Heun functions versus elliptic functions

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Abstract

We present some recent progresses on Heun functions, gathering results from classical analysis up to elliptic functions. We describe Picard's generalization of Floquet's theory for differential equations with doubly periodic coefficients and give the detailed forms of the level one Heun functions in terms of Jacobi theta functions. The finite-gap solutions give an interesting alternative integral representation which, at level one, is shown to be equivalent to their elliptic form.

1 Introduction

Heun functions [9] are defined as a natural generalization of the hypergeometric function, to be the solutions of the Fuchsian differential equation

$$\left\{\begin{array}{l} \dfrac{d^2F}{dw^2}+\left(\dfrac{\gamma}{w}-\dfrac{\delta}{1-w}-\dfrac{\epsilon k^2}{1-k^2w}\right)\dfrac{dF}{dw}+\dfrac{(s+\alpha\beta k^2 w)}{w(1-w)(1-k^2w)}F=0, \\ \gamma+\delta+\epsilon=\alpha+\beta+1, \end{array}\right. \qquad (1)$$

with four regular singularities 0, 1, $1/k^2$, ∞. In [3] the fourth singularity is $a\equiv 1/k^2$ and the auxiliary parameter s is taken as $q\equiv -s/k^2$. We will follow the notation

$$Hn(k^2,s;\alpha,\beta,\gamma,\delta;w), \qquad k^2\in[0,1]. \qquad (2)$$

In [3] one can find a lot of information on these functions and in the remaining chapters of [21] are gathered many results on their confluent limits. Due to limitation of space the former section misses several important items, particularly the relation between Heun functions and elliptic function theory. These last years this field has received new interesting developments from workers in condensed matter physics [5] and integrable systems [15]. Interesting accounts have been given by Smirnov [23] and by Takemura [24].

Our aim is to gather, for the community dedicated to the study of special functions and orthogonal polynomials, these new progresses relating Heun and elliptic functions.

As a preliminary step we will give one further motivation for the study of Heun functions, coming from orthogonal polynomials and birth and death processes.

2 From orthogonal polynomials to Heun functions

Let us consider the three terms recurrence [1]

$$xP_n = b_{n-1}\, P_{n-1} + a_n P_n + b_n\, P_{n+1}, \qquad n \geq 1. \tag{3}$$

We will denote by P_n and Q_n two linearly independent solutions of this recurrence with initial conditions

$$P_0(x) = 1, \qquad P_1(x) = \frac{x - a_0}{b_0}, \qquad\qquad Q_0(x) = 0, \qquad Q_1(x) = \frac{1}{b_0}. \tag{4}$$

The corresponding Jacobi matrix is

$$\begin{pmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{5}$$

If the $b_n > 0$ and $a_n \in \mathbb{R}$ the P_n (resp. the Q_n) will be orthogonal with respect to a positive probabilistic measure ψ (resp $\psi^{(1)}$)

$$\int P_m(x)\, P_n(x)\, d\psi(x) = \delta_{mn}, \qquad\qquad \int Q_m(x)\, Q_n(x)\, d\psi^{(1)}(x) = \delta_{mn}, \tag{6}$$

with the moments

$$s_n = \int x^n\, d\psi(x), \qquad n \geq 1, \qquad s_0 = 1 \tag{7}$$

When considering applications to birth and death processes with killing [14], it is sufficient to consider the special case where

$$a_n = \lambda_n + \mu_n + \gamma_n, \qquad\qquad b_n = \sqrt{\lambda_n \mu_{n+1}}, \quad n \geq 0,$$

$$\pi_0 = 1, \qquad \pi_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}, \quad n \geq 1.$$

It is convenient to introduce new polynomials $F_n(x)$ by

$$P_n(x) = \frac{(-1)^n}{\sqrt{\pi_n}}\, F_n(x), \qquad n \geq 0, \tag{8}$$

The positivity constraints

$$\lambda_n > 0, \qquad \mu_{n+1} > 0 \qquad n \geq 0, \tag{9}$$

will be assumed. From (3) we deduce

$$\begin{aligned} &-xF_n = \mu_{n+1}F_{n+1} - (\lambda_n + \mu_n + \gamma_n)F_n + \lambda_{n-1}F_{n-1}, \\ &F_{-1}(x) = 0, \quad F_0(x) = 1. \end{aligned} \tag{10}$$

The orthogonality relation (6) becomes

$$\int F_m(x)\, F_n(x)\, d\psi(x) = \pi_n \delta_{mn}. \tag{11}$$

The particular case where λ_n, μ_n are quadratic is connected with Heun functions in the following way. Let us take

$$\begin{aligned} &\lambda_n = k^2(n+\alpha)(n+\beta), \qquad \mu_n = n(n+\gamma-1), \qquad \gamma_n = k'^{\,2}\delta n \\ &\alpha > 0,\ \beta > 0,\ \gamma > 0, \quad k^2 \in (0,1), \qquad k'^{\,2} = 1-k^2, \end{aligned} \tag{12}$$

and consider the generating function

$$F(x;w) = \sum_{n\geq 0} F_n(k^2, x; \alpha, \beta, \gamma, \delta) w^n \equiv Hn(k^2, s; \alpha, \beta, \gamma, \delta; w), \qquad s = x - \alpha\beta k^2. \tag{13}$$

Routine computations show that $F(x;w)$ is the solution, analytic in a neighbourhood of $w = 0$ [1], of Heun's differential equation (1). Quite remarkably, the most general quadratic birth and death process with linear killing produces, for its generating function, the most general Fuchsian equation with four regular singularities.

Let us conclude with the proof of:

Proposition 1 *The Hamburger (hence the Stieltjes) moment problem corresponding to the polynomials $F_n(x)$ with recurrence coefficients (12) is determinate.*

Proof :

A theorem by Hamburger [1][p. 84] states that if the series $c_n = P_n(0)^2$ is divergent then the Hamburger moment problem is determinate, i. e. the measure ψ defined in (6) is unique. From relation (8) we can write

$$\frac{c_{n+1}}{c_n} = \frac{\mu_{n+1}}{\lambda_n}\,\frac{F_{n+1}(0)^2}{F_n(0)^2} = \frac{1}{k^2}\,\frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)}\,\frac{F_{n+1}(0)^2}{F_n(0)^2}.$$

Now the generating function $F(x;w)$, introduced in (13), is analytic for $|w| < 1$: hence its radius of convergence is one. It follows that, for $n \to \infty$ the previous ratio has for limit $1/k^2 > 1$ and the series $\{c_n\}$ diverges. $\square$

To have a clear view of the problems encountered in the construction of solutions of Heun's equation, we will introduce some terminology: we will call "generic" solutions the solutions valid for *arbitrary values* of the auxiliary parameter s and "non-generic" solutions

[1] In [21] this function is called "local" Heun. Notice that we have for normalization $Hn(\cdots;0) = 1$.

those which require particular values of s. Let us begin with some results on non-generic solutions.

