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Abstract

In this note, with the help of the boundary classification of diffusions, we derive a criterion of the convergence of perpetual integral functionals of transient real-valued diffusions.

In the particular case of transient Bessel processes, we note that this criterion agrees with the one obtained via Jeulin's convergence lemma.

Keywords: Brownian motion, random time change, exit boundary, local time, additive functional, stochastic differential equation.

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1. Consider a diffusion Y on an open interval $I = (l, r)$ determined by the SDE

$$dY_t = \sigma(Y_t)\,dW_t + b(Y_t)\,dt,$$

where W is a standard Wiener process defined in a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$. It is assumed that σ and b are continuous and $\sigma(x) > 0$ for all $x \in I$. We assume also that Y is transient and

$$\lim_{t\to\zeta} Y_t = r \quad \text{a.s.,} \tag{1}$$

where ζ is the life time of Y. Hence, if $\zeta < \infty$ then

$$\zeta = H_r(Y) := \inf\{t : Y_t = r\}.$$

For the speed and the scale measure of Y we use

$$m^Y(dx) = 2\,\sigma^2(x)\mathrm{e}^{B^Y(x)}\,dx \quad \text{and} \quad S^Y(dx) = \mathrm{e}^{-B^Y(x)}\,dx, \tag{2}$$

respectively, where

$$B^Y(x) = 2\int^x \frac{b(z)}{\sigma^2(z)}\,dz. \tag{3}$$

Let f be a positive and continuous function defined on I, and consider the perpetual integral functional

$$A_\zeta(f) := \int_0^\zeta f(Y_s)\,ds.$$

We are interested in finding neccessary and sufficient conditions for a.s. finiteness of $A_\zeta(f)$. When Y is a Brownian motion with drift $\mu > 0$ such a condition is that the function f is integrable at $+\infty$ (see Engelbert and Senf [4] and Salminen and Yor [9]). This condition is derived in [9] via Ray-Knight theorems and the stationarity property of the local time processes (which makes Jeulin's lemma applicable). In this note a condition (see Theorem 2) valid for general Y is deduced by exploiting the fact that $A_{H_x}(f)$ for $x < r$ can, via random time change, be seen as the first hitting time of a point for another diffusion.

2. The next proposition presents the key result connecting perpetual integral functionals to first hitting times. The result is a generalization of Proposition 2.1 in [8] discussed in Propositions 2.1 and 2.3 in [2].

Proposition 1. *Let Y and f be as above, and assume that there exists a two times continuously differentiable function g such that*

$$f(x) = \big(g'(x)\sigma(x)\big)^2, \quad x \in I. \tag{4}$$

Set for $t > 0$

$$A_t := \int_0^t f(Y_s)\,ds. \tag{5}$$

and let $\{a_t : 0 \le t < A_\zeta\}$ denote the inverse of A, that is,

$$a_t := \min\big\{s : A_s > t\big\}, \qquad t \in [0, A_\zeta).$$

Then the process Z given by

$$Z_t := g\left(Y_{a_t}\right), \qquad t \in [0, A_\zeta),$$

is a diffusion satisfying the SDE

$$dZ_t = d\widetilde{W}_t + G(g^{-1}(Z_t))\, dt, \qquad t \in [0, A_\zeta).$$

where $\widetilde{W}_t$ is a Brownian motion and

$$G(x) = \frac{1}{f(x)} \left(\frac{1}{2}\, \sigma(x)^2\, g''(x) + b(x)\, g'(x) \right).$$

Moreover, for $l < x < y < r$

$$A_{H_y(Y)} = \inf\{t :\, Z_t = g(y)\} =: H_{g(y)}(Z) \quad a.s. \tag{6}$$

with $Y_0 = x$ and $Z_0 = g(x)$.

