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LIMITING LAWS ASSOCIATED WITH BROWNIAN MOTION PERTURBED BY NORMALIZED EXPONENTIAL WEIGHTS, I

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Abstract. Let $(B_t; t \geq 0)$ be a one- dimensional Brownian motion, with local time process $(L_t^x; t \geq 0, x \in \mathbb{R})$. We determine the rate of decay of $Z_t^V(x) := E_x\Big[\exp\Big\{-\dfrac{1}{2}\displaystyle\int_{\mathbb{R}} L_t^y V(dy)\Big\}\Big]$, $t \geq 0, x \in \mathbb{R}$ as t goes to infinity, where $V(dy)$ is a positive Radon measure on $\mathbb{R}$. If $\displaystyle\int_{\mathbb{R}}(1+|y|)V(dy) < \infty$, we prove that $Z_t^V(x) \underset{t\to\infty}{\sim} \varphi_V(x)t^{-1/2}$, where the function φ_V solves the Sturm-Liouville equation $(\varphi_V)"(dx) = \varphi_V(x)V(dx)$, with some boundary conditions. If $\displaystyle\int_{-\infty}^0 (1+|y|)V(dy) < \infty$ and $V(dy)$ is "large" at $+\infty$, the asymptotics of $Z_t^V(x)$ is the same as previously. When $V(dy)$ is "large" at $\pm\infty$, $Z_t^V(x)$ is equivalent to $ke^{-\gamma_0 t}, t \to \infty$. If $V(dy) = [\lambda/(\theta + y^2)]dy$, $\lambda, \theta > 0$, the rate of decay is polynomial : $Z_t^V(x) \underset{t\to\infty}{\sim} k\varphi_V(x/\sqrt{\theta})t^{-n}$, with $n = (1+\sqrt{1+4\lambda})/4$. Taking $V(dy) = [1/(1+|y|^\alpha)]dy, 0 < \alpha < 2$ we only obtain a logarithmic equivalent : $\ln(Z_t^V(x)) \underset{t\to\infty}{\sim} -kt^{-\frac{\alpha-2}{\alpha+2}}$.
Let $Q_{x,t}$ be the probability measure defined on the canonical space $\Omega = \mathcal{C}([0,+\infty[)$, by :

$$Q_{x,t} = \frac{1}{Z_t^V(x)} \exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\Big\}\, W_x,$$

where W_x denotes the Wiener measure. We prove that $Q_{x,t}$ converges as $t \to \infty$ to Q_x and Q_x is the law of the diffusion process X_t^x, solution of the stochastic differential equation :

$$X_t = x + B_t + \int_0^t \frac{\varphi_V'}{\varphi_V}(X_s)ds;\ t \geq 0.$$

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1 Foreword and perspectives

This paper is the first in a series of four related papers, numbered I to IV, a sketchy description of which may be of interest to the reader.
Stimulated by the results obtained in I (see the above abstract), we consided, in II, some asymptotic problems obtained from weighting the Wiener measure with a function of the maximum, or minimum, or local time up to time t, and letting $t \to \infty$. The limit laws exist in some generality; they are not the distribution of a Markov process (X_t), but rather the two dimensional process $\big(X_t, S_t := \sup_{0\le s\le t} X_s; t \geq 0\big)$ are Markovian. We then say that (X_t) is max-Markovian.
In III, we study the existence and characterization of the limit laws for Brownian bridges, on the time interval $[0, t]$, as $t \to \infty$; weighted again by a function of the maximum, or minimum, or local time up to time t.
In IV, we study the variants of the Pitman and Ray-Knight theorems for the max-Markovian process obtained in the previous papers.

2 Introduction

1.1 Consider a general nice Markov process $((X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (P_x)_{x\in E})$ taking values in $(E, \mathcal{E})$, with extended generator L, i.e. : $\varphi \in \mathcal{D}(L)$ iff : $M_t^\varphi := \varphi(X_t) - \varphi(X_0) - \int_0^t L\varphi(X_s)ds,\ t \geq 0$, is a martingale, for some function $L\varphi$, and its operator "carré du champ" $\Gamma(\varphi, \psi)$, defined via :

$$\frac{d < M^\varphi, M^\psi >_t}{dt} = \Gamma(\varphi, \psi)(X_t). \quad (\varphi, \psi \in \mathcal{D}(L)). \tag{2.1}$$

Indeed, by "nice Markov process", we mean in particular that $\mathcal{D}(L)$ is an algebra; hence (cf [8]) it follows that :

$$\Gamma(\varphi, \psi) = L(\varphi\psi) - \varphi L\psi - \psi L\varphi, \quad \varphi, \psi \in \mathcal{D}(L). \tag{2.2}$$

1.2 Associated with the family $(P_x)_{x\in E}$, we shall consider two other families of probabilities constructed from the $(P_x)_{x\in E}$.
a) The first family$(Q^V_{x,t})$. For a potential function $V : E \mapsto \mathbb{R}$, such that :

$$Z_t^V(x) := E_x[\exp\{-\frac{1}{2}\int_0^t V(X_s)ds\}] < \infty, \quad \text{for every} \ \ x \in E, \tag{2.3}$$

we define the family of (normalized) probabilities :

$$Q^V_{x,t} = \frac{\exp\{-\frac{1}{2}\int_0^t V(X_s)ds\}}{Z_t^V(x)} \ P_{x|\mathcal{F}_t}. \tag{2.4}$$

Note that, in general, these laws are not coherent, i.e. for $s < t$,

$$Q^V_{x,t|\mathcal{F}_s} \neq Q^V_{x,s}. \tag{2.5}$$

b) The second family(P_x^φ). Let $\varphi > 0$ be an element of $\mathcal{D}(L)$, then it is well-known that :

$$\frac{\varphi(X_t)}{\varphi(x)} \exp\{-\int_0^t \frac{L\varphi}{\varphi}(X_s)ds\} \quad \text{is a } (\mathcal{F}_t)\text{-local martingale w.r. to any } P_x, \text{for } x \in E. \tag{2.6}$$

We suppose that it is a martingale, so that we can define a second (Markov) family of probabilities :

$$P^\varphi_{x|\mathcal{F}_t} := \frac{\varphi(X_t)}{\varphi(x)} \exp\{-\int_0^t \frac{L\varphi}{\varphi}(X_s)ds\}\ P_{x|\mathcal{F}_t}. \tag{2.7}$$

As a converse to (2.6), if $\varphi \in \mathcal{D}(L)$, and there exists g such that $\left(\varphi(X_t)\exp\{-\int_0^t g(X_u)du\}; t \geq 0\right)$ is a $((P_x), (\mathcal{F}_t))$ local martingale, then $L\varphi = g\varphi$.

c) Let V_φ be the potential function associated with $\varphi \in \mathcal{D}(L), \varphi > 0$:

$$\frac{1}{2}V_\varphi = \frac{L\varphi}{\varphi}. \tag{2.8}$$

It is clear that the following identities hold :

$$V_{c\varphi} = V_\varphi, \qquad Q^{V+c}_{x,t} = Q^V_{x,t}, \qquad \text{for any} \quad c > 0, \tag{2.9}$$

and that the two probabilities introduced in a) and b) are related via :

$$P^\varphi_{x|\mathcal{F}_t} := \frac{Z_t^{V_\varphi}(x)}{\varphi(x)}\varphi(X_t)Q^{V_\varphi}_{x,t}. \tag{2.10}$$

It goes back to [19] that, under (P_x^φ), (X_t) is a Markov process with extended infinitesimal generator

$$L^\varphi = L + \frac{1}{\varphi}\Gamma(\varphi, \cdot). \tag{2.11}$$

There exists a simple relation between the Markovian laws (P_x^φ) (or, rather the associated semi-group (T_t^φ)) and Z^{V_φ}, namely :

$$\varphi(x)T_t^\varphi(\frac{1}{\varphi})(x) = Z_t^{V_\varphi}(x) = E_x[\exp\{-\frac{1}{2}\int_0^t V_\varphi(X_s)ds\}], \tag{2.12}$$

which follows from (2.7).

1.3 We are interested in finding some conditions on V which ensure the weak convergence, as $t \to \infty$, of $Q^V_{x,t}$. We have two possibilities :

- to a given $\varphi > 0$ in $\mathcal{D}(L)$, we may associate the potential function V_φ defined by (2.8).
- conversely, starting from a potential function V, we may look for the solutions φ of the Poisson equation (which is the Sturm-Liouville equation in the Brownian case) :

$$\frac{L\varphi}{\varphi} = \frac{1}{2}V. \tag{2.13}$$

We shall see later that a particular function φ_V plays a central role in our discussion of the convergence.

1.4 A meta-theorem and its "proof".

The following statement shall be rigorously proved under various hypotheses all throughout our paper. It may be used as a guideline for the reader, and shall be refered to as the generic theorem.

Theorem 2.1 *Assume that, for some $k \geq 0$, one has :*

$$\lim_{t\to\infty}\left(t^k Z_t^V(x)\right) = \lim_{t\to\infty}\left(t^k E_x[\exp\{-\frac{1}{2}\int_0^t V(X_s)ds\}]\right) = \varphi_V(x). \tag{2.14}$$

Then, φ_V is a solution of (2.13), $\left\{\frac{\varphi_V(X_t)}{\varphi_V(x)}\exp\{-\frac{1}{2}\int_0^t V(X_u)du\}; t \geq 0\right\}$ is a $((P_x),(\mathcal{F}_t))$ martingale, and for any $\Lambda_s \in \mathcal{F}_s$,

$$\lim_{t\to\infty} Q_{x,t}^V(\Lambda_s) = P_x^{\varphi_V}(\Lambda_s). \tag{2.15}$$

Proof of Theorem 2.1. Let $s > 0$ be fixed, $\Lambda_s \in \mathcal{F}_s$ and $t > s$. We start from (2.4), and we write :

$$Q_{x,t}^V(\Lambda_s) = \frac{E_x\left[1_{\Lambda_s}\exp\{-\frac{1}{2}\int_0^s V(X_u)du\}E_{X_s}\left[\exp\{-\frac{1}{2}\int_0^{t-s}V(X_h)dh\}\right]\right]}{E_x[\exp\{-\frac{1}{2}\int_0^t V(X_u)du\}]}, \tag{2.16}$$

and we multiply both the numerator and denominator by t^k. The result will follow after some justification for the passage to the limit inside the expectation (for the numerator).

■

The "proof" shows that the normalization function $t \mapsto t^k$ in (2.14) may be replaced by any positive and non-decreasing function λ such that $\lim_{t\to\infty}\left(\frac{\lambda(t+s)}{\lambda(t)}\right) = 1$, and also admits some simple extension for $\lambda(u) = ce^{au}$, say.

1.5 Back to the Brownian framework.

Here, $E = \mathbb{R}, L = \frac{1}{2}\frac{d^2}{dx^2}, \Gamma(f,g)(x) = f'(x)g'(x), P_x = W_x$ is the Wiener measure, and we write B_t instead of X_t since, in this case, $\big((B_t; t \geq 0); (P_x; x \in \mathbb{R})\big)$ is a one-dimensional Brownian motion. Hence, we are interested in the Sturm-Liouville equation :

$$\varphi'' = V\varphi, \tag{2.17}$$

and we have $L^\varphi = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{d}{dx}(\log\varphi)\right)\frac{d}{dx}$, $\mu(dx) = \varphi^2(x)dx$ is invariant under (T_t^φ), the semigroup associated with (P_x^φ), and :

$$< T_t^\varphi f, g >_\mu = < f, T_t^\varphi g >_\mu,$$

$$< L^\varphi f, g >_\mu = < f, L^\varphi g >_\mu = -\frac{1}{2} < f', g' >_\mu .$$

Since Brownian motion admits a (bi-continuous) family of local times $(L_t^x; t \geq 0, x \in \mathbb{R})$, we may define the normalization factor $Z_t^V(x)$ (cf (2.3)) when V is a non-negative Radon measure on $\mathbb{R}$, in the following way :

$$Z_t^V(x) = E_x\left[\exp\left\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\right\}\right]. \tag{2.18}$$

Abusing notation, we use the same letter V, whenever V stands for a function or a Radon measure.

1) We investigate the integrable case, i.e. when $V(dx)$ satisfies :

$$\int_{\mathbb{R}}(1+|x|)V(dx) < \infty. \tag{2.19}$$

In Theorem 4.1, we prove that $\sqrt{t}Z_t^V(x)$ converges as $t \to \infty$ to a real number denoted $\varphi_V(x)$. Moreover φ_V is a convex function which takes its values in $]0,\infty[$ and is the unique solution to the Sturm-Liouville equation (2.17), with boundary conditions :

$$\lim_{x\to+\infty}\varphi_V'(x) = -\lim_{x\to-\infty}\varphi_V'(x) = \sqrt{\frac{2}{\pi}}. \tag{2.20}$$

This leads us to relax the assumption (2.19). We shall discuss whether $V(dx)$ is "large" at infinity or not.

2) Let us examine the first case. Suppose for simplicity that V is a function.
We begin with an intermediate case. We say that V is asymmetric if :

$$\int_{-\infty}^{0}(1+|x|)V(x)dx<\infty, \tag{2.21}$$

and

$$\liminf_{x\to\infty}\big(x^{2\alpha}V(x)\big)>0, \qquad \text{for some } \alpha<1. \tag{2.22}$$

Then (cf Theorem 5.1), the rate of decay of $Z_t^V(x)$ is unchanged : $\sqrt{t}Z_t^V(x)$ converges, as $t\to\infty$, to $\varphi_V(x)$: roughly speaking, (2.21) "dominates" (2.22). Again the function φ_V solves the Sturm-Liouville equation (2.17), but with the new boundary conditions :

$$\lim_{x\to-\infty}\varphi_V'(x)=-\sqrt{\frac{2}{\pi}}\ ; \qquad \lim_{x\to+\infty}\varphi_V(x)=0. \tag{2.23}$$

3) Let us now investigate the case where $V(dx)$ is <u>small at infinity</u> but does not satisfy (2.19). We restrict ourselves to two examples:

$$V(x)=\frac{\lambda}{\theta+x^2}, \quad \text{where} \quad \lambda>0,\ \theta\geq 0. \tag{2.24}$$

and

$$V(x)=\frac{\lambda}{1+|x|^\alpha}, \quad \text{where} \quad \lambda>0,\ 0<\alpha<2. \tag{2.25}$$

Suppose V is given by (2.24). If $\theta=0$ then (cf Theorem 7.1) :

$$\lim_{t\to\infty}\Big(t^nE_x\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t\frac{ds}{B_s^2}\Big\}\Big]\Big)=x^{2n}\frac{1}{2^n}\frac{\Gamma(n+\frac12)}{\Gamma(2n+\frac12)}, \tag{2.26}$$

where $n=\dfrac{1+\sqrt{1+4\lambda}}{4}$.
When $\theta>0$ the result looks like the previous one. Let φ_V be the unique smooth function defined on $[0,+\infty[$, solution of $\varphi"(x)=\lambda\dfrac{1}{1+x^2}\varphi(x);\quad x>0$, such that: $\varphi_V(x)\sim x^{2n}, x\to+\infty$. In Theorem 7.3 we give the explicit form of φ_V and we prove :

$$\lim_{t\to\infty}\Big(t^nE_x\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t\frac{ds}{\theta+B_s^2}\Big\}\Big]\Big)=\theta^n\varphi_V(x/\sqrt{\theta})\frac{1}{2^n}\frac{\Gamma(\mu+n+1)}{\Gamma(\mu+2n+1)}, \tag{2.27}$$

where $\mu=-1/2$.
We observe that if formally we take the limit $\theta\to 0$ in (2.27)we recover (2.26).
Note that $(B_s^2; s\geq 0)$ is a squared Bessel process with dimension 1, which led us to generalize the asymptotic results (2.26) and (2.27) to Bessel processes with any positive dimension (see Theorem 7.1 for $\theta=0$ and Theorem 7.3 when $\theta>0$ and $0<\lambda<8\mu^2+6\mu+1$).
Let us deal with the second case : V is given by (2.25). We only obtain in Theorem 8.1, a logarithmic equivalent for $Z_t^V(x)$:

$$\lim_{t\to\infty}\Big(t^{\frac{\alpha-2}{\alpha+2}}\ln\big(Z_t^V(x)\big)\Big)=-\frac{1}{2}\Theta_0(\lambda), \tag{2.28}$$

where

$$\Theta_0(\lambda)=\inf_{\psi\in\mathcal{C}_0}\Big\{\int_0^1\dot{\psi}^2(s)ds+\lambda\int_0^1\frac{ds}{|\psi(s)|^\alpha}\Big\}, \tag{2.29}$$

belongs to $]0,+\infty[$, and $\mathcal{C}_0$ is the set of continuous functions $f:[0,1]\to\mathbb{R}$ vanishing at 0.

4) If $V(dx)$ is large at $\pm\infty$, the asymptotic behaviour of $Z_x^V(t)$ is drastically different. This case was actually considered by Kac [16] and Titchmarsh [30]. More precisely suppose that V is an even function, non-decreasing on $[0,+\infty[$ and converging to a finite limit at infinity. Then we prove in Theorem 6.1 that there exists $\gamma_0>0$ such that

$$\lim_{x\to\infty}\left(e^{\gamma_0 t/2}Z_t^V(x)\right)=\kappa\psi_V(x), \tag{2.30}$$

where $\kappa>0$ and ψ_V is the positive solution to $\psi"(x)=\psi(x)(V(x)-\gamma_0)$, converging to 0 at infinity and such that $\psi_V'(0)=0$.

5) Theorem 2.1 tells us that as soon as we obtain an explicit behaviour of $Z_t^V(x)$ as t runs to infinity, we may proceed further to define new probability measures.
In the Brownian setting, the probability $Q_{x,t}^V$ is defined on $\mathcal{F}_t$ by :

$$Q_{x,t}(\Lambda_t)=\frac{E_x\Big[1_{\Lambda_t}\exp\Big\{-\frac{1}{2}\int_0^t V(B_v)dv\Big\}\Big]}{E_x\Big[\exp\Big\{-\frac{1}{2}\int_0^t V(B_v)dv\Big\}\Big]},\ t>0,\Lambda_t\in\mathcal{F}_t, \tag{2.31}$$

if V is a function. In the case where $V(dy)$ is a Radon measure, we have :

$$Q_{x,t}(\Lambda_t)=\frac{E_x\Big[1_{\Lambda_t}\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\Big\}\Big]}{E_x\Big[\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\Big\}\Big]},\ t>0,\Lambda_t\in\mathcal{F}_t. \tag{2.32}$$

To describe the probability measure $P_x^{\varphi_V}$, we introduce $(X_t^x;t\geq 0)$, the solution of the stochastic differential equation :

$$X_t=x+B_t+\int_0^t\frac{\varphi_V'}{\varphi_V}(X_s)ds,\ t\geq 0 \tag{2.33}$$

The law of $(X_t^x;t\geq 0)$ is $P_x^{\varphi_V}$.

6) Let us briefly detail the organization of the paper. Section 3 presents some preliminaries.
In section 4 we start with a function φ which is locally the difference of two convex functions, hence $\varphi"$ is a Radon measure. We take : $V_\varphi:=\varphi"/\varphi$. Notice that the sign of V_φ is not constant. With some additional assumptions on φ such as : φ "small at infinity" we prove in Theorem 4.1 that $Z_t^{V_\varphi}(x)$ converges, as $t\to\infty$ to $C\varphi(x)$, where C is a suitable constant.
Section 5 is devoted to the proof of the generic Theorem in the integrable case, namely when $V(dy)$ satisfies $\displaystyle\int_{\mathbb{R}}(1+|y|)V(dy)<\infty$. We develop an analytic approach, and two other ones based respectively on the Ray-Knight theorem and excursions.
The asymmetric case (i.e. when $V(x)$ satisfies (2.21) and (2.22)) is discussed in section 6.
We investigate two critical cases (i.e. V satisfying (2.24, 2.25)) in section 7.
In section 8, using the technique of large deviations we deal with V fullfills (2.25).
We end this paper by considering the case where V is large at $\pm\infty$ in section 9.
The results of this paper were announced without proofs in [27].
7) In a subsequent study [28], we consider a similar problem, replacing the exponential weight $\exp\Big\{-\dfrac{1}{2}\displaystyle\int_0^t V(B_v)dv\Big\}$ by $\varphi(A_t)$, where (A_t) may be equal either to the one-sided maximum : $\sup_{0\leq u\leq t}B_u$, or to the one-sided minimum, or to the local time at 0, or to the number of down-crossings from level b to level a.

3 Preliminaries

2.1 Let $\varphi\ :\mathbb{R}\rightarrow]0,+\infty[$ be a function of class C^2 and μ the measure on $\mathbb{R}$ with density $\varphi^2(x)$ with respect to the Lebesgue measure:

$$\mu(dx)=\varphi^2(x)dx. \tag{3.1}$$

We denote by L^φ the differential operator:

$$L^\varphi f(x)=\frac{1}{2}f''(x)+\frac{\varphi'(x)}{\varphi(x)}f'(x), \tag{3.2}$$

defined for every function f of class C^2.
If f and g are two functions of class C^2, with compact support, then by integration by parts we obtain:

$$<L^\varphi f,g>_\mu=<f,L^\varphi g>_\mu=-\frac{1}{2}<f',g'>_\mu=-\frac{1}{2}\int_{\mathbb{R}}f'(x)g'(x)d\mu(x), \tag{3.3}$$

where $<h,k>_\mu=\displaystyle\int_{\mathbb{R}}h(x)k(x)d\mu(x)$.
The relation (3.3) tells us that L^φ is a negative and symmetric operator, defined on $C^2_K(\mathbb{R})$. Thus, it admits a self-adjoint extension, which is the generator of a Markovian semigroup $(T^\varphi_t;t\geq 0)$ of bounded, positive, symmetric operators on $L^p(\mu))$, $1\leq p\leq\infty$ (cf [7]). The norm of T^φ_t is 1 as an operator on any $L^p(\mu)$.

