The quantum Neumann model: asymptotic analysis.

Marc Bellon, Michel Talon

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Abstract

We use semi–classical and perturbation methods to establish the quantum theory of the Neumann model, and explain the features observed in previous numerical computations.

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*LPTHE, CNRS et Universités Paris VI–Paris VII (UMR 7589), Boîte 126, 4 place Jussieu, F-75252 PARIS CEDEX 05

1 Introduction

In a previous publication [1], we have explained how to determine numerically the spectrum of the quantum Neumann model. The Neumann model describes a particle on a sphere subject to a quadratic potential. It has been introduced in the nineteenth century by Neumann [2], who showed that it is integrable. More recently, Moser [3] and Mumford [4] have shown how it fits in the modern approaches to algebraically integrable systems and generalized the model to any number of dimensions. More to our point, the quantum version of the model is also integrable [5, 6]. This means that its solution reduces to the resolution of a one dimensional Schrödinger equation, as it is the generic behaviour for algebraically integrable systems. The notable feature compared to the textbook examples is that the different conserved quantities, which all appear in the same equation, must be solved for simultaneously. This is a non trivial problem which has been attacked in [7]. We will precise and prove some statements made in this paper. Moreover, the Neumann model is related with Stäckel systems as noted by Gurarie and fully developed recently in [8].

We will start with a brief presentation of the model and the equations which have to be solved. The physics of the model is driven by a general scale parameter v, which is proportional both to the strength of the interaction and the radius of the sphere. Small values of v correspond to a quantum regime and are next studied by perturbation theory. Then, we introduce the semi–classical analysis which is appropriate for large values of v. The comparison of energy levels obtained by semi–classical methods and by the numerical solution of the Schrödinger equation shows that it is indeed the case. However, we find a small constant discrepancy between the two. In a last part, we recover this constant by analytic studies of the model at large v according to these two methods.

2 The model.

The Hamiltonian of the system is given in terms of angular momenta $J_{kl} = x_k\dot{x}_l - x_l\dot{x}_k$:

$$H = \frac{1}{4}\sum_{k\neq l} J_{kl}^2 \; + \; \frac{1}{2}\sum_k a_k x_k^2 \tag{1}$$

with the constraint $\sum x_k^2 = r^2$. We assume $a_1 < a_2 < \cdots < a_N$. One can always shift all the a_k by a constant since this merely adds a constant to H. The system is classically and quantum mechanically integrable since the following quantities commute, classically and quantically:

$$F_k = x_k^2 + \sum_{l\neq k} \frac{J_{kl}^2}{a_k - a_l}, \quad H = \frac{1}{2}\sum a_k F_k \tag{2}$$

To solve the equations of motions, one introduces the "separated" variables t_k given as the roots of:

$$\sum_k \frac{x_k^2}{t - a_k} = 0$$

Positivity of x_k^2 implies that there is exactly one root in each interval $[a_k, a_{k+1}]$. It has been shown in [6] that finding the common eigenvector for the F_k with eigenvalue f_k reduces to solving the unique one–dimensional Schrödinger equation:

$$\left[\frac{d^2}{dt^2} + \frac{1}{2}\sum_k \frac{1}{t - a_k}\,\frac{d}{dt} - \frac{1}{4\hbar^2}\sum_k \frac{f_k}{t - a_k}\right]\,\Psi(t) = 0 \tag{3}$$

This is a linear differential equation with singularities at the a_k and at infinity. The situation is explained in more details in [1].

It can be useful to write the potential term in the form:

$$\sum_k \frac{f_k}{t - a_k} = v\,\frac{\prod(t - b_l)}{\prod(t - a_k)} \tag{4}$$

with $v = \sum_k f_k = \sum_k x_k^2 = r^2$. Classically the action S satisfies the separated Hamilton-Jacobi equation:

$$\frac{1}{2}\left(\frac{dS}{dt}\right)^2 = -v\,\frac{\prod(t - b_l)}{\prod(t - a_k)}$$

In Figure 1 we draw the different possibilities for the potential in the Neumann case $N = 3$, with oscillator strengths 0, 1 and y.

