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Abstract

This paper considers the problem of determining the optimal sequence of stopping times for a diffusion process subject to regime switching decisions. This is motivated in the economics literature, by the investment problem under uncertainty for a multi-activity firm involving opening and closing decisions. We use a viscosity solutions approach, and explicitly solve the problem in the two regimes case when the state process is of geometric Brownian nature.

Key words : Optimal switching, system of variational inequalities, viscosity solutions, smooth-fit principle.

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1 Introduction

The theory of optimal stopping and its generalization, thoroughly studied in the seventies, has received a renewed interest with a variety of applications in economics and finance. These applications range from asset pricing (American options, swing options) to firm investment and real options. We refer to [4] for a classical and well documented reference on the subject.

In this paper, we consider the optimal switching problem for an one dimensional stochastic process X. The diffusion process X may take a finite number of regimes that are switched at stopping time decisions. For example in the firm's investment problem under uncertainty, a company (oil tanker, electricity station ....) manages several production activities operating in different modes or regimes representing a number of different economic outlooks (e.g. state of economic growth, open or closed production activity, ...). The process X is the price of input or output goods of the firm and its dynamics may differ according to the regimes. The firm's project yields a running payoff that depends on the commodity price X and of the regime choice. The transition from one regime to another one is realized sequentially at time decisions and incurs certain fixed costs. The problem is to find the switching strategy that maximizes the expected value of profits resulting from the project.

Optimal switching problems were studied by several authors, see [1] or [10]. These control problems lead via the dynamic programming principle to a system of variational inequalities. Applications to option pricing, real options and investment under uncertainty were considered by [2], [5] and [7]. In this last paper, the drift and volatility of the state process depend on an uncontrolled finite-state Markov chain, and the author provides an explicit solution to the optimal stopping problem with applications to Russian options. In [2], an explicit solution is found for a resource extraction problem with two regimes (open or closed field), a linear profit function and a price process following a geometric Brownian motion. In [5], a similar model is solved with a general profit function in one regime and equal to zero in the other regime. In both models [2], [5], there is no switching in the diffusion process : changes of regimes only affect the payoff functions. Their method of resolution is to construct a solution to the dynamic programming system by guessing a priori the form of the strategy, and then validate a posteriori the optimality of their candidate by a verification argument. Our model combines regime switchings both on the diffusion process and on the general profit functions. We use a viscosity solutions approach for determining the solution to the system of variational inequalities. In particular, we derive directly the smooth-fit property of the value functions and the structure of the switching regions. Explicit solutions are provided in the following cases : $\star$ the drift and volatility terms of the diffusion take two different regime values, and the profit functions are identical of power type, $\star$ there is no switching on the diffusion process, and the two different profit functions satisfy a general condition, including typically power functions. The results of our analysis take qualitatively different forms, depending on model parameters values.

The paper is organized as follows. We formulate in Section 2 the optimal switching problem. In Section 3, we state the system of variational inequalities satisfied by the value functions in the viscosity sense. The smooth-fit property for this problem, proved in [9], plays a important role in our subsequent analysis. We also state some useful properties on the switching regions. In Section 4, we explicitly solve the problem in the two-regimes case when the state process is of geometric Brownian nature.

2 Formulation of the optimal switching problem

We consider a stochastic system that can operate in d modes or regimes. The regimes can be switched at a sequence of stopping times decided by the operator (individual, firm, ...). The indicator of the regimes is modeled by a cadlag process I_t valued in $\mathbb{I}_d = \{1, \ldots, d\}$. The stochastic system X (price commodity, salary, ...) is valued in $\mathbb{R}_+^* = (0, \infty)$ and satisfies the s.d.e.

$$dX_t \;=\; b_{I_t} X_t dt + \sigma_{I_t} X_t dW_t, \tag{2.1}$$

where W is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions. $b_i \in \mathbb{R}$, and $\sigma_i > 0$ are the drift and volatility of the system X once in regime $I_t = i$ at time t.

A strategy decision for the operator is an impulse control α consisting of a double sequence $\tau_1, \ldots, \tau_n, \ldots, \kappa_1, \ldots, \kappa_n, \ldots$, $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, where τ_n are stopping times, $\tau_n < \tau_{n+1}$ and $\tau_n \to \infty$ a.s., representing the switching regimes time decisions, and κ_n are $\mathcal{F}_{\tau_n}$-measurable valued in $\mathbb{I}_d$, and representing the new value of the regime at time $t = \tau_n$. We denote by $\mathcal{A}$ the set of all such impulse controls. Now, for any initial condition $(x, i) \in (0, \infty) \times \mathbb{I}_d$, and any control $\alpha = (\tau_n, \kappa_n)_{n \geq 1} \in \mathcal{A}$, there exists a unique strong solution valued in $(0, \infty) \times \mathbb{I}_d$ to the controlled stochastic system :

$$X_0 \;=\; x, \quad I_{0^-} \;=\; i, \tag{2.2}$$

$$dX_t \;=\; b_{\kappa_n} X_t dt + \sigma_{\kappa_n} X_t dW_t, \quad I_t \;=\; \kappa_n, \quad \tau_n \leq t < \tau_{n+1}, \quad n \geq 0. \tag{2.3}$$

Here, we set $\tau_0 = 0$ and $\kappa_0 = i$. We denote by $(X^{x,i}, I^i)$ this solution (as usual, we omit the dependance in α for notational simplicity). We notice that $X^{x,i}$ is a continuous process and I^i is a cadlag process, possibly with a jump at time 0 if $\tau_1 = 0$ and so $I_0 = \kappa_1$.

We are given a running profit function $f : (0, \infty) \times \mathbb{I}_d \to \mathbb{R}$ and we set $f_i(.) = f(., i)$ for $i \in \mathbb{I}_d$. We assume that for each $i \in \mathbb{I}_d$, the function f_i is concave, continuous on $\mathbb{R}_+$, with $f_i(0) = 0$, and the Fenchel-Legendre transform of f_i is finite on $(0, \infty)$:

$$\tilde{f}_i(y) \;:=\; \sup_{x > 0} [f_i(x) - xy] \;<\; \infty, \quad \forall y > 0. \tag{2.4}$$

We also assume Hölder continuity of f_i : there exists $\gamma_i \in (0, 1]$ s.t.

$$|f_i(x) - f_i(\hat{x})| \;\leq\; C|x - \hat{x}|^{\gamma_i}, \quad \forall x, \hat{x} \in (0, \infty), \tag{2.5}$$

for some positive constant C. A typical example satisfying the above two conditions is given by the power utility functions:

$$f_i(x) \;=\; x^{\gamma_i}, \quad 0 < \gamma_i < 1.$$

The cost for switching from regime i to $j \neq i$ is a constant equal to $g_{ij} > 0$, and we assume that

$$g_{ik} \leq g_{ij} + g_{jk}, \quad i \neq j \neq k \neq i \in \mathbb{I}_d. \tag{2.6}$$

This last condition means that it is no more expensive to switch directly in one step from regime i to k than in two steps via an intermediate regime j.

The expected total profit of running the system when initial state is (x, i) and using the impulse control $\alpha = (\tau_n, \kappa_n)_{n \geq 1} \in \mathcal{A}$ is

$$J_i(x, \alpha) = E\left[\int_0^\infty e^{-rt} f(X_t^{x,i}, I_t^i) dt - \sum_{n=1}^{\infty} e^{-r\tau_n} g_{\kappa_{n-1}, \kappa_n}\right].$$

Here $r > 0$ is a positive discount factor, and we use the convention that $e^{-r\tau_n(\omega)} = 0$ when $\tau_n(\omega) = \infty$. We also make the standing assumption :

$$r > b := \max_{i \in \mathbb{I}_d} b_i. \tag{2.7}$$

The objective is to maximize this expected total profit over all strategies α. Accordingly, we define the value functions

$$v_i(x) = \sup_{\alpha \in \mathcal{A}} J_i(x, \alpha), \quad x \in \mathbb{R}_+^*, \ i \in \mathbb{I}_d. \tag{2.8}$$

We shall see in the next section that under (2.4) and (2.7), the expectation defining $J_i(x)$ is well-defined and the value function v_i is finite.

3 System of variational inequalities, switching regions and viscosity solutions

We first state the growth property on the value functions.

Lemma 3.1 *We have for all $i \in \mathbb{I}_d$:*

$$0 \leq v_i(x) \leq \frac{xy}{r - b} + \max_{i \in \mathbb{I}_d} \frac{\tilde{f}_i(y)}{r}, \quad \forall x > 0, \ y > 0. \tag{3.1}$$

In particular, $v_i(0^+) = 0$.

Proof. By considering the particular strategy of no switching from the initial state (x, i), i.e. $\alpha = (\tau_n, \kappa_n)$ with $\tau_n = \infty$, $\kappa_n = i$ for all n, and by noting that the concave, nondecreasing function f_i satisfying $f_i(0) = 0$ is nonnegative, we immediately get the lower bound in assertion (i).

