On geodesic envelopes and caustics

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Abstract

We give a global description of envelopes of geodesic tangents of regular curves in (not necessarily convex) Riemannian surfaces. We prove that such an envelope is the union of the curve itself, its inflectional geodesics and its tangential caustics (formed by the conjugate points to those of the initial curve along the tangent geodesics). Stable singularities of tangential caustics and geodesic envelopes are discussed. We also prove the (global) stability of tangential caustics of close curves in convex closed surfaces under small deformations of the initial curve and of the ambient metric.

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1 Introduction

The *geodesic envelope* of a regular curve in a (complete smooth) Riemannian surface is the envelope of its geodesic tangents. In this note we describe geodesic envelopes from a global viewpoint. For this, we consider the *tangential caustics* of the curve, formed by the conjugate points (of different order) to those of the curve along its tangent geodesics.

Recall that two points are conjugate if the second point is the intersection of two arbitrary close geodesics issuing from the first point; the caustics associated to a point are the sets of the n-th conjugate points to the initial point (the ordering of the conjugate points along a geodesic being well-defined, due to the positiveness of the injectivity radius).

The tangential caustic is not the usal caustic defined in Optics, which is the envelope of the normal geodesics of a curve. It is a different generalisation of the usual caustic of a point (formed by its conjugate points), introduced by Poincaré in [9].

Our first result is the following global description of geodesic envelopes.

Theorem 1. *The geodesic envelope of any regular curve in a complete Riemannian surface is the union of the curve, its inflectional geodesics and its tangential caustics.*

This result generalizes to Riemannian manifolds some of Thom's results about envelopes of 1-parameter families of lines in the projective plane (see [10]). We also prove that the only generic and stable singularities of tangential caustics are semicubic cusps and transversal self-intersections; for geodesic envelopes, we have to add local second-order self-tangencies (which are not stable for general envelopes).

The above theorem and the classification of stable singularities of tangential caustics lead to the following stability statement.

Theorem 2. *Each p-tangential caustic of a regular closed curve in a strictly convex closed Riemannian surface is generically stable under small enough deformations of the initial curve and of the ambient metric (the required smallness depending on the order of the caustic).*

The stability means that there exists a diffeomorphism of the ambient surface transforming the unperturbed n-caustic into the perturbed one. "Generically" means (here and throughout the paper) that the curves for which the statement does not hold form a residual set for Whitney topology in the set of all the curves in the surface (see [3], §3).

The framework here is the same as several classical subjects in Geometry, Optics and Calculus of Variations, going back for instance to Archimedes, Huygens, Barrow and Jacobi, as the study of evolutes of curves and caustics of ray systems. The relation between singularities of ray systems, their caustics, wave fronts, Legendre transformations and reflection groups was discovered by Arnold (see [1]), in the setting of Symplectic and Contact Geometry, and further developed by O.V. Lyashko, A.B. Givental, O.P. Shcherbak (see [4]).

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2 Preliminary definitions

Let (M, g) be a complete smooth Riemannian surface with metric g and let $\gamma : I \to (M, g)$ be a regular curve parameterized by arc-length, where I is an open interval of the circle S^1. We endow as usual the manifold of all such maps with Whitney topology. For each $\xi \in I$, denote by $\Gamma_\xi : \mathbb{R} \to M$ the unit speed geodesic, tangent to γ at $\gamma(\xi)$ (and oriented as γ at this point).

We say that γ is *convex* at ξ if $\ddot{\gamma}(\xi) \neq 0$; in this case, the tangency order of γ with the tangent geodesic Γ_ξ at $\gamma(\xi)$ is 1. If $\ddot{\gamma}(\xi) = 0$, then ξ is said to be an *inflection* of γ. The inflection is *simple* when the curve has a second-order tangency with the inflectional geodesic.

The *graph* of the family of geodesics $\{\Gamma_\xi : \xi \in I\}$ is the immersed surface

$$\Phi := \Big\{ \big(\Gamma_\xi(t), \xi\big) : \xi \in I, t \in \mathbb{R} \Big\} \subset M \times I\ .$$

Definition. The *geodesic envelope* of γ is the apparent contour of the graph in M, that is, the critical value set of the graph's projection $\Phi \to M$ defined by $(P, \xi) \mapsto P$.

Note that the image of γ is contained in the geodesic envelope.

Consider an embedded local component $\bar{\delta} \subset \Phi$ of the critical set of the graph's projection; denote by δ its projection in M. Let $(P, \xi) \in \bar{\delta}$. The branch δ is said to be a *geometric envelope* at $P = \Gamma_\xi(t_0)$ if the curve $t \mapsto (\Gamma_\xi(t), \xi)$ intersects $\bar{\delta}$ only at $t = t_0$ (in a small neighbourhood of t_0).

Definition. The *tangential caustic* of a regular curve γ in a complete Riemannian surface (M, g) is the set Σ formed by the conjugate points to those of γ along its tangent geodesics.

As the caustic of a point, the tangential caustic splits into components Σ_p, formed by the p-th conjugate points. Here p belongs to $\mathbb{Z}$, the conjugate points of negative order lying on the tangent geodesic in the direction opposed to that of γ at the tangency point. The curve γ can be considered as the 0-tangential caustic.

