On a nonlocal equation arising in population dynamics

Jerome Coville, Louis Dupaigne

On a nonlocal equation arising in population dynamics

Jerome Coville[1] and Louis Dupaigne[2]
E-mails: coville@ann.jussieu.fr, dupaigne@math.cnrs.fr
[1] Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie
boîte courrier 187
75252 Paris Cedex 05
France
[2] Laboratoire Amiénois de Mathématique Fondamentale et Appliquée
Université Picardie Jules Verne
Faculté de Mathématiques et d'Informatique
33, rue Saint-Leu 80039 Amiens Cedex 1
France

May 3, 2005

Abstract

We study a one-dimensional nonlocal variant of Fisher's equation describing the spatial spread of a mutant in a given population, and its generalization to the so-called monostable nonlinearity. The dispersion of the genetic characters is assumed to follow a nonlocal diffusion law modelled by a convolution operator. We prove that as in the classical (local) problem, there exist travelling-wave solutions of arbitrary speed beyond a critical value and also characterize the asymptotic behaviour of such solutions at infinity. Our proofs rely on an appropriate version of the maximum principle, qualitative properties of solutions and approximation schemes leading to singular limits.

1 Introduction

In 1930, Fisher [10] suggested to model the spatial spread of a mutant in a given population by the following reaction-diffusion equation :

$$u_t - \Delta u = u(1-u), \tag{1.1}$$

where u represents the gene fraction of the mutant. Dispersion of the genetic characters is assumed to follow a diffusion law while the logistic term $u(1-u)$ takes into account the saturation of this dispersion process.

Since then, much attention has been drawn to reaction-diffusion equations, as they have proved to give a robust and accurate description of a wide variety of phenomena, ranging from combustion to bacterial growth, nerve propagation or epidemiology. We point the interested reader to [9, 14, 12] and their many references.

In this work, we consider a variant of (1.1) where diffusion is modeled by a convolution operator. Going back to the early work of Kolmogorov - Petrovskii-Piskounov (see [13]), dispersion of the gene fraction at point $y \in \mathbb{R}^n$ should affect the gene fraction at $x \in \mathbb{R}^n$ by a factor $J(x,y)u(y)dy$ where $J(x,\cdot)$ is a probability density. Restricting to a one-dimensional setting and assuming that such a diffusion process depends only on the distance between two niches of the population, we end up with the equation

$$u_t - (J \star u - u) = f(u), \tag{1.2}$$

where $J : \mathbb{R} \to \mathbb{R}$ is a nonnegative even function of mass one. More precisely, we assume in what follows that

$$J \in C^1(\mathbb{R}), \qquad J \geq 0, \qquad J(x) = J(-x) \qquad \text{and} \qquad \int_{\mathbb{R}} J = 1, \tag{H1}$$

We make the additional technical assumption

$$\exists \lambda > 0, \ \int_{\mathbb{R}} J(x) e^{\lambda x} dx < +\infty. \tag{H2}$$

For example, (H2) is satisfied if J has compact support or if $J(x) = \frac{1}{2\lambda} e^{-\lambda|x|}$ for some $\lambda > 0$.

The nonlinearity f in (1.2) can be chosen more generally than in equation (1.1). In the literature, three types of nonlinearities appear, according to the underlined application: we always assume that $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$, $f'(1) < 0$ and

- we say that f is of bistable type if there exists $\theta \in (0,1)$ such that

$$f < 0 \text{ in } (0,\theta), \quad f(\theta) = 0 \quad \text{and} \quad f > 0 \text{ in } (\theta, 1)$$

- f is of ignition type if there exists $\theta \in (0,1)$ such that

$$f|_{[0,\theta]} \equiv 0, \qquad f|_{(\theta,1)} > 0 \qquad \text{and } f(1) = 0.$$

- f is of monostable type if

$$f > 0 \text{ in } (0,1)$$

In the present article, we will focus on the monostable nonlinearity. Observe that equation (1.1) falls in this case.

(1.1) can also be seen as a first order approximation of (1.2). Indeed if any given niche of the species is assumed to interact mostly with close-by neighbours, the diffusion term is of the form $J_\epsilon(x) := \frac{1}{\epsilon}J(\frac{1}{\epsilon}x)$, where J is compactly supported and $\epsilon > 0$ is small. We then have

$$
\begin{aligned}
J_\epsilon \star u - u &= \frac{1}{\epsilon}\int J(\frac{1}{\epsilon}y)(u(x-y)-u(x))\,dy = \int J(z)(u(x-\epsilon z)-u(x))\,dz \\
&= -\epsilon\int J(z)u'(x)z\,dz + \epsilon^2\int z^2 J(z)u''(x)\,dz + o(\epsilon^2) = c\epsilon^2 u''(x) + o(\epsilon^2),
\end{aligned}
$$

where we used the fact that J is even in the last equality.

We observe that equation (1.2) can be related to a class of problems studied in [16, 17]. However, our approach differs in at least two ways : firstly, from the technical point of view, inverting the operator $u \to u_t - (J * u - u)$ in any reasonable space yields no *a priori* regularity property on the solution u and the compactness assumptions made in [17] no longer hold in our case.

Secondly, whereas the author favored discrete models over continuous ones to describe the dynamics of certain populations, we remain interested in the latter. In particular, we have in mind the following application to adaptative dynamics : in [11], the authors study a probabilistic model describing the microscopic behavior of the evolution of genetic traits in a population subject to mutation and selection. Averaging over a large number of individuals in the initial state, they derive in the limit a deterministic equation, a special case of which can be written as

$$\partial_t u = [J * u - u] + (1 - K * u)u, \tag{1.3}$$

where $J(x)$ is a kernel taking into account mutation about trait x and $K(x)$ is a competition kernel, measuring the "intensity" of the interaction between x and y. Taking $K(x) = \delta$, we recover equation (1.2) as a special case of (1.3).

The aim of this article is the study of so-called travelling-wave solutions of equation (1.2) i.e. solutions of the form

$$u(x,t) = U(x + ct),$$

where $c \in \mathbb{R}$ is called the wave speed and U the wave profile, which is required to solve the equation

$$
\left\{
\begin{array}{l}
[J \star U - U] - cU' + f(U) = 0 \quad \text{in } \mathbb{R} \\
U(-\infty) = 0 \\
U(+\infty) = 1,
\end{array}
\right. \tag{1.4}
$$

where $U(\pm\infty)$ denotes the limit of $U(x)$ as $x \to \pm\infty$.

Such solutions are expected to give the asymptotic behavior in large time for solutions of (1.2) with say compactly supported initial data : in the Fisher equation, this is equivalent to saying that the mutant propagates (after some time) at constant speed and along the profile U. It is therefore of interest to prove existence of such solutions.

The first results in this direction are due to Schumacher [15], who considered the monostable nonlinearity, under the extra assumption that $f(r) \geq h_0 r - K r^{1+\alpha}$, for some $h_0, K, \alpha > 0$ and all $r \in [0,1]$. In this case, his results imply existence of travelling waves with arbitrary speed $c \geq c^*$, where c^* is the smallest $c \in \mathbb{R}$ such that $\rho_c : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho_c(\lambda) = -\lambda c + \int J(z) e^{\lambda z}\, dz - 1 + f'(0),$$

vanishes for some $\lambda \in \mathbb{R}$.

Furthermore if $c > c^*$ and under some extra assumptions on f, Schumacher shows that the profile U of the associated travelling wave is unique up to translation.

Recently, Carr and Chmaj [3] completed the work of Schumacher. For the "KPP" nonlinearity (i.e. if f is monostable and $f(r) \leq f'(0) r$ for all $r \in [0,1]$) and if J has compact support, they show that the above uniqueness result can be extended to $c = c^*$.

Concerning the bistable nonlinearity, Bates-Fife-Ren-Wang [1] and Chen [4] showed that there exists an increasing travelling wave U with speed c solving (1.4). Furthermore if V is another nondecreasing travelling wave with speed c' then $c = c'$ and $V(x) = U(x + \tau)$ for some $\tau \in \mathbb{R}$.

Coville [5] then looked at the case of ignition nonlinearities and proved again the existence and uniqueness (up to translation) of an increasing travelling wave (U, c). Coville also obtained the existence of at least one travelling-wave solution in the monostable case.

Our first theorem extends some of the afore-mentioned results of Schumacher to the general monostable case:

Theorem 1.1.
Assume (H1) and (H2) hold and assume that f is of monostable type. Then there exists a constant $c^* > 0$ (called the minimal speed of the travelling wave) such that for all $c \geq c^*$, there exists an increasing solution $U \in C^1(\mathbb{R})$ of (1.4) *while no nondecreasing travelling wave of speed $c < c^*$ exists.*

Our second result extends previous work of Coville [5] regarding the behavior of the travelling front U near $\pm\infty$.

Proposition 1.1.
Assume (H1) and (H2) hold. Then given any travelling-wave solution (U, c) of (1.4) *with f monostable, the following assertions hold :*

1. *There exist positive constants A, B, M, λ_0 and δ_0 such that*

$$B e^{-\delta_0 y} \leq 1 - U(y) \leq A e^{-\lambda_0 y} \ \ for \ \ y \geq M.$$

2. *If $f'(0) > 0$ then there exists positive constants K, N and λ_1 such that*

$$U(y) \leq K e^{\lambda_1 y} \ \ for \ \ y \leq -N.$$

The first point is an easy consequence of a similar result when f is of bistable or ignition type, proved in [5].

Regarding Theorem 1.1, our proof is based on the study of two auxiliary problems and the construction of adequate super and subsolutions. We work in three steps.

We start by showing existence and uniqueness of a solution for

$$\left\{\begin{array}{rcll} \mathcal{L}u + f(u) & = & -h_r(x) & \text{in } \Omega \\ u(-r) & = & \theta & \\ u(+\infty) & = & 1, & \end{array}\right. \tag{1.5}$$

where given $\epsilon > 0$, $r \in \mathbb{R}$, $c \in \mathbb{R}$ and $\theta \in (0,1)$,

$$\Omega = (-r, +\infty),$$

$$\mathcal{L}u = \mathcal{L}(\epsilon, r, c)u = \epsilon u'' + \left[\int_{-r}^{+\infty} J(x-y)u(y)dy - u\right] - cu', \tag{1.6}$$

$$h_r(x) = \theta \int_{-\infty}^{-r} J(x-y)dy.$$

The existence is obtained via an iterative scheme using a comparison principle and appropriate sub and supersolutions.

In the second step, with a standard limiting procedure, we prove Theorem 1.1 for the problem

$$\left\{\begin{array}{rcll} \mathcal{M}u + f(u) & = & 0 & \text{in } \mathbb{R} \\ u(-\infty) & = & 0 & \\ u(+\infty) & = & 1, & \end{array}\right. \tag{1.7}$$

where given $\epsilon > 0$, $c \in \mathbb{R}$,

$$\mathcal{M}u = \mathcal{M}(\epsilon, c)u = \epsilon u'' + [J * u - u] - cu'. \tag{1.8}$$

We stress the fact that unlike (1.5), (1.7) does not have an (increasing and smooth) solution u for arbitrary values of $c \in \mathbb{R}$.