Non-generic solutions

1 Special hypergeometric cases

The reason for the restriction $k^2 \in (0,1)$ is that in the two limiting cases $k^2 = 0$ (notice that the positivity conditions (9) require $k^2 > 0$) and $k^2 = 1$ the four singular points reduce to three and therefore Heun functions degenerate into hypergeometric functions. Indeed we have:

1. For $k^2 = 0$ the parameters (α, β) become irrelevant and we have:

$$\begin{cases} Hn(0, s; (\alpha, \beta), \gamma, \delta; w) = {}_2F_1\left(\begin{matrix} r_+, r_- \\ \gamma \end{matrix} ; w\right), \\ r_\pm = \rho \pm \sqrt{\rho^2 + s}, \qquad \rho = \frac{1}{2}(\gamma + \delta - 1). \end{cases} \tag{14}$$

This case, which would correspond to $a \to \infty$, is missing in [3].

2. For $k^2 = 1$ we have the relation:

$$\begin{cases} Hn(1, s; \alpha, \beta, \gamma, \delta; w) = (1-w)^r \, {}_2F_1\left(\begin{matrix} r+\alpha, r+\beta \\ \gamma \end{matrix} ; w\right), \\ r = \rho - \sqrt{\rho^2 - \alpha\beta - s}, \qquad \rho = \frac{1}{2}(\gamma - \alpha - \beta). \end{cases} \tag{15}$$

This case is considered in [3] but only with the extra constraint $s = -\alpha\beta$, i. e. for $r = 0$.

2 The "trivial" solution

This solution corresponds to the special values $s = \alpha\beta = 0$. In this case Heun's differential operator is factorized as

$$\Big(L\, D_w + M\, \mathbb{I}\Big)\, D_w\, F = 0,$$

with obvious L and M. This factorization leads to

$$F_1 = 1 \qquad F_2 = \int e^{-U(w)} dw, \qquad U(w) = \int \frac{dw}{w^\gamma (1-w)^\delta (1-k^2 w)^\epsilon}. \tag{16}$$

Starting from these solutions we can use the change of function

$$F \to \widetilde{F}: \qquad F(w) = w^\rho (1-w)^\sigma (1-k^2 w)^\tau \widetilde{F}(w) \tag{17}$$

which transforms, for special values of the parameters ρ, σ, τ, a Heun function into another Heun function, up to changes in the parameters [3][p.18]. In such a way one generates 7 more Heun functions starting from (16). For all these solutions Heun's differential operator remains obviously factorized and, as shown in [22], this happens only for these cases.

3 Derivatives of Heun functions

In general the derivative of a Heun function cannot be expressed in terms of some Heun function with different parameters. However, as shown in [11], this happens in 4 cases:

$$Hn'(k^2,0;\alpha,\beta,\gamma,\delta;w) = -\frac{\alpha\beta}{\gamma+2}k^2 w\, Hn(k^2,s';\alpha+2,\beta+2,\gamma+2,\delta+1;w),$$
$$-s' = \gamma+\delta+1+(\gamma+\epsilon+1)k^2. \tag{18}$$

$$Hn'(k^2,-\alpha\beta k^2;\alpha,\beta,\gamma,\delta;w) = \frac{\alpha\beta}{\gamma+1}k^2(1-w)\, Hn(k^2,s';\alpha+2,\beta+2,\gamma+1,\delta+2;w),$$
$$-s' = \gamma+\delta+1+(\gamma+\epsilon+\alpha\beta)k^2. \tag{19}$$

$$Hn'(k^2,-\alpha\beta;\alpha,\beta,\gamma,\delta;w) = \frac{\alpha\beta}{\gamma+1}(1-k^2 w)\, Hn(k^2,s';\alpha+2,\beta+2,\gamma+1,\delta+1;w),$$
$$-s' = \gamma+\delta+\alpha\beta+(\gamma+\epsilon+1)k^2. \tag{20}$$

$$Hn'(k^2,s;0,\beta,\gamma,\delta;w) = -\frac{s}{\gamma+1}\, Hn(k^2,s';2,\beta+1,\gamma+1,\delta+1;w),$$
$$s' = s-\gamma-\delta-(\gamma+\epsilon)k^2. \tag{21}$$

Notice that, from (13), the functions $Hn(\cdots;w)$ are analytic around $w=0$.

4 Reduction to hypergeometric functions

For non-generic solutions it was realized some time ago the possibility for Heun functions to reduce to hypergeometric functions. Some relations, using Weierstrass elliptic functions are given in [13]. Later on Kuiken [16] has observed some reductions to hypergeometric functions of some particular rational variable $R(w)$. This may happen only for *polynomial* $R(w)$, with the following list :

$$\begin{aligned}
\mathrm{Hn}\,(-1,0;2a,2b,2c-1,1+a+b-c;w) &= {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};w^2\right),\\
\mathrm{Hn}\,(1/2,-2ab;2a,2b,c,1+2(a+b-c);w) &= {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};w(2-w)\right),\\
\mathrm{Hn}\,(2,-4ab;2a,2b,c,c;w) &= {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};4w(1-w)\right).
\end{aligned} \tag{22}$$

Notice that in all these cases $R(w)$ is a second degree polynomial with 3 free parameters. These results have been completed recently by Maier [17], who discovered new cases [2] with cubic and quartic polynomials $R(w)$ and 2 free parameters:

$$\begin{aligned}
&\mathrm{Hn}\,(1/4,-9ab/4;3a,3b,1/2,2(a+b);w) = {}_2F_1\left(\begin{matrix}a,b\\1/2\end{matrix};w(3-w)^2/4\right),\\
&\mathrm{Hn}\,(1/2,-8ab;4a,4b,a+b+1/2,2(a+b);w) =\\
&\qquad {}_2F_1\left(\begin{matrix}a,b\\a+b+1/2\end{matrix};4w(2-w)(1-w)^2\right).
\end{aligned} \tag{23}$$

[2] We have omitted the cases involving complex values of k^2.

