^{\beta-i}\widehat{f}\Big(\frac{\lambda\pi}{A}\Big)\ \widehat{g}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)\right)^2 \\
&\le \frac{1}{2A}\sum_{\kappa\in\mathbb{Z}}\left(\frac{1}{2A}\sum_{\lambda\in\mathbb{Z}}o(A)\binom{\beta}{\lceil\beta\rceil}\ \Big|\frac{\lambda\pi}{A}\Big|^{\lceil\beta\rceil}\Big|\widehat{f}\Big(\frac{\lambda\pi}{A}\Big)\Big|\cdot\Big|\widehat{g}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)\Big|\right)^2 \\
&= o(A^2)\ \frac{1}{2A}\sum_{\kappa\in\mathbb{Z}}\frac{1}{2A}\sum_{\lambda\in\mathbb{Z}}\binom{\beta}{\lceil\beta\rceil}\ \Big|\frac{\lambda\pi}{A}\Big|^{\lceil\beta\rceil}\Big|\widehat{f}\Big(\frac{\lambda\pi}{A}\Big)\Big|\cdot\Big|\widehat{g}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)\Big|
\end{aligned}
$$

$$\times \frac{1}{2A}\sum_{\mu\in\mathbb{Z}}\binom{\beta}{\lceil\beta\rceil}\ \Big|\frac{\mu\pi}{A}\Big|^{\lceil\beta\rceil}\Big|\widehat{f}\Big(\frac{\mu\pi}{A}\Big)\Big|\cdot\Big|\widehat{g}\Big(\frac{(\kappa-\mu)\pi}{A}\Big)\Big|$$

$$= \ o(A^2)\ \binom{\beta}{\lceil\beta\rceil}^2\ \frac{1}{2A}\sum_{\lambda\in\mathbb{Z}}\Big|\frac{\lambda\pi}{A}\Big|^{\lceil\beta\rceil}\Big|\widehat{f}\Big(\frac{\lambda\pi}{A}\Big)\Big|\ \ \frac{1}{2A}\sum_{\mu\in\mathbb{Z}}\Big|\frac{\mu\pi}{A}\Big|^{\lceil\beta\rceil}\Big|\widehat{f}\Big(\frac{\mu\pi}{A}\Big)\Big|$$

$$\times \frac{1}{2A}\sum_{\kappa\in\mathbb{Z}}\Big|\widehat{g}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)\Big|\cdot\Big|\widehat{g}\Big(\frac{(\kappa-\mu)\pi}{A}\Big)\Big|$$

$$\leq \ o(A^2)\ \binom{\beta}{\lceil\beta\rceil}^2\ \frac{1}{2A}\sum_{\lambda\in\mathbb{Z}}\Big|\Big(\frac{\lambda\pi}{A}\Big)^{\lceil\beta\rceil}\widehat{f}\Big(\frac{\lambda\pi}{A}\Big)\Big|\ \ \frac{1}{2A}\sum_{\mu\in\mathbb{Z}}\Big|\Big(\frac{\mu\pi}{A}\Big)^{\lceil\beta\rceil}\widehat{f}\Big(\frac{\mu\pi}{A}\Big)\Big|$$

$$\times\sqrt{\frac{1}{2A}\sum_{\kappa\in\mathbb{Z}}\Big|\widehat{g}\Big(\frac{(\kappa-\lambda)\pi}{A}\Big)\Big|^2}\sqrt{\frac{1}{2A}\sum_{\kappa\in\mathbb{Z}}\Big|\widehat{g}\Big(\frac{(\kappa-\mu)\pi}{A}\Big)\Big|^2}$$

$$= \ o(A^2)\ \binom{\beta}{\lceil\beta\rceil}^2\ \Big(\frac{1}{2A}\sum_{\lambda\in\mathbb{Z}}\Big|\Big(\frac{\lambda\pi}{A}\Big)^{\lceil\beta\rceil}\widehat{f}\Big(\frac{\lambda\pi}{A}\Big)\Big|\Big)^2\cdot\|g\|_2^2$$