Given an initial state $(X_0, I_{0^-}) = (x, i)$ and an arbitrary impulse control $\alpha = (\tau_n, \kappa_n)$, we get from the dynamics (2.2)-(2.3), the following explicit expression of $X^{x,i}$:

$$\begin{aligned} X_t^{x,i} &= xY_t(i) \\ &:= x\left(\prod_{l=0}^{n-1} e^{b_{\kappa_l}(\tau_{l+1} - \tau_l)} Z_{\tau_l, \tau_{l+1}}^{\kappa_l}\right) e^{b_{\kappa_n}(t - \tau_n)} Z_{\tau_n, t}^{\kappa_n}, \quad \tau_n \leq t < \tau_{n+1}, \quad n \in \mathbb{N}, \end{aligned} \tag{3.2}$$

where

$$Z^j_{s,t} \;=\; \exp\left(\sigma_j(W_t - W_s) - \frac{\sigma_j^2}{2}(t-s)\right), \quad 0 \leq s \leq t, \quad j \in \mathbb{I}_d. \tag{3.3}$$

Here, we used the convention that $\tau_0 = 0$, $\kappa_0 = i$, and the product term from l to $n-1$ in (3.2) is equal to 1 when $n = 1$. We then deduce the inequality $X_t^{x,i} \leq x e^{bt} M_t$, for all t, where

$$M_t \;=\; \left(\prod_{l=0}^{n-1} Z^{\kappa_l}_{\tau_l,\tau_{l+1}}\right) Z^{\kappa_n}_{\tau_n,t}, \quad \tau_n \leq t < \tau_{n+1}, \quad n \in \mathbb{N}. \tag{3.4}$$

Now, we notice that (M_t) is a martingale obtained by continuously patching the martingales $(Z^{\kappa_{n-1}}_{\tau_{n-1},t})$ and $(Z^{\kappa_n}_{\tau_n,t})$ at the stopping times τ_n, $n \geq 1$. In particular, we have $E[M_t] = M_0 = 1$ for all t.

We set $\tilde f(y) = \max_{i\in\mathbb{I}_d} \tilde f_i(y)$, $y > 0$, and we notice by definition of $\tilde f_i$ in (2.4) that $f(X_t^{x,i}, I_t^i) \leq y X_t^{x,i} + \tilde f(y)$ for all t, y. Since the costs g_{ij} are nonnegative, it follows that :

$$\begin{aligned} J_i(x,\alpha) &\leq E\left[\int_0^\infty e^{-rt}\left(yxe^{bt}M_t + \tilde f(y)\right)dt\right] \\ &= \int_0^\infty e^{-(r-b)t} yxE[M_t]dt + \int_0^\infty e^{-rt}\tilde f(y)dt \;=\; \frac{xy}{r-b} + \frac{\tilde f(y)}{r}. \end{aligned}$$

From the arbitrariness of α, this shows the upper bound for v_i.

By sending x to zero and then y to infinity into the r.h.s. of (3.1), and recalling that $\tilde f_i(\infty) = f_i(0) = 0$ for all $i \in \mathbb{I}_d$, we conclude that v_i goes to zero when x tends to zero. □

We next show the Hölder continuity of the value functions.

Lemma 3.2 *For all $i \in \mathbb{I}_d$, v_i is Hölder continuous on $(0,\infty)$:*

$$|v_i(x) - v_i(\hat x)| \;\leq\; C|x-\hat x|^\gamma, \quad \forall x, \hat x \in (0,\infty), \quad \textit{with} \;\; |x - \hat x| \leq 1,$$

for some positive constant C, and where $\gamma = \min_{i\in\mathbb{I}_d} \gamma_i$ of condition (2.5).

Proof. By definition (2.8) of v_i and under condition (2.5), we have for all $x, \hat x \in (0,\infty)$, with $|x - \hat x| \leq 1$:

$$\begin{aligned} |v_i(x) - v_i(\hat x)| &\leq \sup_{\alpha\in\mathcal{A}} |J_i(x,\alpha) - J_i(\hat x,\alpha)| \\ &\leq \sup_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-rt}\left|f(X_t^{x,i}, I_t^i) - f(X_t^{\hat x,i}, I_t^i)\right| dt\right] \\ &\leq C\sup_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-rt}\left|X_t^{x,i} - X_t^{\hat x,i}\right|^{\gamma_{I_t^i}} dt\right] \\ &= C\sup_{\alpha\in\mathcal{A}} \int_0^\infty E\left[e^{-rt}|x-\hat x|^{\gamma_{I_t^i}}|Y_t(i)|^{\gamma_{I_t^i}} dt\right] \\ &\leq C|x-\hat x|^\gamma \sup_{\alpha\in\mathcal{A}} \int_0^\infty e^{-(r-b)t} E|M_t|^{\gamma_{I_t^i}} dt \end{aligned} \tag{3.5}$$

by (3.2) and (3.4). For any $\alpha = (\tau_n, \kappa_n)_n \in \mathcal{A}$, by the independence of $(Z^{\kappa_n}_{\tau_n,\tau_{n+1}})_n$ in (3.3), and since

$$E\left|Z^{\kappa_n}_{\tau_n,\tau_{n+1}}\right|^{\gamma_{\kappa_n}} = \exp\left(\gamma_{\kappa_n}(\gamma_{\kappa_n}-1)\frac{\sigma^2_{\kappa_n}}{2}(\tau_{n+1}-\tau_n)\right) \leq 1,$$

we clearly see that $E|M_t|^{\gamma_{I^i_t}} \leq 1$ for all $t \geq 0$. We thus conclude with (3.5). □

The dynamic programming principle combined with the notion of viscosity solutions are known to be a general and powerful tool for characterizing the value function of a stochastic control problem via a PDE representation, see [6]. We recall the definition of viscosity solutions for a P.D.E in the form

$$H(x, v, D_x v, D^2_{xx} v) = 0, \quad x \in \mathcal{O}, \tag{3.6}$$

where $\mathcal{O}$ is an open subset in $\mathbb{R}^n$ and H is a continuous function and noninceasing in its last argument (with respect to the order of symmetric matrices).

Definition 3.1 *Let v be a continuous function on $\mathcal{O}$. We say that v is a viscosity solution to* (3.6) *on $\mathcal{O}$ if it is*

(i) a viscosity supersolution to (3.6) *on $\mathcal{O}$: for any $\bar{x} \in \mathcal{O}$ and any C^2 function φ in a neighborhood of $\bar{x}$ s.t. $\bar{x}$ is a local minimum of $v - \varphi$, we have :*

$$H(\bar{x}, v(\bar{x}), D_x\varphi(\bar{x}), D^2_{xx}\varphi(\bar{x})) \geq 0.$$

and

(ii) a viscosity subsolution to (3.6) *on $\mathcal{O}$: for any $\bar{x} \in \mathcal{O}$ and any C^2 function φ in a neighborhood of $\bar{x}$ s.t. $\bar{x}$ is a local maximum of $v - \varphi$, we have :*

$$H(\bar{x}, v(\bar{x}), D_x\varphi(\bar{x}), D^2_{xx}\varphi(\bar{x})) \leq 0.$$

Remark 3.1 **1.** By misuse of notation, we shall say that v is viscosity supersolutin (resp. subsolution) to (3.6) by writing :

$$H(x, v, D_x v, D^2_{xx} v) \geq \text{(resp. } \leq) \quad 0, \quad x \in \mathcal{O}, \tag{3.7}$$

2. We recall that if v is a smooth C^2 function on $\mathcal{O}$, supersolution (resp. subsolution) in the classical sense to (3.7), then v is a viscosity supersolution (resp. subsolution) to (3.7).
3. There is an equivalent formulation of viscosity solutions, which is useful for proving uniqueness results, see [3] :
(i) A continuous function v on $\mathcal{O}$ is a viscosity supersolution to (3.6) if

$$H(x, v(x), p, M) \geq 0, \quad \forall x \in \mathcal{O},\ \forall (p, M) \in J^{2,-}v(x).$$

(ii) A continuous function v on $\mathcal{O}$ is a viscosity subsolution to (3.6) if

$$H(x, v(x), p, M) \leq 0, \quad \forall x \in \mathcal{O},\ \forall (p, M) \in J^{2,+}v(x).$$

Here $J^{2,+}v(x)$ is the second order superjet defined by :

$$J^{2,+}v(x) = \Big\{(p,M) \in \mathbb{R}^n \times S^n \; : \; \limsup_{\substack{x' \to x \\ x \in \mathcal{O}}} \frac{v(x') - v(x) - p.(x'-x) - \frac{1}{2}(x'-x).M(x'-x)}{|x'-x|^2} \;\leq\; 0 \Big\},$$

S^n is the set of symmetric $n \times n$ matrices, and $J^{2,-}v(x) = -J^{2,+}(-v)(x)$.

In the sequel, we shall denote by $\mathcal{L}_i$ the second order operator associated to the diffusion X when we are in regime i : for any C^2 function φ on $(0,\infty)$,

$$\mathcal{L}_i\varphi = \frac{1}{2}\sigma_i^2 x^2 \varphi" + b_i x \varphi'.$$

We then have the following PDE characterization of the value functions v_i.

Theorem 3.1 *The value functions v_i, $i \in \mathbb{I}_d$, are the unique viscosity solutions with linear growth condition on $(0,\infty)$ and boundary condition $v_i(0^+) = 0$, to the system of variational inequalities :*

$$\min\Big\{rv_i - \mathcal{L}_i v_i - f_i \;,\; v_i - \max_{j \neq i}(v_j - g_{ij})\Big\} = 0, \quad x \in (0,\infty), \quad i \in \mathbb{I}_d. \tag{3.8}$$

This means
(1) for each $i \in \mathbb{I}_d$, v_i is a viscosity solution to

$$\min\Big\{rv_i - \mathcal{L}_i v_i - f_i \;,\; v_i - \max_{j \neq i}(v_j - g_{ij})\Big\} = 0, \quad x \in (0,\infty). \tag{3.9}$$

(2) if w_i, $i \in \mathbb{I}_d$, are viscosity solutions with linear growth condition on $(0,\infty)$ and boundary condition $w_i(0^+) = 0$, to the system of variational inequalities (3.8), then $v_i = w_i$ on $(0,\infty)$, for all $i \in \mathbb{I}_d$.