These results are not the whole story, since Heun functions may reduce to the product of some function $f(w)$ by some hypergeometric function with variable $R(w)$, not necessary polynomial. A result of this kind was obtained in [12] by Joyce :

$$\mathrm{Hn}\,(1/4,-1/8;1/2,1/2,1,1/2;w)=\sqrt{\sqrt{4-w}-\sqrt{1-w}}\,{}_2F_1\left(\begin{matrix}1/2,1/2\\1\end{matrix};R(w)\right), \quad (24)$$

with

$$R(w)=\frac{1}{4}\left(2-w\sqrt{4-w}-(2-w)\sqrt{1-w}\right), \qquad w\in\,[0,1].$$

Let us turn ourselves to the generic solutions of Heun's equation.

Generic solutions

1 The 192 solutions of Heun's equation

As Heun himself observed [9] there is a set of 24 substitutions of the variable w which produce a transformation of Heun's equation into another Heun's equation with different parameters. This leads to a complete list of 192 solutions. This list has been fully worked out recently by Maier in [18]. We will just quote the generalizations of the Euler transformation of the hypergeometric function:

$$\begin{aligned}&Hn(k^2,s;\alpha,\beta;\gamma,\delta,\epsilon;w)=\\ &\qquad=(1-w)^{1-\delta}Hn(k^2,-s-\gamma(\delta-1);\alpha-\delta+1,\beta-\delta+1;\gamma,2-\delta;w),\end{aligned} \quad (25)$$

and of the Pfaff transformation

$$\begin{aligned}&Hn(k^2,s;\alpha,\beta;\gamma,\delta,\epsilon;w)=\\ &=(1-w)^{-\alpha}\,Hn(-k^2/k'^2,-k^2/k'^2(s+\alpha\gamma);\alpha,\alpha-\delta+1,\gamma,\alpha-\beta+1;w/(w-1)).\end{aligned} \quad (26)$$

2 An integral transform

An integral transform was given in [26]:

$$\begin{aligned}&Hn(k^2,s;\alpha,\beta,\gamma,\delta;w)=\\ &=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\,Hn(k^2,s;\gamma,\beta,\alpha,\delta+\gamma-\alpha;wt)\,dt,\end{aligned} \quad (27)$$

valid for $\mathrm{Re}\,\gamma>\mathrm{Re}\,\alpha>0$ and $w\in\mathbb{C}\backslash\,[1,\infty[$.

Let us use orthogonal polynomials to prove this relation [27]. To this end we define the monic polynomials M_n by

$$M_0=F_0=1, \qquad M_n(P;x)=\mu_1\cdots\mu_n\,F_n(P;x)=n!\,(\gamma)_n\,F_n(P;x), \quad n\geq 1,$$

where P denotes the set of parameters k^2, α, β, γ, δ. These monic polynomials satisfy the recurrence relation:

$$(\lambda_n+\mu_n+\gamma_n-x)M_n = M_{n+1}+\lambda_{n-1}\mu_n\, M_{n-1}, \qquad n\geq 0,$$
$$M_{-1}(x)=0, \qquad M_0(x)=1.$$

If we define the new set of parameters

$$P'=(k^2,\, \alpha'=\gamma,\, \beta'=\beta,\, \gamma'=\alpha,\, \delta'=\delta+\gamma-\alpha), \qquad x'=s+\alpha'\beta'k^2$$

it is easy to check the invariance relation $M_n(P';x')=M_n(P;x)$. This induces

$$x=s+\alpha\beta k^2, \qquad F_n(P;x)=\frac{(\alpha)_n}{(\gamma)_n}\,F_n(P';x'), \qquad x'=s+\alpha'\beta'k^2. \tag{28}$$

For $\mathrm{Re}\,\gamma>\mathrm{Re}\,\alpha>0$ we can write

$$\frac{(\alpha)_n}{(\gamma)_n}=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{n+\alpha-1}(1-t)^{\gamma-\alpha-1},$$

then multiplying both sides of this relation by (28) and summing for $n\geq 0$ gives (27) for $|w|<1$. Analytic continuation extends it to $w\in\mathbb{C}\setminus[1,\infty[$.

Remarks :

1. This is not an integral equation, since the parameters of the Heun function are changed in the transformation. Notice that several integral equations are known [3].

2. For $k^2\to 0$ we recover Bateman's integral relation

$$ {}_2F_1\left(\begin{array}{c} a,b \\ c\end{array};w\right)=\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1}\,{}_2F_1\left(\begin{array}{c} a,b \\ \lambda\end{array};wt\right)dt, \tag{29}$$

valid for $\mathrm{Re}\,c>\mathrm{Re}\,\lambda>0$. Notice that now λ is a *free* parameter, and this enables one to deduce Euler's integral representation. This does not work for Heun functions because the parameter α is not free.

3. For $k^2=1$ we get the relation

$$\begin{aligned}(1-w)^r\,{}_2F_1&\left(\begin{array}{c} r+\alpha, r+\beta \\ \gamma\end{array};w\right)= \\ &=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\,(1-wt)^{\tilde{r}}\,{}_2F_1\left(\begin{array}{c} \tilde{r}+\gamma, \tilde{r}+\beta \\ \alpha\end{array};wt\right)dt,\end{aligned} \tag{30}$$

with

$$r=\rho-\sqrt{\rho^2-\alpha\beta-s}, \quad \rho=\frac{1}{2}(\gamma-\alpha-\beta), \quad \tilde{r}=\tilde{\rho}-\sqrt{\tilde{\rho}^{\,2}-\beta\gamma-s}, \quad \tilde{\rho}=-\frac{1}{2}(\gamma-\alpha+\beta).$$

This relation does not appear in the extensive list of hypergeometric integrals given in [7], so it could be new.