Example 1. A curve in a surface with non-positive curvature (as for instance the Euclidian plane or the Lobachevsky plane) has no tangential caustics of order $p \neq 0$ (see e.g. [8]). In this case, Theorem 1 says that the envelope of the geodesic tangents of a curve is the union of the curve with its inflectional tangents.

Example 2. Let (M, g) be a Riemannian surface whose curvature is everywhere bounded by two given positive constants. Then, every regular curve in M has infinitely many tangential caustics (see [8]); if the curve is closed, then every tangential caustic is also closed. For instance, the p-th tangential caustic of a curve in the standard sphere S^2 is the curve itself for even p and the antipodal curve otherwise.

For $p \in \mathbb{Z}$, let $\tau_p(\xi)$ be the distance along Γ_ξ between $\gamma(\xi)$ and its p-th conjugated point (provided that it exists). In particular, $\tau_0 \equiv 0$. Each map τ_p is continuous, and defined on a subset I_p of I, possibly empty. We have $I_{p+1} \subseteq I_p$ for $p \geq 0$, $I_p \subseteq I_{p+1}$ otherwise.

Lemma. *The map $\varphi_p : I_p \to M$, sending ξ to $\Gamma_\xi(\tau_p(\xi))$, is a smooth parameterization of Σ_p (which is hence a smooth curve).*

Proof. Let P and Q be two consecutive conjugated points along a geodesic c. Then, there exists a non-trivial Jacobi field along c, orthogonal at every point to the geodesic and vanishing at P and Q (see [8]). Consider a smooth deformations $P(\lambda)$ and $c(\lambda)$ of P and c: $c(\lambda)$ is a geodesic issuing from $P(\lambda)$ for every small enough λ. Due to the regularity Theorem about solutions of second order differential equations, depending on parameters, we obtain that, for every λ small enough, there exists a non-trivial orthogonal Jacobi field along $c(\lambda)$, vanishing to $P(\lambda)$ and to some other point $Q(\lambda)$. This point is therefore the conjugate point of $P(\lambda)$ along the geodesic $c(\lambda)$, and it depends smoothly on the parameter λ, provided that the deformation is small enough. □

3 Proof of Theorem 1

The proof of Theorem 1 is subdivided into three steps. In the first two steps we show that the inflectional tangents and the tangential caustics of a curve are contained in its geodesic envelope. Finally we show that this envelope contains nothing else.

Proposition 1. *The geodesic envelope of a regular curve in a Riemannian surface contains its inflectional geodesics (as non geometric components).*

Proof. Let $P = \gamma(\xi_0)$ be a simple inflection of γ and let $Q = \Gamma_{\xi_0}(T)$ be the first conjugate point to P along Γ_{ξ_0} (the argument can be easily adapted to the case where P has no conjugated points). We can suppose $T > 0$, and change the orientation of γ to consider the conjugate points in the other direction.

Fix two geodesic balls B_P and B_Q, centered at P and Q respectively, of radius $r > 0$ arbitrary small. For $\xi \to \xi_0$ with $\xi \neq \xi_0$, the geodesics Γ_ξ meet the geodesic Γ_{ξ_0} in the interior of B_P and then meet the ball's frontier ∂B_P at $\Gamma_\xi(t_1(\xi))$, where $t_1(\xi) = r + o(\xi - \xi_0)$; all these intersections belong to the closure of the same connected component of $\partial B_P \setminus \{\Gamma_{\xi_0}(\pm r)\}$. Next, the geodesics Γ_ξ intersect ∂B_Q at $\Gamma_\xi(t_2(\xi))$, where $t_2(\xi) = T - r + o(\xi - \xi_0)$, and Γ_{ξ_0} in B_Q.

Hence, the geodesic segments $\Gamma_\xi(]t_1(\xi), t_2(\xi)[)$ do not intersect $\Gamma_{\xi_0}(]r, T - r[)$. Thus, their lifting in the graph Φ form a smooth fold-like surface, whose apparent contour in M is $\Gamma_{\xi_0}(]r, T - r[)$ for every $r > 0$ arbitrary small, so $\Gamma_{\xi_0}([0, T])$ is contained in the geodesic envelope. The same argument shows that the envelope contains the Γ_{ξ_0} geodesic segment between any two consecutive conjugate points and, therefore, the whole geodesic $\Gamma_{\xi_0}(\mathbb{R})$.