For v large, we are in case A), b_1 is close to 1 and b_2 is close to y. Classically the allowed region is such that for each t_k one has $P(t) = \prod(t - a_n)\prod(t - b_m) < 0$ so that, since $t_k \in [a_k, a_{k+1}]$ one has $b_1 \leq t_1 \leq 1$ and $b_2 \leq t_2 \leq y$. As v is lowered, both b_1 and b_2 move to the left, and when $v \to 0$, one has $b_1 \to -\infty$ so that one is in case C) or D). One goes from case D) to case A) either through the cases B) or C). In case B), b_1 and b_2 both belong to the interval $[0, 1]$ so that the allowed region corresponds to $b_1 \leq t_1 \leq b_2$ and t_2 describes the whole interval $[1, y]$. In case C) and D), b_1 is negative and b_2 is on either side of 1. One of the t_k describes the whole interval and the other is constrained away from 1 by b_2. It is easy to check that no other configuration is compatible with the existence of t_1 and t_2 satisfying $P(t) < 0$ in the appropriate intervals. In the general case, we will show in the sequel that all the b_i are real and belong to the interval $[a_{i-1}, a_{i+1}]$, with a_0 supposed to be $-\infty$. There is therefore a total of 2^{N-1} possibilities for the relative positions of the b_i and a_k.

At $v = 0$, the system reduces to a free particle on the sphere. The energy levels are given by the usual formula $\frac{1}{2}j(j + 1)$ for spherical harmonics, but with a different choice of basis in the $(2j + 1)$-dimensional eigenspace. These

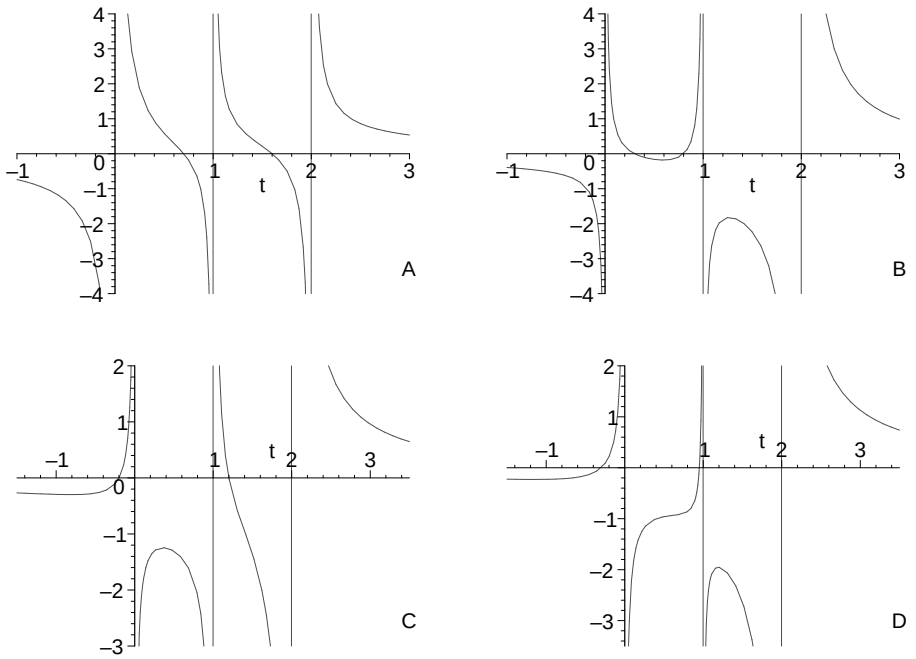

Figure 1: The four different possibilities for t_1, t_2, with $y = 2$.

basic functions are called spheroidal harmonics. These harmonics diagonalize the perturbation by the potential, so that energy levels separate linearly for small v.

For large v, the potential term becomes dominant and the point is restricted to the vicinity of the two opposite poles $x_1 = \pm r$, $x_2 = 0$, $x_3 = 0$. In the next approximation, we can neglect the curvature of the sphere and we have two independent harmonic oscillators in x_2 and x_3. The configuration around the two poles are linked by an exponentially small tunnelling amplitude, so that all states come in almost degenerate pairs.

3 Small v perturbation theory.

In the case $v = 0$, the singularity at infinity of the equation (3) becomes regular. Even if the behaviour of the solution at infinity has no direct bearing on the physical wave function, it appears important for the characterization of the solutions and provides the quantization condition for the energy.