Finally, in the last step we send $\epsilon \to 0$ and extract converging subsequences.

Though elementary in nature, the proofs require a number of lemmas which we list and prove in the Appendix. We construct sub and supersolutions for (1.5) and (1.7) in Section 2. After obtaining some useful *a priori* estimates in Section 3, we prove existence and uniqueness of solutions of (1.5) in Section 4. In Section 5, we show the existence of a speed $c^*(\epsilon) > 0$ such that (1.7) admits a solution for every $c \geq c^*(\epsilon)$. We complete the proof of Theorem 1.1 in Section 6. Section 7 is devoted to the proof of Proposition 1.1.

2 Existence of sub and supersolutions

We start with the construction of a supersolution of (1.7) for speeds $c \geq \kappa(\epsilon)$ for some $\kappa(\epsilon) > 0$.

Lemma 2.1.
Let $\epsilon \geq 0$. There exists a real number $\kappa(\epsilon) > 0$ and an increasing function $\bar{w} \in C^2(\mathbb{R})$ such that, given any $c \geq \kappa(\epsilon)$

$$\begin{cases} \mathcal{M}\bar{w} + f(\bar{w}) \leq 0 \quad in \quad \mathbb{R} \\ \bar{w}(-\infty) = 0 \\ \bar{w}(+\infty) = 1, \end{cases}$$

where $\mathcal{M} = \mathcal{M}(\epsilon, c)$ is defined by (1.8). *Futhermore, $\bar{w}(0) = \frac{1}{2}$*

Proof:

Fix positives constants N, λ, δ such that $\lambda > \delta$ and $(H2)$ holds.
Let $\bar{w} \in C^2(\mathbb{R})$ be a positive increasing function satisfying

- $\bar{w}(x) = e^{\lambda x}$ for $x \in (-\infty, -N]$,
- $\bar{w}(x) \leq e^{\lambda x}$ on $\mathbb{R}$,
- $\bar{w}(x) = 1 - e^{-\delta x}$ for $x \in [N, +\infty)$,
- $\bar{w}(0) = \frac{1}{2}$.

Let $x_0 = e^{-\lambda N}$ and $x_1 = 1 - e^{-\delta N}$. We have $0 < x_0 < x_1 < 1$.
We now construct a positive function g defined on $(0, 1)$ which satisfies $g(w) \geq f(w)$. Since f is smooth near 0 and 1, we have for c large enough, say $c \geq \kappa_0$,

$$\lambda(c - \lambda)s \geq f(s) \;\; for \;\; s \in [0, x_0] \tag{2.1}$$

and

$$\delta(c - \delta)(1 - s) \geq f(s) \;\; for \;\; s \in [x_1, 1]. \tag{2.2}$$

Therefore we can achieve $g(s) \geq f(s)$ for s in [0,1], with g defined by:

$$g(s) = \begin{cases} \lambda(\kappa_0 - \lambda)s & for \;\; 0 \leq s \leq x_0 \\ l(s) & for \;\; x_0 < s < x_1 \\ \delta(\kappa_0 - \delta)(1 - s) & for \;\; x_1 \leq s \leq 1 \end{cases} \tag{2.3}$$

where l is any smooth positive function greater than f on $[x_0, x_1]$ such that g is of class C^1.

According to (2.3), for $x \leq -N$ i.e. for $w \leq e^{-\lambda N}$, we have

$$\begin{aligned} \mathcal{M}\bar{w} + g(\bar{w}) &= \epsilon\bar{w}'' + J \star \bar{w} - \bar{w} - c\bar{w}' + g(\bar{w}) \\ &= \epsilon\lambda^2 e^{\lambda x} + J \star \bar{w} - e^{\lambda x} - \lambda c \, e^{\lambda x} + \lambda(\kappa_0 - \lambda)e^{\lambda x} \\ &\leq \epsilon\lambda^2 e^{\lambda x} + J \star e^{\lambda x} - e^{\lambda x} - \lambda c e^{\lambda x} + \lambda(\kappa_0 - \lambda)e^{\lambda x} \\ &\leq e^{\lambda x}[\int_{\mathbb{R}} J(z)e^{\lambda z}dz - 1 - \lambda(c - \kappa_0) - \lambda^2(1 - \epsilon)] \\ &\leq 0 \end{aligned}$$

for c large enough, say $c \geq \kappa_1 = \frac{\int_{\mathbb{R}} J(z)e^{\lambda z}dz - 1 + \lambda\kappa_0 - \lambda^2(1-\epsilon)}{\lambda}$.
Furthermore for $\bar{w} \geq 1 - e^{-\delta N}$ we have,

$$\begin{aligned}
\mathcal{M}\bar{w} + g(\bar{w}) &= \epsilon\bar{w}'' + J \star \bar{w} - \bar{w} - c\bar{w}' + g(\bar{w}) \\
&= \epsilon\delta^2 e^{-\delta x} + J \star \bar{w} - (1 - e^{-\delta x}) - \delta c\, e^{-\delta x} + \delta(\kappa_0 - \delta)e^{-\delta x} \\
&\leq \epsilon\delta^2 e^{-\delta x} + 1 - 1 + e^{-\delta x} - \delta c e^{-\delta x} + \delta(\kappa_0 - \delta)e^{-\delta x} \\
&\leq e^{-\delta x}[1 - \delta(c - \kappa_0) - \delta^2(1-\epsilon)] \\
&\leq 0
\end{aligned}$$

for c large enough, say $c \geq \kappa_2 = \frac{1 + \delta\kappa_0 - \delta^2(1-\epsilon)}{\delta}$. Thus by taking $c \geq \sup\{\kappa_1, \kappa_2\}$, we achieve

$$\begin{gathered}
g(\bar{w}) \geq f(\bar{w}) \qquad \text{and} \qquad J \star \bar{w} - \bar{w} - c\bar{w}' + g(\bar{w}) \leq 0 \\
\text{for} \quad 0 \leq \bar{w} \leq e^{-\lambda N} \quad \text{and} \quad \bar{w} \geq 1 - e^{-\delta N}.
\end{gathered}$$

For the remaining values of $\bar{w}$, i.e. for $x \in [-N, N]$, $\bar{w}' > 0$ and we may increase c further if necessary, to achieve

$$\epsilon\bar{w}'' + J \star \bar{w} - \bar{w} - c\bar{w}' + g(\bar{w}) \leq 0 \ \ in \ \ \mathbb{R}. \tag{2.4}$$

The result follows for

$$\bar{\kappa}(\epsilon) := \sup\{\kappa_1, \kappa_2, \kappa_3\},$$

where $\kappa_3 = \sup_{x \in [-N,N]}\{\frac{\epsilon|w''| + |J \star w - w| + g(w)}{w'}\}$.

$\square$

Remark 2.1. *$\kappa(\epsilon)$ is a nondecreasing function of ϵ.*

Remark 2.2. *Observe that given any $r \in \mathbb{R}$, then for $c \geq \kappa(\epsilon)$, $\bar{w}$ is also a supersolution of the following problem:*

$$\begin{cases} \mathcal{L}\bar{w} + f(\bar{w}) \leq 0 \qquad in\ \Omega \\ \quad \bar{w}(-r) = 0 \\ \quad \bar{w}(+\infty) = 1, \end{cases} \tag{2.5}$$

where $\mathcal{L} = \mathcal{L}(\epsilon, c, r)$ defined by (1.6).

Next, we construct super and subsolution of (1.5).

Remark 2.3. *Let $\epsilon \geq 0, r \in \mathbb{R}, c \in \mathbb{R}, \theta \in (0,1)$. Then the constant functions $\underline{u} = \theta$ and $\bar{u} = 1$ are respectively a sub- and a supersolution of problem* (1.5), *i.e.*

$$\begin{cases} \mathcal{L}\underline{u} + f(\underline{u}) \geq -h_r(x) \qquad in\ \Omega & (\ resp. \ \ \mathcal{L}\bar{u} + f(\bar{u}) \geq -h_r(x) \ \ in\ \Omega) \\ \quad \underline{u}(-r) \leq \theta & (\ resp. \ \ \bar{u}(-r) \leq \theta) \\ \quad \underline{u}(+\infty) \leq 1, & (\ resp. \ \ \bar{u}(+\infty) \leq 1) \end{cases}$$

We now construct a subsolution of (1.5) satisfying stronger conditions on the boundary of Ω.

Lemma 2.2.
Let $\epsilon > 0, r \in \mathbb{R}, \theta \in (0,1)$. *There exists* $\underline{\kappa}(\epsilon) \in \mathbb{R}$ *and an increasing function* $\underline{w} \in C^2(\mathbb{R})$ *such that, given any* $c \leq \underline{\kappa}(\epsilon)$,

$$\begin{cases} \mathcal{L}\underline{w} + f(\underline{w}) \geq -h_r(x) & \text{in } \Omega \\ \underline{w}(-r) = \theta \\ \underline{w}(+\infty) = 1 \end{cases}$$

Let f_b be a smooth bistable function (e.g $f_b(0) = f_b(1) = 0$ and $\exists \theta \in (0,1)$ such that $f_b < 0$ in $(0,\theta)$, $\quad f_b(\theta) = 0 \quad$ and $\quad f_b > 0$ in $(\theta, 1)$) such that $f_b \leq f$ and $\int_0^1 f_b(s)\,ds > 0$. Let (u_b, c_b) denote the unique (up to translation) increasing solution of (1.7) with f_b instead of f. Such a solution exists, see [1] for details. Moreover $c_b > 0$. Using the translation invariance of (1.7), one can easily show that for any $c \leq c_b$, $u_b^\tau := u_b(. + \tau)$ is a subsolution of (1.5) for some $\tau \in \mathbb{R}$. Namely, choose τ such that $u_b^\tau(-r) = \theta$.

Since u_b^τ is increasing we have

$$h_r(x) = \theta \int_{-\infty}^{-r} J(x-y)dy \geq \int_{-\infty}^{-r} J(x-y)u_b^\tau(y)dy.$$

A simple computation shows that

$$\mathcal{L}u_b^\tau + h_r(x) + f(u_b^\tau) \geq \mathcal{L}u_b^\tau + \int_{-\infty}^{-r} J(x-y)u_b^\tau(y)\,dy + f_b(u_b^\tau) = (u_b^\tau)'(c_b - c) \quad \text{in } \Omega$$

Hence for $c \leq c_b$,

$$\begin{cases} \mathcal{L}u_b^\tau + h_r(x) + f(u_b^\tau) \geq (u_b^\tau)'(c_b - c) \geq 0 & \text{in } \Omega \\ \quad u_b^\tau(-r) = \theta \\ \quad u_b^\tau(+\infty) = 1. \end{cases}$$

$\square$

3 L^2 estimates

In this Section, we obtain L^2 estimates for solutions u of the two following problems:

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + f(u) = 0 & \text{in } \mathbb{R} \\ u \to 0 & x \to -\infty \\ u(x) \to 1 & x \to +\infty, \end{cases} \tag{3.1}$$

and

$$\begin{cases} L_r^c u + h_r(x) + f(u) = 0 & \text{for } x \in (r, +\infty) \\ u(r) = \theta \\ u \to 1 \qquad x \to +\infty, \end{cases}$$

where L_r^c and h_r define as in the previous Section.