Proof. The viscosity property follows from the dynamic programming principle and is proved in [9]. Uniqueness results for switching problems has been proved in [10] in the finite horizon case under different conditions. For sake of completeness, we provide in Appendix a proof of comparison principle in our infinite horizon context, which implies the uniqueness result. □

Remark 3.2 For fixed $i \in \mathbb{I}_d$, we also have uniqueness of viscosity solution to equation (3.9) in the class of continuous functions with linear growth condition on $(0,\infty)$ and given boundary condition on 0. In the next section, we shall use either uniqueness of viscosity solutions to the system (3.8) or for fixed i to equation (3.9), for the identification of an explicit solution in the two-regimes case $d = 2$.

For any regime $i \in \mathbb{I}_d$, we introduce the switching region :

$$\mathcal{S}_i = \left\{ x \in (0,\infty) \ : v_i(x) = \max_{j \neq i}(v_j - g_{ij})(x) \right\}.$$

$\mathcal{S}_i$ is a closed subset of $(0,\infty)$ and corresponds to the region where it is optimal for the operator to change of regime. The complement set $\mathcal{C}_i$ of $\mathcal{S}_i$ in $(0,\infty)$ is the so-called continuation region :

$$\mathcal{C}_i = \left\{ x \in (0,\infty) \ : v_i(x) > \max_{j \neq i}(v_j - g_{ij})(x) \right\},$$

where the operator remains in regime i. In this open domain, the value function v_i is smooth C^2 on $\mathcal{C}_i$ and satisfies in a classical sense :

$$rv_i(x) - \mathcal{L}_i v_i(x) - f_i(x) = 0, \quad x \in \mathcal{C}_i.$$

As a consequence of the condition (2.6), we have the following elementary partition property of the switching regions, see [9] :

$$\mathcal{S}_i = \cup_{j \neq i} \mathcal{S}_{ij}, \quad i \in \mathbb{I}_d,$$

where

$$\mathcal{S}_{ij} = \{ x \in \mathcal{C}_j \ : v_i(x) = (v_j - g_{ij})(x) \}.$$

$\mathcal{S}_{ij}$ represents the region where it is optimal to switch from regime i to regime j and stay here for a moment, i.e. without changing instantaneously from regime j to another regime. The following Lemma gives some partial information about the structure of the switching regions.

Lemma 3.3 *For all $i \neq j$ in $\mathbb{I}_d$, we have*

$$\mathcal{S}_{ij} \subset Q_{ij} := \{ x \in \mathcal{C}_j \ : (\mathcal{L}_j - \mathcal{L}_i) v_j(x) + (f_j - f_i)(x) - r g_{ij} \geq 0 \}.$$

Proof. Let $x \in \mathcal{S}_{ij}$. By setting $\varphi_j = v_j - g_{ij}$, this means that x is a minimum of $v_i - \varphi_j$ with $v_i(x) = \varphi_j(x)$. Moreover, since x lies in the open set $\mathcal{C}_j$ where v_j is smooth, we have that φ_j is C^2 in a neighborhood of x. By the supersolution viscosity property of v_i to the PDE (3.8), this yields :

$$r\varphi_j(x) - \mathcal{L}_i \varphi_j(x) - f_i(x) \geq 0. \tag{3.10}$$

Now recall that for $x \in \mathcal{C}_j$, we have

$$rv_j(x) - \mathcal{L}_j v_j(x) - f_j(x) = 0,$$

so that by substituting into (3.10), we obtain :

$$(\mathcal{L}_j - \mathcal{L}_i) v_j(x) + (f_j - f_i)(x) - r g_{ij} \geq 0,$$

which is the required result. □

We quote the smooth fit property on the value functions, proved in [9].

Theorem 3.2 *For all $i \in \mathbb{I}_d$, the value function v_i is continuously differentiable on $(0, \infty)$, and at $x \in \mathcal{S}_{ij}$, we have $v_i'(x) = v_j'(x)$.*

The next result provides suitable conditions for determining a viscosity solution to the variational inequality type arising in our switching problem.

Lemma 3.4 *Fix $i \in \mathbb{I}_d$. Let $\mathcal{C}$ be an open set in $(0, \infty)$, and w, h two continuous functions on $(0, \infty)$, with $w = h$ on $\mathcal{S} = (0, \infty) \setminus \mathcal{C}$, such that*

$$w \quad \text{is } C^1 \quad \text{on } \partial\mathcal{S} \tag{3.11}$$

$$w \quad \geq \quad h \quad \text{on } \mathcal{C}, \tag{3.12}$$

w is C^2 on $\mathcal{C}$, solution to

$$rw - \mathcal{L}_i w - f_i \;=\; 0 \quad \text{on } \mathcal{C}, \tag{3.13}$$

and w is a viscosity supersolution to

$$rw - \mathcal{L}_i w - f_i \;\geq\; 0 \quad \text{on } \mathrm{int}(\mathcal{S}). \tag{3.14}$$

Here $\mathrm{int}(\mathcal{S})$ is the interior of $\mathcal{S}$ and $\partial\mathcal{S} = \mathcal{S} \setminus \mathrm{int}(\mathcal{S})$ its boundary. Then, w is a viscosity solution to

$$\min\{rw - \mathcal{L}_i w - f_i, w - h\} \;=\; 0 \quad \text{on } (0, \infty). \tag{3.15}$$

Proof. Take some $\bar{x} \in (0, \infty)$ and distinguish the following cases :

$\star$ $\bar{x} \in \mathcal{C}$. Since $w = v$ is C^2 on $\mathcal{C}$ and satisfies $rw(\bar{x}) - \mathcal{L}_i w(\bar{x}) - f_i(\bar{x}) = 0$ by (3.13), and recalling $w(\bar{x}) \geq h(\bar{x})$ by (3.12), we obtain the classical solution property, and so a fortiori the viscosity solution property (3.15) of w at $\bar{x}$.

$\star$ $\bar{x} \in \mathcal{S}$. Then $w(\bar{x}) = h(\bar{x})$ and the viscosity subsolution property is trivial at $\bar{x}$. It remains to show the viscosity supersolution property at $\bar{x}$. If $\bar{x} \in \mathrm{int}(\mathcal{S})$, this follows directly from (3.14). Suppose now $\bar{x} \in \partial\mathcal{S}$, and w.l.o.g. $\bar{x}$ is on the left-boundary of $\mathcal{S}$ so that there exists $\varepsilon > 0$ s.t. $(\bar{x} - \varepsilon, \bar{x}) \subset \mathcal{C}$ on which w is smooth C^2. Take some smooth C^2 function φ s.t. $\bar{x}$ is a local minimum of $w - \varphi$. Since w is C^1 on $\bar{x}$ by (3.11), we have $\varphi'(\bar{x}) = w'(\bar{x})$ and $\varphi"(\bar{x}) \leq w"(\bar{x}^-)$ $(:= \liminf_{x \nearrow \bar{x}} w"(x))$. Now, from (3.13), we have $rw(x) - \mathcal{L}_i w(x) - f_i(x)$ $= 0$ for $x \in (\bar{x} - \varepsilon, \bar{x})$. By sending x to $\bar{x}$, we then obtain :

$$rw(\bar{x}) - \mathcal{L}_i \varphi(\bar{x}) - f_i(\bar{x}) \;\geq\; 0,$$

which is the required supersolution inequality, and ends the proof. □

Remark 3.3 Since $w = h$ on $\mathcal{S}$, relation (3.14) means equivalently that h is a viscosity supersolution to

$$rh - \mathcal{L}_i h - f_i \;\geq\; 0 \quad \text{on } \mathrm{int}(\mathcal{S}). \tag{3.16}$$

Practically, Lemma 3.4 shall be used as follows in the next section : we consider two C^1 functions v and h on $(0,\infty)$ s.t.

$$\begin{aligned} v(x) \;=\; h(x),\; v'(x) &= h'(x), \quad x \in \partial\mathcal{S} \\ v &\geq h \quad \text{on } \mathcal{C}, \end{aligned}$$

v is C^2 on $\mathcal{C}$, solution to

$$rv - \mathcal{L}_i v - f_i \;=\; 0 \quad \text{on } \mathcal{C},$$

and h is a viscosity supersolution to (3.16). Then, the function w defined on $(0,\infty)$ by :

$$w(x) \;=\; \begin{cases} v(x), & x \in \mathcal{C} \\ h(x), & x \in \mathcal{S} \end{cases}$$

satisfies the conditions of Lemma 3.4 and is so a viscosity solution to (3.15). This Lemma combined with uniqueness viscosity solution result may be viewed as an alternative to the classical verification approach in the identification of the value function. Moreover, with our viscosity solutions approach, we shall see in paragraph 4.2 that Lemma 3.3 and smooth-fit property of the value functions in Theorem 3.2 provide a direct derivation for the structure of the switching regions and then of the solution to our problem.