Indeed, recall that around a regular singularity at $x = 0$, a second order differential equation has two independent solutions of the form $x^\alpha(1 + bx + \cdots)$, with the exponents α roots of a second order equation. At the finite singularities a_k, the exponents are 0 and $1/2$ with respective monodromies $+1$ and -1. In our previous work [1], we determined that to produce a single valued wave function on the sphere, the solutions must be of definite monodromy at each of the points

a_k. Monodromy at infinity is the product of the monodromies at the a_k and is therefore ± 1. Since the exponent at infinity is solution of:

$$\alpha^2 + \alpha(\frac{N}{2} - 1) - \frac{E}{2\hbar^2} = 0 \tag{5}$$

and α must be integer or half integer to get the right monodromy, we conclude that the energy E is $\frac{1}{2}\hbar^2 j(j+N-2)$ with any non negative integer $j = 2\alpha$.

For a given energy E, there is a second root α which yields a solution vanishing at infinity. However it cannot be the solution we are seeking with diagonal monodromy at the a_k, since such a solution would be single valued and regular on a compact Riemann surface. In the case where all monodromies are one, the solution has just one singularity, a pole at infinity and is therefore a polynomial of degree α. These functions are known as spheroidal harmonics of first species [9].

In order to find these polynomial solutions, we write them in the form $\prod(t - \lambda_j)$. Inserting in equation (3), we get:

$$\sum_{i \neq j} \frac{1}{(t-\lambda_i)(t-\lambda_j)} + \frac{1}{2}\sum_{k,j} \frac{1}{(t-a_k)(t-\lambda_j)} - \frac{1}{4\hbar^2}\sum_k \frac{f_k}{t-a_k} = 0 \tag{6}$$

As it should be, this equation requires that $\sum_k f_k = 0$ and we recover the value for the energy from the dominant term at $t \to \infty$. This equation is equivalent to the condition that its residues at the a_k and λ_j all vanish. The residues at the λ_j give a system of non-linear equations which determine the λ_j:

$$\sum_k \frac{1}{\lambda_j - a_k} + 4\sum_{h \neq j} \frac{1}{\lambda_j - \lambda_h} = 0 \tag{7}$$

The residues at the a_k then determine the conserved quantities as:

$$f_k = 2\hbar^2 \sum_j \frac{1}{a_k - \lambda_j} \tag{8}$$

The system of equations (7) can be seen as the condition of equilibrium of a system of points with logarithmic repulsive potentials, some of which being fixed. In this formulation, it is therefore clear that there is a unique solution with a definite number of λ_j in each of the intervals $[a_k, a_{k+1}]$. In the case $N = 3$, this gives $d+1$ different solutions of degree d. Similar calculations can be done for solutions which are of the form $\prod_{k\in I}\sqrt{t-a_k}\prod_j(t-\lambda_j)$ where I is a subset of the singularities. This gives spheroidal harmonics of the second, third and fourth species [9] for $N = 3$. The total number of these harmonics with energy $\frac{1}{2}j(j+1)$ is exactly $2j+1$, the known degeneracy of this energy level.

Numerical solutions of the system of equations (7) are easy to get by a multidimensional Newton method and have been used to produce initial solutions in our previous paper.

From the knowledge of this $v = 0$ solutions, we will describe the behaviour of the solutions for small v. As soon as v is different from zero, the solution gets an essential singularity at infinity. We therefore multiply the spheroidal harmonic by the simplest function singular at infinity, namely $\exp(pt)$. In the vicinity of $v = 0$, we expect p to be small and neglect higher powers of p. Inserting in equation (3) gives:

$$\sum_{i\neq j} \frac{1}{(t-\lambda_i)(t-\lambda_j)} + \frac{1}{2}\sum_{k,j} \frac{1}{(t-a_k)(t-\lambda_j)} - \frac{1}{4\hbar^2}\sum_k \frac{f_k}{t-a_k} \\ +2p\sum_j \frac{1}{t-\lambda_j} + \frac{1}{2}p\sum_k \frac{1}{t-a_k} = 0$$

The term proportional to $1/t$ in this equation immediately gives that p is proportional to v, so that the hypothesis that p is small is valid.

$$2np + \frac{1}{2}Np - \frac{1}{4\hbar^2}v = 0$$

Here n is the degree of the polynomial. The roots λ_j of the wave function satisfy equations which are small perturbations of equation (7), hence are close to their values for $v = 0$. The term in $1/t^2$ then gives the perturbation of the energy:

$$E = \hbar^2 n(2n + N - 2) + v\frac{\sum a_k + 4\sum \lambda_j}{2(N + 4n)}$$

Similar calculations can be done for the spheroidal harmonics of the other species and this nicely reproduces the slopes of the numerical results at small v.

4 Semiclassical analysis.

As an integrable system, the Neumann model lends itself to a simple semiclassical analysis writing basically that the Bohr–Sommerfeld conditions quantize the Liouville tori. Equivalently, the separated Schrödinger equation can be analyzed by the WKB method.