3 Carlitz solutions

In his analysis of some orthogonal polynomials of Stieltjes, Carlitz [4] discovered the following remarkable result: the linearly independent Heun functions with parameters $(k^2, s \neq 0; 0, 1/2, 1/2, 1/2)$ are given by

$$\exp\left(\pm 2i\sqrt{s}\, z(w)\right), \qquad z(w) = \int_0^w \frac{du}{2\sqrt{u(1-u)(1-k^2u)}}. \tag{31}$$

To check most simply this result we note that by the inversion theorem of elliptic functions we can write

$$z(w,k^2) = \int_0^w \frac{du}{2\sqrt{u(1-u)(1-k^2u)}} \qquad \Longleftrightarrow \qquad w = \mathrm{sn}^{\,2}(z,k^2).$$

This conformal transformation maps the singular points as

$$\begin{array}{ccccc} w & 0 & 1 & \dfrac{1}{k^2} & \infty \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ z & 0 & K & K+iK' & iK' \end{array}$$

where $K = K(k^2)$ and $K' = K(k'^{\,2})$ are the complete elliptic integrals of first kind. Using z as new variable and setting $F(w) = y(z)$, Heun's equation becomes

$$\begin{array}{c} \dfrac{d^2y}{dz^2} + \left[(2\gamma-1)\dfrac{\mathrm{cn}\,z\,\mathrm{dn}\,z}{\mathrm{sn}\,z} - (2\delta-1)\dfrac{\mathrm{sn}\,z\,\mathrm{dn}\,z}{\mathrm{cn}\,z} - (2\epsilon-1)k^2\dfrac{\mathrm{sn}\,z\,\mathrm{cn}\,z}{\mathrm{dn}\,z}\right]\dfrac{dy}{dz} \\ +4(s+\alpha\beta\,k^2\,\mathrm{sn}^{\,2}z)y = 0. \end{array} \tag{32}$$

At the very symmetric point where $\alpha\beta = 0$ and $\gamma = \delta = \epsilon = 1/2$, it reduces to

$$\frac{d^2y}{dz^2} + 4s\,y = 0,$$

which proves (31).

Using then the transformation (17) we can generate a full set of 8 generic solutions (see [26]) where the detailed list is given). So we realize that Heun's equation, in the elliptic functions setting, lies in the field of differential equations with doubly periodic coefficients. It happens that Floquet's theory for differential equations with periodic coefficients does generalize to the case of doubly periodic coefficients and was derived by Picard. This will be discussed in the next subsection.

4 Elliptic functions of second kind and Picard's theorem

Since this theorem is not easily available in the standard textbooks [2],[28], we will present some background material. Let us just recall the definition and a few basic results from elliptic function theory :

Definition 1 *A function $\Phi(z)$ is elliptic if it is meromorphic and it has two linearly independent periods*

$$\Phi(z+2\omega) = \Phi(z), \qquad \Phi(z+2\omega') = \Phi(z).$$

These two periods define therefore a non-degenerate period parallelogram (usually $\omega = K,\ \omega' = iK'$).

We will need also

Proposition 2 *An elliptic function:*

1. *Has as many poles as zeroes in a period parallelogram.*

2. *Can be written*

$$\Phi(z) = A\frac{H(z-b_1)\cdots H(z-b_n)}{H(z-a_1)\cdots H(z-a_n)}, \qquad a_1+\cdots+a_n = b_1+\cdots+b_n, \tag{33}$$

where $H(z)$ is one of Jacobi's theta function defined as

$$H(z) = 2\sum_{n\geq 0}(-1)^n q^{(n+\frac{1}{2})^2}\sin\left((2n+1)\frac{\pi z}{2K}\right), \qquad q = e^{-\frac{K}{K'}}.$$

3. *Must have at least a pole of multiplicity 2 in a period parallelogram otherwise it is a constant.*

4. *With poles of multiplicity n at $z=a$ can be expanded as:*

$$\Phi(z) = \sum a_0\, g(z-a) + a_1\, g'(z-a) + \cdots a_n\, g^{(n)}(z-a), \qquad g(z) = \frac{H'(z)}{H(z)}, \tag{34}$$

where the sum is extended to all the poles.

To state Picard's theorem we need the definition

Definition 2 *A function $\Phi(z)$ is a an elliptic function of second kind (or a function with constant multipliers) if it is meromorphic and that there exists two non-vanishing constants μ and μ' such that*

$$\Phi(z+2\omega) = e^{\mu}\,\Phi(z) \qquad \Phi(z+2\omega') = e^{\mu'}\,\Phi(z),$$

where 2ω and $2\omega'$ are two linearly independent periods. The constants μ and μ' are called the multipliers.

Obviously for $e^{\mu} = e^{\mu'} = 1$ we recover elliptic functions. We will have to discuss separately the two cases:

1. Generic multipliers for which $\omega\mu' - \omega'\mu \neq 0$,

2. Special multipliers for which $\omega\mu' - \omega'\mu = 0$.

4.1 Theorems for generic multipliers

Let us consider the function

$$f(z) = e^{\lambda z}\frac{H(z-a)}{H(z)}, \tag{35}$$

where a is not homologous to zero. One can check that it is elliptic of second kind with generic multipliers

$$\mu = 2\lambda K, \qquad \mu' = i\left(2\lambda K' - i\pi\frac{a}{K}\right).$$

Let us prove:

Proposition 3 *An elliptic function of second kind with generic multipliers:*

1. *Admits as many zeroes as poles in a period parallelogram.*

2. *Must have at least a simple pole in a period parallelogram, otherwise it vanishes.*

3. *Can be expanded, using $f(z)$ defined in (35), as*

$$\Phi(z) = \sum c_0 f(z-a) + c_1 f'(z-a) + \cdots + c_n f^{(n)}(z-a), \tag{36}$$

where the sum extends to all poles $z = a$ of multiplicity n of Φ.

Proof :
Notice that for any elliptic function of second kind $\Phi(z)$ with given generic multipliers (μ, μ') it is always possible to find λ and a in (35) such that f has the *same* multipliers. It follows that $\Phi(z)/f(z)$ is elliptic. Using (33) we get the most general structure

$$\Phi(z) = A\, f(z)\,\frac{H(z-b_1)\cdots H(z-b_n)}{H(z-a_1)\cdots H(z-a_n)}, \qquad a_1 + \cdots + a_n = b_1 + \cdots + b_n,$$

which proves the first assertion.

For the second assertion, let it be supposed that Φ has no pole: it cannot have any zero and therefore $\Phi(z)$ reduces to $Ae^{\lambda z}$, but in this case the multipliers are not generic, contradicting our hypothesis, hence $A = 0$.