4 Explicit solution in the two regimes case

In this paragraph, we consider the case where $d = 2$. In this two-regimes case, the value functions v_1 and v_2 are the unique continuous viscosity solutions with linear growth, and $v_1(0^+) = v_2(0^+) = 0$, to the system :

$$\min\{rv_1 - \mathcal{L}_1 v_1 - f_1, v_1 - (v_2 - g_{12})\} \;=\; 0 \tag{4.17}$$

$$\min\{rv_2 - \mathcal{L}_2 v_2 - f_2, v_2 - (v_1 - g_{21})\} \;=\; 0. \tag{4.18}$$

Moreover, the switching regions are :

$$\mathcal{S}_i \;=\; \mathcal{S}_{ij} \;=\; \{x > 0 \;: v_i(x) = v_j(x) - g_{ij}\}, \quad i,j = 1,2,\; i \neq j.$$

We set

$$\underline{x}_i^* \;=\; \inf \mathcal{S}_i \qquad \bar{x}_i^* \;=\; \sup \mathcal{S}_i,$$

with the usual convention that $\inf \emptyset = \infty$. By continuity of the value functions on $(0,\infty)$ and since $v_i(0^+) = 0 > -g_{ij} = v_j(0^+) - g_{ij}$, it is clear that

$$\underline{x}_i^* \;>\; 0, \quad i = 1,2.$$

Let us also introduce some other notations. We consider the second order o.d.e for $i = 1,2$:

$$rv - \mathcal{L}_i v - f_i \;=\; 0, \tag{4.19}$$

whose general solution (without second member f_i) is given by :

$$v(x) \;=\; Ax^{m_i^+} + Bx^{m_i^-},$$

for some constants A, B, and where

$$\begin{aligned} m_i^- &= -\frac{b_i}{\sigma_i^2} + \frac{1}{2} - \sqrt{\left(-\frac{b_i}{\sigma_i^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma_i^2}} \;<\; 0 \\ m_i^+ &= -\frac{b_i}{\sigma_i^2} + \frac{1}{2} + \sqrt{\left(-\frac{b_i}{\sigma_i^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma_i^2}} \;>\; 1. \end{aligned}$$

We also denote

$$\hat{V}_i(x) \;=\; E\left[\int_0^\infty e^{-rt} f_i(\hat{X}_t^{x,i})dt\right],$$

with $\hat{X}^{x,i}$ the solution to the s.d.e. $d\hat{X}_t = b_i\hat{X}_t dt + \sigma_i \hat{X}_t dW_t$, $\hat{X}_0 = x$. Actually, $\hat{V}_i$ is a particular solution to ode (4.19). It corresponds to the reward function associated to the no switching strategy from initial state (x,i), and so $\hat{V}_i \leq v_i$.

We now explicit the solution to our problem in the following two situations :
⋆ the diffusion operators are different and the running profit functions are identical.
⋆ the diffusion operators are identical and the running profit functions are different

4.1 Identical profit functions with different diffusion operators

In this paragraph, we suppose that the running functions are identical in the form :

$$f_1(x) \;=\; f_2(x) \;=\; x^\gamma, \quad 0 < \gamma < 1, \tag{4.20}$$

and the diffusion operators are different. A straightforward calculation shows that under (4.20), we have

$$\hat{V}_i(x) \;=\; K_i x^\gamma, \quad \text{with } K_i \;=\; \frac{1}{r - b_i\gamma + \frac{1}{2}\sigma_i^2\gamma(1-\gamma)} \;>\; 0, \;\; i = 1, 2.$$

We show that the structure of the switching regions depends actually only on the sign of $K_2 - K_1$. More precisely, we have the following explicit result.

Theorem 4.3 *Let i, $j = 1, 2$, $i \neq j$.*
1) If $K_i = K_j$, then $\mathcal{S}_i = \mathcal{S}_j = \emptyset$. We have

$$v_i(x) \;=\; \hat{V}_i(x) \;=\; v_j(x) \;=\; \hat{V}_j(x), \qquad x \in (0,\infty),$$

and in both regimes, it is optimal never to switch.
2) If $K_j > K_i$, then $\mathcal{S}_i = [\underline{x}_i^*, \infty)$ with $\underline{x}_i^* \in (0,\infty)$, and $\mathcal{S}_j = \emptyset$. We have

$$v_i(x) \;=\; \begin{cases} Ax^{m_i^+} + \hat{V}_i(x), & x < \underline{x}_i^* \\ v_j(x) - g_{ij}, & x \geq \underline{x}_i^* \end{cases} \tag{4.21}$$

$$v_j(x) \;=\; \hat{V}_j(x), \quad x \in (0,\infty) \tag{4.22}$$

where the constants A and $\underline{x}_i^*$ are determined by the continuity and smooth-fit conditions of v_i at $\underline{x}_i^*$, and explicitly given by :

$$x_i^\gamma = \frac{m_i^+}{m_i^+ - \gamma} \frac{g_{ij}}{K_j - K_i} \tag{4.23}$$

$$A = (K_j - K_i) \frac{\gamma}{m_i^+} x_i^{\gamma - m_i^+}. \tag{4.24}$$

Furthermore, when we are in regime i,it is optimal to switch to regime j whenever the state process X exceeds the threshold $\underline{x}_i^*$, while when we are in regime j, it is optimal never to switch.

Remark 4.4 In the particular case where $\sigma_1 = \sigma_2$, then $K_2 - K_1 > 0$ means that regime 2 provides a higher expected return b_2 than the one b_1 of regime 1 for the same volatility coefficient σ_i. Hence, it is intuitively clear that regime 2 is better than regime 1, which is formalized by the property that $\mathcal{S}_2 = \emptyset$. Similarly, when $b_1 = b_2$, then $K_2 - K_1 > 0$ means that $\sigma_2 < \sigma_1$, i.e. regime 2 is less risky than regime 1 for the same return b_i and so is better. Theorem 4.3 extends these results for general coefficients b_i and σ_i, and show that the critical parameter value determining the form of the optimal strategy is given by the sign of $K_2 - K_1$. The optimal strategy structure is depicted in Figure 1.

Proof of Theorem 4.3.

1) If $K_1 = K_2$, then $\hat{V}_1 = \hat{V}_2$. By the definition of $\hat{V}_i$, and since switching costs are nonnegative, we thus get immediately that $\hat{V}_i$, $i = 1, 2$, are smooth solutions to the system :

$$\min\left\{r\hat{V}_1 - \mathcal{L}_1\hat{V}_1 - f_1, \hat{V}_1 - (\hat{V}_2 - g_{12})\right\} = 0$$
$$\min\left\{r\hat{V}_2 - \mathcal{L}_2\hat{V}_2 - f_2, \hat{V}_2 - (\hat{V}_1 - g_{21})\right\} = 0.$$

Recalling that $\hat{V}_i(0^+) = 0$ and $\hat{V}_i$ satisfy a linear growth condition, and from uniqueness of solution to the PDE system (4.17)-(4.18), we deduce that $v_i = \hat{V}_i$, i.e. $\mathcal{S}_i = \emptyset$, $i = 1, 2$.

2) We now suppose w.l.o.g. that $K_2 > K_1$. We already know that $\underline{x}_1^* > 0$ and we claim that $\underline{x}_1^* < \infty$. Otherwise, v_1 should be equal to $\hat{V}_1$. Since $v_1 \geq v_2 - g_{12} \geq \hat{V}_2 - g_{12}$, this would imply $(\hat{V}_2 - \hat{V}_1)(x) = (K_2 - K_1)x^\gamma \leq g_{12}$ for all $x > 0$, an obvious contradiction.

$\star$ By definition of $\underline{x}_1^*$, we have $(0, \underline{x}_1^*) \subset \mathcal{C}_1$. We prove actually the equality : $(0, \underline{x}_1^*) = \mathcal{C}_1$, i.e. $\mathcal{S}_1 = [\underline{x}_1^*, \infty)$, and also that $\mathcal{C}_2 = (0, \infty)$, i.e. $\mathcal{S}_2 = \emptyset$. To this end, let us consider the function

$$w_1(x) = \begin{cases} Ax^{m_1^+} + \hat{V}_1(x), & 0 < x < x_1 \\ \hat{V}_2(x) - g_{12}, & x \geq x_1, \end{cases}$$

where the positive constants A and x_1 satisfy

$$Ax_1^{m_1^+} + \hat{V}_1(x_1) = \hat{V}_2(x_1) - g_{12} \tag{4.25}$$

$$Am_1^+ x_1^{m_1^+ - 1} + \hat{V}_1'(x_1) = \hat{V}_2'(x_1), \tag{4.26}$$

and are explicitly given by :

$$
\begin{aligned}
(K_2 - K_1)x_1^\gamma &= \frac{m_1^+}{m_1^+ - \gamma} g_{12} && (4.27)\\
A &= (K_2 - K_1)\frac{\gamma}{m_1^+} x_1^{\gamma - m_1^+}. && (4.28)
\end{aligned}
$$

Notice that by construction, w_1 is C^2 on $(0, x_1) \cup (x_1, \infty)$, and C^1 on x_1. By using Lemma 3.4, we now show that w_1 is a viscosity solution to

$$
\min\left\{ r w_1 - \mathcal{L}_1 w_1 - f_1, w_1 - (\hat{V}_2 - g_{12}) \right\} = 0, \quad \text{on } (0, \infty). \tag{4.29}
$$

We first check that

$$
w_1(x) \geq \hat{V}_2(x) - g_{12}, \quad \forall\, 0 < x < x_1, \tag{4.30}
$$

i.e.

$$
G(x) := A x^{m_1^+} + \hat{V}_1(x) - \hat{V}_2(x) + g_{12} \geq 0, \quad \forall\, 0 < x < x_1.
$$

Since $A > 0$, $0 < \gamma < 1 < m_1^+$, $K_2 - K_1 > 0$, a direct derivation shows that the second derivative of G is positive, i.e. G is strictly convex. By (4.26), we have $G'(x_1) = 0$ and so G' is negative, i.e. G is strictly decreasing on $(0, x_1)$. Now, by (4.25), we have $G(x_1) = 0$ and thus G is positive on $(0, x_1)$, which proves (4.30).

By definition of w_1 on $(0, x_1)$, we have in the classical sense

$$
r w_1 - \mathcal{L}_1 w_1 - f_1 = 0, \quad \text{on } (0, x_1). \tag{4.31}
$$

We now check that

$$
r w_1 - \mathcal{L}_1 w_1 - f_1 \geq 0, \quad \text{on } (x_1, \infty), \tag{4.32}
$$

holds true in the classical sense, and so a fortiori in the viscosity sense. By definition of w_1 on (x_1, ∞), and K_1, we have for all $x > x_1$,

$$
r w_1(x) - \mathcal{L}_1 w_1(x) - f_1(x) = \frac{K_2 - K_1}{K_1} x^\gamma - r g_{12}, \quad \forall x > x_1,
$$

so that (4.32) is satisfied iff $\frac{K_2 - K_1}{K_1} x_1^\gamma - r g_{12} \geq 0$ or equivalently by (4.27) :

$$
\frac{m_1^+}{m_1^+ - \gamma} \geq r K_1 = \frac{r}{r - b_1\gamma + \frac{1}{2}\sigma_1^2 \gamma(1-\gamma)} \tag{4.33}
$$

Now, since $\gamma < 1 < m_1^+$, and by definition of m_1^+, we have

$$
\frac{1}{2}\sigma_1^2 m_1^+ (\gamma - 1) < \frac{1}{2}\sigma_1^2 m_1^+ (m_1^+ - 1) = r - b_1 m_1^+,
$$

which proves (4.33) and thus (4.32).