4.1 The WKB treatment.

The presence of a singular factor in the first derivative term of the equation (3) seems to be a problem for the application of the WKB method in our case, but we will show that in contrast to what happens in the spherical symmetry case [10], this does not cause troubles. In a first approach, we write an expansion in powers of $\hbar$ for the action S such that $\Psi = \exp(iS/\hbar)$:

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots \tag{9}$$

As usual the dominant term in $\hbar^{-2}$ does not depend on the first derivative term in the equation and gives for S_0:

$$S_0 = \int dt \, \sqrt{\frac{-v \prod (t - b_i)}{4 \prod (t - a_k)}}$$

The following term gives:

$$iS_0'' - 2S_0'S_1' + \frac{1}{2} iS_1' \sum \frac{1}{t - a_k} = 0$$

Due to the form of S_0, the singular terms at a_k cancel and one gets:

$$\exp(iS_1) = \prod (t - b_j)^{-\frac{1}{4}}$$

At this order, the wave function is regular at the a_k and one can also show that the following correction to S is regular at a_k. In contrast, the first correction is mildly singular at the turning point b_l and the following correction S_2 has so bad singularities that the expansion breaks down. In the cases where the b_l are away from the a_k, the usual connection formulae around the turning point [10] justify an additional $\pi/4$ phase factor in the quantization conditions.

Another approach is to desingularize the equation (3) by introducing a new variable x defined by the (hyper-)elliptic integral [9]:

$$x = \int^t \frac{dt}{\sqrt{\prod (t - a_k)}} \tag{10}$$

The application of the chain rule gives:

$$\frac{d^2\Psi}{dt^2} + \frac{1}{2} \sum_k \frac{1}{t - a_k} \frac{d\Psi}{dt} = \frac{1}{\prod (t - a_k)} \frac{d^2\Psi}{dx^2}$$

so that the Schrödinger equation becomes:

$$\frac{d^2\Psi}{dx^2} - \frac{v}{4\hbar^2} \prod (t - b_l) \Psi = 0 \tag{11}$$

This equation has neither singularities nor first derivative term, so that the usual WKB formula gives:

$$\Psi(x) \simeq \left(\prod (t - b_j) \right)^{-\frac{1}{4}} \exp \left(\frac{1}{\hbar} \int dx \sqrt{\frac{v}{4} \prod (t - b_i)} \right)$$

It should be emphasized that the variation of x is alternatively real and purely imaginary when t crosses the a_k, introducing additional i factors in the action. In the traditional $N = 3$ case, x is an elliptic integral and t can be expressed with Weierstraß elliptic functions. When t goes from $-\infty$ to $+\infty$, x describes the boundary of the half–periods rectangle. In the cases where there are more singularities, it is possible that the t variable is not well defined as an inverse function of x, however this is of no consequence for the study of the Schrödinger equation which proceeds separately on each of the intervals $[a_k, a_{k+1}]$.

4.2 Remarks on the zeroes of the potential.

The above elliptic formulation allows to demonstrate quickly the claim we have made that there must be in each interval $[a_j, a_{j+1}]$ a region where the product $P(t) = \prod(t - a_n) \prod(t - b_m)$ takes negative values. Recall from [1] that the wave function is either even or odd under each parity operation $x_k \to -x_k$. In terms of the t variable, this translates into monodromy around a_k. Now, the x variable of the elliptic parametrization behaves as $x(a_k) + c_k\sqrt{t - a_k}$ in the vicinity of a_k so that the wave function is even or odd as a function of $x - x(a_k)$, which means that either Ψ or its derivative vanishes at each of the $x(a_k)$.

Multiplying eq. (11) by Ψ and integrating by parts, we then get for each k:

$$\int_{x(a_k)}^{x(a_{k+1})} [\Psi'^2(x) + \frac{v}{4\hbar^2} \prod(t(x) - b_l)\Psi^2(x)]dx = 0 \tag{12}$$

Under the elliptic parametrization, the image of the real axis is a sequence of alternatively real and purely imaginary intervals, according to the sign of the product $\prod(t - a_k)$. The $\Psi'^2(x)$ term has therefore the same sign as this product. Since Ψ is real, Eq. (12) cannot be satisfied if the potential remains of the same sign as the kinetic term. This is equivalent to the condition $P(t) < 0$ at some point of the interval $[a_j, a_{j+1}]$. Finally, we prove that this allows to establish that quantum mechanically the b_j obey the constraints we have discussed in the framework of the classical analysis.