3. To fix ideas, assume that the function g as introduced in Proposition 1 is increasing. We define $g(r) := \lim_{x \to r} g(x)$, and use the same convention for any increasing function defined on (l, r). The state space of the diffusion Z is the interval $(g(l), g(r))$ and a.s. $\lim_{t \to \zeta(Z)} Z_t = g(r)$. Clearly, letting $y \to r$ in (6) it follows that

$$A_{H_r(Y)} = \inf\{t :\, Z_t = g(r)\} \quad \text{a.s.}, \tag{7}$$

where both sides in (7) are either finite or infinite. Now we have

Theorem 2. *For Y, A, f and g as above it holds that A_ζ is a.s. finite if and only if for the diffusion Z the boundary point $g(r)$ is an exit boundary, i.e.,*

$$\int^{g(r)} S^Z(d\alpha) \int^{\alpha} m^Z(d\beta) < \infty, \tag{8}$$

where the scale S^Z and the speed m^Z of the diffusion Z are given by

$$S^Z(d\alpha) = \mathrm{e}^{-B^Z(\alpha)}\, d\alpha \quad \text{and} \quad m^Z(d\beta) = 2\, \mathrm{e}^{B^Z(\beta)}\, d\beta$$

with

$$B^Z(\beta) = 2 \int^{\beta} G \circ g^{-1}(z)\, dz.$$

The condition (8) is equivalent with the condition

$$\int^{r} \left(S^Y(r) - S^Y(v) \right)\, f(v)\, m^Y(dv) < \infty. \tag{9}$$

Proof. As is well known from the standard diffusion theory, a diffusion hits its exit boundary with positive probability and an exit boundary cannot be unattainable (see [5] or [1]). This combined with (7) and the characterization of an exit boundary (see [1] No. II.6 p.14) proves the first claim. It remains to show that (8) and (9) are equivalent. We have

$$\begin{aligned} B^Z(\alpha) &= 2\int^{g^{-1}(\alpha)} G(u)\, g'(u)\, du \\ &= 2\int^{g^{-1}(\alpha)} \left(\frac{1}{2}\frac{g''(u)}{g'(u)} + \frac{b(u)}{\sigma^2(u)}\right) du \\ &= \log(g'(g^{-1}(\alpha))) + B^Y(g^{-1}(\alpha)). \end{aligned}$$

Consequently,

$$S^Z(d\alpha) = \mathrm{e}^{-B^Z(\alpha)}\, d\alpha = \frac{1}{g'(g^{-1}(\alpha))}\, \exp\left(-B^Y(g^{-1}(\alpha))\right)\, d\alpha$$

and

$$m^Z(d\alpha) = 2\,\mathrm{e}^{B^Z(\alpha)}\, d\alpha = 2\, g'(g^{-1}(\alpha))\, \exp\left(B^Y(g^{-1}(\alpha))\right)\, d\alpha.$$

Substituting first $\alpha = g(u)$ in the outer integral in (8) and after this $\beta = g(v)$ in the inner integral yield

$$\begin{aligned} \int^{g(r)} S^Z(d\alpha) \int^{\alpha} m^Z(d\beta) &= 2\int^{r} du\, \mathrm{e}^{-B^Y(u)} \int^{u} dv\, (g'(v))^2\, \mathrm{e}^{B^Y(u)} \\ &= 2\int^{r} dv\, (g'(v))^2\, \mathrm{e}^{B^Y(v)} \int_v^{r} du\, \mathrm{e}^{-B^Y(u)} \end{aligned}$$

by Fubini's theorem. Using the expressions given in (2) for the speed and the scale of Y and the relation (4) between f and g complete the proof. □

4. It is easy to derive a condition that the mean of $A_\zeta(f)$ is finite. Indeed,

$$\begin{aligned} \mathbf{E}_x\left(A_\zeta(f)\right) &= \int_0^\infty \mathbf{E}_x\left(f(Y_s)\right)\, ds \\ &= \int_l^r G_0^Y(x,y)\, f(y)\, m^Y(dy) < \infty, \end{aligned} \tag{10}$$

where G_0^Y is the Green kernel of Y w. r. t. m^Y. Under the assumption (1) we may take for $x \geq y$

$$G_0^Y(x,y) = S^Y(r) - S^Y(x).$$

Consequently, the condition (9) may be viewed as a part of the condition (10).

5. Since the exit condition (8) plays a crucial rôle in our approach we discuss here shortly two proofs of this condition, thus making the paper as self-contained as possible.