2.2 Let X^x_t be the solution of the following stochastic differential equation:

$$X_t=x+B_t+\int_0^t\frac{\varphi'}{\varphi}(X_s)ds;\ t\geq 0, \tag{3.4}$$

where $(B_t;t\geq 0)$ is a one-dimensional Brownian motion started at 0.
Since φ is of class C^2 and $\varphi>0$ this stochastic differential equation has a unique strong solution up to an explosion time. We assume that this explosion time is infinite. This occurs for instance if $\dfrac{\varphi'}{\varphi}$ has at most linear growth; for more refined conditions see [21].
Obviously:

$$E[f(X^x_t)]=T^\varphi_t f(x),\ \text{for any}\ f\geq 0, \tag{3.5}$$

and by the Girsanov formula :

$$E_x\Big[f(B_t)\exp\Big\{-\frac{1}{2}\int_0^t\frac{\varphi''}{\varphi}(B_s)ds\Big\}\Big]=\varphi(x)T^\varphi_t(f/\varphi)(x)=\varphi(x)E\Big[\frac{f(X^x_t)}{\varphi(X^x_t)}\Big]. \tag{3.6}$$

In particular choosing $f=1$ we get :

$$Z^{V_\varphi}_t(x)=E_x\Big[\exp\Big\{-\frac{1}{2}\int_0^t\frac{\varphi''}{\varphi}(B_s)ds\Big\}\Big]=\varphi(x)T^\varphi_t(\frac{1}{\varphi})(x)=\varphi(x)E\Big[\frac{1}{\varphi(X^x_t)}\Big]. \tag{3.7}$$

Remark 3.1 *If φ is locally the difference of two convex functions, then it is understood that $\displaystyle\int_0^t\frac{\varphi''}{\varphi}(B_s)ds$ is defined as $\displaystyle\int_{\mathbb{R}}L^x_t\frac{\varphi''(dx)}{\varphi(x)}$.*

4 The case : φ small at infinity

Let $\varphi\ :\mathbb{R}\to]0,+\infty[$ be a function of class C^2. We suppose moreover that φ'/φ is bounded.
We define:

$$V_\varphi(x)=\frac{\varphi''(x)}{\varphi(x)},x\in\mathbb{R}. \tag{4.1}$$

More generally, if φ is locally the difference of two convex functions, we set :

$$V_\varphi(dx)=\frac{\varphi''(dx)}{\varphi(x)},x\in\mathbb{R}. \tag{4.2}$$

In this section, we assume that φ is small at infinity, in the sense that :

$$\int_{\mathbb{R}}\varphi^p(x)dx<\infty,\ \text{for some } 0<p<1, \tag{4.3}$$

and

$$\varphi \text{ is decreasing (resp. increasing) at } +\infty (\text{resp.} -\infty). \tag{4.4}$$

It is clear that the sign of V_φ is not constant.
We note that (4.3) and (4.4) imply that $\int_{\mathbb{R}}\varphi^2(x)dx<\infty$ and the change $\varphi\to\lambda\varphi$, with $\lambda>0$, does not modify V_φ, nor (4.3), nor (4.4).

Theorem 4.1 *We suppose that φ satisfies (4.3), (4.4) and is even, i.e. $\varphi(-x)=\varphi(x),\forall x\in\mathbb{R}$.*

1. *The generic Theorem applies with $k=0$ since :*

$$\lim_{t\to\infty}\Big(T_t^\varphi(1/\varphi)(x)\Big)=\frac{\int_{\mathbb{R}}\varphi(y)dy}{\int_{\mathbb{R}}\varphi^2(y)dy}, \tag{4.5}$$

and

$$\int_{\mathbb{R}}h(x)\varphi^2(x)dx<\infty \qquad \text{where} \qquad h(x)=\sup_{t\ge0}|T_t^\varphi(1/\varphi)(x)|. \tag{4.6}$$

2. *$Q_{x,t}^{V_\varphi}$ converges weakly to P_x^φ, as $t\to\infty$.*
3. *Let $(X_t^x;t\ge0)$ be the solution of the SDE :*

$$X_t=x+B_t+\int_0^t\frac{\varphi'}{\varphi}(X_s)ds,\ t\ge0. \tag{4.7}$$

Then the law of $(X_t^x;t\ge0)$ is P_x^φ.
Moreover $(X_t^x;t\ge0)$ is a recurrent process with finite invariant measure $\mu(dx)=\varphi^2(x)dx$.

Before proving Theorem 4.1, we give five examples numbered from 4.2 to 4.6. For these examples Theorem 4.1 applies because $(T_t^\varphi;t\ge0)$ is an ultracontractive [18] or an hypercontractive semigroup ([22], [14], [23]).

Example 4.2 *Let φ be the function :*

$$\varphi(x) = e^{-\frac{|x|^\alpha}{2}}, x \geq 0,$$

where $\alpha > 2$.
Then φ obeys (4.3) (in fact, for any $p > 0$), (4.4), and

$$V_\varphi(x) = \frac{1}{4}\alpha^2|x|^{2\alpha-2} - \frac{1}{2}\alpha(\alpha-1)|x|^{\alpha-2}, \ x \geq 0.$$

$(T_t^\varphi; t \geq 0)$ is an ultracontractive semigroup (i.e. for any positive t, T_t^φ is a bounded operator from $L^1(\mu)$ to $L^\infty(\mu)$) and this implies directly Theorem 4.1.
More generally, we can take $\varphi(x) = e^{-v(x)/2}$ where $v(x)$ is a convex function for large x and $\int^{+\infty} \frac{1}{v'(x)}dx < \infty$. Theorem 4.1 remains valid since $(T_t^\varphi; t \geq 0)$ is still an ultracontractive semigroup [18].

Example 4.3 *Let $\varphi(x) = e^{-x^2/2}; x \in \mathbb{R}$. Then $V_\varphi(x) = x^2 - 1$ and $(X_t^x; t \geq 0)$ is the Ornstein Uhlenbeck process which solves :*

$$X_t = x + B_t - \int_0^t X_s ds, \ t \geq 0.$$

Notice that $(T_t^\varphi; t \geq 0)$ is an hypercontractive semigroup (cf [22], [14], [23]).

Example 4.4 *Let φ satisfy :*

$$-\Big(\frac{\varphi'}{\varphi}\Big)' = \frac{\varphi'^2 - \varphi\varphi''}{\varphi^2} \geq 2\kappa. \tag{4.8}$$

For every pair of functions f and g of class C^2 with compact support, we recall that :

$$\Gamma^\varphi(f,g) := L^\varphi(fg) - fL^\varphi g - gL^\varphi f,$$

$$\Gamma_2^\varphi(f,g) := L^\varphi(\Gamma(f,g)) - \Gamma(L^\varphi f, g) - \Gamma(f, L^\varphi g).$$

Then ([1]) the operator L^φ enjoys the spectral gap property in $L^2(\mu)$ as soon as

$$\Gamma_2^\varphi(f,f) \geq \kappa\Gamma^\varphi(f,f). \tag{4.9}$$

It is easy to check that (4.8) implies (4.9). Theorem 4.1 follows immediately.

Example 4.5 *Let $a \geq 0$ and φ such that:*

$$\varphi(x) = \begin{cases} e^{-|x|} & \text{if } |x| > a \\ e^{-a}(1 + a - |x|) & \text{otherwise.} \end{cases}$$

Then

$$V_\varphi(dx) = \frac{\varphi''(dx)}{\varphi(x)} = 1_{\{|x|>a\}}dx - \frac{2}{1+a}\delta_0(dx),$$

where $\delta_0(dx)$ denotes the Dirac measure at 0.
Consequently:

$$E_x\Big[\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V_\varphi(dy)\Big\}\Big] = E_x\Big[\exp\Big\{-\frac{1}{2}\int_0^t 1_{\{|B_s|>a\}}ds + \frac{1}{1+a}L_t^0\Big\}\Big]$$

and $(X_t^x; t \geq 0)$ solves:

$$X_t = x + B_t - \int_0^t \mathrm{sgn}(X_s)1_{\{|X_s|>a\}}ds - \int_0^t \frac{\mathrm{sgn}(X_s)}{1+a-|X_s|}1_{\{|X_s|\leq a\}}ds, \quad t \geq 0.$$

Example 4.6 *Let $\varphi(x) = e^{-\lambda|x|}$, with $\lambda > 0$. Then $V_\varphi(dx) = \lambda^2 dx - 2\lambda\delta_0(dx)$,*

$$\lim_{t\to\infty}\left\{\frac{E_x\Big[1_{\Lambda_s}\exp\big\{\lambda L_t^0\big\}\Big]}{E_x\Big[\exp\big\{\lambda L_t^0\big\}\Big]}\right\} = e^{\lambda|x|}E_x\Big[1_{\Lambda_s}\exp\big\{-\lambda|B_s| + \lambda L_s^0 - \frac{\lambda^2 s}{2}\big\}\Big],$$

and $(X_t^x; t\geq 0)$ solves:

$$X_t = x + B_t - \lambda\int_0^t \mathrm{sgn}(X_s)ds.$$

$(X_t^x; t\geq 0)$ is the so-called bang-bang process with parameter $\lambda > 0$ (cf [12], [13], [6]) which satisfies, for $x = 0$:

$$(|X_t^0|; t\geq 0) \overset{(d)}{=} (S_t^{(\lambda)} - B_t^{(\lambda)}; t\geq 0), \tag{4.10}$$

where $B_t^{(\lambda)} = B_t + \lambda t$, $S_t^{(\lambda)} = \sup_{0\leq u\leq t} B_u^{(\lambda)}$.

The proof of Theorem 4.1 is based on two preliminary results which we present in Lemmas 4.7 and 4.8.

Lemma 4.7 *Let $\rho : [0,+\infty[\times\mathbb{R}\to\mathbb{R}$, $\rho(t,x) = T_t^\varphi(1/\varphi)(x)$. Then*

1. *For any $t\geq 0$, $x\to\rho(t,x)$ is even and non-decreasing on $[0,+\infty[$.*
2. *$t\to\rho(t,0)$ is non-decreasing.*

Proof of Lemma 4.7.

i) It is well-known that ρ solves:

$$\begin{cases} \frac{\partial\rho}{\partial t} - \frac{1}{2}\frac{\partial^2\rho}{\partial x^2} - \frac{\varphi'}{\varphi}\frac{\partial\rho}{\partial x} = 0 \\ \rho(0,x) = \frac{1}{\varphi(x)} \end{cases} \tag{4.11}$$

We set : $\theta(t,x) = \dfrac{\partial\rho}{\partial x}(t,x)$.
We take in (4.11) the partial derivative with respect to x :

$$\begin{cases} \frac{\partial\theta}{\partial t} - \frac{1}{2}\frac{\partial^2\theta}{\partial x^2} - \frac{\varphi"\varphi - \varphi'^2}{\varphi^2}\theta - \frac{\varphi'}{\varphi}\frac{\partial\theta}{\partial x} = 0 \\ \theta(0,x) = \frac{\partial}{\partial x}\Big(\frac{1}{\varphi(x)}\Big) \\ \theta(t,0) = 0 \end{cases} \tag{4.12}$$

Since the restriction of φ to $[0,+\infty[$ is non-increasing, $\theta(0,x)\geq 0$ if $x\geq 0$.
It is clear that $\rho(t,.)$ is even, i.e. $\rho(t,-x) = \rho(t,x); \forall x\in\mathbb{R}$, consequently $\theta(t,0) = 0$.
But $w(t,x)\equiv 0$ is a solution to (4.12) on $[0,+\infty[\times[0,+\infty[$; then, the maximum principle implies that $\frac{\partial\rho}{\partial x}(t,x)\geq 0$ if $x\geq 0$. This proves point i).
ii) Since $(X_t^x; t\geq 0)$ is a Markov process and $\rho(t,.)$ is even :

$$\rho(t+s,0) = T_{t+s}^\varphi\big(\frac{1}{\varphi}\big)(0) = E\big[T_t^\varphi\big(\frac{1}{\varphi}\big)(X_s^0)\big] = E\big[\rho(t,|X_s^0|)\big] \geq \rho(t,0).$$

This proves ii). ∎

Lemma 4.8 *Let $h : \mathbb{R} \to \mathbb{R}$ be the function : $h(x) := \sup_{t\geq 0} T_t^\varphi(\frac{1}{\varphi})(x)$, then :*

$$\int_{\mathbb{R}} h(x)\varphi^2(x)dx < \infty. \tag{4.13}$$

Proof of Lemma 4.8 .
Let $x \geq 0$. Due to point 1. of Lemma 4.7, we have:

$$< T_t^\varphi(\frac{1}{\varphi}), 1_{[x,+\infty[} >_\mu \geq \Big(T_t^\varphi(\frac{1}{\varphi})(x)\Big)\mu([x,+\infty[). \tag{4.14}$$

Recall that $\mu(dy) = \varphi^2(y)dy$ and $(T_t^\varphi; t \geq 0)$ is a symmetric semigroup, then

$$< T_t^\varphi(\frac{1}{\varphi}), 1_{[x,+\infty[} >_\mu = < \frac{1}{\varphi}, T_t^\varphi\big(1_{[x,+\infty[}\big) >_\mu = \int_{\mathbb{R}} \frac{T_t^\varphi\big(1_{[x,+\infty[}\big)(y)}{\varphi(y)}\mu(dy).$$

Let $p' = 2 - p > 1$ and q' be the conjugate number $(1/p' + 1/q' = 1)$.
We use Hölder's inequality and the fact that T_t^φ is a bounded operator from $L^{q'}(\mu)$ to itself with norm equal to 1 :

$$T_t^\varphi(\frac{1}{\varphi})(x) \ \leq \tilde{h}(x).$$

where

$$\tilde{h}(x) = C\Big(\mu([x,+\infty[)\Big)^{1/q'-1},$$

and

$$C = \Big(\int_{\mathbb{R}} \frac{1}{\varphi(y)^{p'}}\mu(dy)\Big)^{1/p'} = \Big(\int_{\mathbb{R}} \varphi(y)^p dy\Big)^{1/p'} < \infty,$$

since $2 - p' = p$ and φ satisfies (4.3).
As for (4.13), we have:

$$\frac{1}{C}\int_{\mathbb{R}} \tilde{h}(x)\mu(dx) = \int_{\mathbb{R}} \Big(\mu([x,+\infty[)\Big)^{1/q'-1}\mu(dx) = -q'\Big[\mu([x,+\infty[)^{1/q'}\Big]_{x=-\infty}^{x=+\infty} = q' < \infty.$$

■

Proof of Theorem 4.1.
a) We start with the proof of point 1.
Let us introduce the function $\underline{\theta}$:

$$\underline{\theta}(x) = \underline{\lim}_{t\to\infty}\Big(T_t^\varphi(1/\varphi)(x)\Big).$$

Lemma 4.8 implies that $\underline{\theta}$ is μ-integrable. Moreover:

$$T_s^\varphi(\underline{\theta}) = T_s^\varphi\Big(\liminf_{t\to\infty}\Big\{T_t^\varphi(1/\varphi)\Big\}\Big) \leq \liminf_{t\to\infty}\Big(T_{t+s}^\varphi(1/\varphi)\Big) = \underline{\theta}.$$

Consequently:

$$T_s^\varphi(\underline{\theta}) \leq \underline{\theta}. \tag{4.15}$$

The semigroup $(T_t^\varphi; t \geq 0)$ being μ-symmetric, we have:

$$< T_t^\varphi(\underline{\theta}), 1 >_\mu = < \underline{\theta}, T_t^\varphi(1) >_\mu = < \underline{\theta}, 1 >_\mu = \underline{\theta} \int_{\mathbb{R}} \varphi^2(x) dx.$$

This equality, together with the inequality (4.15) implies that $\underline{\theta} = T_t^\varphi(\underline{\theta})$.
If we take the derivative with respect to t, we obtain: $L^\varphi(\underline{\theta}) = 0$.
Using (3.3) we have:

$$< L^\varphi(\underline{\theta}), \underline{\theta} >_\mu = -\frac{1}{2} \int_{\mathbb{R}} \underline{\theta}'(x)^2 d\mu(x) = 0.$$

Consequently $\underline{\theta} = \underline{C}$, where $\underline{C}$ is a constant.
We introduce:

$$\overline{\theta}(x) = \limsup_{t \to \infty} \Big(T_t^\varphi(1/\varphi)(x) \Big).$$

In the same way as before, we easily check that $\overline{\theta} = \overline{C}$.

b) We now prove that $\overline{\theta} = \underline{\theta} = \dfrac{\int_{\mathbb{R}} \varphi(x) dx}{\int_{\mathbb{R}} \varphi^2(x) dx}$.

Let $(x_n; n \geq 1)$ and $(\varepsilon_n; n \geq 1)$ be two sequences such that $(\varepsilon_n; n \geq 1)$ is positive, decreasing and

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \varepsilon_n = 0.$$

Suppose first that x_n and ε_n are given. Using the definition of $\overline{\theta}$, there exists t_n such that:

$$T_{t_n}^\varphi(1/\varphi)(x_n) \geq \overline{\theta} - \varepsilon_n.$$

Moreover we can choose t_n in such a way that $(t_n; n \geq 1)$ is an increasing sequence converging to $+\infty$ as $n \to \infty$.
Let $x > 0$ fixed. Since $x \to T_t^\varphi(1/\varphi)(x)$ is non-decreasing on $[0, +\infty[$ (cf Lemma 4.7) , if n is large enough:

$$T_{t_n}^\varphi(1/\varphi)(x) \geq T_{t_n}^\varphi(1/\varphi)(x_n).$$

Taking the limsup on both sides we obtain:

$$\lim_{n \to \infty} \Big(T_{t_n}^\varphi(1/\varphi)(x) \Big) = \overline{\theta}.$$

Thanks to Lemma 4.8, we can apply the dominated convergence theorem, hence

$$\lim_{n \to \infty} \Big(< T_{t_n}^\varphi(1/\varphi), 1 >_\mu \Big) = < \overline{\theta}, 1 >_\mu = \overline{\theta} \int_{\mathbb{R}} \varphi^2(x) dx.$$

But recall that since $(T_t^\varphi; t \geq 0)$ is μ-symmetric, then :

$$< T_{t_n}^\varphi(1/\varphi), 1 >_\mu = < \frac{1}{\varphi}, T_{t_n}^\varphi(1) >_\mu = \int_{\mathbb{R}} \varphi(x) dx.$$

Consequently

$$\overline{\theta} = \frac{\int_{\mathbb{R}} \varphi(x) dx}{\int_{\mathbb{R}} \varphi^2(x) dx}.$$

Replacing $(x_n; n \geq 1)$ by $(y_n; n \geq 1)$ such that $y_n < 0$ and

$$\lim_{n \to \infty} y_n = 0,$$

we prove similarly that $\underline{\theta} = \overline{\theta}$.

c) Let $t > 0$. Recall that $Q^{V_\varphi}_{x,t}$ is the probability defined on $\mathcal{F}_t$ by :

$$Q^{V_\varphi}_{x,t}(\Lambda_t) = \frac{E_x\Big[1_{\Lambda_t}\exp\Big\{-\frac{1}{2}\int_0^t V_\varphi(B_r)dr\Big\}\Big]}{E_x\Big[\exp\Big\{-\frac{1}{2}\int_0^t V_\varphi(B_r)dr\Big\}\Big]},\ \Lambda_t \in \mathcal{F}_t. \tag{4.16}$$

Suppose that $s > 0$ is fixed and pick $t > s$; then, replacing in (4.16) Λ_t by $\Lambda_s \in \mathcal{F}_s$, and, using the Markov property at time s together with (3.6), we obtain:

$$Q^{V_\varphi}_{x,t}(\Lambda_s) = \frac{E_x\Big[1_{\Lambda_s}\exp\Big\{-\frac{1}{2}\int_0^s V_\varphi(B_r)dr\Big\}\ \varphi(B_s)T^\varphi_{t-s}(1/\varphi)(B_s)\Big]}{\varphi(x)T^\varphi_t(1/\varphi)(x)}.$$

The numerator can be written as $E_x[1_{\Lambda_s}Y_{s,t}]$, where :

$$Y_{s,t} = \exp\Big\{-\frac{1}{2}\int_0^s V_\varphi(B_r)dr\Big\}\ \varphi(B_s)T^\varphi_{t-s}(1/\varphi)(B_s). \tag{4.17}$$

On one hand, using (4.6), we get an upper bound for $Y_{s,t}$:

$$0 \le Y_{s,t} \le Y_s,\ \text{for any } 0 < s < t$$

where

$$Y_s = \varphi(B_s)h(B_s)\exp\Big\{-\frac{1}{2}\int_0^s V_\varphi(B_r)dr\Big\}.$$

Identity (3.6) tells us that:

$$E_x[Y_s] = \varphi(x)T^\varphi_s h(x) < \infty.$$

On the other hand, (4.5) and (4.6) imply that

$$\lim_{t\to\infty} Y_{s,t} = \lambda\varphi(B_s)1_{\Lambda_s}\exp\Big\{-\frac{1}{2}\int_0^s V_\varphi(B_r)dr\Big\},$$

$$\lim_{t\to\infty} T^\varphi_t(1/\varphi)(x) = \lambda, \quad \text{where} \quad \lambda = \frac{\int_{\mathbb{R}}\varphi(x)dx}{\int_{\mathbb{R}}\varphi^2(x)dx}.$$

Then, for any $s > 0$, $Q^{V_\varphi}_{x,t}(\Lambda_s)$ converges to $P^\varphi_x(\Lambda_s)$ as $t \to \infty$, where P^φ_x is the probability defined on $\mathcal{F}_\infty$ by :

$$P^\varphi_x(\Lambda_s) = \frac{1}{\varphi(x)}\ E_x\Big[1_{\Lambda_s}\varphi(B_s)\exp\Big\{-\frac{1}{2}\int_0^s V_\varphi(B_s)ds\Big\}\Big],$$

for $s > 0$ given and any Λ_s in $\mathcal{F}_s$.
Point iii) of Theorem 4.1 is a direct consequence of Girsanov formula.
Moreover $(X^x_t; t \ge 0)$ is recurrent since μ is its invariant measure. ■

5 The integrable case

Throughout this section, $V(dx)$ shall always denote a finite positive Radon measure on $\mathbb{R}$, different from 0, with finite first moment; hence :

$$\int_{\mathbb{R}} (1+|x|)V(dx) < \infty. \tag{5.1}$$

Recall that in the previous section the initial data was the function φ, whereas now the data is the potential V.