The relative position of the a_k and b_j depends on the signs of the f_k. The only one with a fixed sign is f_N, which is positive. All combinations of the signs of the other ones are possible, which gives 2^{N-1} possibilities. We will show that each of these sign combinations corresponds to a unique set of relative positions of the b_j. The fact that we identify $N - 1$ necessary real zeroes of the potential shows that all b_j must be real.

More precisely, we will show that b_j belongs to the interval $]a_j, a_{j+1}[$ if f_j is positive and to the interval $]a_{j-1}, a_j[$ otherwise. This means that we have to show that in the interval $]a_j, a_{j+1}[$, there is exactly one zero if f_j and f_{j+1} are of the same sign and two if f_j is positive and f_{j+1} is negative. If the f are of the same sign, the potential reaches opposite infinite limits at the ends of the interval, so that it must have a zero in between. In the other case, the potential goes to positive infinity at the two ends of the interval and we proved that the potential must be negative somewhere in the interval. This is only possible if the potential has two zeroes which we identify as b_j and b_{j+1}. The only remaining thing to prove is that there is a zero in the interval $]-\infty, a_1[$ when f_1 is negative. In this case, we use the fact that $\sum f_k$ is positive, so that the dominant term near $-\infty$ is negative. The potential is positive at $0-$ hence there must be a zero in between. Since we identified $N - 1$ zeroes, we have exhausted the whole set of zeroes for the potential and there cannot be any other, proving in particular that all zeroes are real.

4.3 The quantization conditions.

In fact, using any of the two WKB approaches, we get the same semiclassical quantization conditions:

$$\int_{b_j}^{a_{j+1}} dt \, \sqrt{-\frac{v}{4}\frac{\prod(t-b_i)}{\prod(t-a_k)}} = (n_j + \frac{1}{4})\pi\hbar \tag{13}$$

in the case where all the monodromies are one and b_j is in the interval $[a_j, a_{j+1}]$. The integer n_j is the number of zeroes of the wave function in the interval $[a_j, a_{j+1}]$. If the monodromy around a_{j+1} is -1, we should add one half to n_j. In the cases where b_j is near a_j or becomes smaller, the $\pi/4$ phase factor is no more a good approximation and we will postpone to further studies the relevant phase factors. This equation has already appeared in the literature, for example in [6, 7], where the refinement with the $\pi/4$ phase first appears in the last paper. It is introduced there using the machinery of Maslov indices. In this case, the justification through the connection formulae is very direct, it simply uses the linear approximation of the potential around the turning point and consequently the asymptotic analysis of the Airy function:

$$Ai(x) \sim_{+\infty} \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-\frac{2}{3}x^{3/2}}, \qquad Ai(-x) \sim_{+\infty} \frac{1}{\sqrt{\pi}x^{1/4}} \cos(\frac{2}{3}x^{3/2} - \frac{\pi}{4})$$

In order to compare the semiclassical predictions to the exact numerical solutions found in [1], we solve numerically the quantization conditions (13) and plot the resulting expression for the energy as a function of v compared with two exact energy levels which are degenerate in the large v limit. We do this for a small number of energy levels in order to keep the graphs uncluttered and plot them for oscillator strengths equal to $(0, 1, 2)$ in Fig. 2 and $(0, 1, 5)$ in Fig. 2, both in the case $N = 3$. In either figure, the three heavy lines are the WKB predictions. We see that the small v regime is poorly accounted for, which is of no surprise since in this case, we are no longer in the situation A) of Fig. 1 and their should be phase corrections different from $\pi/4$ and moreover, we are in the deep quantum regime, where we do not expect WKB approximation to be good. Nevertheless, we plan to study in further works the different cases of Fig. 1 and the transitions between them.

