Hamiltonian stationary tori in the complex projective plane

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HAMILTONIAN STATIONARY TORI IN THE COMPLEX PROJECTIVE PLANE

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1. *Introduction*

Hamiltonian stationary Lagrangian surfaces are Lagrangian surfaces of a given four-dimensional manifold endowed with a symplectic and a Riemannian structure, which are critical points of the area functional with respect to a particular class of infinitesimal variations preserving the Lagrangian constraint: the compactly supported Hamiltonian vector fields. The Euler–Lagrange equations of this variational problem are highly simplified when we assume that the ambient manifold $\mathcal{N}$ is Kähler. In that case we can make sense of a *Lagrangian angle function* β along any simply-connected Lagrangian submanifold $\Sigma \subset \mathcal{N}$ (uniquely defined up to the addition of a constant). And as shown in [**19**] the mean curvature vector of the submanifold is then $\vec{H} = J\,\nabla\beta$, where J is the complex structure on $\mathcal{N}$ and $\nabla\beta$ is the gradient of β along Σ. It turns out that Σ is Hamiltonian stationary if and only if β is a harmonic function on Σ.

A particular subclass of solutions occurs when β is constant: the Lagrangian submanifold is then simply a minimal one. In the case where $\mathcal{N}$ is a Calabi–Aubin–Yau manifold, such submanifolds admit an alternative characterization as *special Lagrangian*, a notion which has been extensively studied recently because of its connection with string theories and the mirror conjecture, see [**21**].

An analytical theory of two-dimensional Hamiltonian stationary Lagrangian submanifolds was constructed by R. Schoen and J. Wolfson [**19**], proving the existence and the partial regularity of minimizers. In contrast our results in the present paper rest on the fact that, for particular ambient manifolds $\mathcal{N}$, Hamiltonian stationary Lagrangian surfaces are solutions of an integrable system. This was discovered first in the case when $\mathcal{N} = \mathbb{C}^2$ in [**10**] and [**9**]. In a subsequent paper [**11**] we proved that the same problem is also completely integrable if we replace $\mathbb{C}^2$ by any two-dimensional Hermitian symmetric space. Among these symmetric spaces one very interesting example is $\mathbb{C}P^2$, because any simply-connected Lagrangian surface in $\mathbb{C}P^2$ can be lifted into a Legendrian surface in S^5. Furthermore the cone in $\mathbb{C}^3$ over this Legendrian surface is actually a singular Lagrangian three-dimensional submanifold in $\mathbb{C}^3$; and the cone in $\mathbb{C}^3$ is Hamiltonian stationary if and only if the surface in $\mathbb{C}P^2$ is so.

A similar correspondence has been remarked and used in [**12**], [**16**] and [**7**] in the case of minimal Lagrangian surfaces in $\mathbb{C}P^2$ and allows these Authors to connect

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results on minimal Lagrangian surfaces in $\mathbb{C}P^2$ [**20**] to minimal Legendrian surfaces in S^5 [**13**] and special Lagrangian cones in $\mathbb{C}^3$.

Our aim in this paper is the following:

- to expound in details the correspondence between Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}P^2$ and Hamiltonian stationary Legendrian surfaces in S^5 and a formulation using a family of curvature free connections of this integrable system (theorem 2.6). We revisit here the formulation given in [**11**], using *twisted loop groups*. Roughly speaking it rests on the identifications $\mathbb{C}P^2 \simeq SU(3)/S(U(2)\times U(1))$ and $(S^5,\ \text{contact structure}) \simeq \big(U(3)/U(2)\times U(1), A_3^3 = 0\big)$, where A_3^3 is a component of the Maurer–Cartan form. We also show that this problem has an alternative formulation, analogous to the theory of K. Uhlenbeck [**23**] for harmonic maps into $U(n)$, using *based loop groups*.
- to define the notion of *finite type* Hamiltonian stationary Legendrian surfaces in S^5: we give here again two definitions, in terms of twisted loop groups (which is an analogue to the description of finite type harmonic maps into homogeneous manifolds according to [**2**]) and in terms of based loop groups (an analogue to the description of finite type harmonic maps into Lie groups according to [**3**]). We prove the equivalence between the two definitions because we actually need this result for the following. We believe that this fact should be well known to some specialists in the harmonic maps theory, but we did not find it in the literature.
- we prove in theorem 4.1 that all Hamiltonian stationary Lagrangian tori in $\mathbb{C}P^2$ (and hence Hamiltonian stationary Legendrian tori in S^5) are of finite type. This is the main result of this paper. Our proof focuses on the case of Hamiltonian stationary tori which are not minimal, since the minimal case has been studied by many authors ([**3**], [**20**], [**13**], [**7**], [**14**],[**15**], [**16**], [**12**]). The method here is adapted from the similar result for harmonic maps into Lie groups in [**3**]. However the strategy differs slightly: we use actually the two existing formulations of finite type solutions, using twisted or based loop groups. One crucial step indeed is the construction of a *formal Killing field*, starting from a given torus solution. This step can be slightly simplified here in the twisted loop groups formulation, because the semi-simple element we start with is then just constant. However proving that the formal Killing field is *adapted* requires more work in the twisted loop groups formulation (actually we were not able to do it directly) than in the based loop groups formulation; here we take advantage from the two formulations to avoid the difficulties and to conclude.
- lastly we give some examples of Hamiltonian stationary Legendrian tori in S^5: we construct in theorem 5.1 a family of solutions which are equivariant in some sense under the action of the torus, that we call *homogeneous* Hamiltonian stationary tori. These are the simplest examples that one can build.

Let us add that the structure of the integrable system studied here fits in a classification of elliptic integrable systems proposed by C.L. Terng [**22**], as a 2nd $(U(3),\sigma,\tau)$-system[†], where σ is an involution of $U(3)$ such that its fixed set is

[†]We have here exchanged the notations σ and τ with respect to [**22**] in order to be consistent with our notations in [**11**].

$U(3)^\sigma \simeq U(2) \times U(1)$ and $U(3)/U(2)^\sigma \simeq \mathbb{C}P^2$ and τ is a 4th order automorphism (actually $\tau^2 = \sigma$) which encodes the symplectic structure on $\mathbb{C}P^2$ or the Legendrian structure on S^5.

Notations — For any matrix $M \in GL(n, \mathbb{C})$, we denote by $M^\dagger := {}^t\overline{M}$.

2. *Geometrical description of Hamiltonian stationary Lagrangian surfaces in* $\mathbb{C}P^n$

2.1. *The Lagrangian angle*

The complex projective space $\mathbb{C}P^n$ can be identified with the quotient manifold S^{2n+1}/S^1. It is a complex manifold with complex structure J. We denote by $\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$ the canonical projection a.k.a. Hopf fibration, and equip $\mathbb{C}P^n$ with the Fubini-Study Hermitian metric, denoted by $\langle \cdot, \cdot \rangle_{\mathbb{C}P^n} = \langle \cdot, \cdot \rangle - i\omega(\cdot, \cdot)$, where $\langle \cdot, \cdot \rangle$ is a Riemannian metric and ω is the Kähler form†. For each $z \in S^{2n+1}$ we let $\mathcal{H}_z$ be the complex n-subspace in $T_z S^{2n+1} \subset \mathbb{C}^{n+1}$ which is Hermitian orthogonal to z (and hence to the fiber of $d\pi_z$). By construction of the Fubini-Study metric, $d\pi_z : \mathcal{H}_z \longrightarrow T_{\pi(z)}\mathbb{C}P^n$ is an isometry between complex Hermitian spaces. We call the subbundle $\mathcal{H} := \cup_{z \in S^{2n+1}} \mathcal{H}_z$ of TS^{2n+1} the *horizontal distribution*. It defines in a natural way a connection $\nabla^{Hopf} \simeq \nabla^H$ on the Hopf bundle $\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$, whose curvature is $2i\omega$. As a consequence [**18**]:

Proposition 2.1. *Let* Ω *be a* simply connected *open subset of* $\mathbb{R}^n$ *and* $u : \Omega \longrightarrow \mathbb{C}P^n$ *be a smooth Lagrangian immersion, i.e. such that* $u^*\omega = 0$. *Then there exists a lift*

$$\begin{tikzcd} & S^{2n+1} \arrow[d, "\pi"] \\ \Omega \arrow[ur, "\widehat{u}"] \arrow[r, "u"] & \mathbb{C}P^n \end{tikzcd}$$

such that $\left(u^*\nabla^H\right)\widehat{u} = 0$ *(where* $u^*\nabla^H$ *is the pull-back by* u *of the connection* ∇^H*). This lift is unique up to multiplication by a unit complex number. Moreover the pull-back by* $\widehat{u}$ *of the symplectic form* ω *on* $\mathbb{C}^{n+1}$ *vanishes; we say that* $\hat{u}$ *is* Legendrian.

Taking $u, \hat{u}$ as above, we define, for any orthonormal framing $(e_1, \ldots, e_n)$ of $T\Omega$, the Lagrangian angle β by

$$e^{i\beta} = dz^1 \wedge \ldots \wedge dz^{n+1}\big(\hat{u}, d\hat{u}(e_1), \ldots, d\hat{u}(e_n)\big).$$

which makes sense because $(\hat{u}, d\hat{u}(e_1), \ldots, d\hat{u}(e_n))$ is a Hermitian-orthonormal frame, for any $x \in \Omega$. Furthermore, the result is independent from the choice of the framing, and depends on the choice of the lift $\hat{u}$ only through multiplication by a unit complex constant. Hence β is defined up to an additive constant and $d\beta$ is always well-defined along any Lagrangian immersion u. Another characteristic property of

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†note that the sign convention may vary in the literature, e.g. some Authors use $\langle \cdot, \cdot \rangle_{\mathbb{C}P^n} = \langle \cdot, \cdot \rangle + i\omega(\cdot, \cdot)$.

the Lagrangian angle relates it to the mean curvature vector field $\vec{H}$ along u:

$$\vec{H} = \frac{1}{n} J\nabla\beta \tag{2.1}$$

or equivalently $d\beta = -\vec{H} \,\lrcorner\, \omega$ (see [**1**, **4**] for details).

2.2. *Hamiltonian stationary Lagrangian submanifolds*

A *Hamiltonian stationary* Lagrangian submanifold Σ in $\mathbb{C}P^n$ is a Lagrangian submanifold which is a critical point of the n-volume functional $\mathcal{A}$ under first variations which are *Hamiltonian vector fields* with compact support. This means that for any smooth function with compact support $h \in \mathcal{C}_c^\infty(\mathbb{C}P^n, \mathbb{R})$, we have

$$\delta\mathcal{A}_{\xi_h}(\Sigma) := \int_\Sigma \left\langle \vec{H}, \xi_h \right\rangle_E d\mathrm{vol} = 0,$$

where ξ_h is the Hamiltonian vector field of h, i.e. satisfies $\xi_h \,\lrcorner\, \omega + dh = 0$ or $\xi_h = J\nabla h$. We also remark that if $f \in \mathcal{C}_c^\infty(\Sigma, \mathbb{R})$, then there exist smooth extensions with compact support h of f, i.e. functions $h \in \mathcal{C}_c^\infty(\mathbb{C}P^n, \mathbb{R})$ such that $h_{|\Sigma} = f$, and moreover the normal component of $(\xi_h)_{|\Sigma}$ does not depend on the choice of the extension h (it coincides actually with $J\nabla f$, where ∇ is here the gradient with respect to the induced metric on Σ). So we deduce from above that $\delta\mathcal{A}_{\xi_h}(\Sigma) = \frac{1}{n}\int_\Sigma \langle \nabla\beta, \nabla f\rangle_E \, d\mathrm{vol}$. This implies the following.

Corollary 2.2. *Any Lagrangian submanifold Σ in $\mathbb{C}P^n$ is Hamiltonian stationary if and only if β is a harmonic function on Σ, i.e.*

$$\Delta_\Sigma \beta = 0.$$

This theory extends to non simply connected surfaces Σ with the following restrictions. Let γ be a homotopically non trivial loop. The Legendrian lift of γ needs not close, so that in general its endpoints $p_1, p_2 \in S^{2n+1}$ are multiples of each other by a factor $e^{i\theta}$. The same holds for the Lagrangian angle: $\beta(p_2) \equiv \beta(p_1) + (n+1)\theta \mod 2\pi$ (since the tangent plane is also shifted by the Decktransformation $z \longmapsto e^{i\theta} z$). In particular β is not always globally defined on surfaces in $\mathbb{C}P^n$ with non trivial topology, unless the Legendrian lift is globally defined in $S^{2n+1}/\mathbb{Z}_{n+1}$ (here $\mathbb{Z}_{n+1}$ stands for the $n+1$-st roots of unity in $SU(n+1)$).