Theorem 5.1 *Let $V(dx)$ be a finite positive Radon measure on $\mathbb{R}$ fulfilling (5.1).*

1. *The generic Theorem applies with $k = 1/2$.*
2. *φ_V is a convex function which takes its values in $]0,\infty[$ and is the unique solution to the Sturm-Liouville equation*

$$\varphi"(dx) = \varphi(x)V(dx), \tag{5.2}$$

with boundary conditions :

$$\lim_{x\to+\infty} \varphi_V'(x) = -\lim_{x\to-\infty} \varphi_V'(x) = \sqrt{\frac{2}{\pi}}. \tag{5.3}$$

As a consequence

$$\varphi_V(x) \sim_{|x|\to\infty} \sqrt{\frac{2}{\pi}}|x| \tag{5.4}$$

and

$$\varphi_V(x) \le C(1+|x|). \tag{5.5}$$

3. *Let M^{φ_V} be the process:*

$$M^{\varphi_V}(s) = \varphi_V(B_s)\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} L_s^y V(dy)\Big\}, s \ge 0. \tag{5.6}$$

Then $(M^{\varphi_V}(s); s\ge 0)$ is a martingale such that $E[(M^{\varphi_V}(s))^2] < \infty$ for any $s \ge 0$ (recall that $V_{\varphi_V} = V$).

4. *Let $(X_t^x; t\ge 0)$ be the solution to :*

$$X_t = x + B_t + \int_0^t \frac{\varphi_V'}{\varphi_V}(X_s)ds,\ t\ge 0. \tag{5.7}$$

Then the law of $(X_t^x; t \ge 0)$ is $P_x^{\varphi_V}$.

5. *The process $(X_t^x; t\ge 0)$ is transient. More precisely, denoting*

$$\rho = \int_{\mathbb{R}} \frac{dy}{\varphi_V^2(y)} < \infty,$$

then :

$$P\Big(\lim_{t\to\infty} X_t^x = -\infty\Big) = \frac{1}{\rho}\int_x^{+\infty} \frac{dy}{\varphi_V^2(y)}, \tag{5.8}$$

$$P\Big(\lim_{t\to\infty} X_t^x = +\infty\Big) = \frac{1}{\rho}\int_{-\infty}^{x} \frac{dy}{\varphi_V^2(y)}. \tag{5.9}$$

Remark 5.2 *Theorem 5.1 may be generalized replacing the Brownian motion* $(B_t; t \geq 0)$ *by a Bessel process* $(R_t; t \geq 0)$ *of dimension* $0 < d < 2$. *In this case, the generic Theorem applies with a function* V *with compact support and* $k = 1 - \frac{d}{2}$. *On the other hand, we have not been able to settle the case* $d = 2$.

We actually develop two proofs of Theorem 5.1. The first one is based on the study of the function $t \mapsto Z_t^V(x) := E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\big\}\Big]$. The second one relies upon the excursion theory and the Ray-Knight theorem which describes the distribution of $(L_S^y; y \in \mathbb{R})$, where S is an exponential r.v. independent of the Brownian motion.

5.1 An analytical approach

Let us briefly describe our first proof of Theorem 5.1. The crucial point is an a priori inequality concerning $Z_t^V(x)$ stated in Lemma 5.3 below. To demonstrate that $\sqrt{t}\ Z_t^V(x)$ has a limit, when $t \to \infty$ we prove that the normalized Laplace transform $A(\lambda, x)$ of $Z_t^V(x)$ converges as $\lambda \to 0$ (cf Lemma 5.4).This can be done (Lemma 5.5) through properties involving $A(\lambda, x)$ and its derivatives.

Lemma 5.3 *Let* $V(dy) \neq 0$ *be a positive Radon measure on the whole line, satisfying (5.1). Then there exists a constant* C *such that:*

$$\sqrt{1+t}E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\big\}\Big] \leq C(1+|x|),\ t \geq 0,\ x \in \mathbb{R}. \tag{5.10}$$

Proof.

1) We start with $V(dy) = \gamma\delta_x(dy)$, where $\delta_x(dy)$ denotes the Dirac measure at x. We claim that:

$$E_0\big[\exp\{-\gamma L_t^x\}\big] \leq \sqrt{\frac{2}{\pi t}}\big(|x| + \frac{1}{\gamma}\big),\ x \in \mathbb{R}, \gamma > 0, t \geq 0. \tag{5.11}$$

Observing that L_t^x is distributed as $(L_t - |x|)_+$ (cf [26]), then

$$\begin{aligned} E_0[\exp -\gamma L_t^x] = E_0[\exp -\gamma(L_t - |x|)_+)] &= P(L_t \leq |x|) + \sqrt{\frac{2}{\pi t}}\int_{|x|}^{\infty} e^{-\gamma(y-|x|)}e^{\frac{-y^2}{2t}}dy, \\ &= P(|B_1| \leq \frac{|x|}{\sqrt{t}}) + \sqrt{\frac{2}{\pi}}e^{\gamma|x|+\gamma^2 t/2}\int_{\frac{|x|}{\sqrt{t}}+\gamma\sqrt{t}}^{\infty} e^{\frac{-z^2}{2}}dz, \\ &\leq \sqrt{\frac{2}{\pi}}\frac{|x|}{\sqrt{t}} + \sqrt{\frac{2}{\pi}}e^{\gamma|x|+\gamma^2 t/2}e^{-\frac{1}{2}(\frac{|x|}{\sqrt{t}}+\gamma\sqrt{t})^2}\frac{1}{\frac{|x|}{\sqrt{t}}+\gamma\sqrt{t}} \\ &\leq \sqrt{\frac{2}{\pi}}\frac{|x|}{\sqrt{t}} + \sqrt{\frac{2}{\pi}}\frac{1}{\frac{|x|}{\sqrt{t}}+\gamma\sqrt{t}}e^{-\frac{x^2}{2t}} \leq \sqrt{\frac{2}{\pi t}}(|x| + \frac{1}{\gamma}). \end{aligned}$$

2) Let $V(dy) \neq 0$ be a positive Radon measure on $\mathbb{R}$. We choose a and b such that $a < b$ and $\mu = V([a,b])/2 > 0$. We have:

$$\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\big\} \leq \exp\big\{-\frac{1}{2}\int_a^b L_t^y V(dy)\big\}.$$

Since $x \mapsto e^{-\mu x}$ is convex:

$$\exp\big\{-\frac{1}{2}\int_a^b L_t^y V(dy)\big\} \leq \frac{1}{2\mu}\int_a^b \exp\{-\mu L_t^y\}\ V(dy),$$

Taking the expectation and applying (5.11), we obtain

$$E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\big\}\Big] \le \frac{C_1}{\sqrt{1+t}}\int_a^b (|x-y|+\frac{1}{\mu})V(dy).$$

Then (5.10) follows. ■

Lemma 5.4 *Let $V(dx)$ be a finite positive Radon measure as in Theorem 5.1 and A the Laplace transform:*

$$A(\lambda,x) = \int_0^\infty e^{-\lambda t} Z_t^V(x)dt, \tag{5.12}$$

where

$$Z_t^V(x) = E_x\Big[\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\Big\}\Big]. \tag{5.13}$$

Then $\lim_{t\to\infty}\sqrt{t}Z_t^V(x) = \varphi_V(x)$ *if and only if*

$$\lim_{\lambda\to 0}\sqrt{2\lambda}A(\lambda,x) = \sqrt{2\pi}\varphi_V(x). \tag{5.14}$$

Proof. We set

$$\tilde{A}(\lambda,x) = \sqrt{2\lambda}A(\lambda,x). \tag{5.15}$$

The inequality (5.10) implies that

$$\tilde{A}(\lambda,x) \le \kappa(1+|x|), \text{for all } \lambda \ge 0. \tag{5.16}$$

Since $t\mapsto Z_t^V(x)$ is a decreasing function, a classical version of the Tauberian theorem (cf [11], Chap. XIII, section 5) implies that $\lim_{t\to\infty}\sqrt{t}Z_t^V(x) = \varphi_V(x)$ if and only if (5.14) holds. ■

Lemma 5.4 leads us to investigate the asymptotic properties of $\tilde{A}(\lambda,x)$, as $\lambda\to 0$.

Lemma 5.5 *Let $V(dx)$ be a finite positive Radon measure as in Theorem 5.1 and $\tilde{A}$ be the function defined by (5.15).*

1. *The measure $(\tilde{A})"(\lambda,dx) - \tilde{A}(\lambda,x)V(dx)$ admits a density function $\theta(\lambda,x)$ with respect to Lebesgue measure and*

$$\lim_{\lambda\to 0}\Big(\sup_{x\in\mathbb{R}}|\theta(\lambda,x)|\Big) = 0. \tag{5.17}$$

   ($(\tilde{A})'(\lambda,x)$ denotes the first x-derivative of $\tilde{A}(\lambda,\cdot)$ and $(\tilde{A})"(\lambda,dx)$ the second one, in the sense of distributions).

2. *The x-derivative of $\tilde{A}(\lambda,x)$ is bounded:*

$$\sup_{x\in\mathbb{R},\lambda\ge 0}|\tilde{A}'(\lambda,x)| < \infty. \tag{5.18}$$

3. *We have:*

$$\lim_{\lambda\to 0, x\to\pm\infty}(\tilde{A})'(\lambda,x) = \pm 2. \tag{5.19}$$

Proof. a) It is well known that the function $(t,x)\mapsto Z_t^V(x)$ is a solution in the distribution sense to:

$$\begin{cases} \frac{\partial Z}{\partial t}-\frac{1}{2}\frac{\partial^2 Z}{\partial x^2}+\frac{1}{2}VZ=0 \\ \\ Z_0(x)=1, \end{cases} \tag{5.20}$$

and that Z can be expressed through the Brownian motion semigroup $(P_t(x,dy)=p_t(x,y)dy; t\geq 0)$:

$$Z_t^V(x)=P_t(1)-\frac{1}{2}\int_0^t ds\int_{\mathbb{R}} p_{t-s}(x,y)Z_s^V(y)V(dy),$$

$$=1-\frac{1}{2}\int_0^t ds\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{(x-y)^2}{2(t-s)}}Z_s^V(y)V(dy). \tag{5.21}$$

We take the Laplace transform in time on both sides; this yields to

$$A(\lambda,x)=\frac{1}{\lambda}-\frac{1}{2}\int_{\mathbb{R}}A(\lambda,y)V(dy)\Big(\int_0^\infty\frac{1}{\sqrt{2\pi v}}e^{-\frac{(x-y)^2}{2v}}e^{-\lambda v}dv\Big). \tag{5.22}$$

Recall that

$$\int_0^\infty\frac{ds}{\sqrt{s}}\exp\{-\frac{\gamma s}{2}-\frac{a^2}{2s}\}=\sqrt{\frac{2\pi}{\gamma}}e^{-|a|\sqrt{\gamma}};\ \gamma>0, a\in\mathbb{R}.$$

Hence

$$\tilde{A}(\lambda,x)=\sqrt{2\lambda}A(\lambda,x)=\frac{1}{\sqrt{2\lambda}}\big[2-\frac{1}{2}\int_{\mathbb{R}}\tilde{A}(\lambda,y)e^{-|x-y|\sqrt{2\lambda}}V(dy)\big], \tag{5.23}$$

$$\sqrt{2\lambda}\tilde{A}(\lambda,x)=2-\frac{1}{2}\int_{\mathbb{R}}\tilde{A}(\lambda,y)e^{-|x-y|\sqrt{2\lambda}}V(dy). \tag{5.24}$$

Using (5.1), (5.16) and (5.24) we obtain :

$$\lim_{\lambda\to 0}\Big(\int_{\mathbb{R}}\tilde{A}(\lambda,y)e^{-|x-y|\sqrt{2\lambda}}V(dy)\Big)=\lim_{\lambda\to 0}\Big(\int_{\mathbb{R}}\tilde{A}(\lambda,y)V(dy)\Big)=4. \tag{5.25}$$

b) Let h be a smooth function with compact support. As we multiply both sides of (5.23) by $h"(x)$, and integrate with respect to dx, we obtain:

$$(\tilde{A})"(\lambda,h):=\int_{\mathbb{R}}\tilde{A}(\lambda,x)h"(x)dx=-\frac{1}{2\sqrt{2\lambda}}\int_{\mathbb{R}^2}\tilde{A}(\lambda,y)e^{-|x-y|\sqrt{2\lambda}}h"(x)dxV(dy), \tag{5.26}$$

where $(\tilde{A})"(\lambda,dx)$ denotes the second derivative in the distribution sense of $\tilde{A}(\lambda,x)$ with respect to the x variable.
Let $U^\lambda(g)$ be the Brownian λ-potential of the function g :

$$U^\lambda(g)(x)=E_x\big[\int_0^\infty g(B_s)e^{-\lambda s}ds\big]=\frac{1}{\sqrt{2\lambda}}\int_{\mathbb{R}}e^{-|x-y|\sqrt{2\lambda}}g(y)dy.$$

Since $U^\lambda(g)$ solves (cf [17]):

$$U^\lambda(g")(x)=(U^\lambda g)"(x)=-2g(x)+2\lambda U^\lambda(g)(x),$$

then

$$(\tilde{A})"(\lambda,h)-\int_{\mathbb{R}}\tilde{A}(\lambda,y)h(y)V(dy)=-\lambda\int_{\mathbb{R}}\tilde{A}(\lambda,y)U^\lambda(h)(y)V(dy). \tag{5.27}$$

This implies that the distribution $(\tilde{A})"(\lambda, dy) - \tilde{A}(\lambda, y)V(dy)$ is a measure and

$$(\tilde{A})"(\lambda, dy) - \tilde{A}(\lambda, y)V(dy) = \theta(\lambda, y)dy,$$

where:

$$\theta(\lambda, y) = -\sqrt{\frac{\lambda}{2}} \int_{\mathbb{R}} \tilde{A}(\lambda, x) e^{-|x-y|\sqrt{2\lambda}} V(dx). \tag{5.28}$$

Using the inequalities (5.16) and (5.1), we obtain :

$$\sup_{x\in\mathbb{R}} |\theta(\lambda, x)| \leq \sqrt{\frac{\lambda}{2}} \int_{\mathbb{R}} \tilde{A}(\lambda, x)V(dx) \leq k_1\sqrt{\lambda}. \tag{5.29}$$

This proves part 1. of Lemma 5.5 .
c) Obviously, (5.23) can be written as follows:

$$\tilde{A}(\lambda, x) = \frac{1}{\sqrt{2\lambda}}\Big[2 - \frac{1}{2}\Big(e^{-x\sqrt{2\lambda}} \int_{]-\infty,x]} \tilde{A}(\lambda, y)e^{y\sqrt{2\lambda}}V(dy)$$

$$+e^{x\sqrt{2\lambda}} \int_{]x,+\infty[} \tilde{A}(\lambda, y)e^{-y\sqrt{2\lambda}}V(dy)\Big)\Big].$$

Taking the derivatives on both sides with respect to x, we get

$$(\tilde{A})'(\lambda, x) = \frac{1}{2}\int_{]-\infty,x]} \tilde{A}(\lambda, y)e^{-|x-y|\sqrt{2\lambda}}V(dy) - \frac{1}{2}\int_{]x,+\infty[} \tilde{A}(\lambda, y)e^{-|x-y|\sqrt{2\lambda}}V(dy). \tag{5.30}$$

Consequently

$$\sup_{x\in\mathbb{R}} |(\tilde{A})'(\lambda, x)| \leq \int_{\mathbb{R}} \tilde{A}(\lambda, y)V(dy) \leq k_2.$$

d) Due to (5.30), we have :

$$(\tilde{A})'(\lambda, x) = \int_{]-\infty,x]} \tilde{A}(\lambda, y)e^{-|x-y|\sqrt{2\lambda}}V(dy) - \frac{1}{2}\int_{\mathbb{R}} \tilde{A}(\lambda, y)e^{-|x-y|\sqrt{2\lambda}}V(dy). \tag{5.31}$$

Since (5.25) and (5.16) hold,

$$\lim_{\lambda\to 0}\Big(- \frac{1}{2}\int_{\mathbb{R}} \tilde{A}(\lambda, y)e^{-|x-y|\sqrt{2\lambda}}V(dy)\Big) = -2, \tag{5.32}$$

and

$$|\int_{]-\infty,x]} \tilde{A}(\lambda, y)e^{-|x-y|\sqrt{2\lambda}}V(dy)| \leq \int_{]-\infty,x]} \kappa(1 + |y|)V(dy).$$

As a result, $\displaystyle\int_{]-\infty,x]} \tilde{A}(\lambda, y)e^{-|x-y|\sqrt{2\lambda}}V(dy)$ goes to 0 as $x \to -\infty$, uniformly with respect to $\lambda \geq 0$.
Consequently $(\tilde{A})'(\lambda, x)$ converges to -2, as $x \to -\infty, \lambda \to 0$.
In the same way,

$$(\tilde{A})'(\lambda, x) = -\int_{]x,+\infty[} \tilde{A}(\lambda, y)e^{-|x-y|\sqrt{2\lambda}}V(dy) + \frac{1}{2}\int_{\mathbb{R}} \tilde{A}(\lambda, y)e^{-|x-y|\sqrt{2\lambda}}V(dy),$$

$$\lim_{\lambda\to 0, x\to +\infty} (\tilde{A})'(\lambda, x) = 2.$$

This ends the proof of Lemma 5.5. ■

Remark 5.6 *If $V(dx)$ has compact support, say supp($V(dx)$)) $\subset [a,b]$, it is easy to check directly:*

$$\lim_{\lambda\to 0}(\tilde{A})'(\lambda, y) = -\lim_{\lambda\to 0}(\tilde{A})'(\lambda, x) = 2, \text{ for any } x \leq a \text{ and } y \geq b. \tag{5.33}$$

Proof of Remark 5.6.
Let $x \leq a$ and T_a be the stopping time : $T_a = \inf\{t \geq 0; B_t > a\}$.
We have:

$$\begin{aligned} E_x\Big[\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\Big\}\Big] &= E_x\Big[\exp\Big\{-\frac{1}{2}\int_a^b L_t^y V(dy)\Big\}\ 1_{\{T_a>t\}}\Big] \\ &\quad + E_x\Big[\exp\Big\{-\frac{1}{2}\int_a^b L_t^y V(dy)\Big\}\ 1_{\{T_a\leq t\}}\Big], \\ &= P_x(T_a > t) + E_x\Big[\exp\Big\{-\frac{1}{2}\int_a^b L_t^y V(dy)\Big\}\ 1_{\{T_a\leq t\}}\Big]. \end{aligned}$$

Using the strong Markov property at time T_a, and $P_x(T_a \in ds) = \dfrac{|x-a|}{\sqrt{2\pi s^3}} e^{-\frac{(x-a)^2}{2s}}\ 1_{\{s>0\}}ds$ we obtain :

$$E_x\Big[\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\Big\}\Big] = \int_t^\infty \frac{|x-a|}{\sqrt{2\pi s^3}} e^{-\frac{(x-a)^2}{2s}}\ ds + \int_0^t \frac{|x-a|}{\sqrt{2\pi s^3}} e^{-\frac{(x-a)^2}{2s}}\ Z_{t-s}^V(a)ds. \tag{5.34}$$

We take the Laplace transform on both sides with respect to time :

$$A(\lambda, x) = \frac{1}{\lambda}\big(1 - e^{(x-a)\sqrt{2\lambda}}\big) + A(\lambda, a)e^{(x-a)\sqrt{2\lambda}}, x \leq a.$$

Then

$$\begin{aligned} \frac{A(\lambda,x) - A(\lambda,a)}{x-a} &= \frac{1 - e^{(x-a)\sqrt{2\lambda}}}{x-a}\ \frac{1-\lambda A(\lambda,a)}{\lambda}, x < a, \\ (\tilde{A})'(\lambda, a) &= \sqrt{2\lambda}A'(\lambda,a) = -2 + 2\lambda A(\lambda,a),\ x < a. \end{aligned}$$

This proves $\lim_{\lambda\to 0}(\tilde{A})'(\lambda, x) = -2$ for any $x \leq a$. If $x \geq b$, we prove by the same way that $\lim_{\lambda\to 0}(\tilde{A})'(\lambda, x) = 2$. ■

Proof of Theorem 5.1.

1. The Itô-Tanaka formula tells us that :

   $$M^{\varphi}(s) = \varphi(B_s)\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} L_s^y V_\varphi(dy)\Big\}, s \geq 0$$

   is a continuous local martingale.
   As a consequence of (5.5):

   $$M^{\varphi_V}(s) \leq k(1 + \sup_{0\leq u\leq s}|B_u|).$$

   Consequently, there exists $\gamma > 0$, such that:

   $$E\Big[\exp\big\{\gamma\big(\sup_{0\leq u\leq s} M^{\varphi_V}(u)\big)^2\big\}\Big] < \infty.$$

   A fortiori $(M^{\varphi_V}(s); s \geq 0)$ is a continuous martingale such that $E[M^{\varphi_V}(s)^2] < \infty$.

2. The function $\tilde{A}(\lambda,.)$ solves the following ordinary differential equation, depending on the parameter $\lambda > 0$:

$$\begin{cases} (\tilde{A})"(\lambda, dx) - \tilde{A}(\lambda,.)V(dx) = \theta(\lambda, x)dx \\ \lim_{x\to\pm\infty} \Big((\tilde{A})'(\lambda, x)\Big) = \pm 2 + o(\lambda). \end{cases}$$

   Property (5.17) implies that $\dfrac{\tilde{A}(\lambda, x)}{\sqrt{2\pi}}$ converges, as $\lambda \to 0$, to a function φ_V, solution to (5.2), and (5.3). We draw from this three conclusions :

   (a) φ_V is a non-negative function, being a limit of non-negative functions. Since φ_V solves (5.2), φ_V is a convex function.

   (b) Lemma 5.4 implies that $\sqrt{t}Z_t^V(x)$ converges to $\varphi_V(x)$, as $t \to \infty$.