Relations (4.25)-(4.26), (4.30)-(4.31)-(4.32) mean that conditions of Lemma 3.4 are satisfied with $\mathcal{C} = (0, x_1)$, $h = \hat{V}_2 - g_{12}$, and we thus get the required assertion (4.29).

$\star$ On the other hand, we check that

$$\hat{V}_2(x) \quad \geq \quad w_1(x) - g_{21}, \qquad \forall x > 0, \tag{4.34}$$

which amounts to show

$$H(x) \ := \ Ax^{m_1^+} + \hat{V}_1(x) - \hat{V}_2(x) - g_{21} \quad \leq \quad 0, \qquad \forall\, 0 < x < x_1.$$

Since $A > 0$, $0 < \gamma < 1 < m_1^+$, $K_2 - K_1 > 0$, a direct derivation shows that the second derivative of H is positive, i.e. H is strictly convex. By (4.26), we have $H'(x_1) = 0$ and so H' is negative, i.e. H is strictly decreasing on $(0, x_1)$. Now, we have $H(0) = -g_{21} < 0$ and thus H is negative on $[0, x_1)$, which proves (4.34). Recalling that $\hat{V}_2$ is solution to $r\hat{V}_2 - \mathcal{L}_2\hat{V}_2 - f_2 = 0$ on $(0,\infty)$, we deduce obviously from (4.34) that $\hat{V}_2$ is a classical, hence a viscosity solution to :

$$\min\left\{r\hat{V}_2 - \mathcal{L}_2\hat{V}_2 - f_2, \hat{V}_2 - (w_1 - g_{21})\right\} \quad = \quad 0, \quad \text{ on } \ (0,\infty). \tag{4.35}$$

$\star$ Since $w_1(0^+) = \hat{V}_2(0^+) = 0$, w_1, $\hat{V}_2$ satisfy a linear growth condition, we deduce from (4.29), (4.35), and uniqueness to the PDE system (4.17)-(4.18), that

$$v_1 \ = \ w_1, \qquad v_2 \ = \ \hat{V}_2, \quad \text{ on } \ (0,\infty).$$

This proves $\underline{x}_1^* = x_1$, $\mathcal{S}_1 = [x_1, \infty)$ and $\mathcal{S}_2 = \emptyset$, and ends the proof.

4.2 Identical diffusion operators with different profit functions

In this paragraph, we suppose that $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$, i.e. $b_1 = b_2 = b$, $\sigma_1 = \sigma_2 = \sigma > 0$. We then set $m^+ = m_1^+ = m_2^+$, $m^- = m_1^- = m_2^-$, and $\hat{X}^x = \hat{X}^{x,1} = \hat{X}^{x,2}$. Notice that in this case, the set Q_{ij}, i, $j = 1, 2$, $i \neq j$, introduced in Lemma 3.3, satisfies :

$$\begin{aligned} Q_{ij} \quad &= \quad \{x \in \mathcal{C}_j \ : (f_j - f_i)(x) - rg_{ij} \geq 0\} \\ &\subset \quad \hat{Q}_{ij} \ := \ \{x > 0 \ : (f_j - f_i)(x) - rg_{ij} \geq 0\}. \end{aligned} \tag{4.36}$$

Once we are given the profit functions f_i, f_j, the set $\hat{Q}_{ij}$ can be explicitly computed. Moreover, we prove in the next Lemma that the structure of $\hat{Q}_{ij}$ determines the same structure for the switching region $\mathcal{S}_i$.

Lemma 4.5 *Let $i, j = 1, 2$, $i \neq j$.*
1) Assume that

$$\sup_{x>0}(\hat{V}_j - \hat{V}_i)(x) \quad > \quad g_{ij}, \tag{4.37}$$

and there exists $0 < \underline{x}_{ij} < \infty$ such that

$$\hat{Q}_{ij} \quad = \quad [\underline{x}_{ij}, \infty). \tag{4.38}$$

Then $0 < \underline{x}_i^* < \infty$ and

$$\mathcal{S}_i \quad = \quad [\underline{x}_i^*, \infty).$$

2) Assume that there exist $0 < \underline{x}_{ij} < \bar{x}_{ij} < \infty$ *such that*

$$\hat{Q}_{ij} = [\underline{x}_{ij}, \bar{x}_{ij}]. \tag{4.39}$$

Then $0 < \underline{x}_i^* < \bar{x}_i^* < \underline{x}_j^* \wedge \infty$ *and*

$$\mathcal{S}_i = [\underline{x}_i^*, \bar{x}_i^*].$$

Proof. 1) Since $\mathcal{S}_i \subset \hat{Q}_{ij}$ by Lemma 3.3 and (4.36), the condition (4.38) implies $\underline{x}_i^* \geq \underline{x}_{ij}$ > 0. We now claim that $\underline{x}_i^* < \infty$. On the contrary, the switching region $\mathcal{S}_i$ would be empty, and so v_i would satisfy on $(0, \infty)$:

$$rv_i - \mathcal{L}v_i - f_i = 0, \quad \text{on } (0, \infty).$$

Then, v_i would be on the form :

$$v_i(x) = Ax^{m^+} + Bx^{m^-} + \hat{V}_i(x), \quad x > 0.$$

Recalling from Lemma 3.1 that $v_i(0^+) = 0$ and v_i is a nonnegative function satisfying a linear growth condition, and using the fact that $m^- < 0$ and $m^+ > 1$, we deduce that v_i should be equal to $\hat{V}_i$. Now, since we have $v_i \geq v_j - g_{ij} \geq \hat{V}_j - g_{ij}$, this would imply :

$$\hat{V}_j(x) - \hat{V}_i(x) \leq g_{ij}, \quad \forall x > 0.$$

This contradicts condition (4.37) and so $0 < \underline{x}_i^* < \infty$.

By definition of $\underline{x}_i^*$, we already know that $(0, \underline{x}_i^*) \subset \mathcal{C}_i$. We prove actually the equality, i.e. $\mathcal{S}_i = [\underline{x}_i^*, \infty)$ or $v_i(x) = v_j(x) - g_{ij}$ for all $x \geq \underline{x}_i^*$. Consider the function

$$w_i(x) = \begin{cases} v_i(x), & 0 < x < \underline{x}_i^* \\ v_j(x) - g_{ij}, & x \geq \underline{x}_i^* \end{cases}$$

We now check that w_i is a viscosity solution of

$$\min\{rw_i - \mathcal{L}w_i - f_i\ ,\ w_i - (v_j - g_{ij})\} = 0 \quad \text{on} \ \ (0, \infty). \tag{4.40}$$

From Theorem 3.2, the function w_i is C^1 on $(0, \infty)$ and in particular at $\underline{x}_i^*$ where $w_i'(\underline{x}_i^*) = v_i'(\underline{x}_i^*) = v_j'(\underline{x}_i^*)$. We also know that $w_i = v_i$ is C^2 on $(0, \underline{x}_i^*) \subset \mathcal{C}_i$, and satisfies $rw_i - \mathcal{L}w_i - f_i$ $= 0$, $w_i \geq (v_j - g_{ij})$ on $(0, \underline{x}_i^*)$. Hence, from Lemma 3.4, we only need to check the viscosity supersolution property of w_i to :

$$rw_i - \mathcal{L}w_i - f_i \geq 0, \quad \text{on } \ (\underline{x}_i^*, \infty). \tag{4.41}$$

For this, take some point $\bar{x} > \underline{x}_i^*$ and some smooth test function φ s.t. $\bar{x}$ is a local minimum of $w_i - \varphi$. Then, $\bar{x}$ is a local minimum of $v_j - (\varphi + g_{ij})$, and by the viscosity solution property of v_j to its Bellman PDE, we have

$$rv_j(\bar{x}) - \mathcal{L}\varphi(x_0) - f_j(\bar{x}) \geq 0.$$

Now, since $\underline{x}_i^* \geq \underline{x}_{ij}$, we have $\bar{x} > \underline{x}_{ij}$ and so by (4.38), $\bar{x} \in \hat{Q}_{ij}$. Hence,

$$(f_j - f_i)(\bar{x}) - rg_{ij} \quad \geq \quad 0.$$

By adding the two previous inequalities, we also obtain the required supersolution inequality :

$$rw_i(\bar{x}) - \mathcal{L}\varphi(\bar{x}) - f_i(\bar{x}) \quad \geq \quad 0,$$

and so (4.40) is proved.

Since $w_i(0^+) = v_i(0^+)$ $(= 0)$ and w_i satisfies a linear growth condition, and from uniqueness of viscosity solution to PDE (4.40), we deduce that w_i is equal to v_i. In particular, we have $v_i(x) = v_j(x) - g_{ij}$ for $x \geq \underline{x}_i^*$, which shows that $\mathcal{S}_i = [\underline{x}_i^*, \infty)$.