The most notable feature of the curves plotted in Fig. 2 and 3 is that in the large v limit, there is a small constant separation between the exact and WKB curves. This separation is independent both on the energy level and the oscillator strengths. This contradicts the naive expectation that the corrections to the WKB approximation are of order $\hbar^2$ with respect to the order zero solution which has an energy of order $1/\hbar$. This order of the correction is true for the solution away from the turning points, but the difference from the Airy function may create order $\hbar$ correction to the phase as noted in [10]. It is however remarkable

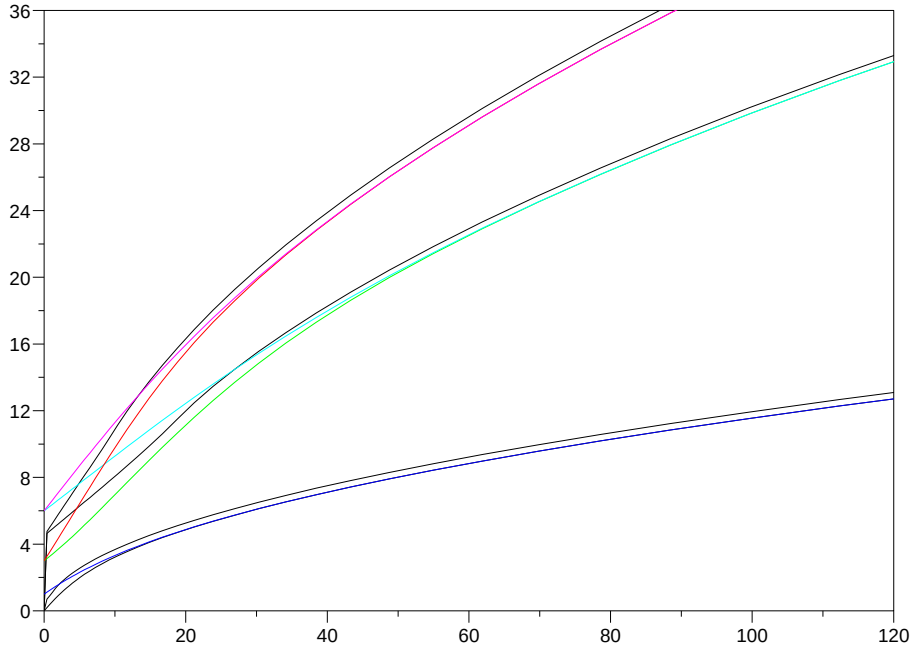

Figure 2: Energy levels for $y = 2$.

that this finite correction to the energy does not depend on the level, so that the transition energies have vanishing errors in the small $\hbar$ regime. It is the purpose of the following section to show through perturbative computations the validity of this result.

5 Large v study.

We first compute perturbatively for large v or equivalently small $\hbar$ the energy levels as obtained from the "enhanced" Bohr–Sommerfeld rules, and then as perturbations of quantum oscillators sitting around the pole of the sphere.

5.1 Semi–classical calculation.

We limit ourselves to the $N = 3$ case and we fix the oscillator strengths to $(0, 1, y)$ as in [1]. Since v is large, the point on the sphere is limited to the vicinity of the points $(\pm 1, 0, 0)$, whose separated variables are $t_1 = 1$ and $t_2 = y$. With b_1 close to 1 and b_2 close to y, t_1 is classically limited to the interval $[b_1, 1]$ and t_2 to the interval $[b_2, y]$, which limits the point to the vicinity of the pole. To do our

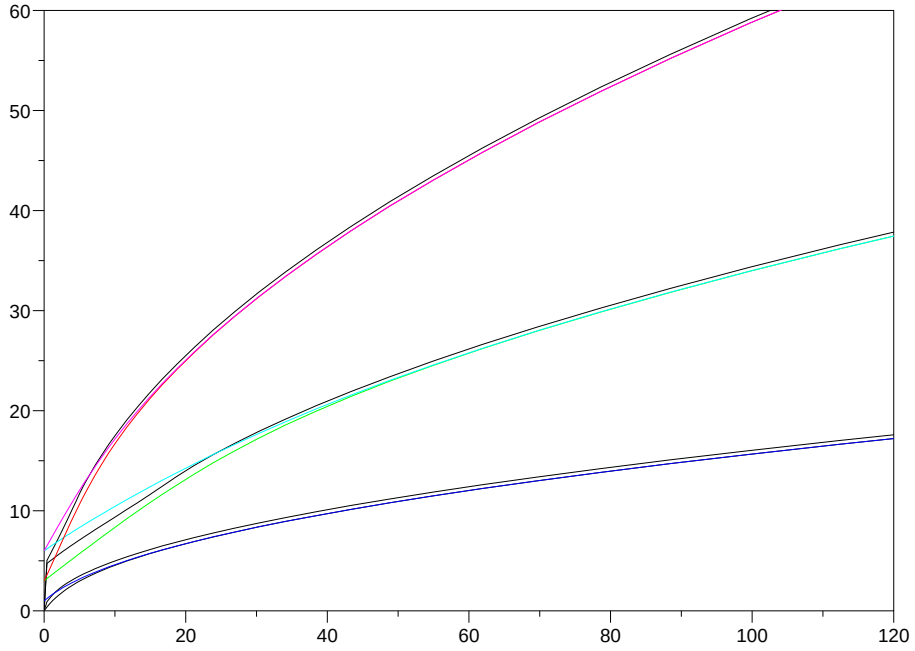

Figure 3: Energy levels for $y = 5$.