2.3. *Conformal Lagrangian immersions into $\mathbb{C}P^2$*

We now set $n = 2$. We suppose that Ω is a simply connected open subset of $\mathbb{R}^2 \simeq \mathbb{C}$ and consider a conformal Lagrangian immersion $u : \Omega \longrightarrow \mathbb{C}P^2$. This implies that we can find a function $\rho : \Omega \longrightarrow \mathbb{R}$ and two sections E_1 and E_2 of $u^*T\mathbb{C}P^2$ such that $\forall (x,y) \in \Omega$, $(E_1(x,y), E_2(x,y))$ is an Euclidean orthogonal basis over $\mathbb{R}$ of $T_{u(x,y)}u(\Omega)$ and

$$du = e^\rho \left(E_1 dx + E_2 dy\right).$$

We observe that, due to the fact that u is Lagrangian, (E_1, E_2) is a also a *Hermitian* basis over $\mathbb{C}$ of $T_{u(x,y)}\mathbb{C}P^2$.

Let $\begin{tikzcd} \Omega \arrow[r, "\widehat{u}"] \arrow[dr, "u"'] & S^5 \arrow[d, "\pi"] \\ & \mathbb{C}P^2 \end{tikzcd}$ be a parallel lift of u as in Proposition 2.1 and (e_1, e_2) be the unique section of $\widehat{u}^*\mathcal{H} \times \widehat{u}^*\mathcal{H}$ which lifts† (E_1, E_2). Then we have

$$d\widehat{u} = e^{\rho} \left(e_1 dx + e_2 dy \right). \tag{2.2}$$

Note that $\forall (x, y) \in \Omega$, $(e_1(x, y), e_2(x, y))$ is a Hermitian basis of $\mathcal{H}_{\widehat{u}(x,y)}$, which is Hermitian orthogonal to $\widehat{u}(x, y)$. Hence $\forall (x, y) \in \Omega$, $(e_1(x, y), e_2(x, y), \widehat{u}(x, y))$ is a Hermitian basis of $\mathbb{C}^3$. Thus this triplet can be identified with some $\widehat{F}(x, y) \in U(3)$. We hence get the diagram

$$\begin{tikzcd} & U(3) \arrow[d, "(\cdot\cdot*)"] \\ \Omega \arrow[ur, "\widehat{F}"] \arrow[r, "\widehat{u}"] \arrow[dr, "u"'] & S^5 \arrow[d, "\pi"] \\ & \mathbb{C}P^2 \end{tikzcd}$$

, where $(\cdot\cdot*)$ is the mapping $(e_1, e_2, e_3) \longmapsto e_3$.

We define the Maurer–Cartan form $\widehat{A}$ to be the 1-form on Ω with coefficients in $\mathfrak{u}(3)$ such that $d\widehat{F} = \widehat{F} \cdot \widehat{A}$. Then we remark that the horizontality assumption $\langle d\widehat{u}, \widehat{u} \rangle_{\mathbb{C}^3} = 0$ exactly means that

$$\widehat{A}^3_3 = 0. \tag{2.3}$$

Moreover the Lagrangian angle function $\beta_{\widehat{u}}$ along $\widehat{u}$ can be computed by

$$e^{i\beta_{\widehat{u}}} = dz^1 \wedge dz^2 \wedge dz^3(e_1, e_2, \widehat{u}) = \det \widehat{F}.$$

As in [**10**] we consider a larger class of framings of u as follows.

Definition 2.3. A *Legendrian framing of u along $\widehat{u}$* is a map $F : \Omega \longrightarrow U(3)$ such that

– $(\cdot\cdot*) \circ F = \widehat{u}$
– $\det F = e^{i\beta_{\widehat{u}}}$.

It is easily seen that the first condition is equivalent to the fact that there exists a smooth map $G : \Omega \longrightarrow U(3)$ (a gauge transformation) of the type

$$G(x, y) = \begin{pmatrix} g(x, y) & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where } g : \Omega \longrightarrow U(2)$$

such that

$$F(x, y) = \widehat{F}(x, y) \cdot G^{-1}(x, y).$$

And then the second one is equivalent to say that g takes values in $SU(2)$.

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†recall that the condition that $v \in (\widehat{u}^*\mathcal{H})_{(x,y)}$ means that v is in the horizontal subspace $\mathcal{H}_{\widehat{u}(x,y)}$

2.4. *A splitting of the Maurer–Cartan form of a Legendrian framing*

Using (2.2) and (2.3) one obtains the following decomposition of $\widehat{A}$:

$$\widehat{A} = \widehat{A}_{\mathfrak{u}(1)} + \widehat{A}_{\mathfrak{su}(2)} + \widehat{A}_{\mathbb{C}^2},$$

with the notations

$$\widehat{A}_{\mathfrak{u}(1)} = \begin{pmatrix} \widehat{\alpha}_{\mathfrak{u}(1)} & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{A}_{\mathfrak{su}(2)} = \begin{pmatrix} \widehat{\alpha}_{\mathfrak{su}(2)} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\text{and} \quad \widehat{A}_{\mathbb{C}^2} = e^{\rho} \begin{pmatrix} 0 & \epsilon dz + \overline{\epsilon} d\bar{z} \\ -\,{}^t(\epsilon dz + \overline{\epsilon} d\bar{z}) & 0 \end{pmatrix},$$

where $\widehat{\alpha}_{\mathfrak{u}(1)}$ is a 1-form on Ω with coefficients in $\mathfrak{u}(1) \simeq \mathbb{R}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, $\widehat{\alpha}_{\mathfrak{su}(2)}$ a 1-form on Ω with coefficients in $\mathfrak{su}(2)$, $\epsilon := \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\overline{\epsilon} := \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}$ (so that $\epsilon dz + \overline{\epsilon} d\bar{z} = \begin{pmatrix} dx \\ dy \end{pmatrix}$). Note that $\det F = e^{i\beta_{\widehat{u}}}$ implies $\widehat{\alpha}_{\mathfrak{u}(1)} = \frac{d\beta_{\widehat{u}}}{2}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$.
Now we let $F : \Omega \longrightarrow U(3)$ be a Legendrian framing and $A := F^{-1} \cdot dF$. The relation $F = \widehat{F} \cdot G^{-1}$ implies that $A = G \cdot \widehat{A} \cdot G^{-1} - dG \cdot G^{-1}$. Hence

$$A = A_{\mathfrak{u}(1)} + A_{\mathfrak{su}(2)} + A_{\mathbb{C}^2},$$

where, using the fact that $\mathfrak{u}(1)$ commutes with $\mathfrak{su}(2)$,

$$A_{\mathfrak{u}(1)} = \begin{pmatrix} \alpha_{\mathfrak{u}(1)} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widehat{\alpha}_{\mathfrak{u}(1)} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} i\frac{d\beta_{\widehat{u}}}{2} & 0 & 0 \\ 0 & i\frac{d\beta_{\widehat{u}}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_{\mathfrak{su}(2)} = \begin{pmatrix} \alpha_{\mathfrak{su}(2)} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} g \cdot \widehat{\alpha}_{\mathfrak{su}(2)} \cdot g^{-1} - dg \cdot g^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$A_{\mathbb{C}^2} = e^{\rho} \begin{pmatrix} 0 & g \cdot (\epsilon dz + \overline{\epsilon} d\bar{z}) \\ -\,(g \cdot (\epsilon dz + \overline{\epsilon} d\bar{z}))^{\dagger} & 0 \end{pmatrix}.$$

We can further split the last term $A_{\mathbb{C}^2}$ along dz and $d\bar{z}$ as $A_{\mathbb{C}^2} = A'_{\mathbb{C}^2} + A''_{\mathbb{C}^2}$ where

$$A'_{\mathbb{C}^2} := e^{\rho} \begin{pmatrix} 0 & g \cdot \epsilon \\ -\,(g \cdot \overline{\epsilon})^{\dagger} & 0 \end{pmatrix} dz \quad \text{and} \quad A''_{\mathbb{C}^2} := e^{\rho} \begin{pmatrix} 0 & g \cdot \overline{\epsilon} \\ -\,(g \cdot \epsilon)^{\dagger} & 0 \end{pmatrix} d\bar{z}.$$

2.5. *Interpretation in terms of an automorphism*

As expounded in [**10**] and [**11**] the key point in order to exploit the structure of an integrable system is to observe that the splitting $A = A_{\mathfrak{u}(1)} + A_{\mathfrak{su}(2)} + A'_{\mathbb{C}^2} + A''_{\mathbb{C}^2}$ corresponds to a decomposition along the eigenspaces of the following automorphism in $\mathfrak{u}(3)^{\mathbb{C}}$, the complexification† of $\mathfrak{u}(3)$. We let $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$\begin{array}{cccc} \tau: & \mathfrak{u}(3)^{\mathbb{C}} & \longrightarrow & \mathfrak{u}(3)^{\mathbb{C}} \\ & M & \longmapsto & -\begin{pmatrix} -J & 0 \\ 0 & 1 \end{pmatrix} \cdot {}^t M \cdot \begin{pmatrix} J & 0 \\ 0 & 1 \end{pmatrix}. \end{array}$$

It is then straightforward that τ is a Lie algebra automorphism, that $\mathfrak{u}(3)$ is stable by τ and that $\tau^4 = Id$. Hence we can diagonalize the action of τ over $\mathfrak{u}(3)^{\mathbb{C}}$ and in the following we denote by $\mathfrak{u}(3)^{\mathbb{C}}_a$ the eigenspace of τ for the eigenvalue i^a, for $a = -1, 0, 1, 2$. We first point out that the eigenspaces $\mathfrak{u}(3)^{\mathbb{C}}_0$ and $\mathfrak{u}(3)^{\mathbb{C}}_2$, with eigenvalues 1 and -1 respectively, are the complexifications of $\mathfrak{u}(3)_0$ and $\mathfrak{u}(3)_2$ respectively, where

$$\mathfrak{u}(3)_0 := \left\{ \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} / g \in \mathfrak{su}(2) \right\} \text{ and } \mathfrak{u}(3)_2 := \left\{ \begin{pmatrix} \lambda i & 0 & 0 \\ 0 & \lambda i & 0 \\ 0 & 0 & \mu i \end{pmatrix} / \lambda, \mu \in \mathbb{R} \right\}.$$

This can be obtained by first computing that

$$\tau \begin{pmatrix} A & X \\ -{}^tY & d \end{pmatrix} = \begin{pmatrix} J\,{}^tAJ & -JY \\ -{}^tXJ & -d \end{pmatrix}, \quad \forall A \in M(2,\mathbb{C}), \forall X, Y \in \mathbb{C}^2, \forall d \in \mathbb{C},$$

and by using the fact that $\forall A \in \mathfrak{sl}(2,\mathbb{C})$, $J\,{}^tAJ = A$. Similarly the eigenspaces $\mathfrak{u}(3)^{\mathbb{C}}_1$ and $\mathfrak{u}(3)^{\mathbb{C}}_{-1}$, with eigenvalues i and $-i$ respectively, are found to be

$$\mathfrak{u}(3)^{\mathbb{C}}_1 = \left\{ \begin{pmatrix} 0 & X \\ -{}^tY & 0 \end{pmatrix} / X, Y \in \mathbb{C}^2, JY = -iX \right\}$$

and

$$\mathfrak{u}(3)^{\mathbb{C}}_{-1} = \left\{ \begin{pmatrix} 0 & X \\ -{}^tY & 0 \end{pmatrix} / X, Y \in \mathbb{C}^2, JY = iX \right\}.$$

Now we have the following

Lemma 2.4. *The eigenspaces $\mathfrak{u}(3)^{\mathbb{C}}_1$ and $\mathfrak{u}(3)^{\mathbb{C}}_{-1}$ can be characterized by*

$$\mathfrak{u}(3)^{\mathbb{C}}_1 = \left\{ \lambda \begin{pmatrix} 0 & h \cdot \overline{\epsilon} \\ -(h \cdot \epsilon)^\dagger & 0 \end{pmatrix} / \lambda \in [0,\infty), h \in SU(2) \right\}$$

and

$$\mathfrak{u}(3)^{\mathbb{C}}_{-1} = \left\{ \lambda \begin{pmatrix} 0 & h \cdot \epsilon \\ -(h \cdot \overline{\epsilon})^\dagger & 0 \end{pmatrix} / \lambda \in [0,\infty), h \in SU(2) \right\}.$$

Proof. This can be proved either by adapting the argument in section 2.4 of [**10**] or by a straightforward computation which exploits the fact that $\forall h \in SU(2)$, $\overline{h}J = Jh$, $J\epsilon = i\epsilon$ and $J\overline{\epsilon} = -i\overline{\epsilon}$. □

We conclude that if, using $\mathfrak{u}(3)^{\mathbb{C}} = \mathfrak{u}(3)^{\mathbb{C}}_{-1} \oplus \mathfrak{u}(3)^{\mathbb{C}}_0 \oplus \mathfrak{u}(3)^{\mathbb{C}}_1 \oplus \mathfrak{u}(3)^{\mathbb{C}}_2$, we decompose

†we can define $\mathfrak{u}(3)^{\mathbb{C}}$ as the set $M(3,\mathbb{C})$ with its standard complex structure and with the conjugation mapping $c : M \longmapsto -M^\dagger$; clearly c is a Lie algebra automorphism, an involution and the set of fixed points of c is $\mathfrak{u}(3)$.
Similarly the complexification $U(3)^{\mathbb{C}}$ is the set $GL(3,\mathbb{C})$ with its standard complex structure and the conjugation map $C : G \longmapsto (G^\dagger)^{-1}$.