For growing A, the Fourier expansion approaches the Fourier transform, and hence

$$\Big\|\sum_{i=\lceil\beta\rceil}^{\infty}\binom{\beta}{i}\ f^{(i)}\cdot g^{(\beta-i)}\Big\|_2^2 \tag{31}$$

$$= \ o(A^2)\ \binom{\beta}{\lceil\beta\rceil}^2\ \Big(\frac{1}{2\pi}\int\Big|\omega^{\lceil\beta\rceil}\widehat{f}(\omega)\Big|d\omega\Big(1+o(1)\Big)\Big)^2\cdot\|g\|_2^2$$

$$= \ o(A^2)\ \Big\|\widehat{f^{\lceil\beta\rceil}}\Big\|_1^2\cdot\|g\|_2^2$$

which is the statement of Lemma 3. □

Proof of **Lemma 2** We start with

$$I_{f_\theta}(\theta_\kappa)$$

$$= \int_{-A}^{A}\frac{(\partial f_\theta(x)/\partial\theta_\kappa)^2}{f_\theta(x)}\ dx$$

$$= \frac{1}{b^2(\theta)}\int_{-A}^{A}\frac{1}{f_\theta(x)}\Big[-f_\theta(x)\int_{-A}^{A}f_0*g_A(y)\varphi_\kappa(y)\ dy+f_0*g_A(x)\varphi_\kappa(x)\Big]^2\ dx$$

$$= \frac{1}{b^2(\theta)}\int_{-A}^{A}\Big[f_\theta(x)\Big(\int_{-A}^{A}f_0*g_A(y)\varphi_\kappa(y)\ dy\Big)^2-2\int_{-A}^{A}f_0*g_A(y)\varphi_\kappa(y)\ dy\ f_0*g_A(x)\varphi_\kappa(x)$$

$$+\frac{1}{f_\theta(x)}\Big(f_0*g_A(x)\varphi_\kappa(x)\Big)^2\Big]\ dx$$

$$= -\frac{1}{b^2(\theta)}\Big[\int_{-A}^{A}f_0*g_A(y)\varphi_\kappa(y)\ dy\Big]^2+\frac{1}{b^2(\theta)}\int_{-A}^{A}\frac{\Big(f_0*g_A(x)\Big)^2\varphi_\kappa^2(x)}{\frac{1}{b(\theta)}f_0*g_A(x)\Big(1+\sum_{\lambda\neq0}\theta_\lambda\varphi_\lambda(x)\Big)}dx$$

$$= -\frac{1}{b^2(\theta)}\Big[\int_{-A}^{A}f_0*g_A(y)\varphi_\kappa(y)\ dy\Big]^2+\frac{1}{b(\theta)}\int_{-A}^{A}\frac{f_0*g_A(x)\varphi_\kappa^2(x)}{1+\sum_{\lambda\neq0}\theta_\lambda\varphi_\lambda(x)}\ dx$$

$$= \; -\frac{1}{b^2(\theta)} \left[\int_{-A}^{A} \frac{\varphi_\kappa(x)}{2A-1}\, dx - 2\int_{A-1}^{A} \left(\frac{1}{2A-1} - f_0 * g_A(x) \right) \varphi_\kappa(x)\, dx \right]^2$$

$$+ \; \frac{1}{b(\theta)} \left[\int_{-A}^{A} \frac{\frac{1}{2A-1}\, \varphi_\kappa^2(x)}{1+\sum\limits_{\lambda\neq 0} \theta_\lambda \varphi_\lambda(x)}\, dx - 2\int_{A-1}^{A} \frac{\left(\frac{1}{2A-1} - f_0 * g_A(x)\right) \varphi_\kappa^2(x)}{1+\sum\limits_{\lambda\neq 0} \theta_\lambda \varphi_\lambda(x)}\, dx \right]$$

The leading term is $\int_{-A}^{A} \frac{\frac{1}{2A-1}\, \varphi_\kappa^2(x)}{1+\sum \theta_\lambda\varphi_\lambda(x)}\, dx$. Due to $|\sum \theta_\lambda\varphi_\lambda(x)| \leq A^{-1/2}\sum |\theta_\lambda| \leq A^{-1/2}\cdot A^{-2\beta+1} < A^{-1/2}$, we know it lies in the interval $((2A-1)^{-1}(1+A^{-1/2})^{-1}, (2A-1)^{-1}(1-A^{-1/2})^{-1})$. Moreover from (20) we have $1 - A^{-3/2} \leq b(\theta) \leq 1 + A^{-3/2}$. For $A \longrightarrow \infty$ we obtain:

$$\begin{aligned} I_{f_\theta}(\theta_\kappa) &= \frac{1}{b^2(\theta)} \left[0 + O\left(\frac{1}{(A-1/2)\sqrt{A}} \right) \right]^2 + \frac{1}{b(\theta)} \left[\frac{1+o(1)}{2A-1} + O\left(\frac{1+o(1)}{(A-1/2)A} \right) \right] \\ &= \Big(1+o(1)\Big) O\left(A^{-3}\right) + \Big(1+o(1)\Big) \left[\frac{1+o(1)}{2A-1} + O\left(A^{-2}\right) \right] \\ &= \frac{1+o(1)}{2A} \end{aligned}$$