For the third assertion, let us consider a pole $z = a$ of multiplicity n. Near to $z = a$ we have the Laurent expansion

$$f^{(k)}(z) = \frac{\xi_k}{(z-a)^{k+1}} + \text{holomorphic},$$

so we can find coefficients $\{c_k,\ k = 0\ldots n\}$ such that

$$\Phi(z) - \sum\left(c_0 f(z-a) + c_1 f'(z-a) + \cdots + c_n f^{(n)}(z-a)\right)$$

has no poles in a period parallelogram, so it must vanish. $\square$

4.2 Theorems for special multipliers

This time let us define

$$f(z) = e^{\lambda z} \frac{H'}{H}(z). \tag{37}$$

One can check that it is elliptic of second kind with special multipliers. We have now

Proposition 4 *Any elliptic function of second kind with special multipliers can be expanded as*

$$\Phi(z) = C\, e^{\lambda z} + \sum \left(c_0 f(z-a) + c_1 f'(z-a) + \cdots c_n\, f^{(n)}(z-a) \right), \tag{38}$$

with the constraint

$$\sum e^{-\lambda a} (c_0 + c_1\, \lambda + \cdots c_n\, \lambda^n) = 0, \tag{39}$$

where the sums extend to all poles of $\Phi(z)$.

Proof:
In this case $\omega\, \mu' - \omega'\, \mu = 0$. So we can find a value of λ such that $F(z) \equiv \Phi(z) e^{-\lambda z}$ is elliptic. Let us consider a pole $z = a$ of order n of Φ. We have for Laurent's series

$$\Phi(z) = \frac{c_0}{z-a} - \frac{c_1}{(z-a)^2} + \cdots + (-1)^n \frac{n!\, c_n}{(z-a)^{n+1}} + \text{holomorphic.}$$

hence we can write

$$F(z) = \frac{a_0}{z-a} - \frac{a_1}{(z-a)^2} + \cdots + (-1)^n \frac{n!\, a_n}{(z-a)^{n+1}} + \text{holomorphic,}$$

where the new residue is

$$a_0 = e^{-\lambda a} (c_0 + \lambda c_1 + \cdots + \lambda^n c_n).$$

Using the expansion theorem for elliptic functions (34) we have

$$\Phi(z) e^{-\lambda z} = C + \sum e^{-\lambda a} \left[a_0\, \frac{H'}{H}(z-a) + a_1\, D_z \left(\frac{H'}{H}(z-a) \right) + \cdots \right.$$
$$\left. \cdots + a_n\, D_z^n \left(\frac{H'}{H}(z-a) \right) \right].$$

Inserting $e^{\lambda z}$ into the right-hand side, and expanding the derivatives according to Leibnitz rule gives (38). The constraint (39) comes from the fact that the sum of the residues in a period parallelogram vanishes for an elliptic function. □

4.3 Picard's theorem

Now we can state Picard's theorem [19]

Proposition 5 *Let us consider the differential equation*

$$\frac{d^nF}{dz^n}+a_1(z)\frac{d^{n-1}F}{dz^{n-1}}+\cdots a_n(z)F=0,$$

with doubly periodic coefficients

$$a_l(z+2\omega)=a_l(z),\qquad a_l(z+2\omega')=a_l(z),\qquad l=1,2,\ldots,n,$$

Any meromorphic *solution $F(z)$ is a linear combination of elliptic functions of the second kind.*

Proof :
To shorten, and in view of application to Heun's case, we will consider a differential equation of second order:

$$\frac{d^2F}{dz^2}+p(z)\frac{dF}{dz}+q(z)F=0, \tag{40}$$

with doubly periodic coefficients p and q. Let us consider a solution $F(z)$ which is not elliptic of second kind: then the ratio $F(z+2\omega)/F(z)$ cannot be a constant. The periodicity of the coefficients implies that the functions $F(z+2\omega)$ and $F(z+4\omega)$ are also solutions, so we must have a linear relation of the form

$$F(z+4\omega)=AF(z)+BF(z+2\omega).$$

Let us now consider the non-vanishing function $\phi(z)=F(z+2\omega)+\rho F(z)$. If we take for ρ a root of $\rho^2+B\rho-A=0$ it is easy to check that $\phi(z+2\omega)=(B+\rho)\phi(z)$.

A similar argument works with respect ot the period $2\omega'$. So we have proved that we can construct a first solution $\phi(z)$ which is elliptic of the second kind.

We have now to prove that a second linearly independent and meromorphic solution, which can be written as

$$\psi(z)=\phi(z)\int G(z)\,dz,\qquad G=\frac{1}{\phi^2(z)}\,e^{-P},\qquad P'=p,$$

is also elliptic of second kind.

The Wronskian

$$\psi\,\phi'-\psi'\,\phi=C\,e^{-P},\qquad C\neq 0,$$

and the meromorphy of ϕ and ψ imply that e^{-P} is meromorphic. Since p is doubly periodic, e^{-P} is elliptic of second kind. So G is a meromorphic elliptic function of second kind.

Let it be supposed first that G has generic multipliers. Using the expansion theorem (36) we can write

$$G(z)=\sum\left[c_0\,f(z-a)+\cdots+c_n\,f^{(n)}(z-a)\right],\qquad f(z)=e^{\lambda z}\frac{H(z-a)}{H(z)},$$

where the sum includes all the poles of multiplicity n of $G(z)$. Since the integral has to be meromorphic all the coefficients c_0 must vanish and we get

$$\mathcal{G}(z)\equiv\int G(z)\,dz=\sum\left[c_1\,f(z-a)+\cdots+c_n\,f^{(n-1)}(z-a)\right],$$

and from (36) it follows that $\mathcal{G}(z)$ and $\psi(z)$ are elliptic of second kind.

Let us now consider the case where G has special multipliers. The expansion theorem (38) gives

$$G(z) = C\,e^{hz} + \sum \left[c_0\, f(z-a) + \cdots + c_n\, f^{(n)}(z-a)\right], \qquad f(z) = e^{\lambda z}\frac{H'}{H}(z),$$

with the constraint

$$\sum \left(c_0 + c_1 h + \cdots + c_n h^n\right) e^{-ha} = 0. \tag{41}$$

If $h \neq 0$ the meromorphy of G requires that all coefficients c_0 vanish, so we can write

$$\mathcal{G}(z) = \frac{C}{h}\,e^{hz} + \sum \left[c_1\, f(z-a) + \cdots + c_n\, f^{(n-1)}(z-a)\right].$$

This relation and the constraint (41), with coefficients c_0 all vanishing, implies that $\mathcal{G}(z)$ is elliptic of second kind with special multipliers, hence ψ is again elliptic of second kind.