A as

$$A = A_{-1} + A_0 + A_1 + A_2,$$

where each A_a is a 1-form with coefficients in the $\mathfrak{u}(3)^{\mathbb{C}}_a$, then we recover the previous splitting by setting $A_0 = A_{\mathfrak{su}(2)}$, $A_2 = A_{\mathfrak{u}(1)}$, $A_{-1} = A'_{\mathbb{C}^2}$ and $A_1 = A''_{\mathbb{C}^2}$. Note that the two last conditions actually reflects the conformality of u.

Remark. Note that by the automorphism property $[\mathfrak{u}(3)_a, \mathfrak{u}(3)_b] \subset \mathfrak{u}(3)_{a+b \bmod 4}$.

2.6. *Legendrian framings of Hamiltonian stationary Lagrangian immersions*

Given the Legendrian framing F of a conformal Lagrangian immersion u in $\mathbb{C}P^2$, we define the family of deformations A_λ of its Maurer–Cartan form A, for $\lambda \in S^1 \subset \mathbb{C}^*$ by

$$A_\lambda := \lambda^{-2} A'_2 + \lambda^{-1} A_{-1} + A_0 + \lambda A_1 + \lambda^2 A''_2, \tag{2.4}$$

where $A'_2 := A_2(\partial/\partial z)dz$ and $A''_2 := A_2(\partial/\partial \bar{z})d\bar{z}$. We then have the following:

Theorem 2.5. *Given a conformal Lagrangian immersion $u : \Omega \longrightarrow \mathbb{C}P^2$ and a Legendrian framing F of u, the Maurer–Cartan form of F satisfies*

$$A_{-1} = A'_{-1} = A'_{\mathbb{C}^2} \text{ and } A_1 = A''_1 = A'_{\mathbb{C}^2}. \tag{2.5}$$

Furthermore u is Hamiltonian stationary if and only if, defining A_λ as in (2.4),

$$dA_\lambda + A_\lambda \wedge A_\lambda = 0, \quad \forall \lambda \in S^1. \tag{2.6}$$

Remark. For $\lambda = 1$, $A_1 = A$ and equation (2.6) is a consequence of its definition $A := F^{-1} \cdot dF$.

Proof. See [**10**] and [**11**]. □

We remark that all the conditions that have been collected about the components A_a can be encoded by the following twisting condition on A_λ:

$$\forall \lambda \in S^1, \quad \tau(A_\lambda) = A_{i\lambda}.$$

Thus we are led to define the following *twisted loop algebra*

$$\Lambda\mathfrak{u}(3)_\tau := \{S^1 \ni \lambda \longmapsto \xi_\lambda \in \mathfrak{u}(3) / \forall \lambda \in S^1, \tau(\xi_\lambda) = \xi_{i\lambda}\},$$

and A_λ is a 1-form on Ω with coefficients in $\Lambda\mathfrak{u}(3)_\tau$.

Actually $\Lambda\mathfrak{u}(3)_\tau$ is the Lie algebra of the following *twisted loop group*

$$\Lambda U(3)_\tau := \{S^1 \ni \lambda \longmapsto g_\lambda \in U(3) / \forall \lambda \in S^1, \tau(g_\lambda) = g_{i\lambda}\},$$

where the Lie algebra automorphism $\tau : \mathfrak{u}(3)^{\mathbb{C}} \longrightarrow \mathfrak{u}(3)^{\mathbb{C}}$ has been extended to the Lie group automorphism by

$$\begin{array}{rccc} \tau : & U(3)^{\mathbb{C}} & \longrightarrow & U(3)^{\mathbb{C}} \\ & M & \longmapsto & \begin{pmatrix} -J & 0 \\ 0 & 1 \end{pmatrix} \cdot {}^tM^{-1} \cdot \begin{pmatrix} J & 0 \\ 0 & 1 \end{pmatrix}. \end{array}$$

Now if we assume that Ω is simply connected, then relation (2.6) allows us to integrate A_λ, i.e. to find a map $F_\lambda : \Omega \longrightarrow U(3)$ for any $\lambda \in S^1$ such that $dF_\lambda =$

$F_\lambda \cdot A_\lambda$. Moreover if we choose some base point $z_0 \in \Omega$, then by requiring further that $F_\lambda(z_0) = Id$, F_λ is unique. A key observation is then that $\tau(A_\lambda) = A_{i\lambda}$ implies $\tau(F_\lambda) = F_{i\lambda}$, $\forall \lambda \in S^1$. Hence, a conformal Lagrangian immersion $u : \Omega \longrightarrow \mathbb{C}P^2$ is Hamiltonian stationary if and only if any Legendrian lift F of it can be deformed into a map $F_\lambda : \Omega \longrightarrow \Lambda U(3)_\tau$, such that $F_\lambda^{-1} \cdot dF_\lambda$ has the form (2.4). Summarizing this result with the observations in the previous section we have:

Theorem 2.6. *Given a simply connected domain $\Omega \subset \mathbb{C}$ and a base point $z_0 \in \Omega$, the set of Hamiltonian stationary conformal Lagrangian immersions $u : \Omega \longrightarrow \mathbb{C}P^2$ such that $u(z_0) = [0 : 0 : 1]$ is in bijection with the set of maps $F_\lambda : \Omega \longrightarrow \Lambda U(3)_\tau$, such that $F_\lambda(z_0) = Id$ and the Fourier decomposition of $A_\lambda := F_\lambda^{-1} \cdot dF_\lambda$, $A_\lambda = \sum_{k\in\mathbb{Z}} \widehat{A}_k \lambda^k$ satisfies*

$$\forall k \in \mathbb{Z}, \quad k \leq -3 \Longrightarrow \widehat{A}_k = 0, \tag{2.7}$$

$$\widehat{A}_{-2} = a(z)dz \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \textit{where } a \in \mathcal{C}^\infty(\Omega, \mathbb{C}), \tag{2.8}$$

$$\widehat{A}_{-1} = \widehat{A}_{-1}(\partial/\partial z)dz, \quad \textit{i.e. } \widehat{A}_{-1}(\partial/\partial \bar{z}) = 0. \tag{2.9}$$

Proof. For any conformal Lagrangian Hamiltonian stationary immersion u the existence of F_λ and the properties (2.7), (2.8) and (2.9) are immediate consequences of Theorem 2.5. Conversely for any map F_λ, conditions (2.7), (2.8) and (2.9) and the reality condition $\overline{A_\lambda} = A_\lambda$ imply that A_λ must satisfy (2.4). In particular we remark that condition (2.8) is a reformulation of (2.3). Thus by theorem 2.5 we deduce that F_1 is the Legendrian lift of some Hamiltonian stationary conformal Lagrangian immersion. □

Remark. ¿From the analysis of the Maurer–Cartan form of a Legendrian lift we know that actually the function a in (2.8) is $\frac{1}{2}\partial\beta/\partial z$, where β is the Lagrangian angle function. In particular since u is Hamiltonian stationary β is harmonic and hence a is holomorphic.

2.7. *An alternative characterization*

We introduce here another construction using based loop groups for characterizing Hamiltonian stationary Lagrangian conformal immersions. Consider

$$E_\lambda := F_\lambda \cdot F^{-1}.$$

We can observe that E_λ is a map with values in the *based loop group*

$$\Omega U(3) := \{S^1 \ni \lambda \longmapsto g_\lambda \in U(3)/g_{\lambda=1} = 1\},$$

since $F_{\lambda=1} = F$. It is easy to check that $\Omega U(3)$ is a loop group, the Lie algebra of which is

$$\Omega\mathfrak{u}(3) := \{S^1 \ni \lambda \longmapsto \xi_\lambda \in \mathfrak{u}(3)/\xi_{\lambda=1} = 0\}.$$

Note that the (formal) Fourier expansion of an element $\xi_\lambda \in \Omega\mathfrak{u}(3)$ can be written $\xi_\lambda = \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{\xi}_k(\lambda^k - 1)$.

The Maurer–Cartan form of E_λ is

$$\begin{array}{rcl}\Gamma_\lambda & := & E_\lambda^{-1}\cdot dE_\lambda \\ & = & F\cdot\left(F_\lambda^{-1}\cdot dF_\lambda - F\cdot dF\right)\cdot F^{-1} = F\cdot(A_\lambda - A)\cdot F^{-1} \\ & = & (\lambda^{-2}-1)\Gamma_2' + (\lambda^{-1}-1)\Gamma_{-1} + (\lambda-1)\Gamma_1 + (\lambda^2-1)\Gamma_2'',\end{array}$$

where $\Gamma_2' := F\cdot A_2'\cdot F^{-1}$, $\Gamma_{-1} := F\cdot A_{-1}\cdot F^{-1}$, $\Gamma_1 := F\cdot A_1\cdot F^{-1}$ and $\Gamma_2'' := F\cdot A_2''\cdot F^{-1}$. We can observe in particular that

$$\Gamma_2' = ia\pi^\perp dz, \quad \text{where } \pi^\perp := F\cdot\begin{pmatrix}1 & & \\ & 1 & \\ & & 0\end{pmatrix}\cdot F^{-1}.$$

Note that $\pi^\perp$ is the Hermitian orthogonal projection in $\mathbb{C}^3$ onto the plane $\widehat{u}^\perp$ (moreover $\pi^\perp$ is actually independent of the lift $\widehat{u}$ chosen for u).

Lastly we point out the following equivariance property with respect to the automorphism τ_u defined[†] by

$$\tau_u(M) = F\cdot\tau(F^{-1}\cdot M\cdot F)\cdot F^{-1}.$$

We have obviously $\tau_u^4 = 1$. Moreover, setting

$$\begin{array}{rl}\gamma_\lambda := & \lambda^{-2}\Gamma_2' + \lambda^{-1}\Gamma_{-1} + \lambda\Gamma_1 + \lambda^2\Gamma_2'' \\ = & F\cdot\left(\lambda^{-2}A_2' + \lambda^{-1}A_{-1} + \lambda A_1 + \lambda^2 A_2''\right)\cdot F^{-1}\end{array}$$

and $\gamma := \gamma_{\lambda=1} = F\cdot(A_2' + A_{-1} + A_1 + A_2'')\cdot F^{-1}$, so that $\Gamma_\lambda = \gamma_\lambda - \gamma$, we have

$$\tau_u(\gamma_\lambda) = \gamma_{i\lambda}.$$

3. *Finite type solutions*

In [**11**] we showed how Theorem 2.6 allows us to adapt the theory of J. Dorfmeister, F. Pedit and H.Y. Wu [**5**], in order to build a Weierstrass type representation theory of *all* conformal Lagrangian Hamiltonian stationary immersions, i.e. using holomorphic data. Here we want to exploit Theorem 2.6 in order to construct a particular class of examples of solutions: the *finite type* ones.

3.1. *Definitions*

We invite the Reader to consult [**3**], [**6**] or [**8**] for more details. We first observe that $U(3)_0 := \{g\in U(3)/\tau(g) = g\}$, the fixed set of τ, is a subgroup of $U(3)$, the Lie algebra of which is $\mathfrak{u}(3)_0$ (same observation about $U(3)_0^{\mathbb{C}}$). Actually $U(3)_0$ is isomorphic to $SU(2)$ so that we make the identifications $U(3)_0\simeq SU(2)$ and $\mathfrak{u}(3)_0\simeq\mathfrak{su}(2)$. We will need an Iwasawa decomposition of $SU(2)^{\mathbb{C}}$ for our purpose: it will be a pair $(SU(2),\mathfrak{B})$ of *subgroups* of $SU(2)^{\mathbb{C}}$, such that $\forall g\in SU(2)^{\mathbb{C}}$, $\exists!(f,b)\in$

[†]We can remark that the definition of τ_u is independent from the choice of the Legendrian framing F of u, and depends only on u. This means that for any pair of Legendrian framings F and $\widehat{F}$ such that $\widehat{F} = F\cdot G$, where $G = \begin{pmatrix} g & \\ & 1\end{pmatrix}$ and $g:\Omega\longrightarrow SU(2)$, we have $\widehat{F}\cdot\tau(\widehat{F}^{-1}\cdot M\cdot\widehat{F})\cdot\widehat{F}^{-1} = F\cdot\tau(F^{-1}\cdot M\cdot F)\cdot F^{-1}$. This can be checked by a computation using the fact that $\begin{pmatrix}\pm J & \\ & 1\end{pmatrix}\cdot G = \overline{G}\cdot\begin{pmatrix}\pm J & \\ & 1\end{pmatrix}$.