3.1 L^2 estimates for solution of (3.1)

We start by some L^2 of solution u of (3.1). For this solutions we have:

Lemma 3.1.
Assume $\epsilon > 0$ and let u be a smooth solution of (3.1) *then*

(i) $u', u'' \in L^2(\mathbb{R})$

(ii) $1 - u \in L^2(\mathbb{R}^+)$.

Proof of Lemma 3.1:

Let u be a smooth increasing solution of (3.1). We start out by showing that u' and u'' vanish at infinity for solution. To this end we use a standard technique.We only prove that $u''(x) \to 0$ as $x \to +\infty$. The proof in the other case is similar. Let us define the following sequence of translate of u, $(u_p)_{p\in\mathbb{N}} = (\bar{u}(.+p))_{p\in\mathbb{N}}$. Then for each p, u_p is a uniformally bounded nondecreasing function and it satisfies

$$\epsilon u_p'' + J \star u_p - u_p - cu_p' + f(u_p) = 0 \qquad \text{in} \qquad \Omega.$$

Using now local $C^{2,\alpha}$ estimates, Helly's Theorem and diagonal extraction, we deduce that there exist a subsequence still denoted u_p which converge pointwise to some nondecreasing function $\widetilde{u}$. Moreover u_p converge to $\widetilde{u}$ in $C^{2,\beta}_{loc}(\Omega)$ topology. Therefore $\widetilde{u}$ satisfies

$$\epsilon \widetilde{u}'' + J \star \widetilde{u} - \widetilde{u} - c\widetilde{u}' + f(\widetilde{u}) = 0 \qquad \text{in} \qquad \Omega. \tag{3.2}$$

From the definition of u_p, u_p converge pointwise to 1, therefore by uniqueness of the limit $\widetilde{u} \equiv 1$. So from (3.2) we deduce that $u_p'', u_p' \to 0$ uniformaly on every compact set. We can repeat this argument with any sequence $(u_p)_{p\in\mathbb{N}} := (u(x_p))_{p\in\mathbb{N}}$ with $x_p \to \infty$. Therefore $u''(x), u'(x) \to 0$ as $x \to +\infty$.

We are now in position to prove *(i)*.
We start by showing that $f(u) \in L^1(\mathbb{R})$. Integrating (3.1) over $(-r, r)$ leads to:

$$\epsilon(u'(r) - u'(-r)) + \int_{-r}^{r} (J \star u - u)\, dx - c(u(r) - u(-r)) = -\int_{-r}^{+r} f(u)$$

Assume for the moment that $J \star u - u \in L^1(\mathbb{R})$, then we can pass to the limit $r \to +\infty$ in the above expression, so we get

$$\int_{-\infty}^{+\infty} (J \star u - u)\, dx - c = -\int_{-\infty}^{+\infty} f(u).$$

Therefore $f(u) \in L^1(\mathbb{R})$. Let now prove that $J \star u - u \in L^1(\mathbb{R})$,

Claim 3.1.
$J \star u - u \in L^1(\mathbb{R})$*, moreover*

$$\|J \star u - u\|_{L^1} \leq \int_{\mathbb{R}} J(z)|z|dz \qquad \textit{and} \qquad \int_{\mathbb{R}} (J \star u - u) = 0$$

proof:

Clearly,

$$\int_{-r}^{r} |(J \star u - u)| \leq \int_{-r}^{r} \int_{\mathbb{R}} J(x-y)|u(y) - u(x)|dy\ dx. \tag{3.3}$$

Using the change of variable in y, $z := y - x$ (3.3) becomes

$$\int_{-r}^{r} |(J \star u - u)| \leq \int_{-r}^{r} \int_{\mathbb{R}} J(z)|u(x+z) - u(x)|dz\ dx. \tag{3.4}$$

Since $u \in C^1(\mathbb{R})$,

$$|u(x+z) - u(x)| = |z| \int_0^1 u'(x+sz)ds.$$

Plug this equality in (3.4) to obtain:

$$\int_{-r}^{r} \int_{\mathbb{R}} J(z)|u(x+z) - u(x)|dydx = \int_{-r}^{r} \int_{\mathbb{R}} J(z)|z| \int_0^1 u'(x+sz)ds\ dz\ dx \tag{3.5}$$

Since all terms are positive, using Tonnelli's Theorem, we can permute the order of integration and obtain

$$\begin{aligned} \int_{-r}^{r} \int_{\mathbb{R}} J(z)|z| \int_0^1 u'(x+sz)ds\ dz\ dx &= \textstyle\int_{\mathbb{R}} J(z)|z| \int_{-r}^{r} \int_0^1 u'(x+sz)\, ds\, dx\ dz \\ &= \textstyle\int_0^1 \int_{\mathbb{R}} J(z)|z|[u(r+sz) - u(-r+sz)]dz\ ds \end{aligned}$$

Hence we have,

$$\int_{-r}^{r} |\int_{\mathbb{R}} J(x-y)(u(y) - u(x))dy|dx \leq \int_0^1 \int_{\mathbb{R}} J(z)|z|[u(r+sz) - u(-r+sz)]dz\ ds$$

Using now Lebesgue dominated convergence, we can pass to the limit in the above expression to get

$$\|J \star u - u\|_{L^1} \leq \int_{\mathbb{R}} J(z)|z|dz \tag{3.6}$$

Let us now compute $\int_{\mathbb{R}} (J \star u - u)\ dx$. Since $J \star u - u \in L^1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} (J \star u - u)dx = \int_{\mathbb{R}^2} J(x-y)(u(y) - u(x))dy\ dx.$$

Since J is symmetric one also have

$$\int_{\mathbb{R}^2} J(x-y)(u(y)-u(x))dy\ dx = \int_{\mathbb{R}^2} J(y-x)(u(y)-u(x))dy\ dx = \int_{\mathbb{R}^2} J(x-y)(u(x)-u(y))dy\ dx.$$

Hence,

$$2\int_{\mathbb{R}^2} J(x-y)(u(y)-u(x))dy\,dx = 0.$$

□

Let us now prove that $u', u'' \in L^2$.

Multiplying (3.1) by u and integrating over $\mathbb{R}$ yield

$$\epsilon\int u''u + \int (J\star u - u)\,u - c\int u'u = -\int f(u)u.$$

Integrating by parts the first term yields to

$$-\epsilon\int (u')^2 + \int (J\star u - u)\,u - \frac{c}{2} = -\int f(u)u.$$

Since u is bounded and $f(u), J\star u - u \in L^1$, from the above expression we have $u' \in L^2$.

We obtain $u'' \in L^2$ similarly. Indeed, multiplying (5.1) by u'' and integrating over $\mathbb{R}$ we get

$$\epsilon\int \left(u''\right)^2 + \int (J\star u - u)\,u'' - c\int u'u'' = \int f(u)u''.$$

Integration by parts and uniform bounds yield

$$\begin{aligned}
\epsilon\int \left(u''\right)^2 &= -\int (J\star u - u)\,u'' - \int f(u)u'' && (3.7)\\
&= \int \left(J\star u' - u'\right)u' + \int f'(u)\left(u'\right)^2 && (3.8)\\
&\leq C_0\int u' + C_1\|u'\|^2_{L^2(\mathbb{R})} && (3.9)
\end{aligned}$$

where C_0 and C_1 are positive constants. This ends the proof of *(i)*.

We can now show that $1-u \in L^2(\mathbb{R}^+)$. Again multiplying (5.1) by $1-u$ and integrating over $\mathbb{R}$ yields to

$$\epsilon\int \left(u'\right)^2 - \int (J\star u - u)u + c/2 + \int f(u)(1-u) = 0.$$

Using now Claim 3.1 and choosing R so large that $f(u) \geq \frac{|f'(1)|}{2}(1-u)$ on $[R,\infty)$, we achieves

$$\frac{|f'(1)|}{2}\int_R^\infty (1-u)^2 \leq \int_{-\infty}^\infty f(u)(1-u) \leq C(\|u'\|^2_{L^2(\mathbb{R})}+1) < \infty, \tag{3.10}$$

which ends the proof of *(ii)*.

□

Remark 3.1. *Note that these estimates trivialy extend to solution of a bistable problem.*

Finally, we obtain some useful L^2 estimates on $J \star u - u$. Namely, we have

Lemma 3.2.
$$\|J \star u - u\|_{L^2} \leq C\|u'\|_{L^2}.$$

proof:

Using the fundamental theorem of calculus, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} J(x-y)u(y)dy - u(x) &= \int_{-\infty}^{+\infty} J(x-y)(u(y)-u(x))dy \\ &= \int_{-\infty}^{+\infty} J(z)z\left(\int_0^1 u'(x+tz)\,dt\right)\,dz.\end{aligned}$$

By standard estimation and the Cauchy-Schwartz inequality, it then follows that

$$\begin{aligned}|\int_{-\infty}^{+\infty} J(x-y)u(y)dy - u(x)|^2 &\leq \left(\int_{-\infty}^{+\infty} J(z)z\left(\int_0^1 u'(x+tz)\,dt\right)\,dz\right)^2 \\ &\leq C\left[\int_{-\infty}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)dt\,dz \cdot \int_{-\infty}^{+\infty} J(z)|z|\,dz\right] \\ &\leq C'\left[\int_{-\infty}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)dt\,dz\right]\end{aligned}$$

Hence, using Tonnelli's Theorem and standard change of variables

$$\begin{aligned}\int_{-\infty}^{+\infty}|\int_{-\infty}^{+\infty} J(x-y)u(y)dy - u(x)|^2\,dx &\leq C'\left[\int_{-\infty}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)dt\,dz\,dx\right] \\ &\leq C''\int_{-\infty}^{+\infty}(u')^2(s)\,ds\end{aligned}$$

$\square$

Remark 3.2. *This last estimates shows in particular that* $f(u) \in L^2(\mathbb{R})$

3.2 L^2 estimates for solution of (3.11)

Next, we obtain some L^2 estimates for solution u of the second problem (3.11)

$$\begin{cases} L_r^c u + h_r(x) + f(u) = 0 & \text{for } x \in (r, +\infty) \\ u(r) = \theta \\ u \to 1 \qquad x \to +\infty, \end{cases} \tag{3.11}$$

More precisely, we have

Lemma 3.3.
Assume $\epsilon > 0$ *and let* u *be a smooth solution of* (3.11) *then*

(iii) $u', u'' \in L^2(\Omega)$

(iv) $1-u \in L^2(\mathbb{R}^+ \cap \Omega)$.