perturbative calculations, we will write $b_1 = 1 - \alpha$ and $b_2 = y - \beta$, so that α and β are small. The integrand in the Bohr–Sommerfeld rules takes the form:

$$\sqrt{-\frac{(t-1+\alpha)\,(t-y+\beta)}{t\,(t-1)\,(t-y)}} \tag{14}$$

When integrating in $[b_1, 1]$, we cannot expand the factors $t-1+\alpha$ and $t-1$, but all the others can be expanded around $t = 1$ and in powers of β. The resulting integrals are easily computable yielding:

$$\frac{1}{2}\,\alpha\,\pi + \frac{1}{16}\,\alpha^2\pi - \frac{1}{4}\,\frac{\alpha\,\beta}{y-1}\,\pi + \cdots$$

Similarly for the integral in $[b_2, y]$ we keep the factors $t - y + \beta$ and $t - y$ and expand the others around $t = y$ in powers of α. We then get the integral:

$$\frac{1}{2}\,\frac{\beta}{\sqrt{y}}\,\pi + \frac{1}{16}\,\frac{\beta^2}{y^{3/2}}\,\pi + \frac{1}{4}\,\frac{\alpha\,\beta}{\sqrt{y}(y-1)}\,\pi + \cdots$$

We write that the first integral is equal to $\frac{\hbar\pi}{\sqrt{v}}(n_1+\frac{1}{2})$ and similarly the second equal to $\frac{\hbar\pi}{\sqrt{v}}(n_2+\frac{1}{2})$ which determines α and β:

$$\begin{aligned}\alpha &= 2\hbar(n_1+\frac{1}{2})\left(\frac{1}{\sqrt{v}}-\frac{\hbar}{v}\left(\frac{1}{4}(n_1+\frac{1}{2})-(n_2+\frac{1}{2})\frac{\sqrt{y}}{(y-1)}\right)+\cdots\right)\\ \beta &= 2\hbar(n_2+\frac{1}{2})\left(\sqrt{\frac{y}{v}}-\frac{\hbar}{v}\left(\frac{1}{4}(n_2+\frac{1}{2})+(n_1+\frac{1}{2})\frac{\sqrt{y}}{(y-1)}\right)+\cdots\right)\end{aligned}$$

Note that this justifies a posteriori the claim we made that α and β are small, and become smaller when v increases. Finally the energy $E=\frac{1}{2}v(1+y-b_1-b_2)=\frac{1}{2}v(\alpha+\beta)$ takes the simple form, in which there is no mixing between n_1 and n_2 excitations:

$$E_{WKB}=\hbar\sqrt{v}\left((n_1+\frac{1}{2})+\sqrt{y}(n_2+\frac{1}{2})\right)-\frac{\hbar^2(n_1+\frac{1}{2})^2}{4}-\frac{\hbar^2(n_2+\frac{1}{2})^2}{4} \qquad (15)$$

Of course the first term corresponds to the energy of two independent harmonic oscillators of respective excitations n_1 and n_2, and nicely fits the $\sqrt{v}$ look of the energy levels at large v apparent in the figures of [1]. The second and third term shift these square root terms towards their correct position, but miss it by a small constant. As we show below this small constant is accessible to a more correct quantum computation.

5.2 Quantum calculation.

The principle of this calculation is to work out a perturbation of harmonic oscillators. We detail it in the case $N=3$ and it is then convenient to go to the elliptic parametrization given by eq. (10). We expand the change of variable in the vicinity of $t=1$ and $t=y$. Taking the origin in $t=y$, we write

$$x=\int\frac{d\tau}{\sqrt{\tau}}\frac{1}{\sqrt{y(y-1)}}\Big(1+\frac{\tau}{2y}+\frac{\tau}{2(y-1)}+\cdots\Big)$$

which can be inverted to yield

$$\tau=y-t=\frac{1}{4}y(y-1)x^2-\frac{1}{48}y(y-1)(2y-1)x^4+\cdots \qquad (16)$$

A similar computation gives the parametrization around $t=1$

$$\tau=1-t=\frac{1}{4}(y-1)x^2-\frac{1}{48}(y-1)(y-2)x^4+\cdots \qquad (17)$$

These values of the variable t will be used in the Schrödinger equation (11) to express in terms of the variable x the potential $\pm(v/4\hbar^2)(t-b_1)(t-b_2)$. The sign

can change since in the calculation of x, we forgot the imaginary unit i which can appear according to the sign of the term under the square root.