If $h = 0$ we have

$$\mathcal{G}(z) = Cz + \sum \left[c_1\, f(z-a) + \cdots + c_n\, f^{(n-1)}(z-a)\right], \qquad \sum c_0 = 0.$$

This implies

$$\mathcal{G}(z+2\omega) = \mathcal{G}(z) + D, \qquad \mathcal{G}(z+2\omega') = \mathcal{G}(z) + D'$$

and so, if (λ_1, λ_2) are the multipliers of ϕ we can write

$$\psi(z+2\omega) = \lambda_1\, \psi(z) + \lambda_1 D\, \phi(z), \qquad \psi(z+2\omega') = \lambda_2\, \psi(z) + \lambda_2 D'\, \phi(z),$$

so we can subtract from ψ a suitable term linear in ϕ for which ψ will be elliptic of second kind. □

5 The meromorphic solutions

So let us look for the necessary conditions on the parameters to get meromorphic solutions in the variable z. The computation of the exponents at the singularities of (32) is quite simple and gives

$$\begin{array}{lll} z = 0 & (\mathrm{sn}\, z)^{2\rho_1} & (0, \gamma - 1) \\ z = K & (\mathrm{cn}\, z)^{2\rho_2} & (0, \delta - 1) \\ z = K + iK' & (\mathrm{dn}\, z)^{2\rho_3} & (0, \epsilon - 1) \\ z = iK' & (\mathrm{sn}\, z)^{-2\rho_3} & (\alpha, \beta) \end{array} \tag{42}$$

So the necessary conditions for meromorphy are

$$\left\{ \begin{array}{llll} \gamma = \frac{1}{2} - m_1, & \delta = \frac{1}{2} - m_2, & \epsilon = \frac{1}{2} - m_3, & M = m_1 + m_2 + m_3, \\ \alpha = -\frac{1}{2}(m_0 + M), & \beta = \frac{1}{2}(m_0 - M + 1), & N = m_0 + M, & \end{array} \right. \tag{43}$$

with the vector $\overline{N} = (m_0,\, m_1,\, m_2,\, m_3) \in \mathbb{Z}^4$. That these conditions are also sufficient is proved in [8]. From Picard's Theorem the solutions will be elliptic functions of the second kind. The differential equation becomes

$$\begin{aligned}\frac{d^2y}{dz^2} + 2\left(-m_1\frac{\mathrm{cn}\, z\,\mathrm{dn}\, z}{\mathrm{sn}\, z} + m_2\frac{\mathrm{sn}\, z\,\mathrm{dn}\, z}{\mathrm{cn}\, z} + m_3\, k^2\frac{\mathrm{sn}\, z\,\mathrm{cn}\, z}{\mathrm{dn}\, z}\right)\frac{dy}{dz}& \\ +(4s + N(N-2m_0-1)\, k^2\,\mathrm{sn}^{\,2} z)y = 0,&\end{aligned} \tag{44}$$

and for $\overline{N} = (n,\, 0,\, 0,\, 0)$ we are back to Lamé's equation.

It is possible, extracting from y suitable factors of w, to relate the negative and positive values of the parameters m_i as summarized in [23][p. 296], so from now on we will consider $\overline{N} \in \mathbb{N}^4$.

It is interesting to get rid of the derivative in (44) by the change of function

$$y(z) = (\mathrm{sn}\, z)^{m_1}\,(\mathrm{cn}\, z)^{m_2}\,(\mathrm{dn}\, z)^{m_3}\, Y(z) \qquad \Longrightarrow \qquad \frac{d^2Y}{dz^2} = (V(z) - A)Y \tag{45}$$

with

$$\begin{aligned} V(z) &= \tfrac{m_1(m_1+1)}{\mathrm{sn}^{\,2} z} + m_2(m_2+1)\tfrac{\mathrm{dn}^{\,2} z}{\mathrm{cn}^{\,2} z} + m_3(m_3+1)\tfrac{k^2\,\mathrm{cn}^{\,2} z}{\mathrm{dn}^{\,2} z} + m_0(m_0+1)\, k^2\,\mathrm{sn}^{\,2} z \\ A &= 4s + (m_1+m_2)^2 + k^2(m_1+m_3)^2. \end{aligned} \tag{46}$$

This equation has been considered by Darboux [6]. In this short article (3 pages) he claims:

- that the product of two solutions is a polynomial, which we denote by $\Psi_{g,N}(\sigma; w)$ of degree N in $w = \mathrm{sn}^{\,2} z$ and of degree g in $\sigma = 4s$. The knowledge of this polynomial is of paramount importance as we will see later. Let us quote Darboux: "Une fois le polynôme Ψ déterminé, l'intégration s'achève, comme on sait, sans aucune difficulté."

- that for half-integer values of $m_1,\, m_2,\, m_3$ this equation can be integrated (for arbitrary A i. e. for what we call generic solutions). But neither detailed proofs nor the explicit forms of the solutions (even for the simpler case of integer m_i) were given.

We would like to point out that even for Lamé's equation with half-integer values of n only *non-generic* solutions, due to Halphen and Brioschi, are known, so that the claim of Darboux seems questionable.

To come back to Lamé equation, let us mention that its solutions are meromorphic for integer n, and their general form, due to Halphen and Hermite, is given for instance in [28][p. 570-575]. However, since one has to solve a set of n linear equations, only for low values of n everything can be made explicit.

Interestingly enough, this differential equation has also appeared in the quite different field of integrable models, particularly KdV equation. Then relation (45) can be interpreted as a Schrödinger eigenvalue problem, with eigenvalue A (equivalent to the auxiliary parameter s) and potential $V(z)$ given by (46).

The solutions corresponding to $m_i \in \mathbb{Z}$ are called "finite-gap" solutions and sometimes $V(z)$ is called a Treibich-Verdier potential [25], after their work on the subject. The name "finite-gap" refers to the appearance of a finite number of energy bands in the Bloch spectrum, a phenomenon discovered a long time ago by Ince [10].