Proof:

We follow the lines of the proof of Lemma 3.1. As in the above proof, we first show that $u', u'' \to 0$ when $x \to +\infty$. Since the argumentation to obtain this behavior is essentially the same as in Lemma 3.1, so we omit it. Next show that $f(u) \in L^1(\Omega)$. Integrating (3.11) over (r, R) leads to:

$$\epsilon(u'(R)-u'(r))+\int_r^R \left(\int_r^{+\infty} J(x-y)u(y)\,dy - u(x)\right)\,dx - c(u(R)-u(r)) = -\int_r^R (f(u)-h_r(x))dx$$

We then obtain $f(u) \in L^1(\Omega)$ by proving that $\int_r^{+\infty} J(x-y)u(y)\,dy - u$ and $h_r(x)$ are in $L^1(\Omega)$ and passing to the limit $R \to +\infty$ in the above expression. We claim that

Claim 3.2.
$\int_r^{+\infty} J(x-y)u(y)\,dy - u$ and $h_r(x)$ are in $L^1(\Omega) \cap L^2(\Omega)$,

Proof:

Let start with $h_r(x)$. From the definition of $h_r(x)$ one have

$$h_r(x) = \theta \int_{-\infty}^{r-x} J(z)dz = \theta j(x).$$

Since $J \geq 0$ and satisfies (H3), a simple computation shows that

$$|j(x)| = \int_{-\infty}^{r-x} J(z)dz \leq e^{\delta(r-x)} \int_{\mathbb{R}} J(z)e^{-\delta z}dz \leq Ke^{\delta(r-x)} \in L^2(\Omega) \cap L^1(\Omega) \quad (3.12)$$

Now, let us prove that $\int_r^{+\infty} J(x-y)u(y)\,dy - u \in L^1(\Omega)$.
Since u is smooth, using uniform bound and the fundamental theorem of calculus, we have

$$\begin{aligned} \left|\int_r^{+\infty} J(x-y)u(y)dy - u(x)\right| &= \left|\int_r^{+\infty} J(x-y)(u(y)-u(x))dy - u(x)\int_{-\infty}^{r} J(x-y)dy\right| \\ &\leq \left|\int_{r-x}^{+\infty} J(z)(u(x+z)-u(x))dz\right| + u(x)\int_{-\infty}^{r-x} J(z)dz \\ &\leq \int_{r-x}^{+\infty} J(z)|z| \left(\int_0^1 u'(x+tz)\,dt\right)\,dz + j(x), \end{aligned}$$

Since $j \in L^1(\Omega)$, we only need to prove that

$$\Gamma(x) = \int_{r-x}^{+\infty} J(z)|z| \left(\int_0^1 u'(x+tz)\,dt\right)\,dz \in L^1(\Omega).$$

Integrating Γ over (r, R) yields to

$$\begin{aligned}\int_r^R \Gamma(x)dx &= \int_r^R \int_{r-x}^{+\infty} J(z)|z| \int_0^1 u'(x+tz)dtdzdx \\ &= \int_r^R \int_0^{+\infty} J(z)|z| \int_0^1 u'(x+tz)dtdzdx + \int_r^R \int_{r-x}^0 J(z)|z| \int_0^1 u'(x+tz)dtdzdx.\end{aligned}$$

Using now Tonnelli's Theorem, we end up with

$$\int_r^R \Gamma(x)dx = \int_0^1 \int_0^{+\infty} J(z)|z| \left(\int_r^R u'(x+tz)dx\right) dzdt$$
$$+ \int_0^1 \int_{r-R}^0 J(z)|z| \left(\int_{r-z}^R u'(x+tz)dx\right) dzdt.$$

Hence, we achieve

$$\int_r^R \Gamma(x)dx = \int_0^1 \int_0^{+\infty} J(z)|z|[u(R+tz) - u(r+tz)]dzdt$$
$$+ \int_0^1 \int_{r-R}^0 J(z)|z|[u(R+tz) - u(r+(t-1)z)]dzdt$$

Using now uniform bounds on u, we end up with,

$$\int_r^R \Gamma(x)dx \leq 4 \int_{-\infty}^{+\infty} J(z)|z|dz,$$

which shows that $\Gamma \in L^1(\Omega)$.

□

To obtain *(iii)* and *(iv)*, we can now use the argumentation of the above subsection.

□

Finally, we obtain some useful L^2 estimates on $\int_r^{+\infty} J(x-y)u(y)dy - u$. More precisely we have,

Lemma 3.4.
$\int_r^{+\infty} J(x-y)u(y)dy - u \in L^2(\Omega)$, *moreover*

$$\| \int_r^{+\infty} J(x-y)u(y)dy - u\|_{L^2(\Omega)} \leq C(\|u'\|_{L^2(\Omega)} + \|j\|_{L^2(\Omega)}).$$

Again, using the fundamental theorem of calculus, we have

$$\int_r^{+\infty} J(x-y)u(y)dy - u(x) = \int_{r-x}^{+\infty} J(z)z \left(\int_0^1 u'(x+tz)\ dt\right)\ dz - u(x)j(x).$$

By standard estimation and the Cauchy-Schwartz inequality, it follows that

$$\begin{aligned}
|\int_r^{+\infty} J(x-y)u(y)dy - u(x)|^2 &\leq 2\left[\left(\int_{r-x}^{+\infty} J(z)z\left(\int_0^1 u'(x+tz)\ dt\right)\ dz\right)^2 + u^2(x)j^2(x)\right] \\
&\leq 2\left[\int_{r-x}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)dt\ dz \cdot \int_{r-x}^{+\infty} J(z)|z|\ dz + u^2j^2\right] \\
&\leq C\left[\int_{r-x}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)dt\ dz + u^2j^2(x)\right]
\end{aligned}$$

Define $\Gamma_1(x) := \int_{r-x}^{+\infty}\int_0^1 J(z)|z|(u')^2(x+tz)dt\ dy$. We then have

$$|\int_r^{+\infty} J(x-y)u(y)dy - u(x)|^2 \leq C\left[\Gamma_1(x) + j^2(x)\right]$$

Since $j \in L^2(\Omega)$, we only need to show that Γ_1 is in $L^1(\Omega)$ and satisfies

$$\|\Gamma_1\|_{L^1(\Omega)} \leq C\|u'\|^2_{L^2(\Omega)}, \tag{3.13}$$

We obtain (3.13) through a direct computation.
By definition of Γ_1 and since all the integrands are positive, using Tonelli's Theorem, we have:

$$\begin{aligned}
\int_r^R \Gamma_1(x)dx = \int_0^{+\infty} J(z)|z|\left(\int_r^R\int_0^1 (u')^2(x+tz)dtdx\right)dz \\
+ \int_{r-R}^0 J(z)|z|\left(\int_{r-z}^R\int_0^1 (u')^2(x+tz)dtdx\right)dz
\end{aligned}$$

Using now standard changes of variables we get,

$$\begin{aligned}
\int_r^R \Gamma_1(x)dx = \int_0^{+\infty} J(z)|z|\left(\int_0^1\int_{r+tz}^{R+tz} (u')^2(s)dsdt\right)dz \\
+ \int_{r-R}^0 J(z)|z|\left(\int_0^1\int_{r+(t-1)z}^{R+tz} (u')^2(s)dsdt\right)dz
\end{aligned}$$

Since $u' \in L^2(\Omega)$ we then have

$$\begin{aligned}
\int_r^R \Gamma_1(x)dx \leq \int_0^{+\infty} J(z)|z|\left(\int_0^1\int_r^{+\infty} (u')^2(s)dsdt\right)dz \\
+ \int_{r-R}^0 J(z)|z|\left(\int_0^1\int_r^{+\infty} (u')^2(s)dsdt\right)dz
\end{aligned}$$

Hence we have

$$\int_r^{+\infty} \Gamma_1(x)dx \leq \left(\int_{-\infty}^{+\infty} J(z)|z|dz\right)\|u'\|^2_{L^2(\Omega)},$$

which is the desired conclusion.

$\square$

4 Construction of a solution of (4.1)

In this section, we will show that for any fixed $r < 0$, $c \in \mathbb{R}$, $\epsilon > 0$ and for any $\theta \in (0,1)$ there exists an increasing solution u_r of Problem (4.1) below. Moreover this solution is unique.

$$\begin{cases} L_r^c u + h_r(x) + f(u) = 0 & \text{for } x \in (r, +\infty) \\ u(r) = \theta \\ u \to 1 \qquad x \to +\infty, \end{cases} \tag{4.1}$$

with $L_r^c u$ defined by (**??**) and $h_r(x)$ by (**??**). For the uniqueness proof see [7]. The existence of a solution is obtained via an iterative scheme using sub and super solutions.

4.1 Preliminaries

Fix c,$r < 0$, $\epsilon > 0$ and $1 > \theta > 0$. Let G be a smooth nondecreasing function such that $G(r) = \theta$, $L_r^c G \in L^2(\Omega)$ and $1 - G \in L^2(\Omega)$. Now for $\lambda > 0$ define

$$\begin{array}{cccc} T_{\lambda,r}: & C_0(\Omega) \cap L^2(\Omega) & \to & C_0(\Omega) \cap L^2(\Omega) \\ & v & \mapsto & z, \end{array}$$

where z is the unique solution of

$$\begin{cases} L_r^c z - \lambda z = F(v,x) & in \ \ \Omega \\ z(r) = 0 \\ z(x) \to 0 \qquad x \to +\infty, \end{cases} \tag{4.2}$$

where $F(v,x) = -f(v+G) - \lambda v - L_r^c G - h_r(x)$. Now, using Lemma A.1, to prove that z is well-defined, it is enough to show that $v \in L^2(\Omega) \cap C_0(\Omega) \implies F(v,x) \in L^2(\Omega) \cap C_0(\Omega)$.

From assumptions on G, to conclude that z solving (4.2) is well-defined, the only things left to prove are $f(v+G) \in L^2(\Omega)$ and $h_r \in L^2(\Omega)$. The latter comes easily form the definition of h_r and the exponential decay of J. Namely, we can bound h_r from above in the following way:

$$0 \le h_r(x) = \theta \int_{-\infty}^{r-x} J(z)dz \le \theta e^{\delta(r-x)} \int_{\mathbb{R}} J(z) e^{-\delta z} dz \le K e^{\delta(r-x)} \in L^2(\Omega)$$

Let us show now that $f(v+G) \in L^2(\Omega)$.

Given $v \in L^2(\Omega) \cap C_0(\Omega)$, since $f(1) = 0$ and $1 - G \in L^2(\Omega)$,

$$|f(v+G)| \le \|f'\|_\infty |v + G - 1| \in L^2(\Omega) \qquad \text{and} \qquad \lim_{+\infty} f(v+G) = 0,$$

so that $f(v+G) \in L^2(\Omega) \cap C_0(\Omega)$.

4.2 Iteration procedure

We claim that there exists a sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ satisfying

$$u_0 = G \text{ and for } n \in \mathbb{N}\setminus\{0\},$$

$$\begin{cases} L_r^c u_{n+1} - \lambda u_{n+1} = -f(u_n) - \lambda u_n - h_r(x) & in\ \ \Omega \\ u_{n+1}(r) = \theta \\ u_{n+1}(x) \to 1 \qquad x \to +\infty. \end{cases} \tag{4.3}$$

We proceed as follows. Using the substitution $v_n = u_n - G$, (4.3) reduces to

$$\begin{cases} L_r v_{n+1} - \lambda v_{n+1} = F(v_n, x) & in\ \ \Omega \\ v_{n+1}(r) = 0 \\ v_{n+1}(x) \to 0 \qquad x \to +\infty, \end{cases} \tag{4.4}$$

where $F(v,x) = -f(v+G) - \lambda v - L_r G - h_r(x)$. Therefore we have $v_{n+1} = T_{\lambda,r} v_n$. Now, using the previous subsection and induction, to prove that v_n is well-defined, it is enough to show that $v_0 \in L^2(\Omega) \cap C_0(\Omega)$ which is trivial since $v_0 = 0$.