Let Y be an arbitrary regular diffusion living on the interval I with the end points l and r. The scale function of Y is denoted by S and the speed measure by m. It is also assumed that the killing measure of Y is identically zero. Recall the definition due to Feller

$$r \text{ is exit} \quad \Leftrightarrow \quad \int^r S(d\alpha) \int^\alpha m(d\beta) < \infty. \tag{11}$$

Note that by Fubini's theorem

$$\int^r S(d\alpha) \int^\alpha m(d\beta) = \int^r m(d\beta)(S(r) - S(\beta)),$$

and, hence, $S(r) < \infty$ if r is exit. Moreover, if r is exit then $H_r < \infty$ with positive probability.

5.1. We give now some details of the proof of (11) following closely Kallenberg [7] (see also Breiman [3]). For $l < a < b < r$ let $H_{ab} := \inf\{t : Y_t = a \text{ or } b\}$. Then for $a < x < b$

$$\mathbf{E}_x\,(H_{ab}) = \int_a^b \widehat{G}_0^Y(x,z)\, m(dz), \tag{12}$$

where $\widehat{G}_0^Y$ is the (symmetric) Green kernel of Y killed when it exits (a,b), i.e.,

$$\widehat{G}_0^Y(x,z) = \frac{(S(b) - S(x))(S(y) - S(a))}{S(b) - S(a)} \qquad x \geq y.$$

If r is exit there exists $h > 0$ such that $\mathbf{P}_x(H_r < h) > 0$ for any fixed $x \in (a,r)$. Using this property it can be deduced (see [7] p. 377) that for any $a \in (l,r)$

$$\mathbf{E}_x\,(H_{ar}) < \infty,$$

which, from (12), is seen to be equivalent with (11).

5.2. Another proof of (11) can be found in Itô and McKean [5] p. 130. To present also this briefly recall first the formula

$$\mathbf{E}_x(\exp(-\lambda\, H_b)) = \frac{\psi_\lambda(x)}{\psi_\lambda(b)}, \tag{13}$$

where $\lambda > 0$ and ψ_λ is an increasing solution of the generalized differential equation

$$\frac{d}{dm}\frac{d}{dS}u = \lambda u. \tag{14}$$

Letting $b \to r$ in (13) it is seen that

$$r \text{ is exit} \quad \Leftrightarrow \quad \lim_{b\to r} \psi_\lambda(b) < \infty.$$

Let ψ_λ^+ denote the (right) derivative of ψ_λ with respect to S. Since ψ_λ is increasing it holds that $\psi_\lambda^+ > 0$. The fact that ψ_λ solves (14) yields for $z < r$

$$\psi_\lambda^+(r) - \psi_\lambda^+(z) = \lambda \int_z^r \psi_\lambda(a)\, m(da).$$

In particular, ψ_λ^+ is increasing and $\psi_\lambda^+(r) > 0$. Hence, assuming now that $\psi_\lambda(r) < \infty$ we obtain $S(r) < \infty$, and, further,

$$\begin{aligned}\lambda\, \psi_\lambda(z) \int_z^r S(d\alpha) \int_z^\alpha m(d\beta) &\leq \lambda \int_z^r S(d\alpha) \int_z^\alpha \psi_\lambda(\beta) m(d\beta) \\ &= \int_z^r S(d\alpha) \left(\psi_\lambda^+(\alpha) - \psi_\lambda^+(z)\right) \\ &= \psi_\lambda(r) - \psi_\lambda(z) - \psi_\lambda^+(z)\,(S(r) - S(z)) < \infty,\end{aligned}$$

which yields the condition on the right hand side of (11). Assume next that the condition on the right hand side of (11) holds, and consider for $z < \beta$

$$0 \leq (\psi_\lambda(\beta))^{-1} \left(\psi_\lambda^+(\beta) - \psi_\lambda^+(z)\right) = (\psi_\lambda(\beta))^{-1} \int_z^\beta \psi_\lambda(\alpha) m(d\alpha).$$

Integrating over β gives

$$\begin{aligned}\log(\psi_\lambda(r)) - \log(\psi_\lambda(z)) - \psi_\lambda^+(z) \int_z^r (\psi_\lambda(\beta))^{-1}\, S(d\beta) & \\ = \int_z^r S(d\beta)\, (\psi_\lambda(\beta))^{-1} \int_z^\beta \psi_\lambda(\alpha) m(d\alpha) & \\ \leq \int_z^r S(d\beta) \int_z^\beta m(d\alpha) < \infty, &\end{aligned}$$

which implies that $\psi_\lambda(r) < \infty$, thus completing the proof.