6 Elliptic level one solutions

Proposition 6 *The level one ($N = 1$) solutions, with $\sigma = 4s$, are given by:*

parameter	solution $y(z)$	constraint	$\Psi(\sigma, w)$
$m_0 = 1$	$e^{zZ(\omega)}\dfrac{H(z-\omega)}{\Theta(z)}$	$\mathrm{dn}^2\omega = \sigma - k^2$	$\sigma + k^2 w - 1 - k^2$
$m_1 = 1$	$e^{zZ(\omega)}\dfrac{\Theta(z-\omega)}{\Theta(z)}$	$\mathrm{dn}^2\omega = \sigma + 1$	$\sigma w + 1$
$m_2 = 1$	$e^{zZ(\omega)}\dfrac{\Theta_1(z-\omega)}{\Theta(z)}$	$\mathrm{dn}^2\omega = \sigma + 1 - k^2$	$\sigma(1-w) + 1 - k^2$
$m_3 = 1$	$e^{zZ(\omega)}\dfrac{H_1(z-\omega)}{\Theta(z)}$	$\mathrm{dn}^2\omega = \sigma$	$\sigma(1 - k^2 w) - 1 + k^2$

(47)

with $Z(z) = \dfrac{\Theta'}{\Theta}(z)$.

Proof:
We will give the detailed proof for $m_1 = 1$, the other cases being analyzed similarly. So we start with

$$\frac{d^2y}{dz^2} - 2\frac{\mathrm{cn}\,z\,\mathrm{dn}\,z}{\mathrm{sn}\,z}\frac{dy}{dz} + \sigma\, y = 0, \tag{48}$$

and we look for a solution $y = e^{\mu z}\dfrac{H(z-\rho)}{\Theta(z)}$. Taking derivatives we get

$$\frac{y'}{y} = \mu + \frac{H'}{H}(z-\rho) - \frac{\Theta'}{\Theta}(z), \qquad \frac{y''}{y} = \left(\frac{y'}{y}\right)^2 - \frac{1}{\mathrm{sn}^2(z-\rho)} + k^2\,\mathrm{sn}^2 z.$$

Let us define the auxiliary function

$$\chi(z) \equiv \left(\frac{y'}{y}\right)^2 - 2\frac{\mathrm{cn}\,z\,\mathrm{dn}\,z}{\mathrm{sn}\,z}\frac{y'}{y} - \frac{1}{\mathrm{sn}^2(z-\rho)} + k^2\,\mathrm{sn}^2 z + \sigma,$$

so that proving that χ vanishes identically is equivalent to proving that y is indeed a solution of (48).

For $z = -iK'$ one can check that the poles in χ cancel automatically. Imposing that the pole at $z = 0$ is absent gives $\mu = \dfrac{H'}{H}(\rho)$. The absence of the pole at $z = \rho$ is easily checked. So we know that χ is bounded in a period parallelogram and since it is doubly periodic, it is bounded in all the complex plane: by Liouville theorem it is a constant. Imposing that χ vanishes for $z = K$ fixes the value of ρ by the equation $\mathrm{sn}^2\rho = -1/\sigma$. Then we switch to the new parameter $\omega = \rho + iK'$ and we express the solution in terms of ω using the transformation theory of theta functions.

We can now determine the spectral polynomial $\Psi(\sigma; z)$ defined, up to an overall constant factor, by the product of the two solutions of Heun's equation. Let us do it first for $m_0 = 1$. The product of the two solutions is computed, using the transformation theory of theta functions, to the identity

$$\frac{H(z-\omega)H(z+\omega)}{\Theta^2(z)\Theta^2(\omega)\Theta^2(0)} = k^2\,\mathrm{sn}^2 z - k^2\,\mathrm{sn}^2\omega, \tag{49}$$

which is a first degree polynomial with respect to $w = \mathrm{sn}^2 z$ and to $\sigma = 1 + k^2 - k^2 \mathrm{sn}^2 \omega$. $\square$

Remarks:

1. Notice that a is uniquely defined up to congruence.

2. All the solutions have genus $g = 1$.

3. The other linearly solution is obtained, for generic $s \neq 0$ by the change $z \to -z$.

4. The solution for $m_0 = 1$ is due to Hermite [28][p.573] and the solution for $m_3 = 1$ is due to Picard [20].

5. The first two formulas for $\Psi(\sigma; w)$ agree with the results given in [23].

6. For special values of σ, or equivalently of a, we may fail to get two linearly independent solutions. In this case factoring out the solution y given previously gives the second solution via a quadrature. Let us give some examples for the solution with $m_0 = 1$:

$$\begin{array}{ll} \sigma = 1 + k^2 & A \,\mathrm{sn}\, z + B \,\mathrm{sn}\, z \left(\dfrac{H'}{H}(z) + \dfrac{E - K}{K}\, z \right) \\ \sigma = 1 & A \,\mathrm{cn}\, z + B \,\mathrm{cn}\, z \left(\dfrac{H_1}{H_1}(z) + \dfrac{E - k'^2 K}{K}\, z \right) \\ \sigma = k^2 & A \,\mathrm{dn}\, z + B \,\mathrm{dn}\, z \left(\dfrac{\Theta_1'}{\Theta_1}(z) + \dfrac{E}{K}\, z \right). \end{array} \qquad (50)$$

7. The general structure of the elliptic solutions, for arbitrary level, are given in [8]. They suffer from the same defect as the general solution of Lamé's equation: they lead to really explicit expressions only for low values of the level N.