   (c) From (5.25), we have : $\sqrt{2\pi}\displaystyle\int_{\mathbb{R}} \varphi_V(y)V(dy) = 4.$

3. We claim that φ_V is strictly positive. Indeed, (5.2) implies that $\varphi_V \not\equiv 0$. As a result $\varphi_V(B_t) \geq 0$, and $\varphi_V(B_t)$ is not a.s. equal to 0. But $(M^{\varphi_V}(s); s \geq 0)$ is a martingale, then

$$\varphi_V(x) = E_x\Big[\varphi_V(B_t)\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\Big\}\Big] > 0.$$

4. The proof of the convergence of $Q_{x,t}(\Lambda_s)$ to $Q_x(\Lambda_s)$ is similar to the one given in section 4.

5. Point 5 is a direct consequence of the Girsanov theorem (cf section 4).

6. The integral $\displaystyle\int_{\mathbb{R}} \frac{dy}{\varphi_V^2(y)}$ is finite because $\varphi_V(y)$ is equivalent to $k|y|$, as $|y| \to \infty$.

   We remark that $\beta(x) = \displaystyle\int_0^x \frac{dy}{\varphi_V^2(y)}$ is a scale function for the diffusion process defined by (5.7) ([17], Chap. 5, section 5). Indeed

$$L^{\varphi_V}(\beta) = \frac{1}{2}\beta" + \frac{\varphi_V'}{\varphi_V}\beta' = \frac{1}{2}\Big(-\frac{2\varphi_V'}{\varphi_V^3}\Big) + \frac{\varphi_V'}{\varphi_V}\Big(\frac{1}{\varphi_V^2}\Big) = 0. \qquad \blacksquare$$

Example 5.7 *Let $V(dx) = \gamma^2 1_{[a,b]}(x)dx$, where $a < b$. Then*

$$\lim_{t\to\infty}\Big(\sqrt{t}\ E_x\Big[\exp\Big\{-\frac{\gamma^2}{2}\int_0^t 1_{[a,b]}(B_s)ds\Big\}\Big]\Big) = \varphi_V(x),$$

with

$$\varphi_V(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\Big(\frac{1}{\gamma\tanh(\gamma\frac{b-a}{2})} + x - b\Big) & \text{if} \quad x > b \\ \sqrt{\frac{2}{\pi}}\Big(\frac{\cosh(\gamma[x-\frac{a+b}{2}])}{\gamma\sinh(\gamma\frac{b-a}{2})}\Big) & \text{if} \quad x \in [a,b] \\ \sqrt{\frac{2}{\pi}}\Big(\frac{1}{\gamma\tanh(\gamma\frac{b-a}{2})} + a - x\Big) & \text{if} \quad x < a. \end{cases}$$

Example 5.8 *Let $V(dx) = \gamma^2(\delta_a(dx) + \delta_b(dx))$, where $a \leq b$. Then*

$$\lim_{t\to\infty}\Big(\sqrt{t}\ E_x\Big[\exp\Big\{-\frac{\gamma^2}{2}(L_t^a + L_t^b)\Big\}\Big]\Big) = \varphi_V(x),$$

with

$$\varphi_V(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\Big(\frac{1}{\gamma^2} + x - b\Big) & \text{if} \quad x > b \\ \sqrt{\frac{2}{\pi}}\frac{1}{\gamma^2} & \text{if} \quad x \in [a,b] \\ \sqrt{\frac{2}{\pi}}\Big(\frac{1}{\gamma^2} + a - x\Big) & \text{if} \quad x < a. \end{cases}$$

In particular if $a = b$,

$$\lim_{t\to\infty}\Big(\sqrt{t}\ E_x\Big[\exp\big\{-\gamma^2 L_t^a\big\}\Big]\Big) = \sqrt{\frac{2}{\pi}}\Big(\frac{1}{\gamma^2}+|x-a|\Big),$$

and the process $(X_t^x; t\geq 0)$ defined by (5.7) solves:

$$X_t = x + B_t + \gamma^2\int_0^t \frac{\operatorname{sgn}(X_s-a)}{1+\gamma^2|X_s-a|}ds.$$

5.2 The Ray-Knight theorem and the excursion theory viewpoints

1) Our first approach is based on the Ray-Knight theorem which gives the law of $(L_S^y; y\in\mathbb{R})$, for S an exponential r.v. independent of the underlying Brownian motion $(B_t; t\geq 0)$. We also use the explicit expressions of Laplace transforms of certain Bessel quadratic functionals in terms of solutions to certain Sturm-Liouville equations. For the convenience of the reader we present the relevant material from [2], [25], [26] without proofs, thus making our exposition self-contained.

Definition 5.9 *1. Let $f:\ \mathbb{R}\mapsto\mathbb{R}$ and v be a positive measure on $\mathbb{R}$. We denote by $< f, v >$ or $v(f)$ the integral of f with respect to v, namely :*

$$< f, v >= v(f) = \int_{\mathbb{R}} f(t)v(dt).$$

We set : $v_+(dx) = 1_{\{x>0\}}v(dx)$ and $v_-(dx)$ the image of $1_{\{x<0\}}v(dx)$ by the map $x\mapsto -x$.

2. Let $\delta\geq 0$. We define $Q_x^{(\delta)}$, the distribution of the square of the δ-dimensional Bessel process, started at x.

We now present some important properties of the family $(Q_x^{(\delta)})$

Proposition 5.10 *1. The family of probability measures $(Q_x^{(\delta)}; \delta, x\geq 0)$ obeys the additivity property :*

$$Q_x^{(\delta)}\star Q_{x'}^{(\delta')} = Q_{x+x'}^{(\delta+\delta')}, \qquad \delta,\delta',x,x'\geq 0. \tag{5.35}$$

2. If $\lambda(ds)$ is a positive Radon measure on $\mathbb{R}_+$, with finite first moment, then [25]:

$$Q_x^{(\delta)}\big[\exp\big\{-\int_0^{+\infty} Y(s)\lambda(ds)\big\}\big] = Q_x^{(\delta)}\big[\exp\big\{-< Y,\lambda >\big\}\big] = \exp\big\{-xM(\lambda)-\delta N(\lambda)\big\}, \tag{5.36}$$

where $x\geq 0$, $(Y(s); s\geq 0)$ denotes the canonical process on $\mathcal{C}(\mathbb{R}_+)$ $(Y(s)(\omega)=\omega(s))$, M and N are the two positive σ-finite measures on $\mathcal{C}(\mathbb{R}_+)$ ([25],[24]) which allow to express the Lévy-Khintchine representation of any $Q_x^{(\delta)}$, i.e. one has :

$$M(\lambda) = \int(1-e^{-<\lambda,\omega>})M(d\omega), \quad N(\lambda) = \int(1-e^{-<\lambda,\omega>})N(d\omega).$$

3. Introducing ϕ_λ the unique solution of :

$$\frac{1}{2}\phi" = \lambda\phi \quad on \quad (0,\infty); \quad \phi(0)=1,\ 0\leq\phi\leq 1, \tag{5.37}$$

we have [25, Theorem (2.1)]:

$$Q_x^{(\delta)}\big[\exp\big\{-\int_0^{+\infty} Y(s)\lambda(ds)\big\}\big] = \big(\phi_\lambda(\infty)\big)^{\delta/2}\exp\big\{\frac{x}{2}\phi_\lambda'(0)\big\}, \tag{5.38}$$

where $\phi_\lambda(\infty)$ and $\phi'_\lambda(0)$ are respectively the limit at ∞, and the right derivative at 0 of ϕ_λ. Comparing (5.36) and (5.38), we have :

$$M(\lambda) = -\frac{1}{2}\phi'_\lambda(0), \qquad N(\lambda) = -\frac{1}{2}\ln(\phi_\lambda(\infty)). \tag{5.39}$$

Our approach is based on the knowledge of the law of $(L^y_S; y \in \mathbb{R})$, for S an exponential r.v. independent of $(B_t; t \geq 0)$. This distribution is given in Proposition 5.12 below, through the family of measures $\big(P^{(\theta)}_{a,l}; a \in \mathbb{R}, l > 0\big)$ defined in the next Definition 5.11.

Definition 5.11 *Let $l \geq 0$ and $a > 0$. We define $P^{(\theta)}_{a,l}$ to be the unique probability measure on $\mathcal{C}(\mathbb{R})$ such that :*

1. $Y_0 = l$,
2. $(Y_{-t}; t \geq 0)$ *is the diffusion process with infinitesimal generator :* $2y\frac{d^2}{dy^2} - 2\theta y \frac{d}{dy}$,
3. $(Y_t; 0 \leq t \leq a)$ *is the diffusion process with infinitesimal generator :* $2y\frac{d^2}{dy^2} - 2\theta y \frac{d}{dy} + 2\frac{d}{dy}$,
4. $(Y_t; t \geq a)$ *is the diffusion process with infinitesimal generator :* $2y\frac{d^2}{dy^2} - 2\theta y \frac{d}{dy}$,

When $a < 0$, $(Y_t; t \in \mathbb{R})$ is distributed under $P^{(\theta)}_{a,l}$ as $(Y_{-t}; t \in \mathbb{R})$ under $P^{(\theta)}_{-a,l}$.

Proposition 5.12 *Suppose that $V(dy)$ is a positive measure on $\mathbb{R}$ satisfying (5.1), S_θ is an exponential r.v.with parameter $\theta^2/2$ (i.e. with expectation $2/\theta^2$) and independent of $(B_t; t \geq 0)$. Then (cf [2, theorem 1]):*

$$E_0\Big[\exp -\{\int_{\mathbb{R}} L^y_{S_\theta} V(dy)\}\Big] = \theta \int_0^\infty e^{-\theta l}\Big(\frac{1}{2}\int_{\mathbb{R}} \theta e^{-\theta|a|} P^{(\theta)}_{a,l}\big[e^{-<Y,V>}\big] da\Big) dl. \tag{5.40}$$

We are now able to state the main result of this subsection.

Proposition 5.13 *Suppose that the positive measure $V(dy)$ has compact support.*

1. *Then :*

$$\lim_{\theta \to 0}\Big(\frac{1}{\theta} E_0\Big[\exp -\{\int_{\mathbb{R}} L^y_{S_\theta} V(dy)\}\Big]\Big) = H(V), \tag{5.41}$$

$$\lim_{t \to \infty}\Big(\sqrt{t} E_0\Big[\exp -\{\int_{\mathbb{R}} L^y_t V(dy)\}\Big]\Big) = \sqrt{\frac{2}{\pi}} H(V), \tag{5.42}$$

where $H(V)$ is defined as :

$$H(V) = \frac{1}{2}\int_0^{+\infty} \Big(Q^{(0)}_l\big[e^{-<Y,V_->}\big] Q^{(2)}_l\big[e^{-<Y,V_+>}\big] + Q^{(2)}_l\big[e^{-<Y,V_->}\big] Q^{(0)}_l\big[e^{-<Y,V_+>}\big]\Big) dl. \tag{5.43}$$

In terms of $M(V_\pm)$ and $N(V_\pm)$ (resp. $\phi_{\lambda_\pm}(\infty)$, $\phi'_{\lambda_\pm}(0)$) , we have :

$$H(V) = \frac{1}{2(M(V_+) + M(V_-))}\big(e^{-2N(V_+)} + e^{-2N(V_-)}\big) = \frac{\phi_{\lambda_+}(\infty) + \phi_{\lambda_-}(\infty)}{\phi'_{\lambda_+}(0) + \phi'_{\lambda_-}(0)}. \tag{5.44}$$

2. *In particular if $V(dx)$ is a symmetric measure (i.e. $V(dx)$ coincides with its image by the map $x \mapsto -x$), then :*

$$H_{sym}(V) := H(V) = \frac{1}{2}\int_0^{+\infty} Q_l^{(2)}\left[e^{-<Y,V_+>}\right]\,dl = \frac{1}{2M(V_+)}e^{-2N(V_+)} = \frac{\phi_{\lambda_+}(\infty)}{\phi'_{\lambda_+}(0)}. \quad (5.45)$$

Proof of Proposition 5.13. We give two proofs of Proposition 5.13; the first one uses the Ray-Knight theorem for Brownian local times up to an exponential time (cf Proposition 5.12); the second one uses excursion theory.

First proof of Proposition 5.13. 1) We set :

$$\Delta = E_0\Big[\exp - \{\int_{\mathbb{R}} L^y_{S_\theta} V(dy)\}\Big].$$

Relation (5.40) implies that Δ may be written as follows :

$$\Delta = \theta\int_0^\infty e^{-\theta l}\Big(\frac{1}{2}\int_{\mathbb{R}} e^{-|b|}\Delta_\theta(b,l)db\Big)dl, \quad (5.46)$$

where :

$$\Delta_\theta(b,l) = P^{(\theta)}_{b/\theta,l}\big[e^{-<Y,V>}\big].$$

Suppose that $b>0$. We decompose $<Y,V>$ in the following way :

$$<Y,V> = \int_{-\infty}^0 Y_z V(dz) + \int_0^{b/\theta} Y_z V(dz) + \int_{b/\theta}^{+\infty} Y_z V(dz).$$

Using Proposition 5.12 and taking the limit, $\theta \to 0$ we obtain :

$$\lim_{\theta\to 0}\Delta_\theta(b,l) = Q_l^{(0)}\big[e^{-<Y,V_->}\big]Q_l^{(2)}\big[e^{-<Y,V_+>}\big].$$

Moreover $0 \le \Delta_\theta(b,l) \le 1$. Applying the same reasoning to the case $b<0$ we obtain :

$$\lim_{\theta\to 0}\Big(\frac{1}{2}\int_{\mathbb{R}} e^{-|b|}\Delta_\theta(b,l)db\Big) = H(V),$$

where $H(V)$ is defined by (5.43).
Identity (5.44) follows directly from (5.36).

2) We now turn to the symmetric case. Since $V_- = V_+$, additivity property (5.35) directly implies that :

$$Q_l^{(0)}\big[e^{-<Y,V_->}\big]Q_l^{(2)}\big[e^{-<Y,V_+>}\big] = Q_{2l}^{(2)}\big[e^{-<Y,V_+>}\big].$$

This proves (5.45).

3) Since S_θ is independent from B, it follows that :

$$E_0\Big[\exp - \{\int_{\mathbb{R}} L^y_{S_\theta} V(dy)\}\Big] = \frac{\theta^2}{2}\int_0^\infty \Big(E_0\Big[\exp - \{\int_{\mathbb{R}} L^y_t V(dy)\}\Big]\Big)e^{-\theta^2 t/2}dt.$$

As $t \mapsto E_0\Big[\exp - \{\int_{\mathbb{R}} L^y_t V(dy)\}\Big]$ is decreasing, we conclude from (5.41) and the Tauberian theorem (cf [11], Chap. XIII, section 5) that (5.42) holds.

Second proof of Proposition 5.13.
We suppose for simplicity that V is a positive function with compact support. We start as in the previous approach considering S_θ an exponential r.v.with parameter $\theta^2/2$ (i.e. with expectation $2/\theta^2$) and independent of $(B_t; t \ge 0)$. We again consider :

$$\Delta = E_0\Big[\exp - \{\int_{\mathbb{R}} L^y_{S_\theta} V(y)dy\}\Big].$$

We express Δ with the help of excursion theory

Lemma 5.14 *Δ is equal to the ratio $\dfrac{\mathcal{N}^{(\theta)}}{\mathcal{D}^{(\theta)}}$, where,*

$$\mathcal{N}^{(\theta)} = \frac{\theta^2}{2}\int\Big[\int_0^{\zeta(\varepsilon)} \exp\Big\{-\frac{\theta^2 t}{2} - \frac{1}{2}\int_0^t V(\varepsilon_s)ds\Big\}\ dt\Big]n(d\varepsilon),$$

$$\mathcal{D}^{(\theta)} = \int\Big[1-\exp\Big\{-\frac{\theta^2}{2}\zeta(\varepsilon) - \frac{1}{2}\int_0^{\zeta(\varepsilon)} V(\varepsilon_s)ds\Big\}\Big]n(d\varepsilon),$$

$n(d\varepsilon)$ denotes Itô's measure of excursions and $\zeta(\varepsilon) = \inf\{s > 0; \varepsilon_s = 0\}$.

Proof of Lemma 5.14. It follows easily from the general integral representation formula (cf [26], Exercise 4.18, Chap. XII):

$$\int_0^\infty P_0^t dt = \int_0^\infty P_0^{\tau_l} dl \ \circ\ \int_0^\infty n^u(\cdot\cap\{u<\zeta\})du, \tag{5.47}$$

where, for any random time T, P_0^T denotes the Wiener measure restricted to the σ-field $\mathcal{F}_T$, n^u denotes the Itô measure restricted to the corresponding σ-field $\mathcal{F}_u^\star = \sigma(\varepsilon_s; 0\le s\le u)$ for excursions ε, and $\circ$ indicates the concatenation, operation acting on measures on path space (see [26], Chap. XII, section 4, for details). Finally $(\tau_l; l\ge 0)$ is the inverse local time at 0.
As a consequence of (5.47), we get $\Delta = \Delta_-\Delta_+$ where :

$$\Delta_+ = \frac{\theta^2}{2}\int_0^\infty dt\Big(\int \exp\Big\{-\frac{\theta^2}{2}t - \int_0^t V(\varepsilon_s)ds\Big\}1_{\{t<\zeta(\varepsilon)\}}n(d\varepsilon)\Big),$$

$$\Delta_- = \int_0^\infty dl E_0[\exp\{-\int_0^{\tau_l} V(B_s)ds\}].$$

Using Fubini's theorem, we find : $\Delta_+ = \mathcal{N}^{(\theta)}$; concerning Δ_-, we get from excursion theory (cf [26], Proposition (2.7), Chap XII) :

$$E_0[\exp\{-\int_0^{\tau_l} V(B_s)ds\}] = \exp\Big\{-l\int n(d\varepsilon)\Big(1-\exp\{-\int_0^{\zeta(\varepsilon)} V(\varepsilon_s)ds\}\Big)\Big\},$$

and consequently $\Delta_- = 1/\mathcal{D}^{(\theta)}$. ■

Let us provide now the second proof of Proposition 5.13.
As $\theta\to 0$, the denominator $\mathcal{D}^{(\theta)}$ tends to :

$$\int\Big[1-\exp\Big\{-\frac{1}{2}\int_0^{\zeta(\varepsilon)} V(\varepsilon_s)ds\Big\}\Big]n(d\varepsilon) = -\frac{1}{2}(\phi'_{V_+}(0_+)+\phi'_{V_-}(0_+)).$$

Let us consider the numerator, which we may write as :

$$\mathcal{N}^{(\theta)} = \frac{\theta^2}{2}\int_0^\infty e^{-\theta^2 t/2}\Big[\int 1_{\{\zeta(\varepsilon)>t\}}\exp\Big\{-\frac{1}{2}\int_0^t V(\varepsilon_s)ds\Big\}n(d\varepsilon)\Big]dt.$$

Now recall that (Ex 4.18, Chap XII in [26]) :

$$1_{\{\zeta(\varepsilon)>t\}}n_\pm(d\varepsilon) := \frac{1}{2}\frac{1}{\sqrt{2\pi t}}M^t(d\varepsilon),$$

where M^t denotes the law of the Brownian meander with length t.
Thus, we find :

$$\mathcal{N}^{(\theta)} = \frac{\theta^2}{2}\int_0^\infty \frac{dt}{\sqrt{2\pi t}}e^{-\theta^2 t/2}\frac{1}{2}\Big\{M^t\Big(\exp\{-\frac{1}{2}\int_0^t V_+(\varepsilon_s)ds\}\Big)$$

$$+M^t\Big(\exp\{-\frac{1}{2}\int_0^t V_-(\varepsilon_s)ds\}\Big)\Big\},$$

(recall that V_+ and V_- are defined by the rules given in Definition 5.9).
For simplicity, we now write $\alpha = \theta^2/2$, and we make the change of variables : $\alpha t = u$; then :

$$\mathcal{N}^{(\theta)} = \sqrt{\alpha}\int_0^\infty \frac{du}{\sqrt{2\pi u}} e^{-u}\frac{1}{2}\Big\{M^{u/\alpha}\Big(\exp\{-\frac{1}{2}\int_0^{u/\alpha} V_+(\varepsilon_s)ds\}\Big)$$

$$+M^{u/\alpha}\Big(\exp\{-\frac{1}{2}\int_0^{u/\alpha} V_-(\varepsilon_s)ds\}\Big)\Big\}. \tag{5.48}$$

Now we use the fact (cf again Ex 4.18, Chap XII in [26]) that :

$$M^t = \sqrt{\frac{\pi}{2}}\frac{\sqrt{t}}{R_t}P^{(3)}_{0|\mathcal{F}_t},$$

where $P_0^{(3)}$ denotes the law of the three dimensional Bessel process started at 0.
It is not difficult to show that :

$$\lim_{t\to\infty} M^t\Big(\exp\{-\frac{1}{2}\int_0^t V_\pm(\varepsilon_s)ds\}\Big) = E_0^{(3)}[\exp\{-\frac{1}{2} < V_\pm, Y >\}], \tag{5.49}$$

where under $P_0^{(3)}$, Y stands for the three dimensional Bessel process starting from 0, and

$$< V_\pm, Y >= \int_0^\infty Y_s V_\pm(s)ds.$$

Hence, from (5.48), we deduce :

$$\mathcal{N}^{(\theta)} \sim \frac{\theta}{2}E_0^{(3)}\Big[\exp\{-\frac{1}{2} < V_+, Y >\} + \exp\{-\frac{1}{2} < V_-, Y >\}\Big], \quad \theta \to 0,$$

and, from the Ray-Knight Theorem for the three dimensional Bessel process, the right hand-side of (5.49) is :

$$\frac{\theta}{2}\Big\{Q_0^{(2)}[\exp\{-\frac{1}{2} < V_+, Y >\}] + Q_0^{(2)}[\exp\{-\frac{1}{2} < V_+, Y >\}]\Big\} = \frac{\theta}{2}(\phi_{V_+}(\infty) + \phi_{V_-}(\infty)), \tag{5.50}$$

from, e.g. [25]. Hence, we have proven (5.41).

