On processes which are infinitely divisible with respect to time

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20th April 2005

Abstract

The aim of this short note is to present the notion of IDT processes, which is a wide generalization of Lévy processes obtained from a modified infinitely divisible property. Special attention is put on a number of examples, in order to clarify how much the IDT processes either differ from, or resemble to, Lévy processes.

Keywords : *Infinitely divisible law, Lévy processes, Gaussian processes.*
AMS 2000 subject classification :*60G48, 60G51, 60G44, 60G10

1 Introduction

Motivated by some one-to-one map from the set of infinitely divisible laws, or rather from the set of their Lévy measures onto itself which was noticed by Barndorff-Nielsen and Thornbjørnsen [BNT02a], it appears to be of some interest to consider the class of stochastic processes $(X_t; t \geq 0)$ which enjoy the following property :

$$\forall n \in \mathbb{N}^*, \ (X_{nt}; t \geq 0) \quad \stackrel{(d)}{=} \quad (X_t^{(1)} + ... + X_t^{(n)}; t \geq 0) \tag{1}$$

where $X^{(1)}$,..., $X^{(n)}$ are independent copies of X.
We shall call such a process an IDT process (which stands for **I**nfinitely **D**ivisible with respect to **T**ime). A simple remark, whose proof is left to the reader, is

Proposition 1.1 *:*
Any Lévy process is IDT

However, there are many other processes than Lévy processes which are IDT, and the purpose of this paper is to exhibit large sets of IDT processes. More precisely, in Section 2, we give some general examples of IDT processes; in Section 3, we characterize the Gaussian processes which are IDT. In Section 4, we focus on the resemblance between a given IDT process and the Lévy process with the same one-dimensional marginals at fixed times. Section 5 is devoted to a description of IDT processes as particular path-valued infinitely divisible variables.

2 Some general examples of IDT processes

The aim of this section is to provide as many examples as possible of IDT processes which are not Lévy processes.

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(2.1) Here is a first simple example; given a strictly stable random variable S_α with parameter α, the process X defined by

$$X_t = t^{1/\alpha} S_\alpha,\ t \in \mathbb{R}^+,$$

is an IDT process. This simply relies on the fact that $n^{1/\alpha} S_\alpha$ is equal in law to the sum of n independent copies of S_α.

(2.2) A more interesting set of examples is obtained by transforming a Lévy process, more generally an IDT process, by integration in time with respect to a measure. Namely, if X is an IDT process and μ a measure on $\mathbb{R}^+$ such that

$$X_t^{(\mu)} = \int \mu(du) X_{ut},\ t \in \mathbb{R}^+,$$

is well defined, then $X^{(\mu)}$ is again an IDT process: this property follows easily from (1).
Some particular cases of interest may be obtained with $\mu(du) = \frac{du}{u} 1_{[a,b]}(u)$ for any $0 < a < b < \infty$. In such a case, $X_t^{(\mu)} = \int_{at}^{bt} \frac{du}{u} X_u$; in fact, this is the original example in [BNT02a] , which motivated us to study IDT processes.
Thus, we note that this procedure allows to construct some quite regular IDT processes, i.e. if X is a Lévy process,then $\int_0^t \frac{du}{u} X_u$ is an absolutely continuous IDT process.

(2.3) Some similar examples can be provided using only the jumps of an IDT process X.
Indeed, if $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ satisfies $f(., 0) \equiv 0$, then

$$X_t^f = \sum_{v \geq 0} f(v, (\Delta X)_{vt})$$

is an IDT process (this easily follows from the fact that, if $(X_t; t \geq 0)$ is an IDT process then the same holds for $((\Delta X)_t; t \geq 0)$), provided it exists.

3 IDT Gaussian processes

One may wonder which among centered Gaussian processes $(G_t; t \geq 0)$ (which, for simplicity, we assume to be centered) are IDT. In order to characterize IDT Gaussian processes, we first recall the Lamperti transformation concerning self-similar processes (see the original paper [Lam62] or [EM02] Theorem 1.5.1 p11) :

Lemma 3.1 *:*
A process $(X_t;\ t \geq 0)$ enjoys the scaling property of order h, i.e.

$$\forall \alpha > 0, \qquad (X_{\alpha t};\ t \geq 0) \overset{(d)}{=} (\alpha^h X_t;\ t \geq 0),$$

if, and only if, its Lamperti transform $(\tilde{X}_y := e^{-hy} X_{e^y};\ y \in \mathbb{R})$ is a strictly stationary process.