6. As an application of Theorem 2, we consider a Bessel process with dimension parameter $\delta > 2$. Let R denote this process. It is well known that $\lim_{t\to\infty} R_t = +\infty$ and that R solves the SDE

$$dR_t = dW_t + \frac{\delta - 1}{2R_t}\, dt,$$

where W is a standard Brownian motion. Here the function B^R (cf. (3)) takes the form

$$B^R(v) = (\delta - 1)\log v,$$

and, consequently,

$$\begin{aligned}\int^\infty dv\,(g'(v))^2\,\mathrm{e}^{B^R(v)} \int_v^\infty du\,\mathrm{e}^{-B^R(u)} \\ &= \int^\infty dv\,(g'(v))^2\,v^{\delta-1} \int_v^\infty du\,u^{-\delta+1} \\ &= \int^\infty dv\,(g'(v))^2\,v^{\delta-1}\,\frac{1}{\delta-2}\,v^{-\delta+2}\end{aligned}$$

leading to

$$\int_0^\infty f(R_t)\,dt < \infty \quad \Leftrightarrow \quad \int^\infty u\,f(u)\,du < +\infty.$$

Another way to derive this condition is via local times and Jeulin's lemma [6]. Indeed, by the occupation time formula and Ray-Knight theorem for the total local times of R (see, e.g. [10] Theorem 4.1 p. 52) we have

$$\begin{aligned}\int_0^\infty f(R_s)\,ds &\stackrel{(\mathrm{d})}{=} \int_0^\infty f(a)\,\frac{\rho_{a^\gamma}}{\gamma\, a^{\gamma-1}}\,da \\ &= \frac{1}{\gamma}\int_0^\infty a\,f(a)\,\frac{\rho_{a^\gamma}}{a^\gamma}\,da\end{aligned}$$

where $\delta = 2+\gamma$ and ρ is a squared 2-dimensional Bessel process. Using the scaling property, it is seen that the distribution of the random variable ρ_{a^γ}/a^γ does not depend on a. Hence, we obtain by Jeulin's lemma that if the function $a \mapsto a\,f(a)$, $a > 0$, is locally integrable on $[0,\infty)$ then

$$\int_0^\infty f(R_s)\,ds < \infty \quad \Leftrightarrow \quad \int^\infty a\,f(a)\,da < \infty. \tag{15}$$

The same argument allows us to recover the result in [9], that is,

$$\int_0^\infty g(W_s^{(\mu)})\,ds < \infty \quad \Leftrightarrow \quad \int^\infty g(x)\,dx < \infty. \tag{16}$$

where g is any non-negative locally integrable function and $W^{(\mu)}$ denotes a Brownian motion with drift $\mu > 0$. To see this, write $g(x) = f(\mathrm{e}^x)$ and use Lamperti's representation

$$\exp(W_s^{(\mu)}) = R^{(\mu)}_{A_s^{(\mu)}}, \quad s \geq 0,$$

where

$$A_s^{(\mu)} = \int_0^s du\, \exp(2W_u^{(\mu)})$$

and $R^{(\mu)}$ is a Bessel process with dimension $d = 2(1+\mu)$ starting from 1, we obtain (cf. [8] Remark 3.3.(3))

$$\int_0^\infty f(\exp(W_s^{(\mu)}))\,ds = \int_0^\infty \left(R_u^{(\mu)}\right)^{-2} f(R_u^{(\mu)})\,du \quad \text{a.s.},$$

and, in order to get (16) it now only remains to use the equivalence (15).

We wish to underline the fact that in Theorem 2 it is assumed that the function f is continuous whereas the approach via Jeulin's lemma, which we developed above, demands only local integrability.

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