■

Remark 5.15 *1)Donati-Martin and Hu [10] prove the convergence in law for the Wiener measure perturbed by the exponential martingale density associated with* $\int_0^t \frac{dB_s}{B_s}1_{\{|B_s|\geq\varepsilon\}}$, *as* $\varepsilon \to 0$*; the limiting law is that of the symmetrized* $BES(3)$ *process, i.e. : a process taking values in* $\mathbb{R}_+$ *with probability* $1/2$, *and in* $\mathbb{R}_-$ *with probability* $1/2$.
2) Let V *be a function with compact support. Y. Hu (private communication) has studied the asymptotic behaviour of* $Z_t^V(x)$, *as* $t \to \infty$, *where in (2.3),* $(X_t; t \geq 0)$ *is a one-dimensional diffusion process . With some additional assumptions, using excursion theory for Brownian motion, Y. Hu has determined the rate of decay of* $Z_t^V(x)$, *as* $t \to \infty$. *In particular, Y. Hu has recovered the result concerning Bessel processes with dimension* $0 < d < 2$.

6 The unilateral case

In this section the given positive Radon measure $V(dy) \neq 0$ on $\mathbb{R}$, is supposed to be strongly asymmetric : it is "small" at $-\infty$ and "big" at $+\infty$. More precisely we suppose :

$$\int_{-\infty}^{0} (1+|y|)V(dy) < \infty. \tag{6.1}$$

$$\liminf_{x\to+\infty} \Big(x^{2\alpha}V_a(x)\Big) > 0, \text{for some } \alpha < 1, \tag{6.2}$$

where $V(dy) = V_a(y)dy + V_s(dy)$ is the Lebesgue decomposition of $V(dy)$, and in (6.2) the liminf may be equal to $+\infty$

We remark that if $\lim_{x\to+\infty} \Big(x^{2\alpha}V_a(x)\Big)$ exists for some $\alpha > 1$, then V fulfills (5.1). This case has been studied in the previous section.

Let us state the main result of this section.

Theorem 6.1 *Let $V(dy)$ be a positive Radon measure on $\mathbb{R}$ fulfilling (6.1) and (6.2).*

1. *The generic Theorem applies with $k = 1/2$, i.e. :*

$$\lim_{t\to\infty} \Big(\sqrt{t}\ E_x\Big[\exp\Big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\Big\}\Big]\Big) := \varphi_V(x) \text{ exists in } \mathbb{R}.$$

2. *φ_V is a convex function which takes its values in $]0,\infty[$ and is the unique solution to the Sturm-Liouville equation :*

$$\varphi\text{"}(dx) = \varphi(x)V(dx), \tag{6.3}$$

with boundary conditions:

$$\lim_{x\to-\infty} \varphi_V'(x) = -\sqrt{\frac{2}{\pi}} \qquad ; \qquad \lim_{x\to+\infty} \varphi_V(x) = 0. \tag{6.4}$$

Moreover there exist two positive constants C, C' such that

$$\varphi_V(x) \le C(1+|x|); \quad x \le 0, \tag{6.5}$$

$$\varphi_V(x) \le Ce^{-C'x^{1-\alpha}}; \quad x \ge 0. \tag{6.6}$$

3. *Let M^φ be the process defined by (5.6), then $(M_t^\varphi; t \ge 0)$ is a continuous martingale.*

4. *Let $(X_t^x; t \ge 0)$ be the solution to (5.7), then the law of $(X_t^x; t \ge 0)$ is $P_x^{\varphi_V}$, $(X_t^x; t \ge 0)$ is transient, i.e.*

$$P\Big(\lim_{t\to\infty} X_t^x = -\infty\Big) = 1. \tag{6.7}$$

Our proof of Theorem 6.1 consists of two main steps. We begin by establishing an a priori upper-bound for $t \mapsto \sqrt{t}Z_t^V(x)$ (cf Lemma 6.2) . In a second step we show that we may reduce the discussion to the case where V has a compact support.

Lemma 6.2 *Let $\tilde{\varphi}$ be the function defined as follows:*

$$\tilde{\varphi}(x) := \limsup_{t\to\infty} \Big(\sqrt{t}\ E_x\Big[\exp\Big\{ -\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy) \Big\} \Big] \Big). \tag{6.8}$$

Then there exist two positive numbers C, C' such that

$$\tilde{\varphi}(x) \le C e^{-C' x^{1-\alpha}},\ \textit{for any } x \ge 0. \tag{6.9}$$

In particular

$$\lim_{x\to+\infty} \tilde{\varphi}(x) = 0. \tag{6.10}$$

Proof of Lemma 6.2. Assumption (6.2) implies that there exist $\kappa, a > 0$ such that

$$\int_{\mathbb{R}} L_t^y V(dy) \ge \frac{1}{2}\frac{\kappa^2}{b^{2\alpha}} \int_a^b L_t^y dy, \quad b > a.$$

Then for any $y \in [a, b]$, Example 5.7 implies that

$$\limsup_{t\to\infty} \Big(\sqrt{t}\ E_y\Big[\exp\Big\{ -\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy) \Big\} \Big] \Big) \le \sqrt{\frac{2}{\pi}} \frac{\cosh\left[\frac{\kappa}{b^\alpha}(y - \frac{a+b}{2})\right]}{\frac{\kappa}{b^\alpha}\sinh\left[\frac{\kappa}{b^\alpha}\frac{b-a}{2}\right]}.$$

Let $y > a$, we choose $b = 2y - a$. This brings

$$\tilde{\varphi}(y) \le \frac{1}{\kappa}\sqrt{\frac{2}{\pi}} \frac{(2y-a)^\alpha}{\sinh\left[\kappa\frac{y-a}{(2y-a)^\alpha}\right]}, \tag{6.11}$$

which proves (6.9). ■

Lemma 6.3 *Let Z^V be the function :*

$$Z_t^V(x) = E_x\Big[\exp\Big\{ -\frac{1}{2}\int_{\mathbb{R}} L_t^z V(dz) \Big\} \Big], t \ge 0, x \in \mathbb{R}. \tag{6.12}$$

Then, if $y > \max\{0, x\}$, $Z_t^V(x) = Z_1(t, y; x) + Z_2(t, y; x)$ where $Z_1(t, y; x)$, $Z_2(t, y; x)$ are two non-negative functions and

$$\limsup_{t\to\infty} \Big(\sqrt{t}\ Z_1(t, y; x) \Big\} \le 2\tilde{\varphi}(y), \tag{6.13}$$

$$\sqrt{t}\ Z_2(t, y; x) \textit{ converges as } t \to \infty, \tag{6.14}$$

where $\tilde{\varphi}$ is the function defined by (6.8).

Proof of Lemma 6.3. We decompose :

$$Z_t^V(x) = Z_1(t, y; x) + Z_2(t, y; x),$$

$$Z_1(t, y; x) = E_x\Big[\exp\Big\{ -\frac{1}{2}\int_{\mathbb{R}} L_t^z V(dz) \Big\}\ 1_{\{T_y < t\}} \Big],$$

$$Z_2(t, y; x) = E_x\Big[\exp\Big\{ -\frac{1}{2}\int_{\mathbb{R}} L_t^z V(dz) \Big\}\ 1_{\{T_y \ge t\}} \Big],$$

where $T_y = \inf\{t \geq 0; B_t = y\}$.

a) We start with the study of $Z_1(t, y; x)$.

Using the strong Markov property at time T_y, we get:

$$Z_1(t, y; x) = E_x\Big[1_{\{T_y<t\}} \exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L^z_{T_y} V(dz)\big\}\ Z^V_{t-T_y}(y)\Big].$$

Since $V(dx)$ is a positive measure, then

$$\sqrt{1+t}\ Z_1(t, y; x) \leq E_x\Big[1_{\{T_y<t\}}\frac{\sqrt{1+t}}{\sqrt{1+t-T_y}}\ \sqrt{1+t-T_y}\ Z^V_{t-T_y}(y)\Big]. \tag{6.15}$$

i) We claim that for fixed y

$$\Big\{\frac{\sqrt{1+t}}{\sqrt{1+t-T_y}}1_{\{T_y<t\}};\ t \geq 1\Big\} \text{ is uniformly integrable.} \tag{6.16}$$

It suffices to prove that $\dfrac{\sqrt{1+t}}{\sqrt{1+t-T_y}}1_{\{T_y<t\}}$ are bounded r.v.'s in L^2, uniformly with respect to $t \geq 1$.

We have:

$$E\Big[\frac{1+t}{1+t-T_y}1_{\{T_y<t\}}\Big] = \int_0^t a_t(s)ds,$$

where:

$$a_t(s) = \frac{1+t}{1+t-s}\frac{y}{\sqrt{2\pi s^3}}\exp\{-\frac{y^2}{2s}\}.$$

We distinguish two cases:

α) $s \in [0, t/2]$, then $\dfrac{1+t}{1+t-s} \leq 2$, hence $a_t(s) \leq \dfrac{2y}{\sqrt{2\pi s^3}}\exp\{-\dfrac{y^2}{2s}\}$ and

$$\int_0^{t/2} a_t(s)ds \leq 2,$$

since $s \mapsto \dfrac{y}{\sqrt{2\pi s^3}}\exp\{-\dfrac{y^2}{2s}\}$ is a density function.

β) $s \in [t/2, t]$, then $a_t(s) \leq \dfrac{1+t}{\sqrt{2\pi(t/2)^3}}\dfrac{y}{1+t-s}$ and

$$\int_{t/2}^t a_t(s)ds \leq \frac{2y}{\sqrt{2\pi}}\frac{1+t}{\sqrt{t^3}}\ln(1+t/2).$$

Finally

$$\sup_{t\geq 1}\Big(\int_0^t a_t(s)ds\Big) < \infty.$$

This proves (6.16).

ii) The definition of $\tilde{\varphi}$(cf (6.8)) implies the existence of a positive number a (depending on y) such that:

$$\sqrt{1+t}Z^V_t(x) \leq 2\tilde{\varphi}(y), \quad \text{for any } t \geq a. \tag{6.17}$$

On the right hand-side of (6.15) the decomposition of $\{T_y < t\}$ as the disjoint union of $\{t - T_y > a\}$ and $\{t - a \leq T_y < t\}$, leads to

$$\sqrt{1+t}\ Z_1(t, y; x) \leq Z_{1,1}(t, y; x) + Z_{1,2}(t, y; x),$$

$$Z_{1,1}(t,y;x) = E_x\Big[1_{\{t-T_y>a\}}\frac{\sqrt{1+t}}{\sqrt{1+t-T_y}}\ \sqrt{1+t-T_y}\ Z^V_{t-T_y}(y)\Big],$$

$$Z_{1,2}(t,y;x) = E_x\Big[1_{\{t-a\le T_y<t\}}\frac{\sqrt{1+t}}{\sqrt{1+t-T_y}}\ \sqrt{1+t-T_y}\ Z^V_{t-T_y}(y)\Big].$$

The inequality (6.17) and the property (6.16) imply

$$\limsup_{t\to\infty}\Big(Z_{1,1}(t,y;x)\Big)\le 2\tilde\varphi(y).$$

As for $Z_{1,2}(t,y;x)$, the function Z^V being less than 1,

$$Z_{1,2}(t,y;x)\le\sqrt{1+t}\ P(t-a\le T_y<t) = \frac{\sqrt{1+t}}{\sqrt{2\pi}}\int_{t-a}^{t} e^{-y^2/2s}\frac{ds}{s^{3/2}},$$

$$Z_{1,2}(t,y;x)\le\frac{\sqrt{1+t}}{\sqrt{2\pi}}\frac{a}{(t-a)^{3/2}}.$$

Consequently:

$$\limsup_{t\to\infty}\Big(Z_{1,2}(t,y;x)\Big)=0,$$

hence, finally :

$$\limsup_{t\to\infty}\Big(Z_{1}(t,y;x)\Big)\le 2\tilde\varphi(y). \qquad (6.18)$$

b) We now prove (6.14).
Recall that $y>x$. The key observation is the following : on $\{T_y\ge t\}$, $V(dz)$ can be replaced by $V^{(y)}(dz)$ where:

$$V^{(y)}(dz) = 1_{]-\infty,y]}(z)V(dz),$$

which allows us to reduce the discussion to the integrable case since $V^{(y)}(dz)$ satisfies (5.1).
More precisely, we have:

$$Z_2(t,y;x) = E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L^z_t V^{(y)}(dz)\big\}\ 1_{\{T_y\ge t\}}\Big].$$

Hence $Z_2(t,y;x) = Z_{2,1}(t,y;x) - Z_{2,2}(t,y;x)$, with

$$Z_{2,1}(t,y;x) = E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L^z_t V^{(y)}(dz)\big\}\Big],$$

and

$$Z_{2,2}(t,y;x) = E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L^z_t Z^{(y)}(dz)\big\}\ 1_{\{T_y< t\}}\Big].$$

Theorem 5.1 tells us that $\sqrt{1+t}\ Z_{2,1}(t,y;x)$ converges as $t\to\infty$.
As for $Z_{2,2}(t,y;x)$, we use the strong Markov property at time T_y:

$$\sqrt{1+t}\ Z_{2,2}(t,y;x) = E_x\Big[1_{\{T_y<t\}}\ \exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L^z_{T_y} V^{(y)}(dz)\big\}\ \sqrt{\frac{1+t}{t+1-T_y}}$$
$$\times\sqrt{t+1-T_y}Z^{V^{(y)}}_{t-T_y}(y)\Big].$$

The conjonction of Theorem 5.1, inequality (5.10) (i) and (6.16) implies that

$$\Big(1_{\{T_y<t\}}\sqrt{\frac{1+t}{t+1-T_y}}\sqrt{t+1-T_y}Z^{V^{(y)}}_{t-T_y}(y); t\ge 1\Big)$$

is a family of uniformly integrable r.v.'s converging a.s. to the constant $\varphi_{V^{(y)}}(y)$, as $t \to \infty$. Hence, it converges in L^1.
As a result, $\sqrt{1+t}\ Z_{2,2}(t,y;x)$ converges as $t \to \infty$.
This ends the proof of Lemma 6.3. ■

Proof of Theorem 6.1. a) Let $y > \max\{0, x\}$. Using Lemma 6.3 we get:

$$\limsup_{t\to\infty}\Big(\sqrt{1+t}\ Z_t^V(x)\Big) \le 2\tilde{\varphi}(y) + \lim_{t\to\infty}\Big(\sqrt{1+t}\ Z_2(t,y;x)\Big),$$

$$\liminf_{t\to\infty}\Big(\sqrt{1+t}\ Z_t^V(x)\Big) \ge \lim_{t\to\infty}\Big(\sqrt{1+t}\ Z_2(t,y;x)\Big).$$

Hence

$$0 \le \limsup_{t\to\infty}\Big(\sqrt{1+t}\ Z_t^V(x)\Big) - \liminf_{t\to\infty}\Big(\sqrt{1+t}\ Z_t^V(x)\Big) \le 2\tilde{\varphi}(y).$$

The parameter y being arbitrary, property (6.10) implies point 1. of Theorem 6.1 :

$$\varphi_V(x) := \lim_{t\to\infty}\big(\sqrt{t}Z_t^V(x)\big) = \lim_{t\to\infty}\Big(\sqrt{t}E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^z V(dz)\big\}\Big]\Big).$$

b)(6.6) (resp. (6.5)) is a direct consequence of (6.9) (resp. the inequality : $V(dz) \ge 1_{]-\infty,0]}(z)V(dz)$ and (5.5)).
c) Obviously, (6.6) implies $\lim_{x\to+\infty}\varphi_V(x) = 0$.
In order to end the proof of Theorem 6.1 we have to check:

$$\lim_{x\to-\infty}\varphi_V'(x) = -\sqrt{\frac{2}{\pi}}. \tag{6.19}$$

Let $x < 0$. We have successively:

$$\varphi_V'(0) - \varphi_V'(x) = \int_x^0 \varphi_V''(dy) = \int_x^0 \varphi_V(y)V(dy),$$

$$|\varphi_V'(0) - \varphi_V'(x)| \le C\int_x^0 (1+|y|)V(dy).$$

Assumption (6.1) implies that $|\varphi_V'(x)|$ is bounded. But φ_V is a convex function, hence $\lim_{x\to-\infty}\varphi_V'(x)$ exists. Moreover

$$\lim_{x\to-\infty}\varphi_V'(x) = \lim_{x\to-\infty}\frac{\varphi_V(x)}{x}. \tag{6.20}$$

Let $V^{[a]}(dy)$ denote the positive measure : $V^{[a]}(dy) = 1_{]-\infty,a[}V(dy)$, with $a < 0$.
It is clear that $V^{[a]}(dy)$ fulfills (6.2) and (6.1). Let $\varphi^{[a]}(x)$ be the limit of $\sqrt{t}\ Z_t^{V^{[a]}}(x)$, as $t \to \infty$, where :

$$Z_t^{V^{[a]}}(x) = E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V^{[a]}(dy)\big\}\Big],$$

Let $x < a$. We have :

$$Z_t^V(x) = E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V(dy)\big\}\ 1_{\{T_a<t\}}\Big] + E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V^{[a]}(dy)\big\}\ 1_{\{T_a\ge t\}}\Big],$$

We use the strong Markov property at time T_a :

$$Z_t^V(x) = E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L_{T_a}^y V(dy)\big\}\ 1_{\{T_a<t\}}Z_{t-T_a}^V(a)\Big] + E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L_t^y V^{[a]}(dy)\big\}\Big]$$

$$-E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L^y_{T_a} V^{[a]}(dy)\big\}\ 1_{\{T_a<t\}} Z^{V^{[a]}}_{t-T_a}(a)\Big].$$

We multiply both sides by $\sqrt{t}$ and we take the limit as $t \to \infty$ to obtain :

$$\varphi_V(x) = \varphi_V(a)h(x) + \varphi^{[a]}(x) + \varphi^{[a]}(a)h^{[a]}(x),$$

where

$$h(x) = E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L^y_{T_a} V(dy)\big\}\Big], \quad h^{[a]}(x) = E_x\Big[\exp\big\{-\frac{1}{2}\int_{\mathbb{R}} L^y_{T_a} V^{[a]}(dy)\big\}\Big].$$

The functions h and $h^{[a]}$ are bounded, and $V^{[a]}(dy)$ satisfies (5.1), hence

$$\lim_{x\to-\infty} \frac{\varphi_V(x)}{x} = \lim_{x\to-\infty} \frac{\varphi^{[a]}(x)}{x} = \lim_{x\to-\infty} (\varphi^{[a]})'(x) = -\sqrt{\frac{2}{\pi}}.$$

■

Example 6.4 *Let $V(dy) = \lambda^2 1_{[0,+\infty[}(y)dy$.*

$$\lim_{t\to\infty} \Big(\sqrt{t}\ E_x\Big[\exp\big\{-\frac{\lambda^2}{2}\int_0^t 1_{\{B_s>0\}}ds\big\}\Big]\Big) = \varphi_V(x),$$

where

$$\varphi_V(x) = \begin{cases} \frac{1}{\lambda}\sqrt{\frac{2}{\pi}}e^{-\lambda x} & \text{if } x \geq 0 \\ \sqrt{\frac{2}{\pi}}(\frac{1}{\lambda} - x) & \text{if } x < 0. \end{cases}$$

Moreover an explicit formula for $E_x\Big[\exp\big\{-\frac{\lambda^2}{2}\int_0^t 1_{\{B_s>0\}}ds\big\}\Big]$ is given in ([3] p 136]):

$$E_x\Big[\exp\big\{-\frac{\lambda^2}{2}\int_0^t 1_{\{B_s>0\}}ds\big\}\Big]$$

$$= \begin{cases} e^{-\frac{\lambda^2 t}{2}}(1 - \mathrm{Erfc}(-\frac{x}{\sqrt{2t}}) + \frac{1}{\pi}\int_0^1 \frac{du}{\sqrt{u(1-u)}}\exp\{-\frac{\lambda^2 tu}{2} - \frac{x^2}{2tu}\} & \text{if } x \geq 0 \\ (1 - \mathrm{Erfc}(-\frac{x}{\sqrt{2t}}) + \frac{1}{\pi}\int_0^1 \frac{du}{\sqrt{u(1-u)}}\exp\{-\frac{\lambda^2 tu}{2} - \frac{x^2}{2t(1-u)}\} & \text{if } x < 0. \end{cases}$$

Example 6.5 *Let $V(dy) = (y^2-1)1_{\{y\geq 1\}}dy$. Then*

$$\lim_{t\to\infty} \Big(\sqrt{t}\ E_x\Big[\exp\big\{-\frac{\lambda^2}{2}\int_0^t (B_s^2-1)1_{\{B_s>1\}}ds\big\}\Big]\Big) = \varphi_V(x),$$

where

$$\varphi_V(x) = \begin{cases} \sqrt{\frac{2}{\pi}}e^{1/2}e^{-x^2/2} & \text{if } x \geq 1 \\ \sqrt{\frac{2}{\pi}}(2-x) & \text{if } x < 1. \end{cases}$$

7 Some critical cases

In this section we consider :

$$V(x) = \frac{\lambda}{\theta + x^2}, \tag{7.1}$$

where $\lambda > 0$ and $\theta \geq 0$.
Denoting by $(R_t; t \geq 0)$ the reflecting Brownian motion : $R_t = |B_t|;\ t \geq 0$, then

$$\int_0^t V(B_s)ds = \lambda \int_0^t \frac{1}{\theta + R_s^2} ds. \tag{7.2}$$

This led us to investigate more generally the asymptotic behaviour of $E_x^\mu\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t \frac{ds}{\theta + R_s^2}\Big\}\Big]$ when $t \to \infty$, where $(R_t; t \geq 0)$ is under P_x^μ, a Bessel process started at x, with index $\mu > -1$ (the dimension is $d_\mu = 2(\mu + 1)$).
Throughout this section, n_μ stands for :

$$n_\mu = \frac{-\mu + \sqrt{\mu^2 + \lambda}}{2} \tag{7.3}$$

This parameter will play a central role in the formulation of our results.
We begin with the case $\theta = 0$.