$SU(2) \times \mathfrak{B}$ with $g = f \cdot b$, a property that we summarize by writing $SU(2)^{\mathbb{C}} = SU(2) \cdot \mathfrak{B}$. Moreover $\mathfrak{B}$ is a solvable Borel subgroup. We can choose for example

$$\mathfrak{B} := \left\{ \begin{pmatrix} T_1^1 & 0 \\ T_1^2 & T_2^2 \end{pmatrix} / T_1^1, T_2^2 \in (0, \infty), T_1^2 \in \mathbb{C}, T_1^1 T_2^2 = 1 \right\}.$$

We denote by $\mathfrak{b}$ the Lie algebra of $\mathfrak{B}$. The Iwasawa decomposition $SU(2)^{\mathbb{C}} = SU(2) \cdot \mathfrak{B}$ immediately implies the vector space decomposition $\mathfrak{su}(2)^{\mathbb{C}} = \mathfrak{su}(2) \oplus \mathfrak{b}$, which leads to the definition of the two projection mappings $(\cdot)_{\mathfrak{su}} : \mathfrak{su}(2)^{\mathbb{C}} \longrightarrow \mathfrak{su}(2)$ and $(\cdot)_{\mathfrak{b}} : \mathfrak{su}(2)^{\mathbb{C}} \longrightarrow \mathfrak{b}$ such that

$$\forall \xi \in \mathfrak{su}(2)^{\mathbb{C}}, \quad \xi = (\xi)_{\mathfrak{su}} + (\xi)_{\mathfrak{b}} \quad \text{with } (\xi)_{\mathfrak{su}} \in \mathfrak{su}(2) \text{ and } (\xi)_{\mathfrak{b}} \in \mathfrak{b}.$$

Then we define the following twisted loop algebras

$$\Lambda \mathfrak{u}(3)_\tau^{\mathbb{C}} := \{ S^1 \ni \lambda \longmapsto \xi_\lambda \in \mathfrak{u}(3)^{\mathbb{C}} / \forall \lambda \in S^1, \tau(\xi_\lambda) = \xi_{i\lambda} \},$$

$$\Lambda_{\mathfrak{b}}^+ \mathfrak{u}(3)_\tau^{\mathbb{C}} := \{ [\lambda \longmapsto \xi_\lambda] \in \Lambda \mathfrak{u}(3)_\tau^{\mathbb{C}} / \forall k \in \mathbb{Z}, k \leq -1 \Longrightarrow \widehat{\xi}_k = 0 \text{ and } \widehat{\xi}_0 \in \mathfrak{b} \},$$

where we use the Fourier decomposition $\xi_\lambda = \sum_{k \in \mathbb{Z}} \widehat{\xi}_k \lambda^k$.

The decomposition $\mathfrak{su}(2)^{\mathbb{C}} = \mathfrak{su}(2) \oplus \mathfrak{b}$ can be extended to loop algebras, i.e. to the splitting $\Lambda \mathfrak{u}(3)_\tau^{\mathbb{C}} = \Lambda \mathfrak{u}(3)_\tau \oplus \Lambda_{\mathfrak{b}}^+ \mathfrak{u}(3)_\tau^{\mathbb{C}}$. This can be checked by using the Fourier expansion of an element $\xi_\lambda \in \Lambda \mathfrak{u}(3)_\tau^{\mathbb{C}}$:

$$\sum_{k \in \mathbb{Z}} \widehat{\xi}_k \lambda^k = \left(\sum_{k<0} \widehat{\xi}_k \lambda^k + (\widehat{\xi}_0)_{\mathfrak{su}} - \sum_{k>0} \left(\widehat{\xi}_{-k} \right)^\dagger \lambda^k \right) + \left((\widehat{\xi}_0)_{\mathfrak{b}} + \sum_{k>0} \left(\widehat{\xi}_k + \left(\widehat{\xi}_{-k} \right)^\dagger \right) \lambda^k \right).$$

We will denote the corresponding projection mappings by $(\cdot)_{\Lambda_{\mathfrak{su}}} : \Lambda \mathfrak{u}(3)_\tau^{\mathbb{C}} \longrightarrow \Lambda \mathfrak{u}(3)_\tau$ and $(\cdot)_{\Lambda_{\mathfrak{b}}^+} : \Lambda \mathfrak{u}(3)_\tau^{\mathbb{C}} \longrightarrow \Lambda_{\mathfrak{b}}^+ \mathfrak{u}(3)_\tau^{\mathbb{C}}$.

We also introduce the following finite dimensional subspaces of $\Lambda \mathfrak{u}(3)_\tau$: for any $p \in \mathbb{N}$ we let

$$\Lambda^{2+4p} \mathfrak{u}(3)_\tau := \left\{ [\lambda \longmapsto \xi_\lambda] \in \Lambda \mathfrak{u}(3)_\tau / \xi_\lambda = \sum_{k=-2-4p}^{2+4p} \widehat{\xi}_k \lambda^k \right\}.$$

We can now define a pair of vector fields $X_1, X_2 : \Lambda^{2+4p} \mathfrak{u}(3)_\tau \longrightarrow \Lambda \mathfrak{u}(3)_\tau$ by

$$X_1(\xi_\lambda) := [\xi_\lambda, (\lambda^{4p} \xi_\lambda)_{\Lambda_{\mathfrak{su}}}], \quad X_2(\xi_\lambda) := [\xi_\lambda, (i\lambda^{4p} \xi_\lambda)_{\Lambda_{\mathfrak{su}}}]. \tag{3.1}$$

Note that $\lambda^{4p} \xi_\lambda$ belongs to $\Lambda \mathfrak{u}(3)_\tau^{\mathbb{C}}$, so that $(\lambda^{4p} \xi_\lambda)_{\Lambda_{\mathfrak{su}}}$ is well defined.

LEMMA 3.1. *Let $p \in \mathbb{N}$ and X_1 and X_2 defined by (3.1). Then*
- *$\forall \xi_\lambda \in \Lambda^{2+4p} \mathfrak{u}(3)_\tau$, $X_1(\xi_\lambda), X_2(\xi_\lambda) \in T_{\xi_\lambda} \Lambda^{2+4p} \mathfrak{u}(3)_\tau \simeq \Lambda^{2+4p} \mathfrak{u}(3)_\tau$, so that X_1 and X_2 are tangent vector fields to $\Lambda^{2+4p} \mathfrak{u}(3)_\tau$.*
- *$|\xi_\lambda|^2$ is preserved by X_1 and X_2. Hence the flow of these vector fields are defined for all time*
- *The Lie bracket of X_1 and X_2 vanishes:*

$$[X_1, X_2] = 0. \tag{3.2}$$

Proof. This result follows by a straightforward adaptation of the analogous results for harmonic maps in [**3**] (see e.g. [**6**] and [**8**]). Note that the proof of (3.2) rests upon the crucial property that $\Lambda\mathfrak{u}(3)_\tau$ and $\Lambda^+_{\mathfrak{b}}\mathfrak{u}(3)^{\mathbb{C}}_\tau$ are Lie algebras (see e.g. [**2**], [**8**]). □

This result allows us to integrate simultaneously X_1 and X_2. So for any $\xi^0_\lambda \in \Lambda^{2+4p}\mathfrak{u}(3)_\tau$ there exists a unique map $\xi_\lambda : \mathbb{R}^2 \longrightarrow \Lambda^{2+4p}\mathfrak{u}(3)_\tau$ such that $\xi_\lambda(z_0) = \xi^0_\lambda$ and

$$\frac{\partial \xi_\lambda}{\partial x}(x,y) = X_1\left(\xi_\lambda(x,y)\right) \quad \text{and} \quad \frac{\partial \xi_\lambda}{\partial y}(x,y) = X_2\left(\xi_\lambda(x,y)\right). \tag{3.3}$$

Denoting by $z = x+iy \in \mathbb{C}$, the system (3.3) can be rewritten

$$\begin{aligned} d\xi_\lambda &= \left[\xi_\lambda, \left(\lambda^{4p}\xi_\lambda\right)_{\Lambda_{\mathfrak{su}}} dx + \left(i\lambda^{4p}\xi_\lambda\right)_{\Lambda_{\mathfrak{su}}} dy\right] \\ &= \left[\xi_\lambda, \left(\lambda^{4p}\xi_\lambda dz\right)_{\Lambda_{\mathfrak{su}}}\right]. \end{aligned}$$

Let us denote by $A_\lambda := (\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}}$. Since the system (3.3) is overdetermined, A_λ should satisfy a compatibility condition. Indeed one can check that

$$dA_\lambda + A_\lambda \wedge A_\lambda = 0. \tag{3.4}$$

This relation can be proved by a method similar to the proof of (3.2) (see [**8**]). It implies that there exists a map $F_\lambda : \mathbb{C} \longrightarrow \Lambda U(3)_\tau$ such that

$$dF_\lambda = F_\lambda \cdot A_\lambda. \tag{3.5}$$

Now observe that $\lambda^{4p}\xi_\lambda = \sum_{k=-2}^{8p+2} \hat{\xi}_{k-4p}\lambda^k$ implies

$$A_\lambda = \lambda^{-2}\hat{\xi}_{-4p-2}dz + \lambda^{-1}\hat{\xi}_{-4p-1}dz + \left(\hat{\xi}_{-4p}dz\right)_{\mathfrak{su}} - \lambda\left(\hat{\xi}_{-4p-1}\right)^\dagger d\bar{z} - \lambda^2\left(\hat{\xi}_{-4p-2}\right)^\dagger d\bar{z}.$$

We recall that $\hat{\xi}_{-4p-2} \in \mathfrak{u}(3)^{\mathbb{C}}_2$ and so has the form $\mathrm{diag}(ia, ia, ib)$. Moreover we have the following result.

Lemma 3.2. *If $\xi_\lambda \longrightarrow \Lambda^{2+4p}\mathfrak{u}(3)_\tau$ and $A_\lambda := (\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}}$ are solutions of $d\xi_\lambda = [\xi_\lambda, A_\lambda]$, then $\hat{\xi}_{-4p-2}$ is constant.*

Proof. The relevant term in the Fourier expansion of $d\xi_\lambda = [\xi_\lambda, A_\lambda]$ gives

$$\begin{aligned} d\hat{\xi}_{-4p-2} &= \left[\hat{\xi}_{-4p-2}, \left(\hat{\xi}_{-4p}dz\right)_{\mathfrak{su}}\right] + \left[\hat{\xi}_{-4p-1}, \hat{\xi}_{-4p-1}\right]dz + \left[\hat{\xi}_{-4p}, \hat{\xi}_{-4p-2}\right]dz \\ &= \left[\left(\hat{\xi}_{-4p}dz\right)_{\mathfrak{b}}, \hat{\xi}_{-4p-2}\right]. \end{aligned}$$

But since the coefficients of $\left(\hat{\xi}_{-4p}dz\right)_{\mathfrak{b}}$ are in $\mathfrak{u}(3)^{\mathbb{C}}_0$ and $\hat{\xi}_{-4p-2}$ takes values in $\mathfrak{u}(3)_2$ we deduce that $d\hat{\xi}_{-4p-2} = 0$, because $\mathfrak{u}(3)^{\mathbb{C}}_0$ and $\mathfrak{u}(3)^{\mathbb{C}}_2$ commute. □

We deduce from this result that if we choose the initial value ξ^0_λ of ξ_λ to be such that $\hat{\xi}^0_{-4p-2} = \mathrm{diag}(ia, ia, 0)$ then $\hat{\xi}_{-4p-2}$ is equal to that value for all (x,y). So in this case the map F_λ obtained by integrating A_λ satisfies all the requirements of Theorem 2.6. It implies that F_λ represents a (conjugate family) of Hamiltonian stationary conformal Lagrangian immersion(s). The category of such F_λ's are exactly characterized by the following definition.

Definition 3.3. Let F_λ be a family of Hamiltonian stationary conformal Lagrangian immersions and let $A_\lambda := F^{-1}_\lambda \cdot dF_\lambda$. Then F_λ is called a family of *finite* *type solutions* if and only if there exists $p \in \mathbb{N}$ and a map $\xi_\lambda : \mathbb{C} \longrightarrow \Lambda^{2+4p}\mathfrak{u}(3)_\tau$ such that $\hat{\xi}_{-4p-2} = \mathrm{diag}(ia, ia, 0)$, for some constant $a \in \mathbb{C}$, and

$$d\xi_\lambda = [\xi_\lambda, A_\lambda] \tag{3.6}$$

$$(\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}} = A_\lambda. \tag{3.7}$$

We also need the following definition in which we introduce an a priori weaker notion of finite type solution.

DEFINITION 3.4. Let F_λ be a family of Hamiltonian stationary conformal Lagrangian immersions and let $A_\lambda := F_\lambda^{-1} \cdot dF_\lambda$. Then F_λ is called a family of *quasi-finite type solutions* if and only if it satisfies the same requirements as in definition 3.3 excepted that condition (3.7) is replaced by

$$\exists B \in \Omega^1 \otimes \mathfrak{u}(3)_0^{\mathbb{C}}, \quad (\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}} = A_\lambda + B. \tag{3.8}$$

We shall see in Section 3.3 that both definitions are actually equivalent.