2) By Lemma 3.3 and (4.36), the condition (4.39) implies $0 < \underline{x}_{ij} \leq \underline{x}_i^* \leq \bar{x}_i^* \leq \bar{x}_{ij} < \infty$. We claim that $\underline{x}_i^* < \bar{x}_i^*$. Otherwise, $\mathcal{S}_2 = \{\bar{x}_i^*\}$ and v_i would satisfy $rv_i - \mathcal{L}v_i - f_i = 0$ on $(0, \bar{x}_i^*) \cup (\bar{x}_i^*, \infty)$. By continuity and smooth-fit condition of v_i at $\hat{x}$, this implies that v_i satisfies actually

$$rv_i - \mathcal{L}v_i - f_i \quad = \quad 0, \quad x \in (0, \infty),$$

and so is in the form :

$$v_i(x) \quad = \quad Ax^{m^+} + Bx^{m^-} + \hat{V}_i(x), \quad x \in (0, \infty)$$

Recalling from Lemma 3.1 that $v_i(0^+) = 0$ and v_i satisfy a linear growth condition, this implies $A = B = 0$. Therefore, v_i is equal to $\hat{V}_i$, which also means that $\mathcal{S}_i = \emptyset$, a contradiction.

We now prove that $\mathcal{S}_i = [\underline{x}_i^*, \bar{x}_i^*]$. Let us consider the function

$$w_i(x) \quad = \quad \begin{cases} v_i(x), & x \in (0, \underline{x}_i^*) \cup (\bar{x}_i^*, \infty) \\ v_j(x) - g_{ij}, & x \in [\underline{x}_i^*, \bar{x}_i^*], \end{cases}$$

which is C^1 on $(0, \infty)$ and in particular on $\underline{x}_i^*$ and $\bar{x}_i^*$ from Theorem 3.2. Hence, by similar arguments as in case 1), using Lemma 3.4, we then show that w_i is a viscosity solution of

$$\min\{rw_i - \mathcal{L}w_i - f_i \ , \ w_i - (v_j - g_{ij})\} \quad = \quad 0. \tag{4.42}$$

Since $w_i(0^+) = v_i(0^+)$ $(= 0)$ and w_i satisfies a linear growth condition, and from uniqueness of viscosity solution to PDE (4.42), we deduce that w_i is equal to v_i. In particular, we have $v_i(x) = v_j(x) - g_{ij}$ for $x \in [\underline{x}_i^*, \bar{x}_i^*]$, which shows that $\mathcal{S}_i = [\underline{x}_i^*, \bar{x}_i^*]$. Finally, since $\mathcal{S}_i \subset \mathcal{C}_j$, this also shows that $\bar{x}_i^* < \underline{x}_j^*$. □

A typical example of different running profit functions is given by

$$f_i(x) \quad = \quad x^{\gamma_i}, \ \ i = 1, 2, \quad \text{with } 0 < \gamma_1 < \gamma_2 < 1. \tag{4.43}$$

Actually, we shall provide explicit solutions to the switching problem for more general different profit functions f_i including (4.43). We assume there exists $\hat{x} \in (0,\infty)$ s.t.

$$\textbf{(HF)} \qquad F := f_2 - f_1 \qquad \text{is strictly decreasing on } (0,\hat{x}), \text{ strictly increasing on } [\hat{x},\infty) \text{ and } \lim_{x\to\infty} F(x) = \infty.$$

Since $F(0) = 0$, notice that $F(\hat{x}) < 0$. Economically speaking, the last condition **(HF)** means that regime 2 is "better" than regime 1 from a certain level $\hat{x}$, and the improvement becomes then better and better.

The next proposition states the form of the switching regions.

Proposition 4.1 *Assume that* **(HF)** *holds.*
1) We have $\underline{x}_1^* \in (0,\infty)$ *and* $\mathcal{S}_1 = [\underline{x}_1^*,\infty)$.
2) i) If $rg_{21} \geq -F(\hat{x})$, *then* $\mathcal{S}_2 = \emptyset$.
ii) If $rg_{21} < -F(\hat{x})$, *then* $0 < \underline{x}_2^* < \bar{x}_2^* < \underline{x}_1^*$, *and* $\mathcal{S}_2 = [\underline{x}_2^*,\bar{x}_2^*]$.

Proof. 1) From Lemma 3.3, we have

$$\hat{Q}_{12} = \{x > 0 \, : F(x) - rg_{12} \geq 0\}. \tag{4.44}$$

Under **(HF)** and since $F(0) - rg_{12} < 0$, $F(\infty) - rg_{12} > 0$, there exists $\hat{x}_{12} \in (0,\infty)$ such that

$$\hat{Q}_{12} = [\underline{x}_{12},\infty). \tag{4.45}$$

Moreover, since

$$(\hat{V}_2 - \hat{V}_1)(x) = E\left[\int_0^\infty e^{-rt} F(\hat{X}_t^x)dt\right], \quad \forall x > 0,$$

and $F(\infty) = \infty$, it is not difficult to see that $\lim_{x\to\infty}(\hat{V}_2 - \hat{V}_1)(x) = \infty$. Hence, conditions (4.37)-(4.38) with $i = 1$, $j = 2$, are satisfied, and we obtain the first assertion by Lemma 4.5 1).
2) From Lemma 3.3, we have

$$\hat{Q}_{21} = \{x > 0 \, : -F(x) - rg_{21} \geq 0\}. \tag{4.46}$$

Under **(HF)**, we distinguish the following cases :

(i1) If $rg_{21} > -F(\hat{x})$, then, $\hat{Q}_{21} = \emptyset$, and so $\mathcal{S}_2 = \emptyset$.
(i2) If $rg_{21} = -F(\hat{x})$, then, $\hat{Q}_{21} = \{\hat{x}\}$ and so $\mathcal{S}_2 \subset \{\hat{x}\}$. In this case, v_2 satisfies $rv_2 - \mathcal{L}v_2 - f_2 = 0$ on $(0,\hat{x}) \cup (\hat{x},\infty)$. By continuity and smooth-fit condition of v_2 at $\hat{x}$, this implies that v_2 satisfies actually

$$rv_2 - \mathcal{L}v_2 - f_2 = 0, \quad x \in (0,\infty),$$

and so is in the form :

$$v_2(x) = Ax^{m^+} + Bx^{m^-} + \hat{V}_2(x), \quad x \in (0,\infty)$$

Recalling from Lemma 3.1 that $v_2(0^+) = 0$ and v_2 satisfy a linear growth condition, this implies $A = B = 0$. Therefore, v_2 is equal to $\hat{V}_2$, which also means that $\mathcal{S}_2 = \emptyset$.

(ii) If $rg_{21} < -F(\hat{x})$. Then there exist $0 < \underline{x}_{21} < \hat{x} < \bar{x}_{21} < \infty$ such that

$$\hat{Q}_{21} = [\underline{x}_{21}, \bar{x}_{21}]. \tag{4.47}$$

We then conclude with Lemma 4.5 2) for $i = 2$, $j = 1$. □

Remark 4.5 In our viscosity solutions approach, the structure of the switching regions is derived from the smooth fit property of the value functions, uniqueness result for viscosity solutions and Lemma 3.3. This contrasts with the classical verification approach where the structure of switching regions should be guessed ad-hoc and checked a posteriori by a verification argument.

We thus finally explicit the value functions and the optimal sequential stopping times. The structure of the optimal strategy is depicted in figure 2.

Theorem 4.4 *Assume that* **(HF)** *holds.*
i) If $rg_{21} \geq -F(\hat{x})$, *then*

$$v_1(x) = \begin{cases} Ax^{m^+} + \hat{V}_1(x), & x < \underline{x}_1^* \\ v_2(x) - g_{12}, & x \geq \underline{x}_1^* \end{cases} \tag{4.48}$$

$$v_2(x) = \hat{V}_2(x) \tag{4.49}$$

where the constants A *and* $\underline{x}_1^*$ *are determined by the continuity and smooth-fit conditions of* v_1 *at* $\underline{x}_1^*$ *:*

$$A(\underline{x}_1^*)^{m^+} + \hat{V}_1(\underline{x}_1^*) = \hat{V}_2(\underline{x}_1^*) - g_{12} \tag{4.50}$$

$$Am^+(\underline{x}_1^*)^{m^+-1} + \hat{V}_1'(\underline{x}_1^*) = \hat{V}_2'(\underline{x}_1^*). \tag{4.51}$$

Furthermore, when we are in regime 1, *it is optimal to switch to regime* 2 *whenever the state process* X *exceeds the threshold* $\underline{x}_1^*$, *while when we are in regime* 2, *it is optimal never to switch.*
ii) If $rg_{21} < -F(\hat{x})$, *then*

$$v_1(x) = \begin{cases} A_1x^{m^+} + \hat{V}_1(x), & x < \underline{x}_1^* \\ v_2(x) - g_{12}, & x \geq \underline{x}_1^* \end{cases} \tag{4.52}$$

$$v_2(x) = \begin{cases} A_2x^{m^+} + \hat{V}_2(x), & x < \underline{x}_2^* \\ v_1(x) - g_{21}, & \underline{x}_2^* \leq x \leq \bar{x}_2^* \\ B_2x^{m^-} + \hat{V}_2(x), & x > \bar{x}_2^* \end{cases} \tag{4.53}$$

where the constants A_1 *and* $\underline{x}_1^*$ *are determined by the continuity and smooth-fit conditions of* v_1 *at* $\underline{x}_1^*$, *and the constants* A_2, B_2, $\underline{x}_2^*$, $\bar{x}_2^*$ *are determined by the continuity and smooth-fit* *conditions of v_2 at $\underline{x}_2^*$ and $\bar{x}_2^*$:*

$$
\begin{aligned}
A_1(\underline{x}_1^*)^{m^+} + \hat{V}_1(\underline{x}_1^*) &= B_2(\underline{x}_1^*)^{m^-} + \hat{V}_2(\underline{x}_1^*) - g_{12} && (4.54)\\
A_1 m^+(\underline{x}_1^*)^{m^+-1} + \hat{V}_1'(\underline{x}_1^*) &= B_2 m^-(\underline{x}_1^*)^{m^--1} + \hat{V}_2'(\underline{x}_1^*) && (4.55)\\
A_2(\underline{x}_2^*)^{m^+} + \hat{V}_2(\underline{x}_2^*) &= A_1(\underline{x}_2^*)^{m^+} + \hat{V}_1(\underline{x}_2^*) - g_{21} && (4.56)\\
A_2 m^+(\underline{x}_2^*)^{m^+-1} + \hat{V}_2'(\underline{x}_2^*) &= A_1 m^+(\underline{x}_2^*)^{m^+-1} + \hat{V}_1'(\underline{x}_2^*) && (4.57)\\
A_1(\bar{x}_2^*)^{m^+} + \hat{V}_1(\bar{x}_2^*) - g_{21} &= B_1(\bar{x}_2^*)^{m^-} + \hat{V}_2(\bar{x}_2^*) && (4.58)\\
A_1 m^+(\bar{x}_2^*)^{m^+-1} + \hat{V}_1'(\bar{x}_2^*) &= B_1 m^-(\bar{x}_2^*)^{m^--1} + \hat{V}_2'(\bar{x}_2^*). && (4.59)
\end{aligned}
$$