Theorem 7.1 *Suppose $\mu > -1$. Then :*

$$\lim_{t\to\infty} \Big(t^{n_\mu} E_x^\mu\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t \frac{ds}{R_s^2}\Big\}\Big]\Big) = x^{2n_\mu}\frac{1}{2^{n_\mu}}\frac{\Gamma(\mu + n_\mu + 1)}{\Gamma(\mu + 2n_\mu + 1)}. \tag{7.4}$$

Remark 7.2 1. *In particular if $\mu = -1/2$, that is if $d_\mu = 1$, then :*

$$\lim_{t\to\infty} \Big(t^{n} E_x\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t \frac{ds}{B_s^2}\Big\}\Big]\Big) = x^{2n}\frac{1}{2^{n}}\frac{\Gamma(n + \frac{1}{2})}{\Gamma(2n + \frac{1}{2})}, \tag{7.5}$$

where $n = n_{-1/2} = \dfrac{1 + \sqrt{1 + 4\lambda}}{4}$.

2. *Taking $\mu = 0$ (i.e. $d_\mu = 2$), we obtain :*

$$\lim_{t\to\infty} \Big(t^{\lambda} E_x\Big[\exp\Big\{-\frac{\lambda^2}{2}\Big(4\int_0^t \frac{ds}{R_s^2}\Big)\Big\}\Big]\Big) = \frac{x^{2\lambda}}{2^\lambda}\frac{\Gamma(\lambda + 1)}{\Gamma(1 + 2\lambda)} = \frac{x^{2\lambda}}{8^\lambda}\frac{\sqrt{\pi}}{\Gamma(1/2 + \lambda)}, \quad \lambda \geq 0. \tag{7.6}$$

To obtain the last equality in (7.6) we have used the Legendre duplication formula ([31], p 240) :

$$\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1}\Gamma(z)\Gamma(z + \frac{1}{2}).$$

The formula (7.6) led Roynette and Yor [29] to define and study a family of positive r.v's $(H_{c,\alpha})$ such that

$$E\Big[\exp\Big\{-\frac{\lambda^2}{2}H_{c,\alpha}\Big\}\Big] = \frac{\Gamma(\alpha)}{\Gamma(\alpha + \lambda)}\exp\{c\lambda\}, \quad \lambda \geq 0,$$

where $\alpha > 0$ and $c \leq \Gamma'(\alpha)/\Gamma(\alpha)$.

We observe that these Laplace transforms appear in (7.6) for $\alpha = 1/2$, and $c = 2\log x - \log 8$.

Proof of Theorem 7.1 Our approach is based on the well-known identity [26, chapter XI, ex 1.22, page 430], or [32].

$$E_x^{\mu}\Big[Y\Big(\frac{x}{R_t}\Big)^{\mu}\exp\Big\{-\frac{\nu^2}{2}\int_0^t\frac{ds}{R_s^2}\Big\}\Big]=E_x^{\nu}\Big[Y\Big(\frac{x}{R_t}\Big)^{\nu}\exp\Big\{-\frac{\mu^2}{2}\int_0^t\frac{ds}{R_s^2}\Big\}\Big],\tag{7.7}$$

for any $\mathcal{F}_t$-measurable r.v. $Y\geq 0$.
Choosing : $\nu=\mu+2n_\mu=\sqrt{\mu^2+\lambda}$ and

$$Y=\Big(\frac{x}{R_t}\Big)^{-\mu}\exp\Big\{\frac{\mu^2}{2}\int_0^t\frac{ds}{R_s^2}\Big\},$$

we get :

$$E_x^{\mu}\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t\frac{ds}{R_s^2}\Big\}\Big]=\frac{x^{2n_\mu}}{t^{n_\mu}}E^{\nu}_{x/\sqrt{t}}\Big[\Big(\frac{1}{R_1^2}\Big)^{n_\mu}\Big].$$

Then, we obtain :

$$\lim_{t\to\infty}\Big(t^{n_\mu}E_x^{\mu}\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t\frac{ds}{R_s^2}\Big\}\Big]\Big)=x^{2n_\mu}E_0^{\nu}\Big[\Big(\frac{1}{R_1^2}\Big)^{n_\mu}\Big].$$

But, under P_0^ν, the distribution of $R_1^2/2$ is gamma$(\nu+1)$. A straightforward calculation yields to :

$$E_0^{\nu}\Big[\Big(\frac{1}{R_1^2}\Big)^{n_\mu}\Big]=\frac{1}{2^{n_\mu}}\frac{\Gamma(\nu-n_\mu+1)}{\Gamma(\nu+1)}=\frac{1}{2^{n_\mu}}\frac{\Gamma(\mu+n_\mu+1)}{\Gamma(\mu+2n_\mu+1)}.\tag{7.8}$$

■

We now investigate the case $\theta>0$. In the sequel, $L^{(\mu)}$ denotes the infinitesimal generator of the Bessel process with index μ :

$$L^{(\mu)}(f)(x)=\frac{1}{2}f''(x)+\frac{2\mu+1}{2x}f'(x).\tag{7.9}$$

Theorem 7.3 *Suppose $\mu\geq -1/2,\lambda>0$ such that :*

$$\lambda<8\mu^2+6\mu+1.\tag{7.10}$$

Let $\varphi_\lambda^{(\mu)}$ be the unique smooth function defined on $[0,+\infty[$, solution of

$$L^{(\mu)}(\varphi)(x)=\frac{1}{2}\varphi''(x)+\frac{2\mu+1}{2x}\varphi'(x)=\frac{\lambda}{2}\frac{1}{1+x^2}\varphi(x);\quad x>0,\tag{7.11}$$

such that:

$$\varphi_\lambda^{(\mu)}(x)\sim x^{2n_\mu},x\to+\infty.\tag{7.12}$$

Then :

$$\lim_{t\to\infty}\Big(t^{n_\mu}E_x^{\mu}\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t\frac{ds}{\theta+R_s^2}\Big\}\Big]\Big)=\theta^{n_\mu}\varphi_\lambda^{(\mu)}(x/\sqrt{\theta})\frac{1}{2^{n_\mu}}\frac{\Gamma(\mu+n_\mu+1)}{\Gamma(\mu+2n_\mu+1)}.\tag{7.13}$$

Remark 7.4 *We observe that if we take the limit $\theta\to 0$ in (7.13)we recover (7.4).*

The function $\varphi_\lambda^{(\mu)}$ is defined in terms of hypergeometric functions. Let $F(\alpha,\beta,\gamma;x)$ be the hypergeometric function with parameters α,β,γ (cf [20]):

$$F(\alpha,\beta,\gamma;x)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}x^k,\tag{7.14}$$

where $(\rho)_k=\rho(\rho+1)\times\cdots\times(\rho+k-1)$.
The series in (7.14) converges for any x such that $|x|<1$.

Lemma 7.5 *The function $\varphi_\lambda^{(\mu)}$ in Theorem 7.3 which solves (7.11) and satisfies (7.12) is given by :*

$$\varphi_\lambda^{(\mu)}(x) = \begin{cases} k_\mu F(n_\mu+\mu, -n_\mu, \mu+1; -x^2) & \text{if } n_\mu \text{ is an integer} \\ k_\mu(1+x^2)^{-n_\mu-\mu} F(n_\mu+\mu, \mu+1+n_\mu, \mu+1; \frac{x^2}{1+x^2}) & \text{otherwise,} \end{cases} \tag{7.15}$$

where

$$k_\mu = \frac{\Gamma(\mu+n_\mu)}{\Gamma(\mu+2n_\mu)} \frac{\Gamma(\mu+n_\mu+1)}{\Gamma(\mu+1)}.$$

Remark 7.6 *We observe that n_μ is a positive real number. If n_μ is an integer, then $\varphi_\lambda^{(\mu)}(x)$ is a polynomial function with degree $2n_\mu$.*

Proof of Lemma 7.5.

1) Recall that $F(\alpha, \beta, \gamma; \cdot)$ fulfills :

$$x(1-x)u"(x) + (\gamma - (\alpha+\beta+1)x)u'(x) = \alpha\beta u(x). \tag{7.16}$$

Let v be the function : $v(t) = F(\alpha, \beta, \gamma; t^2)$. Then [20, p 164], v is a solution to :

$$t(1-t^2)v"(t) + 2(\gamma - \frac{1}{2} - (\alpha+\beta+\frac{1}{2})t^2)v'(t) = 4\alpha\beta v(t).$$

Finally setting $w(t) = v(it)$, we obtain :

$$\frac{1}{2}w"(t) + \frac{1}{t(1+t^2)}\Big[\gamma - \frac{1}{2} - (\alpha+\beta+\frac{1}{2})t^2)\Big]w'(t) = -2\alpha\beta\frac{w(t)}{1+t^2}. \tag{7.17}$$

If we choose $\alpha = n_\mu+\mu, \beta = -n_\mu$ and $\gamma = \mu+1$, then it is easy to check that $F(n_\mu+\mu, -n_\mu, \mu+1; -x^2)$ solves (7.11), with $x \in]0, 1[$.

2) If n_μ is an integer, it is obvious that $F(n_\mu+\mu, -n_\mu, \mu+1; -x^2)$ is a polynomial function with degree $2n_\mu$, and then solves (7.11) for every $x > 0$. Writing :

$$F(n_\mu+\mu, -n_\mu, \mu+1; -x^2) = \sum_{k=0}^{n_\mu} a_k x^{2k}, \tag{7.18}$$

we obtain :

$$a_k = \frac{(n_\mu+\mu)_k \times n_\mu \times \cdots (n_\mu-k+1)}{(\mu+1)_k k!}. \tag{7.19}$$

Hence $a_k > 0$, $F(n_\mu+\mu, -n_\mu, \mu+1; -x^2) > 0$ and

$$a_{n_\mu} = \frac{\Gamma(\mu+2n_\mu)}{\Gamma(n_\mu+\mu)} \frac{\Gamma(\mu+1)}{\Gamma(\mu+n_\mu+1)}. \tag{7.20}$$

Consequently $\varphi_\lambda^{(\mu)}(x)$ satisfies (7.12).

3) Suppose that n_μ is not an integer. To obtain a function defined on the half-line $[0, +\infty[$ we use a fractional linear transformation of hypergeometric functions. Recall [20, (9.5.1)] :

$$F(\alpha, \beta, \gamma; z) = (1-z)^{-\alpha} F(\alpha, \gamma-\beta, \gamma; \frac{z}{z-1}), \quad |\arg(1-z)| < \pi.$$

In our context this identity becomes :

$$F(n_\mu+\mu, -n_\mu, \mu+1; -x^2) = (1+x^2)^{-n_\mu-\mu} F(n_\mu+\mu, \mu+1+n_\mu, \mu+1; \frac{x^2}{1+x^2}), \quad x \in \mathbb{R}. \tag{7.21}$$

An analytic continuation argument shows that $\varphi_\lambda^{(\mu)}$ is a solution of (7.11) for every $x > 0$.
It is easy to check that the coefficients in the series are positive, thus, $\varphi_\lambda^{(\mu)} > 0$.
We conclude from [31, page 297 , ex 8] :

$$F(\alpha,\beta,\gamma;\frac{x^2}{1+x^2}) \sim \frac{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{(1+x^2)^{\gamma-\alpha-\beta}}, \quad x \to +\infty,$$

that (7.12) holds.

■

Lemma 7.7 *1. The function $\varphi_\lambda^{(\mu)}$ defined in Lemma 7.5 fulfills :*

$$(\varphi_\lambda^{(\mu)})'(x) \geq 0, \quad x \geq 0, \tag{7.22}$$

$$2n_\mu - \frac{\rho_\mu}{1+x^2} \leq \frac{x(\varphi_\lambda^{(\mu)})'(x)}{\varphi_\lambda^{(\mu)}(x)} \leq 2n_\mu, \quad x \geq 0, \tag{7.23}$$

where $\rho_\mu > 0$.

2. Let $D_\lambda^{(\mu)}$ be the function defined on $[1,+\infty[\times[0,+\infty[$ by :

$$D_\lambda^{(\mu)}(t,x) = \frac{1}{t^{n_\mu}}\varphi_\lambda^{(\mu)}(x\sqrt{t}). \tag{7.24}$$

Then :

$$\lim_{t\to+\infty} D_\lambda^{(\mu)}(t,x) = x^{2n_\mu}; \quad D_\lambda^{(\mu)}(t,x) \geq \hat{\rho}_\mu x^{2n_\mu}, \forall t, x \geq 0, \quad \textit{for some} \ \ \hat{\rho}_\mu > 0. \tag{7.25}$$

$$0 \leq \frac{1}{D_\lambda^{(\mu)}(t,z)} - \frac{1}{D_\lambda^{(\mu)}(t,y)} \leq k\frac{y-z}{z^{2n_\mu+1}}; 0 < z < y. \tag{7.26}$$

where $k = \dfrac{2n_\mu}{\hat{\rho}_\mu}$.

Proof of lemma 7.7 We give the proof only for the case $n_\mu \in \mathbb{N}$, the other cases are left to the reader.
Property (7.19) gives (7.22).
Using (7.18) we get :

$$2n_\mu\varphi_\lambda^{(\mu)}(x) - x(\varphi_\lambda^{(\mu)})'(x) = 2k_\mu\Big(\sum_{k=0}^{n_\mu}(n_\mu-k)x^{2k}\Big) \geq 0.$$

Hence

$$2n_\mu - x\frac{(\varphi_\lambda^{(\mu)})'(x)}{\varphi_\lambda^{(\mu)}(x)} = \frac{Q(x)}{\varphi_\lambda^{(\mu)}(x)},$$

where Q is a polynomial function with degree less than $2(n_\mu - 1)$. This implies (7.23).
Property (7.25) is due to the fact that $D_\lambda^{(\mu)}(t,\dot{)}$ is a polynomial function with degree $2n_\mu$ and positive coefficients.
As for (7.26), we take the x-derivative of $1/D_\lambda^{(\mu)}(t,x)$, we obtain :

$$\Big|\frac{\partial}{\partial x}\frac{1}{D_\lambda^{(\mu)}(t,x)}\Big| = t^{n_\mu+1/2}\frac{(\varphi_\lambda^{(\mu)})'(x\sqrt{t})}{(\varphi_\lambda^{(\mu)})^2(x\sqrt{t})}.$$

Hence, the inequalities (7.23) and (7.25) directly imply :

$$\Big|\frac{\partial}{\partial x}\frac{1}{D_\lambda^{(\mu)}(t,x)}\Big| \leq \frac{2n_\mu}{\hat{\rho}_\mu}\frac{1}{x^{1+2n_\mu}}.$$

Then (7.26) follows immediately. ■

Lemma 7.8 *For any positive functional F and $x \geq 0,\ t > 0,$ we have :*

$$E_x^\mu\Big[F(R_s; 0 \leq s \leq t)\varphi_\lambda^{(\mu)}(R_t)\exp\Big\{-\frac{\lambda}{2}\int_0^t \frac{ds}{1+R_s^2}\Big\}\Big] = \varphi_\lambda^{(\mu)}(x)E_x\Big[F(X_s; 0 \leq s \leq t)\Big], \quad (7.27)$$

where the function $\varphi_\lambda^{(\mu)}$ is defined in Lemma 7.5 and $(X(t); t \geq 0)$ is the process solution of :

$$X_t = x + B_t + \frac{2\mu+1}{2}\int_0^t \frac{ds}{X_s} + \int_0^t \frac{(\varphi_\lambda^{(\mu)})'}{\varphi_\lambda^{(\mu)}}(X_s)ds, \quad t \geq 0. \quad (7.28)$$

In particular :

$$E_x^\mu\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t \frac{ds}{1+R_s^2}\Big\}\Big] = \varphi_\lambda^{(\mu)}(x)E_x\Big[\frac{1}{\varphi_\lambda^{(\mu)}(X_t)}\Big]. \quad (7.29)$$

Remark 7.9 *Hariya and Yor [15] show the existence, and describe, the limiting measures, as $t \to \infty$, of the laws of $\{B_s + \mu s; 0 \leq s \leq t\}$ perturbed by the Radon-Nikodym density consisting of either the normalized functionals $\exp(-\alpha A_t^{(\mu)})$, or $1/\Big(A_t^{(\mu)}\Big)^m$, where $A_t^{(\mu)} = \int_0^t ds \exp\{2(B_s+\mu s)\}$. The results exhibit different regimes according to whether $\mu \geq 0$ or $\mu < 0$ in the first case, and a partition of the (μ, m)-plane in the second case.*

Proof of lemma 7.8. Let $(R_t; t \geq 0)$ and $(X_t; t \geq 0)$ be defined as solutions of :

$$X_t = x + B_t + \frac{2\mu+1}{2}\int_0^t \frac{ds}{X_s} + \int_0^t \frac{(\varphi_\lambda^{(\mu)})'}{\varphi_\lambda^{(\mu)}}(X_s)ds, \quad t \geq 0, \quad (7.30)$$

$$R_t = x + B_t + \frac{2\mu+1}{2}\int_0^t \frac{ds}{R_s}, \quad t \geq 0, \quad (7.31)$$

with the same underlying Brownian motion $(B_t; t \geq 0)$.
Since $(\varphi_\lambda^{(\mu)})'/\varphi_\lambda^{(\mu)}$ is a bounded function, we may apply Girsanov's theorem :

$$E_x\Big[F(X_s; 0 \leq s \leq t)\Big] = E_x^\mu\Big[Y_t F(R_s; 0 \leq s \leq t)\Big],$$

where

$$Y_t = \exp\Big\{\int_0^t \frac{(\varphi_\lambda^{(\mu)})'}{\varphi_\lambda^{(\mu)}}(R_s)dB_s - \frac{1}{2}\int_0^t \Big(\frac{(\varphi_\lambda^{(\mu)})'}{\varphi_\lambda^{(\mu)}}\Big)^2(R_s)ds\Big\}. \quad (7.32)$$

Applying by now standard arguments (cf, formula (2.6)), we obtain :

$$Y_t = \frac{\varphi_\lambda^{(\mu)}(R_t)}{\varphi_\lambda^{(\mu)}(x)}\exp\Big\{-\int_0^t \frac{L^{(\mu)}(\varphi_\lambda^{(\mu)})}{\varphi_\lambda^{(\mu)}}(R_s)ds\Big\}.$$

We conclude from (7.11) that (7.27) holds. ■

Lemma 7.10 *Let $x \geq 0$. Let us denote by $(X_t; t \geq 0)$ the diffusion :*

$$X_t = x + B_t + \frac{2\mu+1}{2}\int_0^t \frac{ds}{X_s} + \int_0^t \frac{(\varphi_\lambda^{(\mu)})'}{\varphi_\lambda^{(\mu)}}(X_s)ds, \quad t \geq 0, \tag{7.33}$$

and $(R_t^\mu; t \geq 0)$ (resp. $(R_t^{\mu+2n_\mu}; t \geq 0)$ the Bessel processes with index $\nu = \mu$, (resp. $\nu = \mu + 2n_\mu$) solving :

$$R_t^\nu = x + \tilde{B}_t + \frac{2\nu+1}{2}\int_0^t \frac{ds}{R_s^\nu}, \quad t \geq 0, \tag{7.34}$$

where $(\tilde{B}_t; t \geq 0)$ is the Brownian motion : $\tilde{B}_t = \int_0^t sgn(X_s)dB_s,\ t \geq 0$.
Then a.s. for any $t \geq 0$:

$$(R_t^\mu)^2 \leq X_t^2 \leq (R_t^{\mu+2n_\mu})^2. \tag{7.35}$$

Proof of lemma 7.10. Applying Itô's formula to the squares of the processes X and R^ν, we obtain :

$$X_t^2 = x^2 + 2\int_0^t \sqrt{X_s^2}d\tilde{B}_s + \int_0^t \Big(2X_s\frac{(\varphi_\lambda^{(\mu)})'}{\varphi_\lambda^{(\mu)}}(X_s) + 2(\mu+1)\Big)ds, \tag{7.36}$$

$$(R_t^\nu)^2 = x^2 + 2\int_0^t \sqrt{(R_s^\nu)^2}d\tilde{B}_s + 2(\nu+1)t. \tag{7.37}$$

We observe that the function $x \mapsto \dfrac{2x(\varphi_\lambda^{(\mu)})'(x)}{\varphi_\lambda^{(\mu)}(x)} + 2(\mu+1)$ may be written as $h(x^2)$, a function of x^2, hence:

$$X_t^2 = x^2 + 2\int_0^t \sqrt{X_s^2}d\tilde{B}_s + \int_0^t h(X_s^2)ds. \tag{7.38}$$

Inequalities (7.22) and (7.23) imply :

$$2(\mu+1) \leq h(x) \leq 2(2n_\mu + \mu + 1). \tag{7.39}$$

The inequalities (7.35) are a direct consequence of comparison results for solutions of one-dimensional stochastic differential equations [17, Prop 2.18, Chapter 5]. ■

Remark 7.11 *In (7.34) we have chosen the Brownian motion $(\tilde{B})$ instead of (B) in order to obtain the inequalities (7.35)). Replacing $(\tilde{B}_t)$ by (B_t) in (7.34) does not change the law of (R_t^ν).*

Lemma 7.12 *Let $a > 0$ be fixed. Let $(Y_t^a; t \geq 0)$ and $(Z_t^a; t \geq 0)$ denote the processes :*

$$Y_t^a = \frac{1}{a}(R_{at}^\nu)^2; \quad Z_t^a = \frac{1}{a}X_{at}^2, \quad t \geq 0, \tag{7.40}$$

with $\nu = \mu + 2n_\mu$, $(X_t; t \geq 0)$ (resp. $(R_t^\nu; t \geq 0)$) solving (7.33) (resp. (7.34)).
Then for any $1 \leq p < \infty$, we have :

$$\lim_{a\to+\infty} \Big(E[(Y_1^a - Z_1^a)^p]\Big) = 0. \tag{7.41}$$

Proof of Lemma 7.12. From (7.36) and (7.37), we deduce :

$$0 \le E[Y_t^a - Z_t^a] \le E\Big[\int_0^t \Big(4n_\mu - 2X_{as}\frac{(\varphi_\lambda^{(\mu)})'}{\varphi_\lambda^{(\mu)}}(X_{as})\Big)ds\Big].$$

Using successively (7.23) and (7.35), we obtain :

$$E[Y_t^a - Z_t^a] \le 2\theta_\mu E\Big[\int_0^t \frac{1}{1+(X_{as})^2}ds\Big] \le 2\theta_\mu \int_0^t E\Big[\frac{1}{1+(R_{as}^\nu)^2}\Big]ds,$$

The scaling property of Bessel processes yields to :

$$E\Big[\frac{1}{1+(R_{as}^\nu)^2}\Big] = E_x^\nu\Big[\frac{1}{1+(R_{as})^2}\Big] = E_{x/\sqrt{a}}^\nu\Big[\frac{1}{1+a(R_s)^2}\Big].$$

Obviously the right hand-side of the previous inequality tends to 0, as $a \to +\infty$. The dominated convergence theorem implies that :

$$\lim_{a\to+\infty}\Big(E[Y_t^a - Z_t^a]\Big) = 0.$$

Let $1 \le p < \infty$ and $t = 1$. Since $0 \le Y_1^a - Z_1^a$, then $(Y_1^a - Z_1^a)^p$ goes to 0 in probability, as $a \to \infty$. We claim that for any $\alpha \in]1, \infty[$:

$$\sup_{a\ge 1} E[(Y_1^a - Z_1^a)^{p\alpha}] < \infty. \tag{7.42}$$

This will prove (7.41), because (7.42) implies that $((Y_1^a - Z_1^a)^p; a \ge 1)$ is uniformly integrable.
To prove (7.42), we use the definition of Y^a and (7.35). Denoting $\beta = p\alpha$, we have :

$$E[(Y_1^a - Z_1^a)^\beta] \le E[(Y_1^a)^\beta] \le E\Big[\Big(\frac{R_a^\nu}{\sqrt{a}}\Big)^\beta\Big].$$

Recall that with our notations :

$$E\Big[\Big(\frac{R_a^\nu}{\sqrt{a}}\Big)^\beta\Big] = E_x^\nu\Big[\Big(\frac{R_a}{\sqrt{a}}\Big)^\beta\Big] = E_{x/\sqrt{a}}^\nu[(R_1)^\beta].$$

The second equality follows from the scaling property of Bessel processes. Comparison theorem tells us :

$$E_{x/\sqrt{a}}^\nu\Big[(R_1)^\beta\Big] \le E_x^\nu\Big[(R_1)^\beta\Big] < \infty,$$

for any $a \ge 1$. This proves (7.42).