Remark 4.1. *Notice that the behavior of the sequence $(u_n)_{n\in\mathbb{N}}$ strongly depends on the property of u_0. Namely, from the maximum principle property and choosing λ so large that $-f-\lambda$ is nonincreasing, it follows easily by induction that if u_0 is a supersolution, then $(u_n)_{n\in\mathbb{N}}$ is a decreasing sequence. Respectively, u_0 is a subsolution, then $(u_n)_{n\in\mathbb{N}}$ is in increasing sequence.*

4.3 Passing to the limit as $n \to \infty$

Assume that u_0 is either a supersolution or a subsolution satisfying $\theta \le u_0 \le 1$. Recall that the constants θ and 1 are respectively a subsolution and a supersolution of (4.1).

It follows easily from induction and the Maximum Principle (Theorem A.2) that for all $n \in \mathbb{N}\setminus\{0\}$,

$$\theta \le u_n \le 1. \tag{4.5}$$

Choosing $\lambda > 0$ so large that $-f-\lambda$ is nonincreasing, we prove next by induction that

$$x \to u_n(x) \text{ is a nondecreasing function.} \tag{4.6}$$

First define the following sequence of function:

$$\widetilde{u}_n(x) := \begin{cases} \theta & if\ \ x \in \mathbb{R}\setminus\Omega \\ u_n(x) & if\ \ x \in \Omega. \end{cases}$$

We will prove that $(\widetilde{u}_n)_n$ are nondecreasing functions, which implies (4.6). Observe that $\widetilde{u}_{n+1}$ solves the following problem

$$\begin{cases} L_r^c \widetilde{u}_{n+1} - \lambda \widetilde{u}_{n+1} + \int_{-\infty}^{r} J(x-y)\widetilde{u}_{n+1}(y)dy = -(f+\lambda)(\widetilde{u}_n(x)) & in\ \ \Omega \\ \widetilde{u}_{n+1}(r) = \theta \\ \widetilde{u}_{n+1} \to 1 \qquad x \to +\infty, \end{cases} \tag{4.7}$$

which can be rewriten as

$$\begin{cases} \widetilde{u}''_{n+1} - c\widetilde{u}'_{n+1} + J \star \widetilde{u}_{n+1} - \widetilde{u}_{n+1} - \lambda \widetilde{u}_{n+1} = -(f+\lambda)(\widetilde{u}_n(x)) \quad in \ \ \Omega \\ \widetilde{u}_{n+1}(r) = \theta \\ \widetilde{u}_{n+1} \to 1 \qquad x \to +\infty. \end{cases} \tag{4.8}$$

For $n = 0$, we already know that $\widetilde{u}_0$ is nondecreasing. Fix now $n \geq 1$ and assume that $\widetilde{u}_{n-1}$ is nondecreasing. Also given any positive τ, let $w(x) = \widetilde{u}_n(x+\tau) - \widetilde{u}_n(x)$. It follows from Equation (4.8) and the assumption that $\widetilde{u}_{n-1}$ and $f + \lambda$ are nondecreasing that

$$\epsilon w'' + J \star w - cw' - (1+\lambda)w(x) \leq 0 \qquad \text{in } \Omega, \tag{4.9}$$

$$w(x) \geq 0 \ \text{ for } x \in \mathbb{R} \setminus \Omega, \tag{4.10}$$

$$w(\infty) = 0, \tag{4.11}$$

whence by the Maximum Principle $w \geq 0$. In particular, $\widetilde{u}_n(x+\tau) - \widetilde{u}_n(x) \geq 0$ for any positive τ. This shows that $\widetilde{u}_n$ is nondecreasing. Using remark 4.1 and the assumption on u_0, the sequence $(u_n)_{n\in\mathbb{N}}$ is monotone. Hence, using (4.5), (4.6) and Helly's lemma, it follows that $\{u_n\}$ converges pointwise to a nondecreasing function u satisfying

$$\theta \leq u \leq 1.$$

By the dominated convergence theorem, we have for all $x \in \Omega$

$$\int_r^{+\infty} J(x-y)u_n(y)dy - u_n(x) \to \int_r^{+\infty} J(x-y)u(y)dy - u(x), \qquad \text{as} \qquad n \to \infty.$$

Rewriting (4.3) as

$$\epsilon u''_{n+1} - cu'_{n+1} = u_{n+1} - \int_r^{+\infty} J(x-y)u_{n+1}(y)dy - \lambda(u_n - u_{n+1}) - f(u_n) - h_r(x), \tag{4.12}$$

observing that the right-hand side in the above equation is uniformly bounded and using elliptic regularity, we conclude that $\{u_n\}$ is bounded e.g. in $C^{1,\alpha}(\omega)$, where $\alpha \in (0,1)$ and ω is an arbitrary bounded open subset of Ω. Repeating the argument implies that $\{u_n\}$ is bounded in $C^{2,\alpha}(\omega)$. Hence $u \in C^2(\Omega)$ and we can pass to the limit in the equation to obtain that u satisfies

$$\epsilon u'' - cu' + \int_r^{+\infty} J(x-y)u(y)dy - u + f(u) + h_r(x) = 0 \qquad \text{in } \Omega. \tag{4.13}$$

Observing that $u_n(r) = \theta$ and that u_n converges pointwise to u, we easily conclude that $u(r) = \theta$. To complete the construction of the solution, we prove that $u(+\infty) = 1$. Indeed, since u is uniformly bounded and nondecreasing, u achieves its limit at $+\infty$. Using standard estimates we easily get from (4.13) that u satisfies $f(u(+\infty)) = 0$. Hence $u(+\infty) = 1$. We have thus constructed an increasing solution u of (4.1).

□

Remark 4.2. *We may find more easily the boundary condition, when u_0 is a subsolution rather than a supersolution . Indeed, in this case $u_0 \leq u \leq 1$ hence $u \to 1$ since $u_0(x) \to 1$ as $x \to +\infty$.*

The construction of a nondecreasing solution of (4.1) is now reduce to find good sub and supersolution u_0 satisfying $u_0(r) = \theta$, $L_r u_0 \in L^2(\Omega)$ and $1 - u_0 \in L^2(\Omega)$ for fixed $r < 0$, $\theta \in (0,1)$, $\epsilon > 0$ and $c \in \mathbb{R}$.

4.4 Construction of a solution of (4.1) for $c \leq c_b$

Assume that $r < 0$, $\theta \in (0,1)$, $\epsilon > 0$ are fixed and let $c \leq c_b$. Recall that translation of u_b and 1 are respectively a sub and a supersolution of equation (4.1) for any $c \leq c_b$ (see Section 2). From the translation invariance of the bistable problem, we may also assume that $u_b(r) = \theta$. To conclude, it then remains to prove that $L_r^c u_b \in L^2(\Omega)$ and $1 - u_b \in L^2(\Omega)$. Using Lemmas 3.1-3.2 and remark 3.1 yields to

$$u_b'', u_b', (J \star u_b - u_b) \quad \text{in} \quad L^2(\mathbb{R}) \quad \text{and} \quad 1 - u_b \quad \text{in} \quad L^2(\mathbb{R}^+)$$

Hence

$$|L_r^c u_b| \leq \epsilon |u_b''| + |c u_b'| + |\int_{-\infty}^{+\infty} J(x-y) u_b(y)\, dy - u_b| \in L^2(\Omega)$$

We can then applied the previous subsection with $u_0 = u_b$ to obtain a nondecreasing solution of (4.1) for $c \leq c_b$.

4.5 Construction of a solution for $c > c_b$

To obtain solution for $c > c_b$, we argue as follows. Assume as in the previous subsection that $r < 0$, $\theta \in (0,1)$, $\epsilon > 0$ are fixed and choose $c > c_b$. Let u_s be the smooth nondecreasing solution of (4.1) obtained with $c = c_b$ with the above argumentation. Since $c > c_b$ and u_s is increasing, u_s will be a supersolution of (4.1) with speed c. From construction, one have $u_s \geq \theta$ and θ is a subsolution of (4.1). Therefore to obtain a solution of (4.1), it is then sufficient to prove that $L_r^c u_s \in L^2(\Omega)$ and $1 - u_s \in L^2(\Omega)$. The latter is easily obtained using the L^2 estimates (Lemmas 3.3-3.4) obtained in the previous section.

□

The above analysis is independant of the choice of the parameters, since the subsolution u_b exists for any $\epsilon > 0$, $\theta \in (0,1)$, $c \leq c_b$ and $r < 0$. Therefore, there exists a nondecreasing solution of (4.1) for any $\epsilon > 0$, $\theta \in (0,1)$, $c \leq c_b$ and $r < 0$.

□

We can now turn our attention to the construction of a solution of (5.1), which will be proved in the next section.

5 Construction of solutions of (5.1) for all $c \geq c^*(\epsilon)$

In this section, we show that there exists $c^*(\epsilon)$ such that for each $c \geq c^*(\epsilon)$ there exists a positive increasing solution of the following problem

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + f(u) = 0 & in \quad \mathbb{R} \\ \quad u \to 0 & x \to -\infty \\ \quad u(x) \to 1 & x \to +\infty, \end{cases} \tag{5.1}$$

Namely we have the following

Theorem 5.1.
Let $\epsilon > 0$, then there exists a positif real number $c^*(\epsilon)$ such that for all $c \geq c^*(\epsilon)$ there exists a positive smooth increasing solution u_ϵ of (5.1). *Futhermore if $c < c^*(\epsilon)$, then problem* (5.1) *has no increasing solution.*

The proof of Theorem 5.1 will be broken down in two parts. In the first part, Subsection 5.1, we construct a solution of Problem (5.1) for a specific value of the speed $c = \kappa$, using solutions of approximate problems constructed in the previous section and a standard limiting procedure. Then in the second part Subsection 5.2 we define the minimal speed $c^*(\epsilon)$ and construct solutions of (5.1) for speeds $c \geq c^*(\epsilon)$.

5.1 Construction of one solution of (5.1) for $c = \kappa$

For the construction of the solution, we use the approximate problem below

$$\begin{cases} L_r u + h_r(x) + f(u) = 0 & \text{for } x \in (r, +\infty) \\ \quad u(r) = \theta & \\ \quad u(x) \to 1 \quad x \to +\infty, & \end{cases} \tag{5.2}$$

From the previous section, for any real number r and any $\theta \in (0,1)$ there exists a unique solution of (5.2). For fixed $r < 0$, we claim that the solution of (5.2) satisfies the following normalization.