Furtheremore, when we are in regime 1, it is optimal to switch to regime 2 whenever the state process X exceeds the threshold $\underline{x}_1^*$, while when we are in regime 2, it is optimal to switch to regime 1 whenever the state process lies between $\underline{x}_2^*$ and $\bar{x}_2^*$.

Proof. 1. From Proposition 4.1, we have $\mathcal{S}_1 = [\underline{x}_1^*, \infty)$, which means that when we are in regime 1, it is optimal to switch to regime 2 whenever the state process exceeds $\underline{x}_1^*$. Moreover, we have $v_1 = v_2 - g_{12}$ on $[\underline{x}_1^*, \infty)$ and v_1 is solution to $rv_1 - \mathcal{L}v_1 - f_1 = 0$ on $(0, \underline{x}_1^*)$. Since $v_1(0^+) = 0$, v_1 should have the form expressed in (4.48) or (4.52).

2. The form of v_2 and $\mathcal{S}_2$ depends on the two following cases :

(i) If $rg_{21} \geq -F(\hat{x})$, then from Proposition 4.1, $\mathcal{S}_2$ is empty, which means that when we are in regime 1, it is never optimal to switch of regime. This also means that v_2 is equal to $\hat{V}_2$, the unique solution with linear growth condition on $(0,\infty)$ to $rv_2 - \mathcal{L}v_2 - f_2 = 0$, with $v_2(0^+) = 0$. The constants A and $\underline{x}_1^*$ expliciting completely v_1 are then determined by the two relations (4.50)-(4.51) resulting from the continuity and smooth-fit conditions of v_1 at x_1^*.

(ii) If $rg_{21} < -F(\hat{x})$, then from Proposition 4.1, $\mathcal{S}_2 = [\underline{x}_2^*, \bar{x}_2^*]$, which means that when we are in regime 2, it is optimal to switch to regime 1 whenever the state process lies between $[\underline{x}_2^*, \bar{x}_2^*]$. Moreover, v_2 satisfies on $\mathcal{C}_2 = (0, \underline{x}_2^*) \cup (\bar{x}_2^*, \infty)$: $rv_2 - \mathcal{L}v_2 - f_2 = 0$. Recalling again that $v_2(0^+) = 0$ and v_2 satisfies a linear growth condition, we deduce that v_2 has the form expressed in (4.53). Finally, the constants A_1, $\underline{x}_1^*$ expliciting completely v_1, and the constants A_2, B_2, $\underline{x}_2^*$, $\bar{x}_2^*$ expliciting v_2 are determined by the six relations (4.54)-(4.55)-(4.56)-(4.57)-(4.58)-(4.59) resulting from the continuity and smooth-fit conditions of v_1 at $\underline{x}_1^*$ and v_2 at $\underline{x}_2^*$ and $\bar{x}_2^*$. □

Remark 4.6 In the classical approach, for instance in the case ii) $rg_{21} \leq -F(\hat{x})$, we construct a priori a candidate solution in the form (4.52)-(4.53), and we have to check the existence of a sixtuple solution to (4.54)-(4.55)-(4.56)-(4.57)-(4.58)-(4.59), which may be somewhat tedious! Here, our viscosity solutions approach, and since we already state the smooth-fit C^1 property of the value functions, we know a priori the existence of a sixtuple solution to (4.54)-(4.55)-(4.56)-(4.57)-(4.58)-(4.59).

Appendix: proof of comparison principle

In this section, we prove a comparison principle for the system of variational inequalities (3.8). The comparison result in [10] for switching problems in finite horizon does not apply in our context. Inspired by [8], we first produce some suitable perturbation of viscosity supersolution to deal with the switching obstacle, and then follow the general viscosity solution technique, see e.g. [3].

Theorem 4.5 *Suppose u_i, $i \in \mathbb{I}_d$, are continuous viscosity subsolutions to the system of variational inequalities (3.8) on $(0,\infty)$, and w_i, $i \in \mathbb{I}_d$, are continuous viscosity supersolutions to the system of variational inequalities (3.8) on $(0,\infty)$, satisfying the boundary conditions $u_i(0^+) \leq w_i(0^+)$, $i \in I_d$, and the linear growth condition :*

$$|u_i(x)| + |w_i(x)| \quad \leq \quad C_1 + C_2 x, \quad \forall x \in (0,\infty), \; i \in \mathbb{I}_d, \tag{A.1}$$

for some positive constants C_1 and C_2. Then,

$$u_i \quad \leq \quad w_i, \quad \text{on } (0,\infty), \;\; \forall i \in \mathbb{I}_d.$$

Proof. Step 1. Let u_i and w_i, $i \in \mathbb{I}_d$, as in Theorem 4.5. We first construct strict supersolutions to the system (3.8) with suitable perturbations of w_i, $i \in \mathbb{I}_d$. We set

$$h(x) \quad = \quad C_1' + C_2' x^p, \quad x > 0,$$

where C_1', $C_2' > 0$ and $p > 1$ are positive constants to be determined later. We then define for all $\lambda \in (0,1)$, the continuous functions on $(0,\infty)$ by :

$$w_i^\lambda = (1-\lambda) w_i + \lambda h, \quad i \in \mathbb{I}_d.$$

We then see that for all $\lambda \in (0,1)$, $i \in \mathbb{I}_d$:

$$\begin{aligned} w_i^\lambda - \max_{j \neq i}(w_j^\lambda - g_{ij}) &= (1-\lambda) w_i - \max_{j\neq i}[(1-\lambda)(w_j - g_{ij}) - \lambda g_{ij}] \\ &\geq (1-\lambda)[w_i - \max_{j \neq i}(w_j - g_{ij})] + \lambda \min_{j \neq i} g_{ij} \\ &\geq \lambda \underline{g}, \end{aligned} \tag{A.2}$$

where $\underline{g} := \min_{i \in \mathbb{I}_d} \min_{j \neq i} g_{ij} > 0$ is a positive constant independent of i. By definition of the Fenchel Legendre in (2.4), and by setting $\tilde{f}(1) = \max_{i \in \mathbb{I}_d} \tilde{f}_i(1)$, we have for all $i \in \mathbb{I}_d$,

$$f_i(x) \quad \leq \quad \tilde{f}(1) + x \; \leq \tilde{f}(1) + 1 + x^p, \quad \forall x > 0.$$

Moreover, recalling that $r > b := \max_i b_i$, we can choose $p > 1$ s.t.

$$\rho \; := \; r - pb - \frac{1}{2}\sigma^2 p(p-1) \quad > \quad 0,$$

where we set $\sigma := \max_i \sigma_i > 0$. By choosing

$$C_1' \; \geq \; \frac{2 + \tilde{f}(1)}{r}, \qquad C_2' \; \geq \; \frac{1}{\rho},$$

we then have for all $i \in \mathbb{I}_d$,

$$\begin{aligned} rh(x) - \mathcal{L}_i h(x) - f_i(x) &= rC_1' + C_2' x^p [r - pb_i - \frac{1}{2}\sigma_i^2 p(p-1)] - f_i(x) \\ &\geq rC_1' + \rho C_2' x^p - f_i(x) \\ &\geq 1, \quad \forall x > 0. \end{aligned} \tag{A.3}$$

From (A.2) and (A.3), we then deduce that for all $i \in \mathbb{I}_d$, $\lambda \in (0,1)$, w_i^λ is a supersolution to

$$\min\left\{ rw_i^\lambda - \mathcal{L}_i w_i^\lambda - f_i, w_i^\lambda - \max_{j\neq i}(w_j^\lambda - g_{ij}) \right\} \;\geq\; \lambda\delta, \quad \text{on } (0,\infty), \tag{A.4}$$

where $\delta = \underline{g} \wedge 1 > 0$.