$\mathrm{Re}\widehat{f_\theta}\left(\frac{\kappa\pi}{A}\right)$ can be expressed as $A^{1/2}\int f_\theta(x)\varphi_{|\kappa|}(x)dx$, yielding

$$\begin{aligned} &\frac{\partial\, \mathrm{Re}\widehat{f_\theta}\left(\frac{\kappa\pi}{A}\right)}{\partial\theta_{|\kappa|}} \\ &= \frac{\partial}{\partial\theta_{|\kappa|}} \sqrt{A} \int_{-A}^{A} f_\theta(x)\varphi_{|\kappa|}(x)dx \\ &= \frac{\sqrt{A}}{b(\theta)} \left[-\int_{-A}^{A} f_\theta(x) \int_{-A}^{A} f_0 * g_A(y)\varphi_{|\kappa|}(y)dy\ \varphi_{|\kappa|}(x)dx + \int_{-A}^{A} f_0 * g_A(x)\varphi_{|\kappa|}^2(x)dx \right] \\ &= \frac{\sqrt{A}}{b(\theta)} \left[-\frac{\mathrm{Re}\widehat{f_\theta}\left(\frac{\kappa\pi}{A}\right)}{\sqrt{A}} \int_{-A}^{A} f_0 * g_A(y)\varphi_{|\kappa|}(y)dy + \int_{-A}^{A} f_0 * g_A(x)\varphi_{|\kappa|}^2(x)dx \right] \\ &= \frac{\mathrm{Re}\widehat{f_\theta}\left(\frac{\kappa\pi}{A}\right)}{b(\theta)} \left[-\int_{-A}^{A} \frac{1}{2A-1}\ \varphi_{|\kappa|}(y)dy + 2\int_{A-1}^{A} \left(\frac{1}{2A-1} - f_0 * g_A(y) \right) \varphi_{|\kappa|}(y)dy \right] \\ &\quad + \frac{\sqrt{A}}{b(\theta)} \left[\int_{-A}^{A} \frac{1}{2A-1}\ \varphi_{|\kappa|}^2(x)dx - 2\int_{A-1}^{A} \left(\frac{1}{2A-1} - f_0 * g_A(x) \right) \varphi_{|\kappa|}^2(x)dx \right] \\ &= \frac{\mathrm{Re}\widehat{f_\theta}\left(\frac{\kappa\pi}{A}\right)}{1+o(1)} \left[0 + O\left(\frac{1}{(A-1/2)\sqrt{A}} \right) \right] + \frac{\sqrt{A}}{1+o(1)} \left[\frac{1}{2A-1} + O\left(\frac{1}{(A-1/2)A} \right) \right] \\ &= \frac{1+o(1)}{2\sqrt{A}} \end{aligned}$$

A similar result is obtained for $\mathrm{Im}\widehat{f_\theta}\left(\frac{\kappa\pi}{A}\right)$, whereby $\int f_0 * g_A(y)\varphi_{-|\kappa|}(y)dy = 0$ simplifies the task, because the sine function is anti-symmetric. □

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References

[1] Dalelane, C. (2005). Data driven kernel choice in non-parametric curve estimation. *Ph.D. dissertation.* TU Braunschweig. (available at http://opus.tu-bs.de/opus/volltexte/2005/659/)

[2] Efroimovich, S.Yu. and Pinsker, M.S. (1983). Estimation of square-integrable probability density of a random variable. *Probl. Inf. Transm.* **18**, 175-189.

[3] Gill, R.D. and Levit, B.Y. (1995). Applications of the van Trees inequality: A Bayesian Cramér-Rao bound. *Bernoulli* **1** No.1-2, 59-79.

[4] Golubev, G.K. (1991). LAN in problems of nonparametric estimation of functions and lower bounds for quadratic risk. *Theory Probab. Appl.* **36** No.1, 152-157.

[5] Golubev, G.K. (1992). Nonparametric estimation of smooth probability densities in L_2. *Probl. Inf. Transm.* **28** No.1, 44-54.

[6] Golubev, G.K. and Levit, B.Y. (1996). On the second order minimax estimation of distribution functions. *Math. Methods Stat.* **5** No.1, 1-31.

[7] Rigollet, P. (2004). Adaptive density estimation using Stein's blockwise method. Preprint PMA-913 (available at www.proba.jussieu.fr)

[8] Samarov, A.M. (1976). Minimax bound on the risk of nonparametric density estimates. *Probl. Inf. Transm.* **12** No. 3, 108-111

[9] Samko, G. (1987). *Fractional integrals and derivatives: theory and applications.* Gordon and Breach Science Publishers. Yverdon.

[10] Schipper, M. (1996). Optimal rates and constants in L_2-minimax estimation of probability density functions. *Math. Methods Stat.* **5** No.3, 253-274.