The next proposition emphasizes again how much more general IDT processes are than Lévy processes (since the only Gaussian Lévy processes are Brownian motions with drifts).

Proposition 3.2 *:*
Let $(G_t;\ t \geq 0)$ be a centered Gaussian process, which is assumed to be continuous in probability (which is equivalent to its covariance function c being continuous). Then the following properties are equivalent :

1. *$(G_t;\ t \geq 0)$ is an IDT process.*

2. *The covariance function* $c(s,t) := \mathbb{E}[G_s G_t]$, $0 \le s \le t$, *satisfies*

$$\forall \alpha > 0, \qquad c(\alpha s, \alpha t) = \alpha c(s,t), \qquad 0 \le s \le t$$

3. *The process* $(G_t;\ t \ge 0)$ *satisfies the "Brownian scaling property", namely*

$$\forall \alpha > 0, \qquad (G_{\alpha t}; t \ge 0) \overset{(d)}{=} (\sqrt{\alpha} G_t; t \ge 0)$$

4. *The process* $(\tilde{G}_y := e^{-y/2} G_{e^y};\ y \in \mathbb{R})$ *is stationary.*

5. *The covariance fonction* $\tilde{c}(y,z) := \mathbb{E}[\tilde{G}_y \tilde{G}_z]$, $y, z \in \mathbb{R}$, *is of the form*

$$\tilde{c}(y,z) = \int \mu(du) e^{iu|y-z|}, \qquad y, z \in \mathbb{R}$$

   where μ *is a positive, finite, symmetric measure on* $\mathbb{R}$.

Then, under these equivalent conditions, the covariance function c *of* $(G_t;\ t \ge 0)$ *is given by*

$$c(s,t) = \sqrt{st} \int \mu(da) e^{ia|\ln(\frac{s}{t})|}$$

Remark 3.3 *(Private communication from F. Hirsch to M. Yor):*
The conditions found in Proposition 3.2 are also equivalent to the positivity of the quadratic form Q *defined as*

$$Q(f) = \int_0^1 du\, f(u) \int_0^1 ds\, f(su) c(1,s), \qquad f \in L^2([0,1])$$

Proof :

(1 ⇔ 2) The IDT property (1) is equivalent to

$$\forall n \in \mathbb{N}, \qquad c(ns, nt) = nc(s,t), \qquad 0 \le s \le t$$

and also to

$$\forall q \in \mathbb{Q}_+, \qquad c(qs, qt) = qc(s,t), \qquad 0 \le s \le t$$

The result is then obtained using the density of $\mathbb{Q}_+$ in $\mathbb{R}_+$ and the continuity of c (deduced from the continuity in probability, hence in L^2, of $(G_t;\ t \ge 0)$).

(2 ⇔ 3) Simple, since the law of a centered Gaussian process is determined by its covariance function.

(3 ⇔ 4) Lamperti's transformation (Lemma 3.1) of order $h = 1/2$.

(4 ⇔ 5) Bochner's theorem for definite positive functions.

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Example 3.4 *:*
Let $\varphi \in L^2(\mathbb{R}^+, du)$*; then the process* $G^{(\varphi)}$ *defined by*

$$\forall t > 0,\ G_t^{(\varphi)} = \int_0^\infty \varphi\left(\frac{u}{t}\right) dB_u, \qquad \text{and} \qquad G_0^{(\varphi)} = 0 \tag{2}$$

is an IDT process.
Indeed, it suffices to note that $G^{(\varphi)}$ *is a well defined Gaussian process with covariance function*

$$c(s,t) = s \int_0^\infty dv\ \varphi\left(\frac{s}{t} v\right) \varphi(v) = \sqrt{st} \int \mu(dy) e^{iy|\ln(\frac{s}{t})|}, \qquad s,\ t > 0$$

with $\int \mu(dy)e^{iyx} = e^{-|x|/2}\int_0^\infty dv\, \varphi(e^{-|x|}v)\varphi(v),\ x \in \mathbb{R}$.
In particular, we can compute μ *in the following simple cases :*