Step 2. In order to prove the comparison principle, it suffices to show that for all $\lambda \in (0,1)$:

$$\max_{j\in\mathbb{I}_d} \sup_{(0,+\infty)} (u_j - w_j^\lambda) \;\leq\; 0$$

since the required result is obtained by letting λ to 0. We argue by contradiction and suppose that there exists some $\lambda \in (0,1)$ and $i \in \mathbb{I}_d$ s.t.

$$\theta := \max_{j\in\mathbb{I}_d} \sup_{(0,+\infty)} (u_j - w_j^\lambda) \;=\; \sup_{(0,+\infty)} (u_i - w_i^\lambda) \;>\; 0. \tag{A.5}$$

From the linear growth condition (A.1), and since $p > 1$, we observe that $u_i(x) - w_i^\lambda(x)$ goes to $-\infty$ when x goes to infinity. By choosing also $C_1' \geq \max_i w_i(0^+)$, we then have $u_i(0^+) - w_i^\lambda(0^+) = u_i(0^+) - w_i(0^+) + \lambda(w_i(0^+) - C_1') \leq 0$. Hence, by continuity of the functions u_i and w_i^λ, there exists $x_0 \in (0,\infty)$ s.t.

$$\theta \;=\; u_i(x_0) - w_i^\lambda(x_0).$$

For any $\varepsilon > 0$, we consider the functions

$$\begin{aligned}\Phi_\varepsilon(x,y) &= u_i(x) - w_i^\lambda(y) - \phi_\varepsilon(x,y),\\ \phi_\varepsilon(x,y) &= \frac{1}{4}|x - x_0|^4 + \frac{1}{2\varepsilon}|x-y|^2,\end{aligned}$$

for all $x, y \in (0,\infty)$. By standard arguments in comparison principle, the function Φ_ε attains a maximum in $(x_\varepsilon, y_\varepsilon) \in (0,\infty)^2$, which converges (up to a subsequence) to (x_0, x_0) when ε goes to zero. Moreover,

$$\lim_{\varepsilon\to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \;=\; 0. \tag{A.6}$$

Applying Theorem 3.2 in [3], we get the existence of M_ε, $N_\varepsilon \in \mathbb{R}$ such that:

$$\begin{aligned}(p_\varepsilon, M_\varepsilon) &\in J^{2,+}u_i(x_\varepsilon),\\ (q_\varepsilon, N_\varepsilon) &\in J^{2,-}w_i^\lambda(y_\varepsilon)\end{aligned}$$

where

$$\begin{aligned}p_\varepsilon &= D_x\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\\ q_\varepsilon &= -D_y\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)\end{aligned}$$

and

$$\begin{pmatrix} M_\varepsilon & 0 \\ 0 & -N_\varepsilon \end{pmatrix} \;\leq\; D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) + \varepsilon\left(D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon)\right)^2 \tag{A.7}$$

with

$$D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \begin{pmatrix} 3(x_\varepsilon - x_0)^2 + \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \\ -\frac{1}{\varepsilon} & \frac{1}{\varepsilon} \end{pmatrix},$$

By writing the viscosity subsolution property (3.9) of u_i and the viscosity strict supersolution property (A.4) of w_i^λ, we have the following inequalities:

$$\min\Big\{ru_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right) b_i x_\varepsilon - \frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon - f_i(x_\varepsilon)\,, \\ u_i(x_\varepsilon) - \max_{j\neq i}(u_j - g_{ij})(x_\varepsilon)\Big\} \;\leq\; 0 \qquad (A.8)$$

$$\min\Big\{rw_i^\lambda(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) b_i y_\varepsilon - \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon - f_i(y_\varepsilon)\,, \\ w_i^\lambda(y_\varepsilon) - \max_{j\neq i}(w_j^\lambda - g_{ij})(y_\varepsilon)\Big\} \;\geq\; \lambda\delta \qquad (A.9)$$

We then distinguish the following two cases :

(1) $u_i(x_\varepsilon) - \max_{j\neq i}(u_j - g_{ij})(x_\varepsilon) \leq 0$ in (A.8).
By sending $\varepsilon \to 0$, this implies

$$u_i(x_0) - \max_{j\neq i}(u_j - g_{ij})(x_0) \;\;\leq\;\; 0. \qquad (A.10)$$

On the other hand, we have by (A.9) :

$$w_i^\lambda(y_\varepsilon) - \max_{j\neq i}(w_j^\lambda - g_{ij})(y_\varepsilon) \;\;\geq\;\; \lambda\delta,$$

so that by sending ε to zero :

$$w_i^\lambda(x_0) - \max_{j\neq i}(w_j^\lambda - g_{ij})(x_0) \;\;\geq\;\; \lambda\delta. \qquad (A.11)$$

Combining (A.10) and (A.11), we obtain :

$$\begin{aligned} \theta \;=\; u_i(x_0) - w_i^\lambda(x_0) \;&\leq\; -\lambda\delta + \max_{j\neq i}(u_j - g_{ij})(x_0) - \max_{j\neq i}(w_j^\lambda - g_{ij})(x_0) \\ &\leq\; -\lambda\delta + \max_{j\neq i}(u_j - w_j^\lambda)(x_0) \\ &\leq\; -\lambda\delta + \theta, \end{aligned}$$

which is a contradiction.
(2) $ru_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right) b_i x_\varepsilon - \frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon - f_i(x_\varepsilon) \leq 0$ in (A.8).
Since by (A.9), we also have :

$$rw_i^\lambda(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) b_i y_\varepsilon - \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon - f_i(y_\varepsilon) \;\;\geq\;\; \lambda\delta,$$

this yields by combining the above two inequalities :

$$ru_i(x_\varepsilon) - rw_i^\lambda(y_\varepsilon) - \frac{1}{\varepsilon} b_i (x_\varepsilon - y_\varepsilon)^2 - (x_\varepsilon - x_0)^3 b_i x_\varepsilon \\ + \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon - \frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon + f_i(y_\varepsilon) - f_i(x_\varepsilon) \;\;\leq\;\; -\lambda\delta. \qquad (A.12)$$

Now, from (A.7), we have :

$$\frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon - \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon \leq \frac{3}{2\varepsilon}\sigma_i^2(x_\varepsilon - y_\varepsilon)^2 + \frac{3}{2}\sigma_i^2 x_\varepsilon^2 (x_\varepsilon - x_0)^2 \left(3\varepsilon(x_\varepsilon - x_0)^2 + 2\right),$$

so that by plugging into (A.12) :

$$\begin{aligned} r\left(u_i(x_\varepsilon) - w_i^\lambda(y_\varepsilon)\right) \quad \leq \quad & \frac{1}{\varepsilon} b_i (x_\varepsilon - y_\varepsilon)^2 + (x_\varepsilon - x_0)^3 b_i x_\varepsilon + \frac{3}{2\varepsilon}\sigma_i^2 (x_\varepsilon - y_\varepsilon)^2 \\ & + \frac{3}{2}\sigma_i^2 x_\varepsilon^2 (x_\varepsilon - x_0)^2 \left(3\varepsilon(x_\varepsilon - x_0)^2 + 2\right) + f_i(y_\varepsilon) - f_i(x_\varepsilon) - \lambda\delta \end{aligned}$$

By sending ε to zero, and using (A.6), continuity of f_i, we obtain the required contradiction: $r\theta \leq -\lambda\delta < 0$. This ends the proof of Theorem 4.5. □

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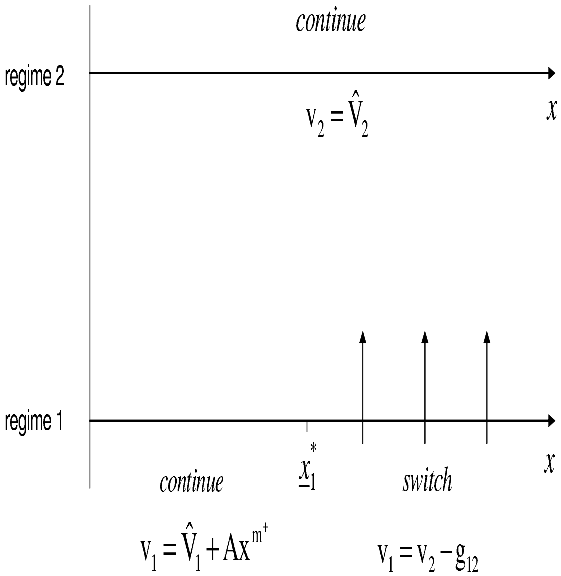

Figure 1: Identical profit functions: $f_1 = f_2,\ K_2 > K_1$

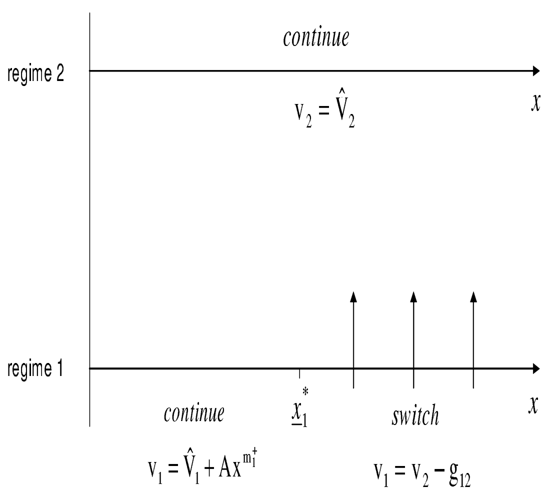

Figure 2a: $rg_{21} \geq -F(\hat{x})$

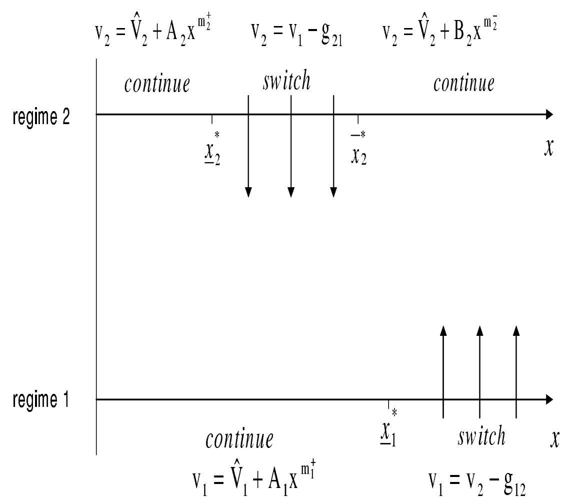

Figure 2b: $rg_{21} < -F(\hat{x})$