7 Finite-gap solutions

In [23], the following facts were used:

1. Let us call $\Psi(\sigma = 4s; w)$ the product of two solutions of Heun's equation and let us define p (resp. q) as the coefficient of the derivative (resp. of the function) in (1). Then Ψ must be a solution of the third order differential equation

$$\Psi''' + 3p\Psi'' + (p' + 2p^2 + 4q)\Psi' + 2(q' + 2pq)\Psi = 0. \qquad (51)$$

For the meromorphic solutions of Heun's equations, with parameters

$$\left\{ \begin{array}{llll} \gamma = \frac{1}{2} - m_1, & \delta = \frac{1}{2} - m_2, & \epsilon = \frac{1}{2} - m_3, & M = m_1 + m_2 + m_3, \\ \alpha = -\frac{1}{2}(m_0 + M), & \beta = \frac{1}{2}(m_0 - M + 1), & N = m_0 + M, & \end{array} \right.$$

the product $\Psi_{g,N}(\sigma; w)$ is a polynomial of degree N with respect to the variable w:

$$\Psi_{g,N}(\sigma, w) = a_0(\sigma)\, w^N + a_1(\sigma)\, w^{N-1} + \cdots + a_N(\sigma), \qquad (52)$$

and a polynomial of degree g in the parameter σ:

$$\Psi_{g,N}(\sigma, w) = b_0(w)\,\sigma^g + b_1(w)\,\sigma^{g-1} + \cdots + b_g(w). \tag{53}$$

The leading term b_0 is given by a first order differential equation due to the fact that σ appears linearly only in the coefficients of Ψ' and Ψ of equation (51). We have taken for normalization

$$b_0(w) = w^{m_1}(1-w)^{m_2}(1-k^2w)^{m_3},$$

as can be checked from (47).

2. Then, following a method due to Lindemann and Stieltjes [28][p. 420] adapted to Heun's case, one looks for a solution of the form

$$F(w) = \sqrt{\Psi}\exp\left(\pm i\frac{\nu(\sigma)}{2}\int \frac{N(w)}{\Psi(w)}\,dw\right).$$

Inserting this ansatz into Heun's equation gives on the one hand

$$\frac{N'}{N} = -p \qquad \Longrightarrow \qquad N(w) = \frac{w^{m_1}(1-w)^{m_2}(1-k^2w)^{m_3}}{\sqrt{w(1-w)(1-k^2w)}}$$

(recall that all the m_i are positive) and on the other hand:

$$\nu^2(\sigma) = \frac{2\Psi\Psi'' - \Psi'^{\,2} + 2p\Psi\Psi' + 4q\Psi^2}{N^2}, \tag{54}$$

showing that ν^2 is of degree $2g+1$ in σ. The fact that ν is a constant is easily checked by differentiating relation (54) and using the differential equation (51).

So we conclude that Heun's functions (the so-called “finite-gap” solutions), with the parameters given above, have for integral representation:

$$\sqrt{\Psi_{g,N}(\lambda; w)}\,\exp\left(\pm i\frac{\nu(\lambda)}{2}\int \frac{w^{m_1}(w-1)^{m_2}(1-k^2w)^{m_3}\,dw}{\Psi_{g,N}(\lambda; w)\sqrt{w(w-1)(1-k^2w)}}\right). \tag{55}$$

For the level one solutions already considered the polynomial $\nu^2(\sigma)$ is given by

$$\begin{array}{ll} m_0 = 1 & (\sigma-1)(\sigma-k^2)(\sigma-1-k^2) \\ m_1 = 1 & \sigma(\sigma+1)(\sigma+k^2) \\ m_2 = 1 & \sigma(\sigma-k^2)(\sigma+1-k^2) \\ m_3 = 1 & \sigma(\sigma-1)(\sigma-1+k^2) \end{array} \tag{56}$$

A partial list of the finite-gap solutions, up to level $N = 5$, has been given in [23].

These results show that there do exist integral representations for Heun functions, but they are not “cheap”. Furthermore, as we will show now on a particular example, these integral representations are just a different dressing of the elliptic solutions: they are a kind of “algebraization” of the elliptic solutions, but are exactly the same analytic objects.

8 Finite-gap versus elliptic solutions

We will show, for the level one finite-gap integral representation with $m_1 = 1$, that it does coincide with its corresponding elliptic solution given by (47). The other cases can be relateded using completely similar arguments. For this identification it is sufficient to consider $w \in [0,1]$. We start from the data

$$\sigma = -k^2 \,\mathrm{sn}^2\omega, \qquad \nu^2(\sigma) = \sigma(\sigma+1)(\sigma+k^2), \qquad \Psi(\sigma; w) = \sigma\, w + 1.$$

So the factor appearing in the exponential of relation (55) is

$$\pm\frac{1}{2}\int \frac{k^2\,\mathrm{sn}\,\omega\,\mathrm{cn}\,\omega\,\mathrm{dn}\,\omega\;\; w\,dw}{(1+\sigma\,w)\sqrt{w(1-w)(1-k^2\,w)}}.$$

Under the change of variable $w = \mathrm{sn}^2 z$, with $z \in [0, K]$, it becomes an elliptic integral of the third kind, computed in [28][p. 523]:

$$\pm\int \frac{k^2 \mathrm{sn}\,\omega\,\mathrm{cn}\,\omega\,\mathrm{dn}\,\omega\,\mathrm{sn}^2 z\,dz}{1-k^2\,\mathrm{sn}^2\omega\,\mathrm{sn}^2 z} = \pm\frac{1}{2}\left(\ln\frac{\Theta(z-\omega)}{\Theta(z+\omega)} + z\,Z(\omega)\right).$$

So, keeping the plus sign, we get for the exponential term in (55)

$$e^{zZ(\omega)}\,\sqrt{\frac{\Theta(z-\omega)}{\Theta(z+\omega)}}. \tag{57}$$

And we need to multiply this by the square root of

$$\Psi \propto 1 - k^2\,\mathrm{sn}^2\omega\,\mathrm{sn}^2 z = \mathrm{sn}^2\omega\left(k^2\,\mathrm{sn}^2(\omega - iK') - k^2\,\mathrm{sn}^2 z\right),$$

and upon use of the identity (49) we are first left with

$$\Psi \propto \frac{H(-\omega+z+iK')\,H(-\omega-z+iK')}{\Theta^2(z)},$$

and after use of the transformation theory for the theta functions we end up with

$$\Psi \propto \frac{\Theta(z+\omega)\,\Theta(z-\omega)}{\Theta^2(z)}. \tag{58}$$

Gathering (57) and (58) we conclude that the finite-gap solution (55) reduces to

$$e^{zZ(\omega)}\frac{\Theta(z-\omega)}{\Theta(z)},$$

which is nothing that the elliptic solution given in (47).

9 Conclusion

The finite-gap solutions or their elliptic counterparts solve Heun's equation for all cases where these solutions are meromorphic functions of the variable z. It is interesting to notice that these progresses have been evolving in relation with integrability considerations. It is quite difficult to say what new ideas will require the generalization of these results to cover the non-meromorphic solutions. These much more difficult problems are left for the future.

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