3.2. *An alternative description of quasi-finite type solutions*

We may as well characterize such finite type solutions in terms of $E_\lambda = F_\lambda \cdot F^{-1}$. For that purpose we need to introduce the untwisted loop Lie algebra

$$\Lambda^+\mathfrak{u}(3)^{\mathbb{C}} := \{S^1 \ni \lambda \longmapsto \xi_\lambda \in \mathfrak{u}(3)^{\mathbb{C}} / \xi_\lambda = \sum_{k=0}^{\infty} \widehat{\xi}_k \lambda^k\}$$

and observe that any $\xi_\lambda = \sum_{k=-\infty}^{\infty} \widehat{\xi}_k \lambda^k \in \Lambda\mathfrak{u}(3)^{\mathbb{C}}$ can be split as

$$\xi_\lambda = \left(\sum_{k=-\infty}^{-1} \widehat{\xi}_k(\lambda^k - 1) - (\widehat{\xi}_k)^\dagger(\lambda^{-k} - 1)\right) + \left(\sum_{k=0}^{\infty} \widehat{\xi}_k \lambda^k + \sum_{k=1}^{\infty} \widehat{\xi}_{-k} + (\widehat{\xi}_{-k})^\dagger(\lambda^k - 1)\right)$$

and hence $\Lambda\mathfrak{u}(3)^{\mathbb{C}} = \Omega\mathfrak{u}(3) \oplus \Lambda^+\mathfrak{u}(3)^{\mathbb{C}}$. This defines a pair of projection mappings $(\cdot)_\Omega : \Lambda\mathfrak{u}(3)^{\mathbb{C}} \longrightarrow \Omega\mathfrak{u}(3)$ and $(\cdot)_{\Lambda^+} : \Lambda\mathfrak{u}(3)^{\mathbb{C}} \longrightarrow \Lambda^+\mathfrak{u}(3)^{\mathbb{C}}$.

Now consider a family F_λ of quasi-finite type, let $A_\lambda := F_\lambda^{-1} \cdot dF_\lambda$, $A := F^{-1} \cdot dF$ (where $F = F_{\lambda=1}$) and ξ_λ be a solution of (3.6). We let

$$\eta_\lambda := F \cdot \xi_\lambda \cdot F^{-1} = \sum_{k=-2-4p}^{2+4p} F \cdot \widehat{\xi}_k \cdot F^{-1}\lambda^k.$$

Then (3.6) implies by a straightforward computation that

$$\begin{aligned} d\eta_\lambda &= F \cdot (d\xi_\lambda + [A, \xi_\lambda]) \cdot F^{-1} \\ &= F \cdot ([\xi_\lambda, A_\lambda] - [\xi_\lambda, A]) \cdot F^{-1} = [\eta_\lambda, \Gamma_\lambda], \end{aligned}$$

where $\Gamma_\lambda := E_\lambda^{-1} \cdot dE_\lambda$. Now setting $R_\lambda := \sum_{k=-4p}^{2+4p} F \cdot \widehat{\xi}_k \cdot F^{-1}\lambda^k$, we have

$$\begin{aligned} \left(\lambda^{4p}\eta_\lambda dz\right)_\Omega &= \left(\lambda^{-2}F \cdot \widehat{\xi}_{-2-4p} \cdot F^{-1}dz + \lambda^{-1}F \cdot \widehat{\xi}_{-1-4p} \cdot F^{-1}dz + \lambda^{4p}R_\lambda dz\right)_\Omega \\ &= (\lambda^{-2} - 1)F \cdot \widehat{\xi}_{-2-4p} \cdot F^{-1}dz + (\lambda^{-1} - 1)F \cdot \widehat{\xi}_{-1-4p} \cdot F^{-1}dz \\ &\quad -(\lambda - 1)\left(F \cdot \widehat{\xi}_{-1-4p} \cdot F^{-1}\right)^\dagger d\bar{z} - (\lambda^2 - 1)\left(F \cdot \widehat{\xi}_{-2-4p} \cdot F^{-1}\right)^\dagger d\bar{z}. \end{aligned}$$

But relation (3.8) implies in particular that $\widehat{\xi}_{-2-4p} = A_2'(\partial/\partial z)$ and $\widehat{\xi}_{-1-4p} = A_{-1}(\partial/\partial z)$. So we deduce that

$$\left(\lambda^{4p}\eta_\lambda dz\right)_\Omega = F\cdot(A_\lambda - A)\cdot F^{-1} = \Gamma_\lambda.$$

Hence E_λ can be constructed by solving a system analogous to (3.6), (3.7), i.e.

$$d\eta_\lambda + [\Gamma_\lambda,\eta_\lambda] = 0 \quad \text{and} \quad \Gamma_\lambda = \left(\lambda^{4p}\eta_\lambda dz\right)_\Omega. \tag{3.9}$$

Conversely a similar computation shows that a solution of (3.9) gives rise to a quasi-finite type family of solutions by an inverse transformation, but we shall prove more in the next section.

Note that system (3.9) can also be interpreted as a pair of commuting ordinary differential equations in the finite dimensional space $\Lambda^{2+4p}\mathfrak{u}(3) := \{S^1 \ni \lambda \longmapsto \eta_\lambda \in \mathfrak{u}(3)/\eta_\lambda = \sum_{k=-2-4p}^{2+4p}\widehat{\eta}_k\lambda^k\}$. It is the analogue of the definition of a finite type solution according to [**3**].

3.3. *Quasi-finite type solutions are actually finite type*

We show here the following

THEOREM 3.5. *For any family F_λ of Hamiltonian stationary Lagrangian conformal immersions of quasi-finite type, i.e. such that there exists $\xi_\lambda : \Omega \longrightarrow \Lambda^{2+4p}\mathfrak{u}(3)_\tau$ which satisfies (3.6) and (3.8), there exists a gauge transformation $F_\lambda \longmapsto F_\lambda^G := F_\lambda\cdot G$, where $G\in\mathcal{C}^\infty(\Omega, U(3)_0)$, such that F_λ^G is of finite type. More precisely, denoting by $A_\lambda^G := G^{-1}\cdot A_\lambda\cdot G + G^{-1}\cdot dG$ and $\xi_\lambda^G := G^{-1}\cdot\xi_\lambda\cdot G$, then $d\xi_\lambda^G + \left[A_\lambda^G,\xi_\lambda^G\right] = G^{-1}\cdot(d\xi_\lambda + [A_\lambda,\xi_\lambda])\cdot G = 0$ and $(\lambda^{4p}\xi_\lambda^G dz)_{\Lambda_{\mathfrak{su}}} = A_\lambda^G$.*

Proof. We set $E_\lambda := F_\lambda\cdot F^{-1}$, $\Gamma_\lambda := E_\lambda^{-1}\cdot dE_\lambda$ and $\eta_\lambda := F\cdot\xi_\lambda\cdot F^{-1}$ and will use the results of the previous section.

A constant in $\Lambda^{2+4p}\mathfrak{u}(3)_\tau$ associated to the quasi-finite type family — First (3.9), which is a reformulation of (3.6), implies

$$d\left(E_\lambda\cdot\eta_\lambda\cdot E_\lambda^{-1}\right) = E_\lambda\cdot(d\eta_\lambda + [\Gamma_\lambda,\eta_\lambda])\cdot E_\lambda^{-1} = 0.$$

Hence

$$\eta_\lambda^0 := E_\lambda\cdot\eta_\lambda\cdot E_\lambda^{-1}$$

is a constant in $\Lambda\mathfrak{u}(3)$. Moreover

$$\eta_\lambda^0 = E_\lambda(z_0)\cdot\eta_\lambda(z_0)\cdot E_\lambda^{-1}(z_0) = \eta_\lambda(z_0) = F(z_0)\cdot\xi_\lambda(z_0)\cdot F^{-1}(z_0) = \xi_\lambda(z_0),$$

which proves that $\eta_\lambda^0 \in \Lambda^{2+4p}\mathfrak{u}(3)_\tau$.

An auxiliary map into $\Lambda^+U(3)^{\mathbb{C}}$ — We let

$$\Theta_\lambda := \left(\lambda^{4p}\eta_\lambda dz\right)_{\Lambda^+} = \lambda^{4p}\eta_\lambda dz - \left(\lambda^{4p}\eta_\lambda dz\right)_\Omega.$$

Then using (3.9) we have $\Theta_\lambda = \lambda^{4p}\eta_\lambda dz - \Gamma_\lambda$ and so

$$d\Gamma_\lambda + \Gamma_\lambda\wedge\Gamma_\lambda + d\Theta_\lambda - \Theta_\lambda\wedge\Theta_\lambda = -\lambda^{4p}\left(d\eta_\lambda + [\Gamma_\lambda,\eta_\lambda]\right)\left(\frac{\partial}{\partial\bar z}\right)dz\wedge d\bar z = 0.$$

But since $d\Gamma_\lambda + \Gamma_\lambda \wedge \Gamma_\lambda = 0$ this implies that $d\Theta_\lambda - \Theta_\lambda \wedge \Theta_\lambda = 0$. Hence $\exists! V_\lambda : \Omega \longrightarrow \Lambda^+ U(3)^{\mathbb{C}}$ such that

$$dV_\lambda = \Theta_\lambda \cdot V_\lambda \quad \text{and} \quad V_\lambda(z_0) = 1.$$

Now, starting from $\lambda^{4p}\eta_\lambda dz = \Gamma_\lambda + \Theta_\lambda$, we deduce that

$$\begin{array}{rcl} \lambda^{4p}\eta^0_\lambda dz & = & E_\lambda \cdot \Gamma_\lambda \cdot E_\lambda^{-1} + E_\lambda \cdot \Theta_\lambda \cdot E_\lambda^{-1} \\ & = & dE_\lambda \cdot E_\lambda^{-1} + E_\lambda \cdot dV_\lambda \cdot V_\lambda^{-1} \cdot E_\lambda^{-1} \\ & = & d\left(E_\lambda \cdot V_\lambda\right)\left(E_\lambda \cdot V_\lambda\right)^{-1}, \end{array}$$

which can be integrated into the relation

$$E_\lambda \cdot V_\lambda = e^{\lambda^{4p}(z-z_0)\eta^0_\lambda}.$$

An Iwasawa decomposition of $e^{\lambda^{4p}(z-z_0)\eta^0_\lambda}$ — The latter implies

$$e^{\lambda^{4p}(z-z_0)\eta^0_\lambda} = F_\lambda \cdot F^{-1} \cdot V_\lambda.$$

¿From this relation and the fact that η^0_λ and F_λ are twisted we deduce that $W_\lambda := F^{-1} \cdot V_\lambda$ is twisted. It is also a map with values in $\Lambda^+ U(3)^{\mathbb{C}}_\tau$. However it may not be not in $\Lambda^+_{\mathfrak{B}} U(3)^{\mathbb{C}}_\tau$ in general, because in the development

$$F^{-1} \cdot V_\lambda = \widehat{W}_0 + \sum_{k=1}^{\infty} \widehat{W}_k \lambda^k,$$

we are not sure that $\widehat{W}_0$ takes values in $\mathfrak{B}$. But it takes values in $U(3)^{\mathbb{C}}_0$, so by using the Iwasawa decomposition $U(3)^{\mathbb{C}}_0 = U(3)_0 \cdot \mathfrak{B}$ we know that $\exists! G \in U(3)_0$, $\exists! \widehat{B}_0 \in \mathfrak{B}$, $\widehat{W}_0 = G \cdot \widehat{B}_0$. Hence

$$G^{-1} \cdot F^{-1} \cdot V_\lambda = \widehat{B}_0 + \sum_{k=1}^{\infty} G^{-1} \cdot \widehat{W}_k \lambda^k$$

takes values in $\Lambda^+_{\mathfrak{B}} U(3)^{\mathbb{C}}_\tau$. So the splitting

$$e^{\lambda^{4p}(z-z_0)\eta^0_\lambda} = \left(F_\lambda \cdot G\right)\left(G^{-1} \cdot F^{-1} \cdot V_\lambda\right)$$

exactly reproduces the Iwasawa decomposition $\Lambda U(3)^{\mathbb{C}}_\tau = \Lambda U(3)_\tau \cdot \Lambda^+_{\mathfrak{B}} U(3)^{\mathbb{C}}_\tau$ proved in [**5**].