$\varphi(x)$	$\mu(dy)$
$x^{-\alpha}1_{x\geq 1}$ *with* $\alpha > 1/2$; $x^{-\alpha}1_{x\leq 1}$ *with* $\alpha < 1/2$	$\dfrac{1}{2\pi(y^2+(1/2-\alpha)^2)}dy$
$(1-x)^{-\alpha}1_{x\leq 1}$ *with* $\alpha < 1/2$	$\sum_{n=0}^{\infty} \dfrac{\Gamma(1-\alpha)}{\Gamma(\alpha)} \dfrac{\Gamma(n+\alpha)}{\Gamma(n+2-\alpha)} \dfrac{n+1/2}{\pi(y^2+(n+1/2)^2)}\, dy$

More generally, if $\varphi(x) = \dfrac{1}{x^\alpha}\left(\sum_{n=0}^{\infty} a_n x^n\right)$ *with some sequence of positive numbers* $(a_n;\ n \geq 0)$, *then*

$$\mu(dy) = \sum_{n=0}^{\infty} a_n \int dv\ \varphi(v)\ v^{n-\alpha}\ \frac{n-\alpha+1/2}{\pi(y^2+(n-\alpha+1/2)^2)}\, dy$$

Therefore, the quantity $\int \mu(dy)e^{iy\sqrt{2\lambda}}$ *may be interpreted as the Laplace transform of a positive random variable* A *and* μ *is the law of* β_A, *with* $(\beta_t;\ t \geq 0)$ *a standard Brownian motion.*

4 IDT and Lévy processes with the same marginals

First we note that the laws of finite dimensional marginals of an IDT process X are infinitely divisible. In particular, for any fixed t, the law of X_t is infinitely divisible.

Proposition 4.1 *:*
Let $(X_t; t \geq 0)$ *be a right-continuous IDT process. Denote by* $(\tilde{X}_t; t \geq 0)$ *the unique Lévy process such that*

$$X_1 \overset{(d)}{=} \tilde{X}_1$$

Then $(X_t; t \geq 0)$ *and* $(\tilde{X}_t; t \geq 0)$ *have the same one-dimensional marginals,*
i.e. for any fixed $t \geq 0$,

$$X_t \overset{(d)}{=} \tilde{X}_t \tag{3}$$

Proof :
Since, for any $k \in \mathbb{N}$, $X_k \overset{(d)}{=} X_1^{(1)} + ... + X_1^{(k)}$, it follows that $X_k \overset{(d)}{=} \tilde{X}_k$.
Identity (3) can then be obtained for any rational time since the n-th power of the characteristic functions of $X_{k/n}$ and of $\tilde{X}_{k/n}$ are equal and non-vanishing (because the laws of these variables are infinitely divisible).
We conclude by using the right-continuity of paths of both X and $\tilde{X}$.

■

Remark 4.2 *:*
Proposition 4.1 just states that IDT processes "mimick" Lévy processes in the sense of [Gyö86] who exhibits other examples of various processes with the same 1-dimensional marginals. Further studies in this direction can be found in [MY02] or [FWY00].

We now illustrate Proposition 4.1 by computing the Lévy measure of $X_1^{(\mu)} = \int_0^\infty d\mu(u) X_u$, where X is a Lévy process (hence $X^{(\mu)}$ is an IDT process; see (2.2)), or equivalently of the Lévy process $(\tilde{X}_t^{(\mu)}, t \geq 0)$ related to $X^{(\mu)}$ by (3).

Proposition 4.3 *:*
Let X be a Lévy process and ν its Lévy measure. Let $\nu^{(\mu)}$ be the Lévy measure of the infinitely divisible variable $X_1^{(\mu)} = \int_0^\infty \mu(du) X_u$ for a "good" measure μ. Then for any non-negative Borel function f, we have

$$\int \nu^{(\mu)}(dy) f(y) \quad = \quad \int_0^\infty dh \int \nu(dx) f(\mu([h,\infty))x) \tag{4}$$

Proof :
First, integration by part implies

$$\int_0^\infty \mu(du) X_u = \int_0^\infty \mu([h,\infty)) dX_h$$

so that

$$\mathbb{E}\left[\exp\left(-\lambda \int_0^\infty \mu(du) X_u\right)\right] \quad = \quad \exp\left(-\int_0^\infty dh \int \nu(dx) \left(1 - e^{-\lambda \mu([h,\infty))x}\right)\right)$$

from which we immediately deduce (4).