In the large v limit, b_1 is near 1 and b_2 is near y and the lowest relevant order for the potential for t around y is:

$$\frac{1}{4\hbar^2}v(y-1)\Big((y-b_2)-\frac{1}{4}y(y-1)x^2\Big)$$

which is the potential for a harmonic oscillator with energy $(v/4\hbar^2)(y-1)(y-b_2)$ and oscillator strength $(v/4\hbar^2)y(y-1)^2$. This gives $y-b_2 = 2\hbar\sqrt{y/v}(n_2+1/2)$, which is exactly the semiclassical result at this order. Similarly, $1-b_1 = 2\hbar/\sqrt{v}(n_1+1/2)$. In order to choose the terms which will contribute to the next order, we should remember that for the harmonic oscillator, x^2 is of order $v^{-1/2}$. We therefore get corrections stemming from the variation of the oscillator strength, from the factor in front of $y-b_2$ and from anharmonic terms in x^4 which stem in part from the one in τ and in part from the product of the two x^2 terms occurring in both factors $t-b_i$. The order one corrected potential is:

$$\frac{1}{4\hbar^2}v\ \Big((y-b_2)(y-1+\frac{2\hbar}{\sqrt{v}}(n_1+\frac{1}{2}))-\frac{1}{4}y(y-1)^2x^2 \\ -\frac{\hbar}{2\sqrt{v}}\Big(n_1+\frac{1}{2}+(n_2+\frac{1}{2})\sqrt{y}\Big)y(y-1)x^2+\frac{1}{48}y(y-1)^2(5y-1)x^4\Big)$$

With the anharmonic oscillator Schrödinger equation written as $-\Psi''+1/4(x^2+\lambda x^4)\Psi = E\Psi$, first order perturbation theory yields the energy levels $E_n = n+1/2+3/4\lambda(2n^2+2n+1)$. This result is easily obtained writing x^4 in terms of creation and annihilation operators. Putting these corrections together, we get:

$$y-b_2 = 2\hbar\sqrt{\frac{y}{v}}(n_2+\frac{1}{2})-\frac{\hbar^2}{v}\Big(\frac{2\sqrt{y}}{(y-1)}(n_1+\frac{1}{2})(n_2+\frac{1}{2})+\frac{1}{2}(n_2+\frac{1}{2})^2+\frac{1}{8}\frac{5y-1}{y-1}\Big)$$

Note that this only differs from the semiclassical result by the last term, independent on the n_i.

A similar computation yields the potential developed around $t=1$ and the first order correction to the energy of the corresponding anharmonic oscillator gives

$$1-b_1 = 2\hbar\frac{1}{\sqrt{v}}(n_1+\frac{1}{2})+\frac{\hbar^2}{v}\Big(\frac{2\sqrt{y}}{(y-1)}(n_1+\frac{1}{2})(n_2+\frac{1}{2})-\frac{1}{2}(n_1+\frac{1}{2})^2-\frac{1}{8}\frac{y-5}{y-1}\Big)$$

Whatever happens outside of these two regions is exponentially suppressed for large v, since the wave function of the harmonic oscillator are already exponentially small there. Combining the values of b_1 and b_2, we obtain for the energy at this order in $1/v$ a result which differs from the semiclassical one of eq. (15) only by a constant

$$E_Q = E_{WKB}-\frac{3}{8}\hbar^2$$

This nicely reflects our findings on Figs. 2 and 3 where $\hbar = 1$.

6 Conclusion.

The interaction of the study of the Neumann model and the generalized Lamé equation allowed us to give precise results on the behaviour of the spectrum in some limits. The generalized Lamé equation has independent interest since it is the most general linear differential equation of second order with only regular singularities at finite distance.

We were able to precisely count the different cases for the positions of the zeroes of the potential, showing in particular that they are all real. It is also remarkable that there is a constant error in the WKB approximation. This error is however independent on the level, so that it does not appear in transition energies. In this integrable case, the WKB approximation gives pretty good results even for low excitation numbers and the further study of its behaviour around turning points at low energy could make it even more useful.

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