Claim 5.1.
There exists $\theta_0 \in (0,1)$ such that the corresponding solution $u_r^{\theta_0}$ of (5.2) *with $\theta = \theta_0$ satisfies the normalization $u_r^{\theta_0}(0) = \frac{1}{2}$.*

Remark 5.1. *This normalization has no meaning when r is no longer negative.*

Proof of Claim 5.1

We start with the definition of the following set of acceptable values of θ.

$$\Theta = \{\theta | u_r^\theta(0) > \frac{1}{2}\}$$

Choosing any $\theta \geq \frac{1}{2}$ and observing that u_r^θ is increasing we have $[\frac{1}{2}, 1) \subset \Theta$. The uniqueness of the solution u_r^θ and standard a priori estimates imply that $\theta \to u_r^\theta(0)$ is a continuous over $[0,1]$. By continuity, we can therefore conclude that

- Either there exists a positive θ_0 such that $u_r^{\theta_0}(0) = \frac{1}{2}$
- Or $(0, 1) \subset \Theta$.

We show that the latter case can not occur which will prove the claim. For this, we argue by contradiction. Suppose that $(0,1) \subset \Theta$. Let $(\theta_n)_{n\in\mathbb{N}}$ a sequence such that $\theta_n \to 0$. Let $(u_n)_{n\in\mathbb{N}}$ be the corresponding sequence of solution of (5.2) with $\theta = \theta_n$. Using Helly's Lemma and standard *a priori* estimates, we can extract a subsequence, still denoted $(u_n)_{n\in\mathbb{N}}$ which converges to a nondecreasing function u. Since $u_n(0) > \frac{1}{2}$, $u(0) \geq \frac{1}{2}$ and u is thus a non-trivial function, satisfying the following equation

$$\begin{cases} L_r u + f(u) = 0 & \text{for } x \in (r, +\infty) \\ \quad u(r) = 0 & \\ \quad u(x) \to 1 & x \to +\infty. \end{cases} \tag{5.3}$$

Observe that the function w constructed in Section 2 is a subsolution of (5.3). One can show that $w > u$, which provides a contradiction since $\frac{1}{2} \leq u(0) < w(0) = \frac{1}{2}$. See the appendix for details.

□

With the latter normalization, we are ready for the construction of a solution of (5.1). Let $(r_n)_{n\in\mathbb{N}} = (-n)_{n\in\mathbb{N}}$ and $(u_n^{\theta_n})_{n\in\mathbb{N}}$ be the sequence of solutions of the corresponding approximate problem (5.2) with r replaced by r_n and $\theta = \theta_n$, where $(\theta_n)_{n\in\mathbb{N}}$ is such that we have the normalization $u_n^{\theta_n}(0) = \frac{1}{2}$. Define $(h_n)_{n\in\mathbb{N}}$ by

$$h_n(x) = \theta_n \int_{-\infty}^{r_n} J(x-y)dy. \tag{5.4}$$

From Claim 5.1 and the previous section such sequences are well defined. Clearly, $h_n \to 0$ pointwise, as $n \to \infty$. Observe now that $(u_n^{\theta_n})_{n\in\mathbb{N}}$ is a uniformly bounded sequence of increasing functions, therefore using Helly's lemma, there exists a subsequence which converges pointwise to a nondecreasing function u. Since $\epsilon > 0$, using local $C^{2,\alpha}$ estimates, up to extraction, the subsequence converge in $C^{2,\alpha}_{loc}$. Therefore $u \in C^{2,\alpha}$ and satisfies

$$\epsilon u'' + J \star u - u - cu' + f(u) = 0 \qquad \text{in} \qquad \mathbb{R}. \tag{5.5}$$

From the normalization and the fact that $f(\frac{1}{2}) \neq 0$, u is not trivial. Since u is increasing and bounded, u achieves its limits $l^{\pm}$ at $\pm\infty$. A standard argument implies that $f(l^-) = 0$ therefore $l^- = 0$ since f is a nonnegative function and $l^- \leq \frac{1}{2}$. Similarly $l^+ = 1$. Therefore we have constructed a non trivial solution of (5.1).

Remark 5.2. *Observe that the supersolution is only needed in the normalization process. Therefore, the previous construction will holds with any other supersolution ψ of* (5.1) *such that $\psi(0) = \frac{1}{2}$.*

Let us now turn our attention to the second part of the proof.

5.2 Definition of $c^*(\epsilon)$

Define

$$c^*(\epsilon) := \inf\{c > 0 : \quad \text{(5.1) admits an increasing solution}\} \tag{5.6}$$

By the previous section, $c^*(\epsilon)$ is well defined. Obviously, from the definition of $c^*(\epsilon)$, there is no increasing solution to (5.1) for speeds $c < c^*(\epsilon)$. Our goal in this subsection is to provide a solution of (5.1) for all $c \geq c^*(\epsilon)$.

First we observe that (5.1) has a solution for $c = c^*(\epsilon)$. Indeed, by definition of $c^*(\epsilon)$, there exists a sequence of speeds c_n converging to $c^*(\epsilon)$. The corresponding solutions u_n of (5.1) are increasing (and uniformly bounded by 1) so that we may apply Helly's lemma and elliptic regularity as in the previous section to conclude that u_n converges to an increasing solution of (5.1) for $c = c^*(\epsilon)$, which we denote by u_ϵ. Boundary conditions for u_ϵ are obtained as in Subsection 5.1.

Fix now $c > c^*(\epsilon)$ and observe that $w := u_\epsilon$ is a smooth increasing supersolution of (5.1) (with speed c). Assume for a moment that u_ϵ satisfies $u_\epsilon(0) = \frac{1}{2}$, then by Remark 5.2 the construction of Subsection 5.1 applies. Therefore we get a solution of (5.1) for all $c \geq c^*(\epsilon)$ which ends the proofs of Theorem 5.1.

6 Existence of a solution for $\epsilon = 0$

In the previous section, we were able to prove that for every positive ϵ, the following problem:

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + f(u) = 0 & in\ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty, \end{cases} \tag{6.1}$$

admits a semi infinite interval of solution, i.e for $c \geq c^*(\epsilon)$ there exists a positive increasing solution of (6.1). We will see that the same holds true of the following problem.

$$\begin{cases} J \star u - u - cu' + f(u) = 0 & in\ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty, \end{cases} \tag{6.2}$$

The idea is to let $\epsilon \to 0$ in the equation and to extract a converging sequence of solutions. The main problem is to control $c^*(\epsilon)$ when $\epsilon \to 0$. We prove the following:

Lemma 6.1.
For every positive ϵ_0, there exists $\nu_0 > 0$ such that $c^*(\epsilon) \leq \nu_0$ for all $\epsilon \in [0, \epsilon_0)$

Proof:

According to Remark 2.1, $\kappa(\epsilon)$ is an nondecreasing function of ϵ, therefore $\kappa(\epsilon) \leq \kappa(\epsilon_0)$. The conclusion easily follows from the definition of $c^*(\epsilon)$, i.e. $c^*(\epsilon) \leq \kappa(\epsilon)$.

$\square$

We can now derive existence of solution for (6.2) for every speed c greater than ν_0. More precisely we have the following:

Theorem 6.1.
There exists ν_0 such that for every speed c greater than ν_0, there exists a solution u with speed c of the equation (6.2)*.*

Proof:

According to the previous lemma, for ϵ small, say $\epsilon \leq \epsilon_0$, equation (6.1) has a solution u_ϵ for every c greater than ν_0 and $\epsilon \leq \epsilon_0$. Without loss of generality we assume that for all ϵ, $u_\epsilon(0) = \frac{1}{2}$. From standard a-priori estimates, u_ϵ is a bounded smooth increasing function. Let $\epsilon \to 0$ along a sequence. As in the previous section, uniform *a priori* estimates and Helly's theorem applied to u_ϵ, provide the existence of a monotone increasing solution u of

$$J \star u - u - cu' + f(u) = 0 \ \ in \ \ \mathbb{R}. \tag{6.3}$$

The solution cannot be trivial, according to the normalisation $\frac{1}{2} = u_\epsilon(0) \to u(0)$. Boundary conditions are obtained as in Section 5 .

□

We can define another minimal speed

$$c^{**} = \inf\{c | \forall c' \geq c \ \text{(6.2) has a positive increasing solution of speed} \ c'\}. \tag{6.4}$$

This minimal speed is well defined according to the previous theorem.

Remark 6.1. *A quick computation, shows that*

$$c^{**} \leq \liminf_{\epsilon \to 0} c^*(\epsilon).$$

Nevertheless to complete the characterization of the set of solutions of (6.2), we have to prove that there exists no travelling-wave solutions of speed c less than c^{**}. In other words, if we defined :

$$c^* = \inf\{c| \ \text{(6.2) has a positive increasing solution of speed} \ c\}, \tag{6.5}$$

we have to show that $c^* = c^{**}$. Clearly we have $c^{**} \geq c^*$, the main problem is to prove $c^{**} \leq c^*$. This will be done with the aid of the monotony of the speed of truncated problems $c_\theta(\epsilon)$ and its continuous behavior at zero. More precisely, consider equation (6.6) below

$$\begin{cases} \epsilon u'' + J \star u - u - cu' + (f\chi_\theta)(u) = 0 & in \ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty, \end{cases} \tag{6.6}$$

where $\epsilon \geq 0$, $\theta > 0$ and let χ_θ be such that

- $\chi_\theta \in C_0^\infty(\mathbb{R})$,
- $0 \leq \chi_\theta \leq 1$,
- $\chi_\theta(s) \equiv 0$ for $s \leq \theta$ and $\chi_\theta(s) \equiv 1$ for $s \geq 2\theta$.

We have the following existence and uniqueness theorem

Theorem 6.2.
There exists a unique smooth increasing solution u_θ with speed $c_\theta(\epsilon)$ to (6.6)*. Moreover the speed $c_\theta(\epsilon)$ is positive and satisfies*

$$c_\theta(\epsilon) < c^*(\epsilon) \tag{6.7}$$

$$\lim_{\theta\to 0} c_\theta(\epsilon) = c^*(\epsilon). \tag{6.8}$$

A proof of Theorem 6.2 can be found in [5, 8], so we do not include it. A natural corollary of this theorem is the continuity of the speed $c_\theta(\epsilon)$ with respect to ϵ and θ. Namely, we have

Corollary 6.1.
Under the above assumptions, the following application

$$\begin{array}{ccc} (0,1)\times[0,1] & \to & \mathbb{R}^+ \\ (\theta,\epsilon) & \mapsto & c_\theta(\epsilon) \end{array}$$

is continuous.