■

Example 4.4 *:*
With $\mu(du) = \frac{du}{u} 1_{[0,1]}(u)$, the identity (4) becomes :

$$\nu^{(\mu)}(dv) = \left(\int \frac{\nu(dx)}{x} e^{-v/x}\right) .dv$$

In particular, for $\nu(dx) = \frac{dx}{\sqrt{2\pi x}} 1_{x>0}$, then $\nu^{(\mu)}(dv) = \frac{dv}{\sqrt{2v}} 1_{v>0}$.

5 A link with path-valued Lévy processes

It follows from the very definition of an IDT process that, viewed as a random variable taking values in path-space (i.e. $D = D([0,\infty))$, the space of right continuous paths over $[0,\infty)$), this random variable, which we shall denote by $\bar{X}$ is infinitely divisible. The Lévy-Khintchine representation theorem for such variables has been discussed in [PY82], [AG78] or [Lin86] Chapter 5, among others.
First of all, we remark that there exist D-valued infinitely divisible variables which are not IDT. For example, consider the random variable R associated with the path of a squared Bessel process of dimension 1. Although the distribution of R is known to be infinitely divisible (See [SW73] or [RY99] theorem 1.2 p440), it is not the law of an IDT process. Indeed, if it were so, the identity (1) combined with the scaling property of Bessel processes would entail that a squared Bessel process of dimension n is the square of a one-dimensional Brownian motion multiplied by n, which is absurd.
Now, we try to understand some properties of IDT Lévy measures. To avoid some confusion, the expression "IDT Lévy measure" will indicate the Lévy measure of the IDT process considered as an infinitely divisible D-valued variable (not the Lévy measure of the mimicked Lévy process as in Proposition 4.1).

Lemma 5.1 *:*
Let X be a D-valued infinitely divisible variable. X is an IDT process if, and only if,

- *its Lévy measure M over D satisfies, for any non-negative functional F on D,*

$$\int_D M(dy)F(y(n\cdot)) \quad = \quad n\int_D M(dy)F(y(\cdot)) \tag{5}$$

- *its Gaussian measure ρ over D satisfies, for any $t,u \in \mathbb{R}$,*

$$\int_D y(nu)y(nt)\rho(dy) \quad = \quad n\int_D y(u)y(t)\rho(dy) \tag{6}$$

Proof :
The IDT property (1) admits the following equivalent formulation : for any $f \in C_c(\mathbb{R}_+,\mathbb{R}_+)$,

$$\mathbb{E}\left[\exp - \int_0^\infty dt f(t)X_{nt}\right] \quad = \quad \left(\mathbb{E}\left[\exp - \int_0^\infty dt f(t)X_t\right]\right)^n$$

That is

$$\int_D M(dy)\left(1 - e^{-\int_0^\infty dt f(t)y(nt)}\right) \quad = \quad n\int_D M(dy)\left(1 - e^{-\int_0^\infty dt f(t)y(t)}\right)$$

and

$$\int_D \int_0^\infty du f(u)y(nu)\int_0^\infty dt f(t)y(nt)\rho(dy) \quad = \quad n\int_D\int_0^\infty du f(u)y(u)\int_0^\infty dt f(t)y(t)\rho(dy)$$

from which (5)-(6) follow in a straightforward manner.

■

This lemma provides us with some "new" constructions of IDT processes :

Proposition 5.2 *:*
Let N be a Lévy measure on path space D. Define M as follows

$$\int_D M(dy)F(y(\cdot)) \quad = \quad \int_0^\infty du \int_D N(dy)F(y(\frac{\cdot}{u})) \tag{7}$$

Then M is an IDT Lévy measure.

Proof :
From Lemma 5.1, all we need to show is that, for any $n \in \mathbb{N}$,

$$n\int_D M(dy)F(y(\cdot)) \quad = \quad \int_D M(dy)F(y(n\cdot))$$

This follows from the obvious change of variable $nv = u$

$$\int_0^\infty du \int_D N(dy)F(y(\frac{n}{u}\cdot)) \quad = \quad n\int_0^\infty dv \int_D N(dy)F(y(\frac{\cdot}{v}))$$