■

Proof of Theorem 7.3 1) Using the scaling property of Bessel processes, we get :

$$E_x^\mu\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t \frac{ds}{\theta + R_s^2}\Big\}\Big] = E_{x/\sqrt{\theta}}^\mu\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^{t/\theta}\frac{ds}{1+R_s^2}\Big\}\Big]. \tag{7.43}$$

Thus, it suffices to prove (7.13) when $\theta = 1$.

2) Let $\theta = 1$. From (7.29), it remains to prove :

$$\lim_{t\to\infty}\Big(t^{n_\mu} E_x\Big[\frac{1}{\varphi_\lambda^{(\mu)}(X_t)}\Big]\Big) = \frac{1}{2^{n_\mu}}\frac{\Gamma(\mu + n_\mu + 1)}{\Gamma(\mu + 2n_\mu + 1)}. \tag{7.44}$$

By (7.40) and (7.24), we get :

$$E_x\Big[\frac{1}{\varphi_\lambda^{(\mu)}(X_t)}\Big] = \frac{1}{t^{n_\mu}} E_x\ \Big[\frac{1}{D_\lambda^{(\mu)}(t,\sqrt{Z_1^t})}\Big],$$

3) Let us prove :

$$\lim_{t\to\infty} E_x\Big[\frac{1}{D_\lambda^{(\mu)}(t,\sqrt{Z_1^t})} - \frac{1}{D_\lambda^{(\mu)}(t,\sqrt{Y_1^t})}\Big] = 0. \tag{7.45}$$

Using (7.26), we obtain :

$$E_x\Big[\frac{1}{D_\lambda^{(\mu)}(t,\sqrt{Z_1^t})} - \frac{1}{D_\lambda^{(\mu)}(t,\sqrt{Y_1^t})}\Big] \le CE_x\Big[\frac{\sqrt{Y_1^t}-\sqrt{Z_1^t}}{(Z_1^t)^{n_\mu+1/2}}\Big] \le CE_x\Big[\frac{\sqrt{Y_1^t - Z_1^t}}{(Z_1^t)^{n_\mu+1/2}}\Big].$$

Since $\lambda < 8\mu^2+6\mu+1$ and $n_\mu = \frac{-\mu+\sqrt{\mu^2+\lambda}}{2}$, then $n_\mu + 1/2 - \mu < 1$. Hence we may find $\varepsilon > 0$ such that :

$$n_\mu + 1/2 + \varepsilon - \mu < 1. \tag{7.46}$$

Let $q = q(\varepsilon) = \dfrac{n_\mu + 1/2 + \varepsilon}{n_\mu + 1/2} > 1$ and p be the conjugate exponent of q.
Applying Hölder's inequality leads to :

$$E_x\Big[\frac{1}{D_\lambda^{(\mu)}(t,\sqrt{Z_1^t})} - \frac{1}{D_\lambda^{(\mu)}(t,\sqrt{Y_1^t})}\Big] \le C\Big\{E_x\big[(Y_1^t - Z_1^t)^{p/2}\big]\Big\}^{1/p}\Big\{E_x\big[\frac{1}{(Z_1^t)^{n_\mu+1/2+\varepsilon}}\big]\Big\}^{1/q}.$$

Property (7.45) will be a direct consequence of (7.41), once we have proved that : $t \to E_x\big[\frac{1}{(Z_1^t)^{n_\mu+1/2+\varepsilon}}\big]$ is bounded.
Using the definition of Z_1^t, (7.35) and the scaling property of Bessel processes, we obtain :

$$E_x\big[\frac{1}{(Z_1^t)^{n_\mu+1/2+\varepsilon}}\big] = E_x^\mu\big[\Big(\frac{t}{R_t^2}\Big)^{n_\mu+1/2+\varepsilon}\big] = E_{x/\sqrt{t}}^\mu\big[\frac{1}{(R_1^2)^{n_\mu+1/2+\varepsilon}}\big].$$

Comparison theorem implies that :

$$E_{x/\sqrt{t}}^\mu\big[\frac{1}{(R_1^2)^{n_\mu+1/2+\varepsilon}}\big] \le E_0^\mu\big[\frac{1}{(R_1^2)^{n_\mu+1/2+\varepsilon}}\big].$$

Under P_0^μ, the distribution of $R_1^2/2$ is gamma$(\mu+1)$. Hence $E_0^\mu\big[\frac{1}{(R_1^2)^{n_\mu+1/2+\varepsilon}}\big] < \infty$ as soon as :

$$\int_0^1 \frac{y^\mu}{y^{n_\mu+1/2+\varepsilon}}dy < \infty.$$

This integral is finite since condition (7.46) holds.
4) Due to the scaling property of Bessel processes,

$$E_x\Big[\frac{1}{D_\lambda^{(\mu)}(t,\sqrt{Y_1^t})}\Big] = E_{x/\sqrt{t}}^\nu\Big[\frac{1}{D_\lambda^{(\mu)}(t,R_1)}\Big],$$

where $\nu = \mu + 2n_\mu$.
Applying (7.25), we obtain :

$$\lim_{t\to\infty} E_{x/\sqrt{t}}^\nu\Big[\frac{1}{D_\lambda^{(\mu)}(t,R_1)}\Big] = E_0^\nu\Big[\frac{1}{R_1^{2n_\mu}}\Big].$$

Relation (7.44) now follows from (7.8). ■

Remark 7.13 *Note that condition (7.10) has only been used in the last part of the proof of Theorem 7.3. It may not be necessary but we have not been able to justify this.*

Theorem 7.14 *Assume that $\lambda,\ \theta > 0$ obey (7.10). Let $Q_{x,t}$ be the probability defined on $\mathcal{F}_t$ via :*

$$Q_{x,t}(\Lambda_t) = \frac{E_x\Big[1_{\Lambda_t}\ \exp\Big\{-\frac{\lambda}{2}\int_0^t \frac{ds}{\theta+R_s^2}\Big\}\Big]}{E_x\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t \frac{ds}{\theta+R_s^2}\Big\}\Big]},\ \Lambda_t \in \mathcal{F}_t. \tag{7.47}$$

Then, for any Λ_s in $\mathcal{F}_s$, $Q_{x,t}(\Lambda_s)$ converges to $P_x^{\varphi_\lambda^{(\mu)}}(\Lambda_s)$ as $t \to \infty$, where $P_x^{\varphi_\lambda^{(\mu)}}$ is the probability defined on $\mathcal{F}_\infty$ by :

$$P_x^{\varphi_\lambda^{(\mu)}}(\Lambda_s) = \frac{1}{\varphi_\lambda^{(\mu)}(x/\sqrt{\theta})} E_x\Big[1_{\Lambda_s}\varphi_\lambda^{(\mu)}(R_s/\sqrt{\theta})\exp\Big\{-\frac{1}{2}\int_0^s \frac{dv}{\theta+R_v^2}\Big\}\Big], \tag{7.48}$$

for any $s > 0$ and $\Lambda_s \in \mathcal{F}_s$.
Let $(X_t^x; t \geq 0)$ be the solution to :

$$X_t = x + B_t + \frac{2\mu+1}{2}\int_0^t \frac{ds}{X_s} + \frac{1}{\sqrt{\theta}}\int_0^t \frac{(\varphi_\lambda^{(\mu)})'}{\varphi_\lambda^{(\mu)}}(X_s/\sqrt{\theta})ds, \quad t \geq 0. \tag{7.49}$$

Then the law of $(X_t^x; t \geq 0)$ is $P_x^{\varphi_\lambda^{(\mu)}}$.

The proof of Theorem 7.14 is based on Theorem 7.3 and the estimate (7.25), the details are left to the reader.

Let us mention two consequences of Theorem 7.14.

Corollary 7.15 *Let $\mu \geq -1/2, \theta > 0$ satisfying (7.10). Then*

$$\lim_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon^{n_\mu}}E^\mu_{x\sqrt{\varepsilon}}\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^1 \frac{ds}{\theta\varepsilon+R_s^2}\Big\}\Big]\Big) = \theta^{n_\mu}\varphi_\lambda^{(\mu)}(x/\sqrt{\theta})\frac{1}{2^{n_\mu}}\frac{\Gamma(\mu+n_\mu+1)}{\Gamma(\mu+2n_\mu+1)}. \tag{7.50}$$

Corollary 7.16 *Assume $\mu \geq -1/2, \theta > 0$ and (7.10) holds. Let ψ_λ be the unique solution of :*

$$\begin{cases} \frac{\partial\psi}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2\psi}{\partial x^2}(t,x) + \frac{2\mu+1}{2x}\frac{\partial\psi}{\partial x}(t,x) - \frac{\lambda}{2}\frac{\psi(t,x)}{\theta+x^2}, \quad t > 0, x \geq 0, \\ \psi(0,x) = 1. \end{cases} \tag{7.51}$$

Then

$$\lim_{t\to\infty}\Big(t^{n_\nu}\psi_\lambda(t,x)\Big) = \theta^{n_\mu}\varphi_\lambda^{(\mu)}(x/\sqrt{\theta})\frac{1}{2^{n_\mu}}\frac{\Gamma(\mu+n_\mu+1)}{\Gamma(\mu+2n_\mu+1)}. \tag{7.52}$$

8 On the use of large deviations

In this section we will be concerned with λV, where $\lambda > 0$ and :

$$V(x) = \frac{1}{1+|x|^\alpha}; \qquad x \in \mathbb{R},\ 0 < \alpha < 2. \tag{8.1}$$

We investigate the asymptotic behaviour of :

$$Z_t^{\lambda V}(x) = E_x\Big[\exp\Big\{-\frac{\lambda}{2}\int_0^t \frac{ds}{1+|B_s|^\alpha}\Big\}\Big], \tag{8.2}$$

when $t \to \infty$.
Notice that if $\alpha > 2$ then $\lambda\int_{\mathbb{R}} V(x)|x|dx < \infty$, hence we may apply the results of section 3. The critical case $\alpha = 2$ has been treated in the previous section.

Theorem 8.1 *Let $0 < \alpha < 2$, $\lambda > 0$. Then*

$$\lim_{t\to\infty} \Big(t^{\frac{\alpha-2}{\alpha+2}} \ln\big(Z_t^{\lambda V}(x)\big)\Big) = -\frac{1}{2} I_0(\lambda), \tag{8.3}$$

where

$$I_0(\lambda) = \inf_{\psi\in\mathcal{C}_0} \Big\{ \int_0^1 \dot\psi^2(s)ds + \lambda \int_0^1 \frac{ds}{|\psi(s)|^\alpha} \Big\}, \tag{8.4}$$

belongs to $]0,+\infty[$, and $\mathcal{C}_0$ is the set of continuous functions $f : [0,1] \to \mathbb{R}$ vanishing at 0.

Remark 8.2 *We observe that the limit in (8.3) does not depend on x, which may be due to the fact this result only gives a logarithmic equivalent to $Z_t^V(x)$. Indeed, consider the equivalent of $Z_t^V(x)$ given by Theorem 5.1 : $\lim_{t\to\infty}\Big(\sqrt{t} Z_t^V(x)\Big) = \varphi_V(x)$. Then $\dfrac{\ln(Z_t^V(x))}{\ln t}$ converges to the constant $-1/2$, as $t \to \infty$.*

Lemma 8.3 *Let $\eta > 0$. Let us denote :*

$$I_\eta(\lambda) = \inf_{\psi\in\mathcal{C}_0} \Big\{ \int_0^1 \dot\psi^2(s)ds + \lambda \int_0^1 \frac{ds}{\eta + |\psi(s)|^\alpha} \Big\}. \tag{8.5}$$

Then $I_\eta(\lambda)$ is a positive real number, $\eta \mapsto I_\eta(\lambda)$ is decreasing and

$$\lim_{\eta\to 0} I_\eta(\lambda) = I_0(\lambda). \tag{8.6}$$

Proof of Lemma 8.3. Let ψ_η be a function in $\mathcal{C}_0$ such that :

$$I_\eta(\lambda) = \int_0^1 \dot\psi_\eta^2(s)ds + \lambda \int_0^1 \frac{ds}{\eta + |\psi_\eta(s)|^\alpha}. \tag{8.7}$$

Then $\psi_\eta \geq 0$ and the Euler equation associated with ψ_η is :

$$\begin{cases} 2\ddot\psi_\eta + \frac{\alpha\lambda\psi_\eta^{\alpha-1}}{(\eta+\psi_\eta^\alpha)^2} = 0 \\ \psi_\eta(0) = 0, \qquad \dot\psi_\eta(1) = 0. \end{cases} \tag{8.8}$$

Consequently ψ_η is a positive, increasing and convex function. Multiplying the first line of (8.8) by $\dot\psi_\eta$ and integrating, we obtain :

$$\dot{\psi_\eta}^2(t) = \lambda\Big(\frac{1}{\eta + \psi_\eta(t)^\alpha} - \frac{1}{\eta+\psi_\eta(1)^\alpha}\Big). \tag{8.9}$$

Let H_η be the function :

$$H_\eta(C, x) = \frac{1}{\sqrt{\lambda}} \int_0^x \frac{dy}{\sqrt{\frac{1}{\eta+y^\alpha} - \frac{1}{\eta+C^\alpha}}}, \quad x \in [0, C],\ C > 0.$$

Since the derivative of ψ_η is positive, the relation (8.9) is equivalent to :

$$\frac{d}{dt} H_\eta(C, \psi_\eta(t))) = 1; \quad 0 \leq t \leq 1,$$

or

$$H_\eta(C, \psi_\eta(t)) = t, \quad 0 \leq t \leq 1,$$

with $C = \psi_\eta(1)$.

This implies that ψ_η is the inverse of $t\ (\geq 0) \mapsto H_\eta(C,t)$. As for C, we observe that it remains to take into account the condition : $C = \psi_\eta(1)$. Let C_η be the unique solution in $]0,+\infty[$ of :

$$\frac{1}{\sqrt{\lambda}} \int_0^{C_\eta} \frac{dy}{\sqrt{\frac{1}{\eta+y^\alpha} - \frac{1}{\eta+C_\eta^\alpha}}} = 1. \tag{8.10}$$

Taking $C = C_\eta$, we have $C = \psi_\eta(1)$.

■

Lemma 8.4 *Let $\alpha \in]0,2[$ and ψ_0 be defined by the relation (8.7), with $\eta = 0$. Then*

$$\liminf_{\varepsilon\to 0} \Big(\varepsilon \ln \big(P\big\{\sqrt{\varepsilon}|B_s|^\alpha + \varepsilon^{\frac{2\alpha}{2-\alpha}} \geq \psi_0(s)^\alpha;\ \forall s \in [0,1]\big\}\big)\Big) \geq -\frac{1}{2}\int_0^1 \dot{\psi}_0^2(s)ds. \tag{8.11}$$

Proof of Lemma 8.4. We suppose for simplicity $\alpha = 1$, the general case being only slightly more complicated.
Let us introduce the set :

$$\Gamma_\varepsilon = \big\{\sqrt{\varepsilon}|B_s| \geq \psi_0(s) - \varepsilon^2;\ \forall s \in [0,1]\big\}.$$

Since ψ_0 is an increasing and positive function, there exists $\delta(\varepsilon) > 0$ such that $\psi_0(s) \geq \varepsilon^2$ if and only if $s \geq \delta(\varepsilon)$ and $\delta(\varepsilon)$ goes to 0 as $\varepsilon \to 0$. Consequently :

$$\Gamma_\varepsilon = \big\{\sqrt{\varepsilon}B_s \geq \psi_0(s) - \varepsilon^2;\ \forall s \in [\delta(\varepsilon),1]\big\} \cup \big\{\sqrt{\varepsilon}B_s \leq -\psi_0(s) + \varepsilon^2;\ \forall s \in [\delta(\varepsilon),1]\big\}.$$

Then, computing the probability of Γ_ε, we obtain :

$$P(\Gamma_\varepsilon) = P\big\{B_s \geq \frac{\psi_0(s)}{\sqrt{\varepsilon}} - \varepsilon^{3/2};\ \forall s \in [\delta(\varepsilon),1]\big\},$$

$$\geq P\big\{B_s - \frac{\psi_0(s)}{\sqrt{\varepsilon}} \geq -\varepsilon^{3/2};\ \forall s \in [0,1]\big\}.$$

Let us denote by Λ_ε the set : $\Lambda_\varepsilon = \{\inf_{s\in[0,1]} B_s \geq -\varepsilon^{3/2}\}$. Using Girsanov's theorem, we have :

$$P(\Gamma_\varepsilon) \geq E\Big[1_{\Lambda_\varepsilon}\ \exp\Big\{-\frac{1}{\sqrt{\varepsilon}}\int_0^1 \dot{\psi}_0(s)dB_s - \frac{1}{2\varepsilon}\int_0^1 \dot{\psi}_0(s)^2 ds\Big\}\Big],$$

$$\geq \exp\Big\{-\frac{1}{2\varepsilon}\int_0^1 \dot{\psi}_0(s)^2 ds\Big\}\ E\Big[1_{\Lambda_\varepsilon}\ \exp\Big\{-\frac{1}{\sqrt{\varepsilon}}\int_0^1 \dot{\psi}_0(s)dB_s\Big\}\Big].$$

Jensen's inequality applied to $x \mapsto e^{-x}$ leads to :

$$\frac{1}{P(\Lambda_\varepsilon)}E\Big[1_{\Lambda_\varepsilon}\ \exp\Big\{-\frac{1}{\sqrt{\varepsilon}}\int_0^1 \dot{\psi}_0(s)dB_s\Big\}\Big] \geq \exp\Big\{-\frac{1}{P(\Lambda_\varepsilon)}E\Big[1_{\Lambda_\varepsilon}\frac{1}{\sqrt{\varepsilon}}\int_0^1 \dot{\psi}_0(s)dB_s\Big]\Big\}.$$

Holder's inequality yields to :

$$E\Big[1_{\Lambda_\varepsilon}\Big|\int_0^1 \dot{\psi}_0(s)dB_s\Big|\ \Big] \leq C(p)P(\Lambda_\varepsilon)^{1/p},$$

where $p > 1$ and

$$C(p) = \Big(E\Big[\big|\int_0^1 \dot{\psi}_0(s)dB_s\big|^q\Big]\Big)^{1/q} = c_q\Big(\int_0^1 (\dot{\psi}_0(s))^2 ds\Big)^{1/2}, \quad \frac{1}{p}+\frac{1}{q} = 1,$$

where $c_q = \Big(E[|B_1|^q]\Big)^{1/q}$ is a universal constant.

Recall that $(-\inf_{s\in[0,1]} B_s)$ is distributed as $|B_1|$, hence, there exist two positive constants c_0 and c_1 such that :

$$c_0\varepsilon^{3/2} \le P(\Lambda_\varepsilon) = P(|B_1| \le \varepsilon^{3/2}) \le c_1\varepsilon^{3/2}, \quad \varepsilon \in]0,1].$$

Let $0 < \delta < 1/2$, we choose $p > 1$ such that $1 - \dfrac{1}{p} = \dfrac{2}{3}\delta$, then :

$$\frac{1}{P(\Lambda_\varepsilon)} E\Big[1_{\Lambda_\varepsilon} \frac{1}{\sqrt{\varepsilon}} \int_0^1 \dot{\psi}_0(s) dB_s\Big] \le \frac{c'_q\Big(\int_0^1 (\dot{\psi}_0(s))^2 ds\Big)^{1/2}}{\varepsilon^{\delta+1/2}},$$

$$P(\Gamma_\varepsilon) \ge c_0\varepsilon^{3/2} \exp\Big\{-\frac{1}{2\varepsilon}\int_0^1 \dot{\psi}_0(s)^2 ds\Big\}\ \exp\Big\{-\frac{c'_q\Big(\int_0^1 (\dot{\psi}_0(s))^2 ds\Big)^{1/2}}{\varepsilon^{\delta+1/2}}\Big\},$$

where $c'_q = \dfrac{c_q}{c_0^{2\delta/3}}$. Then (8.11) follows immediately.