Suppose, for a moment that the continuity in θ and ϵ holds, then we can easily conclude the proof of $c^* = c^{**}$. Namely, suppose that $c^* < c^{**}$. Then choose c such that $c^* < c < c^{**}$. Since $c_\theta < c^*$ for every positive θ, we have $c_\theta < c^* < c$. Fix $\theta > 0$: since $c_\theta(\epsilon)$ is a continuous function of ϵ, one has on the one hand $c_\theta(\epsilon) < c$ for ϵ small, say $\epsilon \in [0, \epsilon_0]$. On the other hand, according to Remark 6.1, we may achieve,

$$c_\theta(\epsilon) < c < c^*(\epsilon) \forall \epsilon \in [0, \epsilon_0]. \tag{6.9}$$

From this last inequality, and according to (6.8), for each $\epsilon \in (0, \epsilon_0]$ there exists a positive $\theta(\epsilon) \le \theta$ such that $c = c_{\theta(\epsilon)}(\epsilon)$. Let $u_{\theta(\epsilon)}$ be the normalized associated solution.
Now we take a sequence θ_n which goes to 0. From the above construction for each n there exists $\epsilon_n \le \theta_n$, and $\theta(\epsilon_n) \le \theta_n$ such that $c = c_{\theta(\epsilon_n)}(\epsilon_n)$ and $u_{\theta(\epsilon_n)}$ is the corresponding normalized solution. From our construction we have,

$$\theta(\epsilon_n) \to 0.$$

Use now, as usual, uniform *a priori* estimates and Helly's theorem to get a solution $\bar u$ of the following problem

$$\begin{cases} J \star \bar u - \bar u - c\bar u' + f(\bar u) = 0 & in\ \mathbb{R} \\ \bar u(x) \to 0 & x \to -\infty \\ \bar u(x) \to 1 & x \to +\infty, \end{cases} \tag{6.10}$$

with $c > c^*$. So we get a non trivial solution of (6.2) for the speed c. Since c is arbitrary, there exists a non trivial solution of (6.2) for any speed $c > c^*$, which contradicts the definition of c^{**}.

□

Now, let us turn our attention to the continuity of $c_\theta(\epsilon)$, which will complete the proof.

Proof of Corollary 6.1

We know from Theorem 6.2 that for every $\epsilon \geq 0$ and $\theta > 0$ there exists a unique solution $(u_\theta^\epsilon, c_\theta^\epsilon)$ to the following problem,

$$\begin{cases} \epsilon(u_\theta^\epsilon)'' + J \star u_\theta^\epsilon - u_\theta^\epsilon - c(u_\theta^\epsilon)' + f_\theta(u_\theta^\epsilon) = 0 & in \ \mathbb{R} \\ u_\theta^\epsilon \to 0 & x \to -\infty \\ u_\theta^\epsilon \to 1 & x \to +\infty, \end{cases} \tag{6.11}$$

Fix $\epsilon_0 \geq 0$ and $\theta_0 > 0$, we will show that for any sequence $(\epsilon_n, \theta_n) \to (\epsilon_0, \theta_0)$, we have $c_{\theta_n}^{\epsilon_n} \to c_{\theta_0}^{\epsilon_0}$. This will show the continuity of the speed. Let $u_{\theta_n}^{\epsilon_n}$ be the normalized associated solution, i.e $u_{\theta_n}^{\epsilon_n}(0) = \frac{1}{2}$. Since $c_\theta(\epsilon) > 0$ and using (6.7), we have $c_{\theta_n}^{\epsilon_n}$ bounded as $(\epsilon_n, \theta_n) \to (\epsilon_0, \theta_0)$. We can extract a sequence of speeds, which converges to some value γ. From the *a priori* estimates on $u_{\theta_n}^{\epsilon_n}$, there also exists a subsequence which converges to a smooth function u solution of the following problem with speed γ.

$$\begin{cases} \epsilon_0 u'' + J \star u - u - \gamma u' + f_{\theta_0}(u) = 0 & in \ \mathbb{R} \\ u \to 0 & x \to -\infty \\ u \to 1 & x \to +\infty, \end{cases} \tag{6.12}$$

According to Theorem 6.2, the speed and the profile are unique. Therefore, $\gamma = c_{\theta_0}^{\epsilon_0}$ and $u(x) = u_{\theta_0}^{\epsilon_0}(x + \tau)$. Since $c_{\theta_n}^{\epsilon_n}$ is precompact and has a unique accumulation point, the sequence $c_{\theta_n}^{\epsilon_n}$ must converge to $c_{\theta_0}^{\epsilon_0}$. This ends the proof of the continuity and by means the characterization of the minimal speed c^*.

□

7 Asymptotic behavior of solutions

In this section we establish the asymptotic behavior of the solution u near $\pm\infty$ provided J satisfies $(H3)$. The behavior of the function near $+\infty$ has been already obtained in a previous work by one of the authors [5], therefore we only deal with the behavior of u near $-\infty$.

Remark 7.1. *The behavior of u near $\pm\infty$ for bistable and ignition type nonlinearities was also obtained in [5].*

We use the same strategy as in [2] and start by proving the following lemma

Lemma 7.1. *Assume that (H1) and (H3) hold. Also assume that f is of KPP-type i.e. $f'(0) > 0$. Let u be an increasing solution of problem (P). Then there exists $\epsilon > 0$ such that*

$$\int_{-\infty}^{\infty} u(x) e^{-\epsilon x} \, dx < \infty.$$

Proof

Let $\zeta \in C^\infty(\mathbb{R})$ be a nonnegative nondecreasing function such that $\zeta \equiv 0$ in $(-\infty, -2]$ and $\zeta \equiv 1$ in $[-1, \infty)$. For $N \in \mathbb{N}$, let $\zeta_N = \zeta(x/N)$. Multiplying (P) by $e^{-\epsilon x}\zeta_N$ and integrating over $\mathbb{R}$, we get

$$\int (J * u - u)(e^{-\epsilon x}\zeta_N) - \int cu'(e^{-\epsilon x}\zeta_N) + \int f(u)(e^{-\epsilon x}\zeta_N) = 0 \tag{7.1}$$

Since J is even,

$$\begin{aligned}\int (J * u - u)(e^{-\epsilon x}\zeta_N) &= \int (J * (e^{-\epsilon x}\zeta_N) - e^{-\epsilon x}\zeta_N)u \\ &= \int u(x)e^{-\epsilon x}\left(\int J(y)e^{\epsilon y}\zeta_N(x-y)\,dy - \zeta_N(x)\right)\,dx \\ &= \int u(x)e^{-\epsilon x}\left(\int J(y)e^{-\epsilon y}\zeta_N(x+y)\,dy - \zeta_N(x)\right)\,dx \\ &\geq \int u(x)e^{-\epsilon x}\left(\int_{-R}^{\infty} J(y)e^{-\epsilon y}\,dy\;\zeta_N(x-R) - \zeta_N(x)\right)\,dx,\end{aligned} \tag{7.2}$$

where we used the monotone behaviour of ζ_N in the last inequality and where $R > 0$ is chosen as follows : first pick $0 < \alpha < f'(0)$ and $R > 0$ so large that

$$f(u)(x) \geq \alpha u(x) \qquad \text{for } x \leq -R. \tag{7.3}$$

Next, one can increase R further if necessary so that $\int_{-R}^{\infty} J(y)\,dy > (1 - \alpha/2)$. By continuity we obtain for some $\epsilon_0 > 0$ and all $0 < \epsilon < \epsilon_0$,

$$\int_{-R}^{\infty} J(y)e^{-\epsilon y}\,dy \geq (1 - \alpha/2)e^{\epsilon R}. \tag{7.4}$$

Collecting (7.2) and (7.4), we then obtain

$$\begin{aligned}\int (J * u - u)(e^{-\epsilon x}\zeta_N) &\geq \int u(x)e^{-\epsilon x}\left((1-\alpha/2)e^{\epsilon R}\zeta_N(x-R) - \zeta_N(x)\right)\,dx \\ &\geq (1-\alpha/2)\int u(x+R)e^{-\epsilon x}\zeta_N(x)\,dx - \int u(x)e^{-\epsilon x}\zeta_N(x)\,dx \\ &\geq -\alpha/2\int u(x)e^{-\epsilon x}\zeta_N(x)\,dx,\end{aligned} \tag{7.5}$$

where we used the monotone behaviour of u in the last inequality.

We now estimate the second term in (7.1) :

$$\begin{aligned}\int u'\zeta_N e^{-\epsilon x}\,dx &= \epsilon\int u\zeta_N e^{-\epsilon x} \; - \int u\zeta_n' e^{-\epsilon x}\,dx \\ &\leq \epsilon\int u\zeta_N e^{-\epsilon x}\;.\end{aligned} \tag{7.6}$$

Finally using (7.3), the last term in (7.1) satisfies

$$\int f(u)\zeta_N e^{-\epsilon x}\,dx \geq \alpha\int_{-\infty}^{-R} u\zeta_N e^{-\epsilon x}\,dx - C. \tag{7.7}$$

By (7.1), (7.5), (7.6) and (7.7) we then obtain

$$(\alpha/2 - c\epsilon)\int_{-\infty}^{-R} u\zeta_N e^{-\epsilon x}\,dx \leq C.$$

Choosing $\epsilon < \alpha/(2c)$ and letting $N \to \infty$ proves the lemma.

□

Using Lemma 7.1 it is now easy to see that $u(x) \leq Ce^{\epsilon x}$ for all $x \in \mathbb{R}$. Suppose indeed this is not the case and let $x_n \in \mathbb{R}$ be such that $u(x_n) > ne^{\epsilon x_n}$. Since $0 \leq u \leq 1$, we may pick a subsequence x_{n_k} such that $x_{n_{k+1}} < x_{n_k} - 1$. But since u is nondecreasing,

$$\begin{aligned}\int u(x)e^{-\epsilon x}\,dx &\geq \sum_{k\geq 1}\int_{x_{n_k}}^{x_{n_{k-1}}} u(x)e^{-\epsilon x}\,dx\\ &\geq \sum_{k\geq 1} n_k\int_{x_{n_k}}^{x_{n_k}-1} e^{\epsilon(x_{n_k}-x)}\,dx\\ &\geq \sum_{k\geq 1} n_k/\epsilon\left(1-e^{-\epsilon}\right) = \infty.\end{aligned}$$

□

Acknowledgements. We are very thankfull to Henri Berestycki who has supported this work through the "Dynamic of reactif fronts" semester organized at the Institut Henri Poincare in september 2002. We also thank Adam Chmaj for pointing us Schumacher's reference. We would also thank Augusto Ponce and Pascal Autissier for useful discussion.

A Appendix

Here we prove some maximum principles and existence results for solutions of linear problems associated to the operator $\mathcal{L}$ defined by (1.6).

Theorem A.1. *Strong Maximum Principle for $\mathcal{L}$*
Let $\epsilon \geq 0$, $r > 0$, $c \in \mathbb{R}$ and $\mathcal{L}$ defined by (1.6) *on $\Omega = (-r, +\infty)$.*
Assume further that Int(supp J) $\cap\, \Omega^- \neq \emptyset$, where $\Omega^- = (-r, 0)$.
Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy

$$\mathcal{L}u \geq 0 \; in \;\; \Omega \qquad (resp.\;\; \mathcal{L}u \leq 0 \;\; in \;\; \Omega). \tag{A.1}$$

Then u may not achieve a positive maximum (resp. negative minimum) inside Ω without being constant.