Conclusion — Let us denote by $F^G_\lambda := F_\lambda \cdot G$, $A^G_\lambda := \left(F^G_\lambda\right)^{-1} \cdot dF^G_\lambda = G^{-1} \cdot A_\lambda \cdot G + G^{-1} \cdot dG$ and $B^G_\lambda := G^{-1} \cdot F^{-1} \cdot V_\lambda$ and let us introduce

$$\xi^G_\lambda := \left(F^G_\lambda\right)^{-1} \cdot \eta^0_\lambda \cdot F^G_\lambda.$$

(These definitions imply immediately $d\xi^G_\lambda + [A^G_\lambda, \xi^G_\lambda] = 0$.) The first main observation is that the relation $\eta^0_\lambda = E_\lambda \cdot \eta_\lambda \cdot E_\lambda^{-1} = F_\lambda \cdot F^{-1} \cdot \eta_\lambda \cdot F \cdot F_\lambda^{-1} = F_\lambda \cdot \xi_\lambda \cdot F_\lambda^{-1}$ implies

$$\xi^G_\lambda = G^{-1} \cdot F_\lambda^{-1} \cdot \eta^0_\lambda \cdot F_\lambda \cdot G = G^{-1} \cdot \xi_\lambda \cdot G. \tag{3.10}$$

Second, from the relation

$$\begin{array}{rcl} \lambda^{4p}\eta^0_\lambda dz & = & d\left(e^{\lambda^{4p}(z-z_0)\eta^0_\lambda}\right) \cdot e^{-\lambda^{4p}(z-z_0)\eta^0_\lambda} \\ & = & d\left(F^G_\lambda \cdot B^G_\lambda\right) \cdot \left(F^G_\lambda \cdot B^G_\lambda\right)^{-1} \\ & = & dF^G_\lambda \cdot \left(F^G_\lambda\right)^{-1} + F^G_\lambda \cdot dB^G_\lambda \cdot \left(B^G_\lambda\right)^{-1} \cdot \left(F^G_\lambda\right)^{-1}, \end{array}$$

we deduce that

$$\lambda^{4p}\xi_\lambda^G dz = \left(F_\lambda^G\right)^{-1}\cdot\left(\lambda^{4p}\eta_\lambda^0 dz\right)\cdot F_\lambda^G = \left(F_\lambda^G\right)^{-1}\cdot dF_\lambda^G + dB_\lambda^G\cdot\left(B_\lambda^G\right)^{-1}.$$

Hence, since B_λ^G takes values in $\Lambda^+_{\mathfrak{B}}U(3)^{\mathbb{C}}_\tau$,

$$A_\lambda^G = \left(F_\lambda^G\right)^{-1}\cdot dF_\lambda^G = \left(\lambda^{4p}\xi_\lambda^G dz\right)_{\Lambda_{\mathfrak{su}}}. \tag{3.11}$$

And relations (3.10) and (3.11) lead to the conclusion. □

4. *All Hamiltonian stationary Lagrangian tori are of finite type*

The subject of this section is to prove the following:

THEOREM 4.1. *Let $u:\mathbb{C}\longrightarrow\mathbb{C}P^2$ be a doubly periodic Hamiltonian stationary Lagrangian conformal immersion. Then u is of finite type.*

We will actually prove a slightly more general result, since we can replace the doubly periodicity assumption by the hypothesis that the Maurer–Cartan form of any *Legendrian framing* of u is doubly periodic. This result of course implies immediately that Hamiltonian stationary Lagrangian tori are of finite type, since they always can be covered conformally by the plane.

Note also that the study of Hamiltonian stationary Lagrangian tori splits into exactly two subcases: the *minimal* Lagrangian tori and the *non minimal* Hamiltonian stationary Lagrangian ones. The first case occurs when the Lagrangian angle function along any Legendrian lift is locally constant, the second one when this function is harmonic and non constant. In the case of minimal Lagrangian surfaces, Theorem 4.1 is a special case of the result in [**3**], since in this case u is a harmonic map into $\mathbb{C}P^2$, as discussed in [**14**], [**15**], [**16**] and [**12**]. The non minimal case however is not covered by the theory in [**3**] and is the subject of this section.

Let $F:\mathbb{C}\longrightarrow U(3)$ be a Legendrian framing of u, $A := F^{-1}\cdot dF$ its Maurer–Cartan form and A_λ the family of deformations of A as defined by (2.4). The first basic observation is that $A_2\left(\frac{\partial}{\partial z}\right)$ is holomorphic and doubly periodic on $\mathbb{C}$, hence constant. Thus two cases occur: either $A_2\left(\frac{\partial}{\partial z}\right)=0$, which corresponds to the minimal case that we exclude here, or $A_2\left(\frac{\partial}{\partial z}\right)$ is a constant different from 0, the case that we consider next.

In order to show Theorem 4.1 we need to prove that there exists some $p\in\mathbb{N}$ and a map $\xi_\lambda:\mathbb{C}\longrightarrow\Lambda^{2+4p}\mathfrak{u}(3)_\tau$ such that $d\xi_\lambda=[\xi_\lambda,A_\lambda]$ and $A_\lambda=(\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}}$. But thanks to Theorem 3.5 it will enough to prove that $A_\lambda-(\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}}$ is a 1-form with coefficients in $\mathfrak{u}(3)_0$. Our proof here follows a strategy inspired from [**3**]: a first step consists in building a formal series $Y_\lambda=\sum_{k=-2}^{\infty}\widehat{Y}_k\lambda^k$ which is a solution of $dY_\lambda=[Y_\lambda,A_\lambda]$. Such a series is called a *formal Killing field*. We will also require Y_λ to be *quasi-adapted*, i.e. is such that

$$(Y_\lambda dz)_{\Lambda_{\mathfrak{su}}} = A_\lambda + B,\quad\text{where the coefficients of } B \text{ are in } \mathfrak{u}(3)_0. \tag{4.1}$$

This is achieved through a recursion procedure.

In a second step we will show that the coefficients of Y_λ form a countable collection of doubly periodic functions satisfying an elliptic PDE and hence, by using a compactness argument, we conclude that they are contained in a finite dimensional space. Then we deduce the existence of ξ_λ using linear algebra.

4.1. *Construction of an adapted formal Killing field*

We first introduce some notations. We denote by

$$\pi_0^\perp := \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

and $a := \frac{1}{2}\frac{\partial\beta}{\partial z}$ (here a constant different from 0). Then $A_2' = ia\pi_0^\perp dz$. We will also set $X := A_{-1}\left(\frac{\partial}{\partial z}\right)$ and $C := A_0\left(\frac{\partial}{\partial z}\right)$, so that

$$A_\lambda = \lambda^{-2} ia\pi_0^\perp dz + \lambda^{-1} X dz + C dz - C^\dagger d\bar{z} - \lambda X^\dagger d\bar{z} + \lambda^2 i\bar{a}\pi_0^\perp d\bar{z}.$$

We also introduce the linear map $\operatorname{ad}\pi_0^\perp : \mathfrak{u}(3)^{\mathbb{C}} \longrightarrow \mathfrak{u}(3)^{\mathbb{C}}$, acting by $\xi \longmapsto [\pi_0^\perp, \xi]^\dagger$. We observe that $\pi_0^\perp$ commutes with the elements in $\mathfrak{u}(3)_0^{\mathbb{C}}$ and $\mathfrak{u}(3)_2^{\mathbb{C}}$. Moreover

$$\forall a, b \in \mathbb{C},\quad \left[\pi_0^\perp, \begin{pmatrix} & & a \\ & & b \\ \mp ib & \pm ia & \end{pmatrix}\right] = \begin{pmatrix} & & a \\ & & b \\ \pm ib & \mp ia & \end{pmatrix},$$

that is $\operatorname{ad}\pi_0^\perp$ maps $\mathfrak{u}(3)_{\mp 1}^{\mathbb{C}}$ to $\mathfrak{u}(3)_{\pm 1}^{\mathbb{C}}$. ¿From that we deduce that $V := \operatorname{Ker}\operatorname{ad}\pi_0^\perp$ coincides with $\mathfrak{u}(3)_0^{\mathbb{C}} \oplus \mathfrak{u}(3)_2^{\mathbb{C}}$ and $V^\perp := \operatorname{Im}\operatorname{ad}\pi_0^\perp$ coincides with $\mathfrak{u}(3)_{-1}^{\mathbb{C}} \oplus \mathfrak{u}(3)_1^{\mathbb{C}}$ (note that $V^\perp$ is actually the orthogonal subspace to V in $\mathfrak{u}(3)^{\mathbb{C}}$). In our construction we will use extensively the following properties:

- the map $\operatorname{ad}\pi_0^\perp\big|_{V^\perp \to V^\perp}$ is a vector space isomorphism (it is actually a involution on $V^\perp$),
- the inclusions $VV \subset V$, $VV^\perp \subset V^\perp$, $V^\perp V \subset V^\perp$ and $V^\perp V^\perp \subset V$. These properties can be checked by a direct computation using the fact that matrices in V are diagonal by blocks and the matrices in $V^\perp$ are off-diagonal by blocks. (The three first properties can also be deduced from the definition of V and $V^\perp$ and the fact that ad is a derivation).

We look for a formal Killing field Y_λ, i.e. a solution of the equation

$$dY_\lambda = [Y_\lambda, A_\lambda], \tag{4.2}$$

of the form $Y_\lambda = (1 + W_\lambda)^{-1}\lambda^{-2} ia\pi_0^\perp(1 + W_\lambda)$, where $W_\lambda = \sum_{k=0}^\infty \widehat{W}_k \lambda^k$ as in [**3**]. In order to have a well-posed problem (and in particular to guarantee the existence of an unique solution of this type) we assume that W_λ takes values in $V^\perp$. We start by evaluating (4.2) along $\partial/\partial z$. It gives, after conjugation by $1 + W_\lambda$:

$$\begin{aligned}\lambda^{-2}\frac{\partial a}{\partial z}\pi_0^\perp &+ \lambda^{-2} a\big[\pi_0^\perp, \\ &\frac{\partial W_\lambda}{\partial z}(1 + W_\lambda)^{-1} - (1 + W_\lambda)(\lambda^{-2} ia\pi_0^\perp + \lambda^{-1}X + C)(1 + W_\lambda)^{-1}\big] = 0\end{aligned} \tag{4.3}$$

Here the fact that a is a constant leads to an immediate simplification, namely that the bracket in the left hand side of (4.3) is 0. Thus equation (4.3) implies that

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[†]Actually the map $\xi \longmapsto i[\pi_0^\perp, \xi]$ corresponds to the complex structure on the Legendrian distribution.

$\frac{\partial W_\lambda}{\partial z}(1+W_\lambda)^{-1}-(1+W_\lambda)(\lambda^{-2}ia\pi_0^\perp+\lambda^{-1}X+C)(1+W_\lambda)^{-1}$ lies in V, hence there exists a map $\varphi_\lambda:\mathbb{C}\longrightarrow V$ such that

$$\frac{\partial W_\lambda}{\partial z}(1+W_\lambda)^{-1}-(1+W_\lambda)(\lambda^{-2}ia\pi_0^\perp+\lambda^{-1}X+C)(1+W_\lambda)^{-1}=\varphi_\lambda$$

or

$$\frac{\partial W_\lambda}{\partial z}-(1+W_\lambda)(\lambda^{-2}ia\pi_0^\perp+\lambda^{-1}X+C)=\varphi_\lambda(1+W_\lambda),$$

which can be projected according to the splitting $V\oplus V^\perp$ as

$$\left\{\begin{array}{l}\lambda^{-2}ia\pi_0^\perp+\lambda^{-1}W_\lambda X+C=-\varphi_\lambda\in V\\ \dfrac{\partial W_\lambda}{\partial z}-\lambda^{-2}iaW_\lambda\pi_0^\perp-\lambda^{-1}X-W_\lambda C=\varphi_\lambda W_\lambda\in V^\perp.\end{array}\right.$$

Substituting φ_λ,

$$\frac{\partial W_\lambda}{\partial z}-\lambda^{-2}iaW_\lambda\pi_0^\perp-\lambda^{-1}X-W_\lambda C+\lambda^{-2}ia\pi_0^\perp W_\lambda+\lambda^{-1}W_\lambda XW_\lambda+CW_\lambda=0$$

or

$$[ia\pi_0^\perp,W_\lambda]+\lambda(W_\lambda XW_\lambda-X)+\lambda^2[C,W_\lambda]+\lambda^2\frac{\partial W_\lambda}{\partial z}=0$$

or

$$\begin{aligned}ia\sum_{n\geq 0}[\pi_0^\perp,\widehat{W}_n]\lambda^n+\sum_{n\geq 1}\left(\sum_{k=0}^{n-1}\widehat{W}_kX\widehat{W}_{n-1-k}\right)\lambda^n-\lambda X&\\ +\sum_{n\geq 2}\left([C,\widehat{W}_{n-2}]+\frac{\partial\widehat{W}_{n-2}}{\partial z}\right)\lambda^n&=0.\end{aligned}$$

Hence

$$\left\{\begin{array}{ll}n=0, & ia[\pi_0^\perp,\widehat{W}_0]=0\\ n=1, & ia[\pi_0^\perp,\widehat{W}_1]+\widehat{W}_0X\widehat{W}_0-X=0\\ n\geq 2, & ia[\pi_0^\perp,\widehat{W}_n]+\displaystyle\sum_{k=0}^{n-1}\widehat{W}_kX\widehat{W}_{n-1-k}+[C,\widehat{W}_{n-2}]+\frac{\partial\widehat{W}_{n-2}}{\partial z}=0\end{array}\right.$$

and thus

$$\left\{\begin{array}{l}\widehat{W}_0=0\\ \widehat{W}_1=-ia^{-1}[\pi_0^\perp,X]\\ \widehat{W}_n=ia^{-1}\left[\pi_0^\perp,\displaystyle\sum_{k=0}^{n-1}\widehat{W}_kX\widehat{W}_{n-1-k}+[C,\widehat{W}_{n-2}]+\frac{\partial\widehat{W}_{n-2}}{\partial z}\right]\end{array}\right.$$