■

Proof of Theorem 8.1 Suppose that $x = 0$. Setting $\varepsilon = t^{\frac{\alpha-2}{2+\alpha}}$ and using the scaling property of Brownian motion and definition (8.2), we have :

$$Z_t^{\lambda V}(0) = E_0\Big[\exp\Big\{-\frac{\lambda t}{2}\int_0^1 \frac{ds}{1+t^{\alpha/2}|B_s|^\alpha}\Big\}\Big] = E_0\Big[\exp\Big\{-\frac{\lambda}{2\varepsilon}\int_0^1 \frac{ds}{\varepsilon^{\frac{2\alpha}{2-\alpha}} + |\sqrt{\varepsilon}B_s|^\alpha}\Big\}\Big]. \quad (8.12)$$

1)We first prove :

$$\limsup_{t\to+\infty}\Big(t^{\frac{\alpha-2}{\alpha+2}}\ln\big(Z_t^{\lambda V}(0)\big)\Big) \le -\frac{1}{2}I_0(\lambda), \quad (8.13)$$

where $I_0(\lambda)$ is defined by (8.4).
Let $\eta > 0$ be a fixed real number, and $\varepsilon > 0$ such that $\varepsilon^{\frac{2\alpha}{2-\alpha}} < \eta$. Hence :

$$Z_t^{\lambda V}(0) \le \exp\big\{-\frac{1}{2\varepsilon}\Phi_\eta(\lambda, \sqrt{\varepsilon}B.)\big\}, \quad (8.14)$$

where

$$\Phi_\eta(\lambda, f) = \lambda\int_0^1 \frac{ds}{\eta + |f(s)|^\alpha}. \quad (8.15)$$

Varadhan's theorem [9] yields to:

$$\lim_{\varepsilon\to 0}\Big(\varepsilon\ln\Big(E_0\Big[\exp\big\{-\frac{1}{2\varepsilon}\Phi_\eta(\lambda, \sqrt{\varepsilon}B.)\big\}\Big]\Big)\Big) = -\frac{1}{2}I_\eta(\lambda),$$

where $I_\eta(\lambda)$ is defined by (8.5).
Consequently,

$$\limsup_{t\to+\infty}\Big(t^{\frac{\alpha-2}{\alpha+2}}\ln\big(Z_t^{\lambda V}(0)\big)\Big) \le -\frac{1}{2}I_\eta(\lambda),$$

for any $\eta > 0$.
Lemma 8.3 implies (8.13).

2) We claim that :

$$\liminf_{t\to+\infty}\Big(t^{\frac{\alpha-2}{\alpha+2}}\ln\big(Z_t^{\lambda V}(0)\big)\Big) \ge -\frac{1}{2}I_0(\lambda), \quad (8.16)$$

Starting from (8.12), we have :

$$Z_t^{\lambda V}(x) \geq E_0\Big[1_{\Gamma_\eta} \exp\Big\{ -\frac{\lambda}{2\varepsilon}\int_0^1 \frac{ds}{\varepsilon^{\frac{2\alpha}{2-\alpha}} + |\sqrt{\varepsilon}B_s|^\alpha}\Big\}\Big],$$

with $\eta > 0$ and

$$\Gamma_\eta = \big\{\sqrt{\varepsilon}|B_s|^\alpha + \varepsilon^{\frac{2\alpha}{2-\alpha}} \geq \psi_0(s)^\alpha;\ \forall s \in [0,1]\big\}\Big).$$

Hence,

$$Z_t^{\lambda V}(x) \geq \exp\Big\{ -\frac{\lambda}{2\varepsilon}\int_0^1 \frac{ds}{\psi_0(s)^\alpha}\Big\} P(\Gamma_\eta).$$

Relation (8.16) is a direct consequence of Lemma 8.4 and (8.7). ■

9 The bilateral case

In this section it is required that $V(x)$ goes to $+\infty$ as $|x| \to \infty$. The asymptotic behaviour of $Z_t^V(x) := E_x\Big[\exp\big\{-\frac{1}{2}\int_0^t V(B_s)ds\big\}\Big]$, when $t \to \infty$ has been initiated by Kac [16] . Let us briefly recall (cf [30]) the main result. Let us consider the second order differential equation :

$$\frac{1}{2}\psi" - V\psi = -\lambda\psi. \tag{9.1}$$

Then there exist a sequence $(\lambda_n)_{n\geq 1}$ of positive numbers, $0 < \lambda_1 < \lambda_2 < \cdots$ and an orthonormal basis of functions $(\psi_n)_{n\geq 1}$ in $L^2(\mathbb{R})$ such that for any n, ψ_n is a solution to (9.1) with $\lambda = \lambda_n$, and the ground state $\psi_1 > 0$. This spectral gap property (i.e. $\lambda_1 > 0$, cf [30]) plays a central role. With some additional assumptions, Kac proved :

$$E_x\Big[\exp\big\{-\frac{1}{2}\int_0^t V(B_s)ds\big\}\Big] \sim \rho e^{-\lambda_1 t} \quad t \to \infty, \tag{9.2}$$

where

$$\rho = \sum_{\lambda_j = \lambda_1} \psi_j(0)\int_{\mathbb{R}} \psi_j(x)dx.$$

R. Carmona [4], [5] generalized this result to the case where V may be written as the sum $V_1 + V_2$, where $V_2 \in L^p(\mathbb{R})$, V_1 being larger than a constant and fulfilling for any $\beta > 0$:

$$\lim_{|x|\to\infty}\Big(\int_{x-\alpha}^{x+\alpha} e^{-\beta V_1(y)}dy = 0\Big), \text{ for some } \alpha > 0.$$

The proof is based on the compactness of the family of operators $(T_t)_{t\geq 0}$, and the discrete spectrum of the generator L of the semi group $(T_t)_{t\geq 0}$:

$$L(f) = \frac{1}{2}f" - Vf. \tag{9.3}$$

Here, we investigate the case where

$$V : \mathbb{R} \to \mathbb{R} \text{ is an even function, non-decreasing on } [0, +\infty[, \tag{9.4}$$

$$V(x) \text{ converges to a real number } \bar{V}, \text{ as } |x| \to \infty. \tag{9.5}$$

Notice that, to our knowledge, this setting was neither considered by Kac nor Carmona.

Our approach is direct. We prove that there exists a unique γ_0 such that a solution $\varphi_{V-\gamma_0}$ to the Sturm-Liouville equation $\varphi'' = (V-\gamma_0)\varphi$ with suitable boundary conditions, satisfies the condition of Section (4). This allows us to prove the exponential decay of $Z_t^V(x)$, as $t \to \infty$.
In order to present our main result in Theorem 9.1 below,we need to define the parameter γ_0.
We start with the following definitions :

$$\underline{V} = \inf_{x\in\mathbb{R}} V(x) = V(0)\ ,\ \bar{V} = \sup_{x\in\mathbb{R}} V(x) < \infty. \tag{9.6}$$

Notice that we do not require that $V(x)$ is non-negative.
If V is constant, the result is obvious. Therefore we suppose in the sequel that : $\underline{V} < \bar{V}$.
The function V being monotone on $[0,\infty[$ (resp. $]-\infty,0]$), coincides with its right continuous modification V_0, except on an at most countable set. Then, a.s. :

$$\int_0^t V(B_s)ds = \int_0^t V_0(B_s)ds, \qquad \text{for any } t \geq 0.$$

Then V may be assumed to be right continuous.
Let V^{-1} be the right inverse of the restriction of V to $[0,\infty[$:

$$V^{-1}(\gamma) = \inf\{t \geq 0; V(t) > \gamma\}, \quad \gamma \in]\underline{V}, \bar{V}[.$$

Then:

$$V(V^{-1}(\gamma)) \geq \gamma, \quad \underline{V} < \gamma < \bar{V}. \tag{9.7}$$

For any $\gamma \in]\underline{V}, \bar{V}[$, let F_γ (resp. G_γ) be the unique solution to

$$Y" = (V-\gamma)Y, \tag{9.8}$$

on $[0, V^{-1}(\gamma)]$ (resp. $[V^{-1}(\gamma), \infty[$) with the boundary conditions

$$F_\gamma(V^{-1}(\gamma)) = 1, \qquad F'_\gamma(0) = 0. \tag{9.9}$$

$$G_\gamma(V^{-1}(\gamma)) = 1, \qquad G_\gamma(+\infty) := \lim_{x\to\infty} G_\gamma(x) = 0. \tag{9.10}$$

We set:

$$\varphi_{V-\gamma}(x) = \begin{cases} F_\gamma(x) & \text{if} \quad x \in [0, V^{-1}(\gamma)] \\ G_\gamma(x) & \text{if} \quad x \geq V^{-1}(\gamma). \end{cases} \tag{9.11}$$

We extend $\varphi_{V-\gamma}$ to the whole line, setting : $\varphi_{V-\gamma}(-x) = \varphi_{V-\gamma}(x)$. Then $\varphi_{V-\gamma}$ is a continuous and even function defined on $\mathbb{R}$. Notice that $\varphi_{V-\gamma}(x)$ is differentiable for any $x \neq \pm V^{-1}(\gamma)$.

Theorem 9.1 *Let V be a function fulfilling (9.4) and (9.5).*

1. *There exists a unique $\gamma_0 \in]\underline{V}, \bar{V}[$ such that the function $\varphi_{V-\gamma_0}$ defined by (9.11) is differentiable on $\mathbb{R}$.*
2. *The quantity:*

$$e^{\gamma_0 t/2}\ E_x\Big[\exp\Big\{-\frac{1}{2}\int_0^t V(B_s)ds\Big\}\Big],$$

converges as $t \to \infty$, to $\big(\int_{\mathbb{R}} \varphi_{V-\gamma_0}(y)dy\big)\varphi_{V-\gamma_0}(x)$.

3. *Let us define the probability* $Q^V_{x,t}$ *on* $\mathcal{F}_t$ *via :*

$$Q^V_{x,t}(\Lambda_t) = \frac{E_x\Big[1_{\Lambda_t}\ \exp\Big\{-\frac{1}{2}\int_0^t V(B_h)dh\Big\}\Big]}{E_x\Big[\exp\Big\{-\frac{1}{2}\int_0^t V(B_h)dh\Big\}\Big]},\ \Lambda_t \in \mathcal{F}_t. \tag{9.12}$$

Then, for any Λ_s *in* $\mathcal{F}_s$, $Q^V_{x,t}(\Lambda_s)$ *converges to* $P_x^{\varphi_{V-\gamma_0}}(\Lambda_s)$ *as* $t \to \infty$, *where* $P_x^{\varphi_{V-\gamma_0}}$ *is the probability defined on* $\mathcal{F}_\infty$ *by :*

$$P_x^{\varphi_{V-\gamma_0}}(\Lambda_s) = \frac{e^{\gamma_0 s/2}}{\varphi_{V-\gamma_0}(x)} E_x\Big[1_{\Lambda_s}\varphi_{V-\gamma_0}(B_s)\exp\Big\{-\frac{1}{2}\int_0^s V(B_h)dh\Big\}\Big], \tag{9.13}$$

for any $s > 0$ *and* $\Lambda_s \in \mathcal{F}_s$.

4. *Let* $(X^x_t; t \geq 0)$ *be the solution to :*

$$X_t = x + B_t + \int_0^t \frac{\varphi'_{V-\gamma_0}}{\varphi_{V-\gamma_0}}(X_s)ds,\ t \geq 0. \tag{9.14}$$

Then the law of $(X^x_t; t \geq 0)$ *is* $P_x^{\varphi_{V-\gamma_0}}$.

5. *The process* $(X^x_t; t \geq 0)$ *is recurrent with invariant finite measure* $(\varphi_{V-\gamma_0})^2(x)dx$.

We begin by proving two preliminary results in the form of the next Lemmas 9.2 and 9.3.

Lemma 9.2 *Let* θ_1, θ_2 *be two functions defined on* $[a, b)$, $\theta_1 \geq \theta_2 \geq 0$, φ_i *a solution of* $\varphi_i'' = \theta_i\varphi_i, i = 1, 2$ *on* $[a, b)$, *such that* $\varphi_1(a) = \varphi_2(a)$ *and* $\varphi_1(b) = \varphi_2(b) \geq 0$ *(If* $b = +\infty$, $\varphi_i(b)$ *has to be understood as* $\lim_{x\to\infty}\varphi_i(x)$*). Then* $\varphi_2 \geq \varphi_1$.

Proof of Lemma 9.2. Suppose there exists x_0 in (a, b) such that $\varphi_2(x_0) < \varphi_1(x_0)$. We can find a non-empty interval $[\alpha, \beta[$ included in $[a, b)$ such that $\varphi_1(\alpha) = \varphi_2(\alpha)$, $\varphi_1(\beta) = \varphi_2(\beta)$ and $\varphi_1 > \varphi_2$ on $]\alpha, \beta[$.
Let $h = \varphi_1 - \varphi_2$. Then $h'' = \theta_1\varphi_1 - \theta_2\varphi_2$.
But on $[\alpha, \beta[$:

$$\theta_1 \geq \theta_2 \geq 0, \quad \varphi_1 \geq \varphi_2 \geq 0, \quad \Rightarrow h'' \geq 0.$$

This generates a contradiction because then h is a non-constant and non-negative convex function on $[\alpha, \beta[$ such that $h(\alpha) = h(\beta) = 0$. ■

Lemma 9.3 *Let* $\gamma \in]\underline{V}, \bar{V}[$.

1. *There exists two positive constants* k_1, k_2 *such that*

$$\varphi_{V-\gamma}(x) \leq k_1 e^{-k_2|x|}. \tag{9.15}$$

2. *The function :* $\gamma :\in]\underline{V}, \bar{V}[\mapsto \varphi'_{V-\gamma}(V^{-1}(\gamma)_-)$ *is continuous, increasing and*

$$\lim_{\gamma\to\underline{V}} \varphi'_{V-\gamma}(V^{-1}(\gamma)_-) = 0, \qquad \liminf_{\gamma\to\bar{V}} \big(-\varphi'_{V-\gamma}(V^{-1}(\gamma)_-)\big) > 0 \tag{9.16}$$

($\varphi'_{V-\gamma}(V^{-1}(\gamma)_-)$ *denotes the left derivative of* $\varphi'_{V-\gamma}$ *at point* $V^{-1}(\gamma)$*).*

3. *We have :*

$$\lim_{\gamma\to\bar{V}} \varphi'_{V-\gamma}(V^{-1}(\gamma)_+) = 0, \quad \lim_{\gamma\to\underline{V}} \varphi'_{V-\gamma}(V^{-1}(\gamma)_+) < 0. \tag{9.17}$$

Proof of Lemma 9.3. 1) Let $\underline{u} < \gamma < \gamma' < \bar{u}$ and θ be the solution to $\theta'' = (\gamma' - \gamma)\theta$, on $[V^{-1}(\gamma'), +\infty[$, with the boundary conditions : $\theta(V^{-1}(\gamma')) = \varphi_{V-\gamma}(V^{-1}(\gamma'))$, $\theta(+\infty) = \varphi_{V-\gamma}(+\infty) = 0$.
Because $0 < \gamma' - \gamma \le V(x) - \gamma$, for any $x \ge V^{-1}(\gamma')$, Lemma 9.2 implies $\varphi_{V-\gamma} \le \theta$.
But

$$\theta(x) = \varphi_{V-\gamma}(V^{-1}(\gamma'))e^{-k_1(x-V^{-1}(\gamma'))},$$

with $k_1 = \sqrt{\gamma' - \gamma}$.
This proves (9.15).
2) Since $\varphi_{V-\gamma}$ coincides with F_γ on $[0, V^{-1}(\gamma)]$,

$$\varphi'_{V-\gamma}(V^{-1}(\gamma)_-) = \int_0^{V^{-1}(\gamma))} (V(x) - \gamma)\varphi_{V-\gamma}(x)dx.$$

The function $\varphi_{V-\gamma}$ is concave on $[0, V^{-1}(\gamma)]$, consequently, if $x \in [0, V^{-1}(\gamma)]$, then

$$\varphi_{V-\gamma}(x) \ge 1, \tag{9.18}$$

$$|\varphi'_{V-\gamma}(V^{-1}(\gamma)_-)| = -\varphi'_{V-\gamma}(V^{-1}(\gamma)_-), \tag{9.19}$$

$$|\varphi'_{V-\gamma}(V^{-1}(\gamma)_-)| = \int_0^{V^{-1}(\gamma))} (\gamma - V(x))\varphi_{V-\gamma}(x)dx, \tag{9.20}$$

$$|\varphi'_{V-\gamma}(V^{-1}(\gamma)_-)| \ge \int_0^{V^{-1}(\gamma))} (\gamma - V(x))dx, \tag{9.21}$$

$$\varphi_{V-\gamma}(x) \le 1 + \varphi'_{V-\gamma}(V^{-1}(\gamma)_-)(x - V^{-1}(\gamma)),$$
$$\varphi_{V-\gamma}(x) \le 1 + |\varphi'_{V-\gamma}(V^{-1}(\gamma)_-)|(V^{-1}(\gamma) - x).$$

Minoring $\varphi_{V-\gamma}(x)$ in (9.20), we obtain:

$$|\varphi_{V-\gamma}(V^{-1}(\gamma)_-)|\Big(1 - \int_0^{V^{-1}(\gamma))} (\gamma - V(x))(V^{-1}(\gamma) - x)dx\Big) \le \int_0^{V^{-1}(\gamma))} (\gamma - V(x))dx. \tag{9.22}$$

From now on, we suppose for simplicity that the restriction of V to $[0, +\infty[$ is strictly increasing. In particular V^{-1} is a continuous function. Relation (9.22) implies that

$$\lim_{\gamma \to \underline{V}} \varphi'_{V-\gamma}(V^{-1}(\gamma)_-) = 0.$$

From (9.21), it easily follows that

$$\liminf_{\gamma \to \bar{V}} \varphi'_{V-\gamma}(V^{-1}(\gamma)_-) > 0.$$

3) Let h be the solution to $h'' = (\bar{V} - \gamma)h$, on $[V^{-1}(\gamma), +\infty[$, with the boundary conditions : $h(V^{-1}(\gamma)) = \varphi_{V-\gamma}(V^{-1}(\gamma))$, $h(+\infty) = \varphi_{V-\gamma}(+\infty) = 0$.
Since $0 < V - \gamma \le \bar{V} - \gamma$, for any $x \ge V^{-1}(\gamma)$, Lemma 9.2 shows that $\varphi_{V-\gamma} \ge h$.
But $h(V^{-1}(\gamma)) = \varphi_{V-\gamma}(V^{-1}(\gamma))$, hence

$$|\varphi'_{V-\gamma}(V^{-1}(\gamma)_+)| = -\varphi'_{V-\gamma}(V^{-1}(\gamma)_+) \le -h'(V^{-1}(\gamma)) = \sqrt{\bar{V} - \gamma},$$

$$\lim_{\gamma\to\bar{V}} \varphi'_{V-\gamma}(V^{-1}(\gamma)_+) = 0.$$

It is easy to check that :

$$\lim_{\gamma\to\underline{V}} \varphi'_{V-\gamma}(V^{-1}(\gamma)_+) = H'(0),$$

where H is the solution to $H"(x) = (V(x)-\gamma)H(x)\ x \geq 0$, with the boundary conditions : $H(0) = 1$ and $H(\infty) = 0$.
Because $H'(\infty) = 0$,

$$-H'(0) = \int_0^\infty (V(y)-\gamma)H(y)dy > 0.$$

This gives (9.17). ■

Proof of Theorem 9.1.
The existence of γ_0 such that $\varphi_{V-\gamma_0}$ is of class C^1 can be derived using the continuity of the functions $\gamma \mapsto \varphi'_{V-\gamma}(V^{-1}(\gamma)_+)$ and $\gamma \mapsto \varphi'_{V-\gamma}(V^{-1}(\gamma)_-)$, and properties (9.16), (9.17).
$\varphi_{V-\gamma_0}$ is an even function, non-decreasing on $(0,\infty[$. Inequality (9.15) leads to $\int_{\mathbb{R}} (\varphi_{V-\gamma_0}(x))^p dx < \infty$, for any $p > 0$. Then we may apply Theorem 4.1 :

$$\lim_{t\to\infty} E_x\Big[\exp\Big\{-\frac{1}{2}\int_0^t (V(B_s)-\gamma_0)ds\Big\}\Big] = \frac{\int_{\mathbb{R}} \varphi_{V-\gamma_0}(y)dy}{\int_{\mathbb{R}} (\varphi_{V-\gamma_0})^2(y)}\varphi_{V-\gamma_0}(x). \tag{9.23}$$

Consequently point 2 of Theorem 9.1 holds.
Obviously the probability measure $Q^V_{x,t}$ defined by relation (9.12) is also given by the following :

$$Q^V_{x,t}(\Lambda_t) = \frac{E_x\Big[1_{\Lambda_t}\ \exp\Big\{-\frac{1}{2}\int_0^t (V(B_h)-\gamma_0)dh\Big\}\Big]}{E_x\Big[\exp\Big\{-\frac{1}{2}\int_0^t (V(B_h)-\gamma_0)dh\Big\}\Big]},\ \Lambda_t \in \mathcal{F}_t.$$

From Theorem 4.1 we may conclude that $Q^V_{x,t}(\Lambda_s)$ converges to $Q^V_x(\Lambda_s)$, for any positive s and Λ_s in $\mathcal{F}_s$, and $P_x^{\varphi_{V-\gamma_0}}$ is defined by (9.13).
Parts 3. and 4. of Theorem 9.1 are direct consequences of Theorem 4.1.

■

Example 9.4 *We will denote by V the function : $V(x) = 1_{\{|x|>a\}}$, where $a > 0$. Let γ_0 be the unique solution in $]0, 1 \wedge (\frac{\pi^2}{4})[$ to*

$$\tan(a\sqrt{\gamma}) = \sqrt{\frac{1-\gamma}{\gamma}}.$$

Then :

$$\varphi_{V-\gamma_0}(x) = \begin{cases} e^{-\sqrt{1-\gamma_0}|x-a|} & \text{if } |x| > a \\ \dfrac{\cos(\sqrt{\gamma_0}x)}{\cos(\sqrt{\gamma_0}a)} & \text{if } |x| \leq a. \end{cases} \tag{9.24}$$

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