Similarly we have

Theorem A.2. *Strong Maximum Principle for $\mathcal{L} + h_r(x)$.*
Let $\epsilon \geq 0$, $r > 0$, $c \in \mathbb{R}$, $\theta \in (0,1)$ and $\mathcal{L}$, $h_r(x)$ defined by (1.6) *on $\Omega = (-r, +\infty)$. Assume further that Int(supp J) $\cap\, \Omega^- \neq \emptyset$, where $\Omega^- = (-r, 0)$.*

Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy

$$\begin{cases} \mathcal{L}u & \geq & -h_r(x) & \text{in } \Omega & (\text{resp. } \mathcal{L}u \leq -h_r(x) \text{ in } \Omega) \\ u(-r) & = & \theta & & \\ u & \geq & \theta & \text{in } \Omega & (\text{resp. } u \leq \theta \text{ in } \Omega). \end{cases} \tag{A.2}$$

Then u may not achieve a positive maximum (resp. negative minimum) inside Ω without being constant.

Proof of Theorem A.1:

We argue by contradiction and assume that u is nonconstant and achieves a positive maximum at some point $x_0 \in \Omega$. Since $\int_{\mathbb{R}} J(z)dz = 1$ we can rewrite (1.6) as

$$\mathcal{L}u = \epsilon u'' + \int_{-r}^{+\infty} J(x-y)[u(y) - u(x)]dy - cu' - d(x)u, \tag{A.3}$$

with $d(x) = \int_{-\infty}^{-r} J(x-y)dy$.

At the point x_0 of (positive) maximum, we have on the one hand

$$\epsilon u''(x_0) \leq 0, \quad \int_{-r}^{+\infty} J(x_0 - y)[u(y) - u(x_0)]dy \leq 0 \quad \text{and} \quad -d(x_0)u(x_0) \leq 0. \tag{A.4}$$

On the other hand by (A.1),

$$\epsilon u''(x_0) + \int_{-r}^{+\infty} J(x_0 - y)[u(y) - u(x_0)]dy - \bar{d}(x_0)u(x_0) \geq 0 \tag{A.5}$$

Hence $\epsilon u''(x_0) = d(x_0)u(x_0) = 0$ and

$$\int_{-r}^{\infty} J(x_0 - y)[u(y) - u(x_0)]\, dy = 0. \tag{A.6}$$

If $J > 0$ in $\mathbb{R}$, we conclude directly that $u(y) = u(x_0)$ for all $y \in \Omega$, contradicting our original assumption.

In general, J is a continuous nonnegative even function with $\overset{\circ}{supp}(J) \cap \Omega^- \not\equiv \emptyset$. In particular, there exist constants $0 < a < b$ such that $[-b, -a] \cup [a, b] \subset supp(J)$ and $[a, b] \subset \Omega$. We deduce from (A.6) that

$$u(y) = u(x_0) \quad \text{for all } \; y \in (x_0 + [-b, -a] \cup [a, b]) \cap \Omega$$

Let $z = x_0 + b$ and observe that $u(z) = u(x_0)$. We may thus argue as above and conclude that $u(y) = u(z)$ for all $y \in (z + [-b, -a] \cup [a, b]) \cap \Omega$. In particular,

$$u(y) = u(x_0) \quad \text{for all } y \in (x_0 + [0, b-a]) \cap \Omega.$$

Repeating the argument with $z = x_0 + a$, we obtain that $u(y) = u(x_0)$ for all $y \in (x_0 + [-(b-a), 0]) \cap \Omega$. Thus,

$$u(y) = u(x_0) \quad \text{ for all } y \in (x_0 + [-(b-a), b-a]) \cap \Omega.$$

Applying the above successively with x_0+b-a and $x_0-(b-a)$ in place of x_0, we obtain that $u(y) = u(x_0)$ for all $y \in x_0+[-2(b-a), 2(b-a)]\cap\Omega$. Working inductively, we conclude that $u \equiv u(x_0)$ in Ω, which contradicts our original assumption.

$\square$

Proof of Theorem A.2

Define

$$\widetilde{u}(x) := \begin{cases} u(x) & in \ \ \Omega \\ \theta & in \ \ \mathbb{R} \setminus \Omega \end{cases}$$

and observe that we can rewrite (A.2) as

$$\begin{cases} \mathcal{M}\widetilde{u} \geq 0 & in \ \ \Omega \\ \widetilde{u}(x) \geq \theta & in \ \ \Omega, \end{cases}$$

where $\mathcal{M}\widetilde{u} = \epsilon\widetilde{u}'' + [J \star \widetilde{u} - \widetilde{u}] - c\widetilde{u}'$.

We argue by contradiction and assume that $\widetilde{u}$ achieves a positive maximum at some point $x_0 \in \Omega$ and is nonconstant. Since $u(x) \geq \theta$ in Ω we have $u(x_0) > \theta$. Working as in the proof of Theorem A.1 we obtain that $u \equiv u(x_0)$ on $\bar{\Omega}$, which is a contradiction.

$\square$

Remark A.1. *Theorems A.2 and A.1 remain valid when replacing $\mathcal{L}$ by $\mathcal{L} - d_0$, where d_0 is any positive constant.*

Next, we provide an elementary lemma to construct solutions of Dirichlet problems associated to $\mathcal{L}$.

Lemma A.1. *Let $d_0 > 0$, $\epsilon > 0$, $r > 0$, $c \in \mathbb{R}$ and $\mathcal{L}$ defined by* (1.6) *on $\Omega = (-r, +\infty)$. Assume further that Int(supp J) $\cap\, \Omega^- \neq \emptyset$, where $\Omega^- = (-r, 0)$.*

Given $f \in C_0(\Omega) \cap L^2(\Omega)$, there exists a unique solution $u \in C^2(\Omega) \cap L^2(\Omega)$ of

$$\begin{cases} \mathcal{L}u - d_0 u = f & in \ \ \Omega \\ u(-r) = 0 \\ u(+\infty) = 0 \end{cases} \tag{A.7}$$

Proof

Uniqueness follows from the maximum principle. Let $X = H_0^1(\Omega)$ and define the bilinear form $\mathcal{A}(u, v)$ for $u, v \in X$ by

$$\mathcal{A}(u,v) = \epsilon \int_\Omega u'v' + \frac{1}{2}\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))(v(y)-v(x))dydx - c\int_\Omega u'v + \int_\Omega d(x)uv,$$

where $d(x) = \int_{-\infty}^{-r} J(x-y)dy + d_0$. To solve (A.7), we just need to find $u \in X$ such that $\mathcal{A}(u,v) = \int_\Omega uv$ for all $v \in X$. We will show that $\mathcal{A}$ is coercive and continuous in X. Existence will then be given by the Lax-Milgram Lemma. Clearly,

$$\mathcal{A}(u,u) \geq \epsilon \int_\Omega (u')^2 - c\int_\Omega u'u + d_0 \int_\Omega u^2 = \epsilon \int_\Omega (u')^2 + d_0 \int_\Omega u^2$$

Thus $\mathcal{A}$ is coercive in X. It remains to prove the continuity of $\mathcal{A}$. Let ϕ and ψ be two smooth functions with compact support in Ω.

$$|\mathcal{A}(\phi,\psi)| \leq \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)|\phi(y)-\phi(x)||\psi(y)-\psi(x)|dydx$$

By the Fundamental Theorem of Calculus and the Cauchy-Schwartz inequality we obtain:

$$\begin{aligned}
|\mathcal{A}(\phi,\psi)| &\leq \int_{\mathbb{R}^2}\int_0^1\int_0^1 J(z)z^2|\phi'(x+tz)||\psi'(x+sz)|dz\, dx\, dt\, ds \\
&\leq \int_{\mathbb{R}}\int_{[0,1]^2} J(z)z^2 \int_{\mathbb{R}} |\phi'(h)||\psi'(h+(s-t)z)|dh\, ds\, dz\, dt \\
&\leq \int_{\mathbb{R}}\int_{[0,1]^2} J(z)z^2\, dz\, dt\, ds \|\phi'\|_{L^2(\mathbb{R})}\|\psi'\|_{L^2(\mathbb{R})} \\
&\leq (\int_{\mathbb{R}} J(z)z^2\, dz)\|\phi'\|_{L^2(\mathbb{R})}\|\psi'\|_{L^2(\mathbb{R})},
\end{aligned}$$

which shows the continuity of $\mathcal{A}$.

□

References

[1] P. W. Bates, P. C. Fife, X. Ren, X. Wang *Travelling Waves in a convolution model for phase transition,* Arch. Rational Mech. Anal. 138 (1997), 105–136.

[2] H. Berestycki, L. Nirenberg *Travelling fronts in cylinder,* Ann. Inst. Henri Poincaré 9 (1992), 497–572.

[3] J. Carr, A. Chmaj *Uniqueness of travelling waves for nonlocal monostable equations,* Preprint

[4] X. Chen *Existence, Uniqueness and asymptotic Stability of Travelling Fronts in non-local Evolution Equations,* Adv. Differential. Equation 2 (1997), 125–160.

[5] J. Coville *Travelling wave in non-local reaction diffusion equation with ignition non-linearity,* Preprint

[6] J. Coville *On the uniqueness and monotonicity of solution of non-local reaction-diffusion equation,* Preprint

[7] J. Coville *Équations de réaction-diffusion non-locale,* PhD Thesis, Univ. Paris 6, 2003.

[8] J. Coville, L. Dupaigne *Min-Max formula for travelling front speed of Non-local reaction-diffusion equation,* Preprint

[9] P. Fife *Mathematical aspects of reacting and diffusing systems,* Lecture Notes in Biomathematics, 28. Springer-Verlag, Berlin-New York, (1979).

[10] R. A. Fisher, *The genetical theory of natural selection,* (English. English summary) A complete variorum edition. Revised reprint of the 1930 original. Edited, with a foreword and notes, by J. H. Bennett.Oxford University Press, Oxford, (1999).

[11] N. Fourier, S. Méléard *A microscopic probabilistic description of a locally regulated population and macroscopic approximations* Prépublication du Laboratoire de probabilités et modèles aléatoires, 798 (Feb. 2003)

[12] Ya.I. Kanel' *Certain problems on equations in the theory of burning,* Dokl. Akad. Nauk SSSR 136 277–280 (Russian); translated as Soviet Math. Dokl. 2 1961 48–51.

[13] A. N. Kolmogorov, I. G. Petrovsky, N. S. Piskunov, *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique,* Bulletin Université d'État à Moscow (Bjul. Moskowskogo Gos. Univ), Série Internationale, Section A. 1, 1937, pp. 1-26, English Translation:Study of the Difussion Equation with Growth of the Quantity of Matter and its Application to a Biological Problem. In *Dynamics of curved front,* R. PELCÉ Ed., Perspective in Physics Series, Academic Press, New York, (1988), 105-130.

[14] J. D. Murray, *Mathematical biology,* Third edition. Interdisciplinary Applied Mathematics, 17.Springer-Verlag, New York, (2002).

[15] K. Schumacher *Travelling-front solutions for integro-differential equations. I. ,* J. Reine Angew. Math. 316 (1980), 54–70.

[16] H. F. Weinberger *Long-time behavior of a class of biological models,* SIAM J. Math. Anal. 13 (1982), no. 3, 353–396.

[17] H. F. Weinberger *Asymptotic behavior of a model in population dynamics* Nonlinear partial differential equations and applications (Proc. Special Sem., Indiana Univ., Bloomington, Ind., 1976–1977), pp. 47–96. Lecture Notes in Math., Vol. 648, Springer, Berlin, 1978.