We observe that the formal Killing field is *quasi-adapted* in the sense that the two first coefficients are the right ones:

$$\begin{aligned}Y_\lambda=ia\lambda^{-2}(1+W_\lambda)^{-1}\pi_0^\perp(1+W_\lambda)&=ia\lambda^{-2}\left(\pi_0^\perp-\lambda[\widehat{W}_1,\pi_0^\perp]+\mathcal{O}(\lambda^2)\right)\\ &=\lambda^{-2}ia\pi_0^\perp+\lambda^{-1}X+\mathcal{O}(1).\end{aligned}$$

Another pleasant property is that this formal field is automatically twisted (as in the case of $\mathbb{C}^2$, see [**10**]). Indeed using the fact that τ is an automorphism for the product of matrices as well as for the Lie bracket (and so $[\mathfrak{u}(3)_a^{\mathbb{C}},\mathfrak{u}(3)_b^{\mathbb{C}}]\subset\mathfrak{u}(3)_{a+b}^{\mathbb{C}}$ and $\mathfrak{u}(3)_a^{\mathbb{C}}\mathfrak{u}(3)_b^{\mathbb{C}}\subset\mathfrak{u}(3)_{a+b}^{\mathbb{C}}$), we obtain that

$$\tau(Y_\lambda)=\tau(1+W_\lambda)^{-1}(-\lambda^2)ia\pi_0^\perp\tau(1+W_\lambda).$$

Thus it is enough to show that $1 + W_\lambda$ is twisted, i.e. $\tau(W_\lambda) = W_{i\lambda}$. In terms of the Fourier decomposition of W_λ this is equivalent to proving that $\widehat{W}_n$ belongs to $\mathfrak{u}(3)^{\mathbb{C}}_n$. Let us prove it by recursion. We already know that $\widehat{W}_1 = -ia^{-1}[\pi_0^\perp, X]$ is in $\mathfrak{u}(3)^{\mathbb{C}}_1$. Assume that the result is true up to $n-1$,then

$$\sum_{p=0}^{n-1} \widehat{W}_p X \widehat{W}_{n-1-p} + [C, \widehat{W}_{n-2}] + \frac{\partial \widehat{W}_{n-2}}{\partial z}$$

belongs to $\mathfrak{u}(3)^{\mathbb{C}}_{n-2}$. And since $\pi_0^\perp \in \mathfrak{u}(3)^{\mathbb{C}}_2$, $\widehat{W}_n$ is in $\mathfrak{g}^{\mathbb{C}}_n$.

We now prove that (4.2) is also true along $\partial/\partial\bar{z}$. We follow here the same kind of arguments as in [**3**] slightly simplified†. We want to show that

$$\frac{\partial Y_\lambda}{\partial \bar{z}} + \left[A_\lambda\left(\frac{\partial}{\partial \bar{z}}\right), Y_\lambda\right] = 0$$

and for that purpose we rather consider the conjugate of the left hand side $\zeta_\lambda = (1 + W_\lambda)(\frac{\partial Y_\lambda}{\partial \bar{z}} + [A_\lambda\left(\frac{\partial}{\partial \bar{z}}\right), Y_\lambda])(1 + W_\lambda)^{-1}$. We then prove two facts

– ζ_λ takes its values in $V^\perp$: this follows from the identity

$$\zeta_\lambda = \left[\lambda^{-2} ia\pi_0^\perp, \frac{\partial W_\lambda}{\partial \bar{z}}(1 + W_\lambda)^{-1} - (1 + W_\lambda)A_\lambda\left(\frac{\partial}{\partial \bar{z}}\right)(1 + W_\lambda)^{-1}\right].$$

Note that since ζ_λ is twisted the fact that $\zeta_\lambda \in V^\perp$ implies also that ζ_λ is an odd function of λ and so that

$$\zeta_\lambda = \sum_{k=0}^{\infty} \widehat{\zeta}_{2k-1}\lambda^{2k-1}. \tag{4.4}$$

– the relation

$$\frac{\partial \zeta_\lambda}{\partial z} = [\varphi_\lambda, \zeta_\lambda]. \tag{4.5}$$

Indeed $d + \mathrm{ad}\, A_\lambda$ is a flat connection and in particular $\frac{\partial}{\partial z} + \mathrm{ad}\, A_\lambda(\frac{\partial}{\partial z})$ commutes with $\frac{\partial}{\partial \bar{z}} + A_\lambda(\frac{\partial}{\partial \bar{z}})$. Hence

$$\left(\frac{\partial}{\partial z} + \mathrm{ad}\, A_\lambda\left(\frac{\partial}{\partial z}\right)\right)\left(\frac{\partial}{\partial \bar{z}} + \mathrm{ad}\, A_\lambda\left(\frac{\partial}{\partial \bar{z}}\right)\right) Y_\lambda = 0$$

i.e.

$$\left(\frac{\partial}{\partial z} + \mathrm{ad}\, A_\lambda\left(\frac{\partial}{\partial z}\right)\right)\left((1 + W_\lambda)^{-1}\zeta_\lambda(1 + W_\lambda)\right) = 0$$

Thus (4.5) follows from a computation which uses $\varphi_\lambda = \frac{\partial W_\lambda}{\partial z}(1+W_\lambda)^{-1} - (1 + W_\lambda)A_\lambda\left(\frac{\partial}{\partial z}\right)(1 + W_\lambda)^{-1}$.

Now assume by contradiction that $\zeta_\lambda \neq 0$: in view of (4.4) there exists an integer $k \in \mathbb{N}$ such that $\widehat{\zeta}_{2k-1} \neq 0$ and $\widehat{\zeta}_{2k-3} = 0$. By substituting the Fourier decompositions in (4.5) and observing that the Fourier series expansion of φ_λ starts by $\lambda^{-2} ia\pi_0^\perp$, we deduce that $0 = \partial\widehat{\zeta}_{2k-3}/\partial z = [ia\pi_0^\perp, \widehat{\zeta}_{2k-1}]$; but $\mathrm{ad}\,\pi_0^\perp$ is invertible on $V^\perp$ and hence $\widehat{\zeta}_{2k-1} = 0$. So we get a contradiction.

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†essentially the simplifications occur because the semi-simple term B of [**3**] is here $ia\pi_0^\perp$ which is constant for d, so we do not need to introduce a flat connection.

4.2. *Polynomial Killing fields*

We now deduce the existence of a non-trivial polynomial Killing field.

A first easy consequence of the results of the previous section is that, for all $n \in \mathbb{N}$ and for all polynomial of the form $P(\lambda) = a_n\lambda^{-4n} + a_{n-1}\lambda^{-4(n-1)} + \cdots + a_0$, where $a_0, a_1, \cdots, a_n \in \mathbb{C}$ and $a_n \neq 0$, then $Z_\lambda := P(\lambda)Y_\lambda$ is again formal Killing field. Moreover it is quasi-adapted (modulo the multiplicative factor $a_n\lambda^{-4n}$), i.e. the lower degree terms are $a_n\lambda^{-4n}\left(ia\lambda^{-2}\pi_0^\perp + \lambda^{-1}X + O(\lambda^0)\right)$. Let us consider

$$Z_\leq := \sum_{k=-2-4n}^{0} \widehat{Z}_k\lambda^k, \quad \text{and} \quad Z_> := \sum_{k=1}^{\infty} \widehat{Z}_k\lambda^k,$$

so that $Z_\lambda = Z_\leq + Z_>$. We study

$$R_\lambda := dZ_\leq + [A_\lambda, Z_\leq]. \tag{4.6}$$

We first remark that R_λ is necessarily of the form $R_\lambda = \sum_{k=-4-4n}^{2} \widehat{R}_k\lambda^k$. But because of $dZ_\lambda + [A_\lambda, Z_\lambda] = 0$, we also have

$$R_\lambda = -dZ_> - [A_\lambda, Z_>], \tag{4.7}$$

which implies $R_\lambda = \sum_{k=-1}^{\infty} \widehat{R}_k\lambda^k$. Hence finally

$$R_\lambda = \lambda^{-1}\widehat{R}_{-1} + \widehat{R}_0 + \lambda^1\widehat{R}_1 + \lambda^2\widehat{R}_2.$$

Each term $\widehat{R}_k$ can be evaluated through two different ways: by using (4.6) or (4.7). From (4.6) we obtain

$$\begin{cases} \widehat{R}_{-1}(\partial_z) = \partial_z\widehat{Z}_{-1} + [A_{-1}(\partial_z), \widehat{Z}_0] + [A_0(\partial_z), \widehat{Z}_{-1}] \\ \widehat{R}_0(\partial_z) = \partial_z\widehat{Z}_0 + [A_0(\partial_z), \widehat{Z}_0] \\ \widehat{R}_1(\partial_z) = 0 \\ \widehat{R}_2(\partial_z) = 0 \end{cases} \tag{4.8}$$

and

$$\begin{cases} \widehat{R}_{-1}(\partial_{\bar z}) = \partial_{\bar z}\widehat{Z}_{-1} + [A_0(\partial_{\bar z}), \widehat{Z}_{-1}] + [A_1(\partial_{\bar z}), \widehat{Z}_{-2}] + [A_2''(\partial_{\bar z}), \widehat{Z}_{-3}] \\ \widehat{R}_0(\partial_{\bar z}) = \partial_{\bar z}\widehat{Z}_0 + [A_0(\partial_{\bar z}), \widehat{Z}_0] + [A_1(\partial_{\bar z}), \widehat{Z}_{-1}] + [A_2''(\partial_{\bar z}), \widehat{Z}_{-2}] \\ \widehat{R}_1(\partial_{\bar z}) = [A_1(\partial_{\bar z}), \widehat{Z}_0] + [A_2''(\partial_{\bar z}), \widehat{Z}_{-1}] \\ \widehat{R}_2(\partial_{\bar z}) = [A_2''(\partial_{\bar z}), \widehat{Z}_0]. \end{cases} \tag{4.9}$$

¿From (4.7) we get

$$\begin{cases} \widehat{R}_{-1}(\partial_z) = -[A_2'(\partial_z), \widehat{Z}_1] \\ \widehat{R}_0(\partial_z) = -[A_2'(\partial_z), \widehat{Z}_2] - [A_{-1}(\partial_z), \widehat{Z}_1] \\ \widehat{R}_1(\partial_z) = -\partial_z\widehat{Z}_1 - [A_2'(\partial_z), \widehat{Z}_3] - [A_{-1}(\partial_z), \widehat{Z}_2] - [A_0(\partial_z), \widehat{Z}_1] \\ \widehat{R}_2(\partial_z) = -\partial_z\widehat{Z}_2 - [A_2'(\partial_z), \widehat{Z}_4] - [A_{-1}(\partial_z), \widehat{Z}_3] - [A_0(\partial_z), \widehat{Z}_2] \end{cases} \tag{4.10}$$

and

$$\begin{cases} \widehat{R}_{-1}(\partial_{\bar z}) = 0 \\ \widehat{R}_0(\partial_{\bar z}) = 0 \\ \widehat{R}_1(\partial_{\bar z}) = -\partial_{\bar z}\widehat{Z}_1 - [A_0(\partial_{\bar z}), \widehat{Z}_1] \\ \widehat{R}_2(\partial_{\bar z}) = -\partial_{\bar z}\widehat{Z}_2 - [A_0(\partial_{\bar z}), \widehat{Z}_2] - [A_1(\partial_{\bar z}), \widehat{Z}_1]. \end{cases} \tag{4.11}$$

Thus in order to obtain an expression of R_λ which does depend only on $\widehat{Z}_{-1}$ and

$\widehat{Z}_0$, we exploit (4.8) and the two last equations in (4.9). But instead of using the two first equations of (4.9) we take the two first ones of (4.11). This gives us

$$R_\lambda(\partial_z) = \lambda^{-1}\left(\partial_z \widehat{Z}_{-1} + [A_{-1}(\partial_z), \widehat{Z}_0] + [A_0(\partial_z), \widehat{Z}_{-1}]\right) + \left(\partial_z \widehat{Z}_0 + [A_0(\partial_z), \widehat{Z}_0]\right), \tag{4.12}$$

$$R_\lambda(\partial_{\bar z}) = \lambda\left([A_1(\partial_{\bar z}), \widehat{Z}_0] + [A_2''(\partial_{\bar z}), \widehat{Z}_{-1}]\right) + \lambda^2[A_2''(\partial_{\bar z}), \widehat{Z}_0]. \tag{4.13}$$

These relations will imply that $\widehat{Z}_{-1}$ and $\widehat{Z}_0$ satisfy a second order elliptic equation. In order to prove that we need to establish another relation between $R_\lambda(\partial_z)$ and $R_\lambda(\partial_{\bar z})$. For that purpose recall that $dA_\lambda + A_\lambda \wedge A_\lambda = 0$, which means that the connection $d + \mathrm{ad}\, A_\lambda$ has a vanishing curvature. In particular

$$0 = (d + \mathrm{ad}\, A_\lambda) \circ (d + \mathrm{ad}\, A_\lambda)\, Z_{\leq} = dR_\lambda + [A_\lambda \wedge R_\lambda].$$

This implies

$$\frac{\partial R_\lambda(\partial_z)}{\partial \bar z} - \frac{\partial R_\lambda(\partial_{\bar z})}{\partial z} = [A_\lambda(\partial_z), R_\lambda(\partial_{\bar z})] - [A_\lambda(\partial_{\bar z}), R_\lambda(\partial_z)]. \tag{4.14}$$

A substitution of (4.12) and (4.13) in (4.14) gives a system of linear elliptic equations on $\widehat{Z}_{-1}$ and $\widehat{Z}_0$. Since the space of solutions to this system which are periodic is finite dimensional, it turns out that $\widehat{Z}_{-1}$ and $\widehat{Z}_0$ belong to a finite dimensional vector space. Hence relations (4.12) and (4.13) force $R_\lambda(\partial_z)$ and $R_\lambda(\partial_{\bar z})$ to stay in a finite dimensional vector space.