■

Example 5.3 *:*
In order to illustrate this Proposition 5.2, let us compute the IDT Lévy measure of some IDT processes construted in paragraph (2.2). Let X be a subordinator without drift, ν its Lévy measure, φ a regular function and define

$$X_t^{(\varphi)} \quad = \quad \int_0^\infty du\ \varphi(u)X_{ut}$$

Then its IDT Lévy measure satisfies for any functional F over D

$$\int_D M^{(\varphi)}(dy)F(y(\cdot)) \quad = \quad \int_0^\infty du \int \nu(dx)F(x\Phi(\frac{u}{\cdot})) \tag{8}$$

where Φ is the tail of the integral of φ : $\Phi(u) = \int_u^\infty dv\varphi(v)$.
To make a close link with Proposition 5.2, the measure N is now the image of ν by $x \mapsto x\Phi\left(\frac{1}{x}\right)$.

6 Links with temporal self-decomposability

In [BNMS05], Barndorff-Nielsen, Maejima and Sato introduce the notion of temporally self decomposable processes. These processes turn out to be deeply linked with IDT processes.
We first recall the definition of temporal self-decomposability as presented in [BNMS05] :

Definition 6.1 *:*

- *A real-valued process* $(X_t;\ t \geq 0)$ *is temporally self decomposable of order 1 if for any* $c \in (0,1)$*, there exists a process* $(U_t^{(c)};\ t \geq 0)$*, called the c-residual (of* $(X_t;\ t \geq 0)$*), such that* X *and* $(X_{ct} + U_t^{(c)};\ t \geq 0)$*, with the obvious independence assumption, have the same finite dimensional marginals, i.e. are identical in law.*
- $(X_t;\ t \geq 0)$ *is temporally self decomposable of order* $n > 1$ *if for any* $c \in (0,1)$*, the c-residual* $(U_t^{(c)};\ t \geq 0)$ *is temporally self decomposable of order* $n-1$*.*
- $(X_t;\ t \geq 0)$ *is temporally self decomposable of infinite order if* $(X_t;\ t \geq 0)$ *is temporally self decomposable of order* n *for every* $n \in \mathbb{N}^*$*.*

Proposition 6.2 *:*
A right-continuous IDT process is temporally self decomposable of infinite order.

Proof :
Let $(t_1, ..., t_n) \in \mathbb{R}_+^n$ and $z = (z_1, ..., z_n) \in \mathbb{R}^n$.
Consider the characteristic function of the $(t_1, ..., t_n)$-marginal of a right-continuous IDT process $(X_t;\ t \geq 0)$

$$\hat{\mu}_{t_1,....,t_n}(z) := \mathbb{E}\left[\exp\left(i\sum_{j=1}^{n} z_j X_{t_j}\right)\right]$$

For any $r \in \mathbb{Q}$, the IDT property implies

$$\hat{\mu}_{t_1,....,t_n}(z) \quad = \quad (\hat{\mu}_{rt_1,....,rt_n}(z))^{\frac{1}{r}} \tag{9}$$

In particular, we deduce that, for $c \in \mathbb{Q}\bigcap(0,1)$:

$$\begin{aligned} \hat{\mu}_{t_1,....,t_n}(z) &= \hat{\mu}_{ct_1,....,ct_n}(z)\,(\hat{\mu}_{ct_1,....,ct_n}(z))^{\frac{1}{c}-1} \\ &= \hat{\mu}_{ct_1,....,ct_n}(z)\hat{\mu}_{\frac{c}{c'}t_1,....,\frac{c}{c'}t_n}(z) \qquad \text{with } \frac{1}{c'} = \frac{1}{c} - 1 \end{aligned}$$

where the last equality is obtained using (9).
With the continuity assumption on $(X_t;\ t \geq 0)$, we deduce that $(X_t;\ t \geq 0)$ is temporally self decomposable and that, for any $c \in (0,1)$, the finite dimensional marginals of the associated c-residual fit with the marginals of X suitably rescaled; hence the result.

■

7 Conclusion

In this short note, we have introduced and discussed the notion of IDT processes, and related this notion with the temporally self decomposable processes (Section 6). We have shown that :

$$\boxed{\mathcal{L} \subset \mathcal{I} \subset \mathcal{S}_\infty}$$

Where $\mathcal{L}$ is the family of Lévy processes, $\mathcal{I}$ the family of IDT processes assumed to be right-continuous and $\mathcal{S}_\infty$ the family of temporally self decomposable processes of infinite order.

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