We can conclude: let us consider

$$\mathcal{R} := \{R_\lambda / R_\lambda(\partial_z), R_\lambda(\partial_{\bar z}) \text{ are given by (4.12) and (4.13) and satisfy (4.14)}\}.$$

It is a complex finite dimensional vector space. Let us also denote by $\mathcal{P}_n := \{P(\lambda) = a_n\lambda^{-4n} + a_{n-1}\lambda^{-4(n-1)} + \cdots + a_0 / (a_0, \cdots, a_n) \in \mathbb{C}^{n+1}\}$ and $\mathcal{P}_\infty := \cup_{n\in\mathbb{N}}\mathcal{P}_n$.

The linear map $\mathcal{P}_\infty \ni P(\lambda) \longmapsto dZ_{\leq} + [A_\lambda, Z_{\leq}]$, where $Z_{\leq} = (P(\lambda)Y_\lambda)_{\leq}$ takes values in $\mathcal{R}$ and so has a finite rank, say n. Then since $\dim_{\mathbb{C}}\mathcal{P}_n = n+1$, the map $\mathcal{P}_n \ni P(\lambda) \longmapsto dZ_{\leq} + [A_\lambda, Z_{\leq}]$ has a non trivial kernel: let $P(\lambda) = \sum_{k=0}^n a_k\lambda^{-4k}$ be a non trivial polynomial in this kernel. Let $4p$ be the degree of P in λ^{-1}, i.e. such that $P(\lambda) = \sum_{k=0}^p a_k\lambda^{-4k}$ and $a_p \neq 0$. Without loss of generality we can assume that $a_p = 1$. Then $\xi_\lambda := (P(\lambda)Y_\lambda)_{\leq} - (P(\lambda)Y_\lambda)_{\leq}^\dagger$ is a solution of (3.6) and (3.8).

5. *Homogeneous tori in* $\mathbb{C}P^2$

We describe here the simplest examples of Hamiltonian stationary Lagrangian tori in $\mathbb{C}P^2$: the *homogeneous* Hamiltonian stationary Lagrangian tori, i.e. immersions u of $S^1 \times S^1$ into $\mathbb{C}P^2$ such that $u(x+t, y) = e^{tA}u(x,y)$ and $u(x, y+t) = e^{tB}u(x,y)$ for some skew-Hermitian matrices A and B. Notice that A and B are only defined up to addition with a multiple of iId. The simplest example is the Clifford torus, namely the image by the Hopf map π of the product torus $\{z = (z^1, z^2, z^3); |z^1| = |z^2| = |z^3| = 1/\sqrt{3}\}$. This torus is minimal. The main result states that all homogeneous Hamiltonian stationary Lagrangian tori are similar to the Clifford torus.

THEOREM 5.1. *Any homogeneous Hamiltonian stationary Lagrangian torus in* $\mathbb{C}P^2$ *is the image by the Hopf map of some Cartesian product* $r_1S^1 \times r_2S^1 \times r_3S^1 =$

$\{z = (z^1, z^2, z^3); |z^1| = r_1, |z^2| = r_2, |z^3| = r_3\}$ *where* $r_1^2 + r_2^2 + r_3^2 = 1$*, up to* $U(3)$ *congruence. Moreover, the torus is special Lagrangian if and only if* $r_1 = r_2 = r_3 = \sqrt{3}$.

Proof. Let us first see why $\pi(T)$ is a Hamiltonian stationary Lagrangian torus in $\mathbb{C}P^2$, where $T = r_1S^1 \times r_2S^1 \times r_3S^1$. Indeed it suffices to show that $\pi(T)$ admits a Legendrian preimage. Let

$$f(x, y) := \left(r_1 e^{i((1-r_1^2)x - r_2^2 y)}, r_2 e^{i(-r_1^2 x + (1-r_2^2)y)}, r_3 e^{i(-r_1^2 x - r_2^2 y)}\right).$$

Then the orbit under the Hopf action of the image of f is exactly the 3-torus T above and $\pi \circ f$ is doubly periodic with periods $(2\pi, 0)$ and $(0, 2\pi)$. Note that this immersion is not conformal but there exists an orthonormal Hermitian moving frame (e_1, e_2) such that $\frac{\partial f}{\partial x} = r_1\sqrt{1 - r_1^2}\, e_1$ and $\frac{\partial f}{\partial y} = \frac{r_2}{\sqrt{1-r_1^2}}(r_3 e_2 - r_1 r_2 e_1)$. And it is easy to check that f is Legendrian (and flat). Its Lagrangian angle function is

$$\beta(x, y) = x(1 - 3r_1^2) + y(1 - 3r_2^2) + \pi$$

and since the metric is flat, β is clearly harmonic, and constant if and only if $r_1 = r_2 = r_3 = 1/\sqrt{3}$. Notice that many of these tori do not lift up to S^5 as Legendrian tori (they do not close up). Indeed the Maslov class is not always an integer: for the implicit homology basis, $t \longmapsto (2\pi t, 0)$ and $t \longmapsto (0, 2\pi t)$, it is $(1 - 3r_1^2, 1 - 3r_2^2)$. However, if all r_i^2 are rational, the torus in $\mathbb{C}P^2$ possesses a Legendrian toric multiple cover.

Suppose now that $u : S^1 \times S^1 \longrightarrow \mathbb{C}P^2$ is a homogeneous Lagrangian immersion. According to our definition u has a lift $\hat{u}$ such that $\pi(\hat{u}(x + t, y)) = \pi(e^{tA}\hat{u}(x, y))$ and $\pi(\hat{u}(x, y+t)) = \pi(e^{tB}\hat{u}(x, y))$. In particular $\pi(e^{xA}e^{yB}p) = \pi(e^{yB}e^{xA}p)$, for any $p \in S^5$ in the image. However the image is never contained in a complex subspace of $\mathbb{C}^3$, hence $[A, B] \in i\mathbb{R}Id$. Since $[A, B]$ is traceless, A and B commute.

The obvious (non Legendrian) lift in S^5 is $(x, y) \longmapsto e^{xA}e^{yB}p$ where now $p = (p_1, p_2, p_3)$ is a fixed point mapped by the Hopf map π to $u(0, 0)$. A Legendrian lift $\hat{u}$ takes the following form: $\hat{u}(x, y) = e^{i\theta(x,y)}e^{xA}e^{yB}p$ for some function θ. The horizontality condition implies $\langle(i\frac{\partial\theta}{\partial x}Id + A)p, p\rangle_{\mathbb{C}^3} = 0$ so that $\frac{\partial\theta}{\partial x} = i\frac{\langle Ap,p\rangle_{\mathbb{C}^3}}{|p|^2}$ is a constant. The same holds in the y direction so we can define the lift $\hat{u}(x, y) = e^{x\hat{A}+y\hat{B}}p$ where $\hat{A} = A + i\frac{\partial\theta}{\partial x}Id$ and $\hat{B} = B + i\frac{\partial\theta}{\partial y}Id$ are two commuting skew-symmetric matrices. (Notice that $\hat{u}$ is only defined on the universal cover $\mathbb{R}^2$.) The base point p depends of course on the choice of origin and is only defined up to multiplication by a complex unit number. Nevertheless it plays an important role.

Consider now the metric induced by $\hat{u}$. Due to homogeneity, it is a constant metric on the (x, y)-plane. By doing a simple change in variables, we may as well assume that the metric is the standard plane metric, in other words the immersion is isometric. (Of course that will change the matrices $\hat{A}$ and $\hat{B}$, but since they are replaced by some real linear combination of themselves, the properties mentioned above still hold.) Henceforth we suppose that $\hat{u}$ is an isometric homogeneous Legendrian immersion of the plane.

Up to a unitary rotation in $\mathbb{C}^3$ we may suppose that $\hat{A}$ is diagonal, and write $\hat{A} = i\,\mathrm{diag}(a_1, a_2, a_3)$ with real coefficients a_1, a_2, a_3. We will now consider three cases and show that only case (i) is possible.

(i) Suppose $B = i\mathrm{diag}(b_1, b_2, b_3)$ is diagonal. Then the surface lies inside the three torus $T = |p_1|S^1 \times |p_2|S^1 \times |p_3|S^1$. Necessarily it lifts $\pi(T)$. Isometry will constrain the coefficients to be as above.

(ii) One and only one of the off-diagonal coefficients of B is non zero. We can assume it is b_{12} up to permutation of the coordinates. Commutation of $\hat{A}$ and $\hat{B}$ forces $a_1 = a_2$, while $a_3 \neq a_1$, otherwise we would get a contradiction: A cannot be a multiple of iId. Let us first look at equations involving A. The immersion being isometric in S^5, $|p| = |Ap| = 1$

$$1 = |p_1|^2 + |p_2|^2 + |p_3|^2 = a_1^2(|p_1|^2 + |p_2|^2) + a_3^2|p_3|^2$$

but it is also Legendrian, so

$$\omega(Ap, p) = \langle iAp, p\rangle_{\mathbb{C}^3} = a_1(|p_1|^2 + |p_2|^2) + a_3|p_3|^2 = 0.$$

Hence

$$|p_1|^2 + |p_2|^2 = \frac{a_3}{a_3 - a_1}\ , \quad |p_3|^2 = -\frac{a_1}{a_3 - a_1} \text{ and } a_1 a_3 = -1$$

excluding thus $a_1 = 0$, and finally

$$|p_1|^2 + |p_2|^2 = \frac{1}{1 + a_1^2} \quad , \quad |p_3|^2 = \frac{a_1^2}{1 + a_1^2} \quad .$$

Take now into account the Legendrian constraints on B:

$$B = \begin{pmatrix} ib_1 & b_{12} & 0 \\ -\overline{b_{12}} & ib_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}$$

$$0 = \langle Bp, p\rangle_{\mathbb{C}^3} = i\left(\sum_1^3 b_j|p_j|^2 + 2\mathrm{Im}(b_{12}\overline{p_1}p_2)\right)$$

$$0 = \langle Bp, Ap\rangle_{\mathbb{C}^3} = \sum_1^3 a_j b_j|p_j|^2 + 2a_1\mathrm{Im}(b_{12}\overline{p_1}p_2).$$

Uniting both, we deduce

$$a_1b_1|p_1|^2 + a_1b_2|p_2|^2 + a_1b_3|p_3|^2 = a_1b_1|p_1|^2 + a_1b_2|p_2|^2 + a_3b_3|p_3|^2.$$

Since $|p_3| \neq 0$ and $a_1 \neq a_3$, b_3 vanishes. The remaining equations are:

$$|p_1|^2 + |p_2|^2 = \frac{1}{1 + a_1^2}$$

$$b_1|p_1|^2 + b_2|p_2|^2 + 2\mathrm{Im}(b_{12}\overline{p_1}p_2) = 0 \tag{5.1}$$

$$(b_1^2 + |b_{12}|^2)|p_1|^2 + (b_2^2 + |b_{12}|^2)|p_2|^2 + 2(b_1 + b_2)\mathrm{Im}(b_{12}\overline{p_1}p_2) = 1. \tag{5.2}$$

Finally we infer a contradiction: indeed equation (5.1) amounts to the existence of an isotropic vector (p_1, p_2) for the skew-hermitian matrix $\begin{pmatrix} ib_1 & b_{12} \\ -\overline{b_{12}} & ib_2 \end{pmatrix}$, and that requires its determinant $|b_{12}|^2 - b_1b_2$ to vanish. Plugging this into (5.2), we obtain

$$(b_1 + b_2)\big(b_1|p_1|^2 + b_2|p_2|^2 + 2\mathrm{Im}(b_{12}\overline{p_1}p_2)\big) = 1,$$

obviously contradicting (5.1).

(iii) If at least two off-diagonal coefficients of B are non-zero, then $A = ia_1 Id$. But that contradicts $(Ap|p) = 0$. So that case is also excluded.

Notice that in the language of integrable systems, homogeneous tori correspond to vacuum solutions and are of finite type for $p = 0$. □

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