Finally, if γ has non-simple inflections, we consider a deformation γ^λ of the curve γ, having only simple inflections for $\lambda \neq 0$. The statement holding for the perturbed envelopes, by continuity we get the claim for the unperturbed envelope. This ends the proof, since the non geometricity of the inflectional geodesic is clear. □

Proposition 2. *The tangential caustic of a regular curve is contained in its geodesic envelope.*

Proof. Fix $p \in \mathbb{Z}$ such that I_p is not empty. Let $\xi_0 \in I$, such that $\tau_p(\xi_0) \in I_p$. By the very definition of conjugate point, for every $\varepsilon > 0$ there exists a p-depending constant C such that for every $|\xi - \xi_0| < C$ the geodesics Γ_{ξ_0} and Γ_ξ intersect each other at some points $\Gamma_{\xi_0}(t_i)$, $i = 0, \ldots, p$, where $|t_i - \tau_i(\xi_0)| < \varepsilon$. By continuity, $t_i \to \tau_i(\xi_0)$ for $\xi \to \xi_0$. Hence, every point of the p-th tangential caustic Σ_p is the intersection of infinitesimally close curves of the family of tangent geodesics. Therefore, Σ_p is contained in the geodesic envelope. □

Remarks. (1) Let us recall that there exists a different definition of envelope: a point belongs to the "naif envelope" of a family of curves if it is the intersection of infinitesimally close curves of the family. In the preceding proof, we have actually shown that Σ_p is contained in the "naif envelope" of the geodesic tangents of γ. But it is well known (see e.g. [5], §5) that the "naif envelope" is contained in Thom's envelope (being sometimes different).
(2) Each tangential caustic is a geometric branch of the geodesic envelope.
(3) The caustic of a point can be viewed as the envelope of the geodesics issuing from it. A similar characterization holds for the tangential caustics of a curve. Indeed, $\Sigma_p \cup \Sigma_{-p}$ is the envelope of the pencil, parameterized by ξ, of the usual $|p|$-caustics of the points $\gamma(\xi)$.

The proof of Theorem 1 is completed by the following fact.

Proposition 3. *Every point of the geodesic envelope of a regular curve γ belongs to its tangential caustics or to an inflectional geodesic tangent.*

Proof. Fix a point $\Gamma_\xi(t)$ of the geodesic envelope of γ. Thus, $(\Gamma_\xi(t), \xi)$ is a critical point of the graph's projection π. Suppose that this point belongs to a regular geometric branch of the envelope (if the branch is not regular, we can make it regular by an arbitrary small deformation of γ). In this case the geodesics Γ_ξ and $\Gamma_{\xi+\varepsilon}$ intersect each other at two points $\Gamma_\xi(o(\varepsilon))$ and $\Gamma_\xi(t + o(\varepsilon))$ for ε arbitrary small. Passing to the limit for $\varepsilon \to 0$ we obtain either that $\Gamma_\xi(0)$ and $\Gamma_\xi(t)$ are conjugated or that $t = 0$. In both cases $\Gamma_\xi(t)$ belongs to a tangential caustic of γ.

On the other hand, if the envelope is not geometric at $(\Gamma_\xi(t), \xi)$, then it is clear that the whole geodesic Γ_ξ is contained in the geodesic envelope of γ. As in the proof of Proposition 1, this is possible if and only if γ has an inflection at ξ. □

4 Proof of Theorem 2

We start describing the singularities of tangental caustics.

Proposition 4. *Each tangential caustic Σ_p of a regular curve γ is regular outside a subset $X_p \subset I_p$ of I_p, which is generically discrete. The generic singularities of the tangential caustics are semicubic cusps and transversal self-intersections; these singularities are stable under small enough deformations of the curve γ and of the ambient metric g.*

Proof. Recall that a *system of rays* emanating from a given embedded curve is the pencil of the normal geodesics of the curve. The envelope of these normal geodesics is called the *caustic of the system of rays.*

Fix a point $\xi_0 \in I$, such that $\ddot{\gamma}(\xi_0)$ is not vanishing. Then there exists some evolvent σ of γ, such that the tangent geodesics to γ, Γ_ξ, form a system of rays emanating from σ, provided that ξ is close enough to ξ_0 (see fig. 1). The generic singularities of the caustics of systems of rays are semicubic cusps and transversal self-intersections (see [4]). These singularities are stable under small deformations of the system of rays. Thus, each curve φ_p is regular outside a generically discrete subset of I_p. □

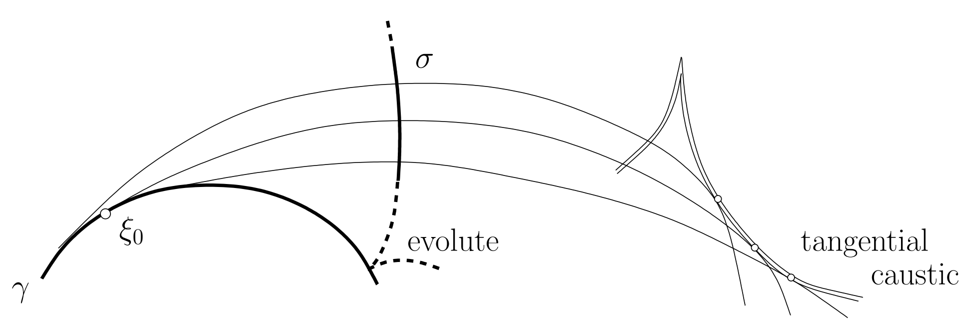

Figure 1: The tangent geodesics of a curve as a system of rays.

The genericity required in the Lemma provides the genericity of the Lagrangian surface associated to the system of rays, see [4] and therein references.

Proposition 5. *Let $\gamma(\xi_0)$ be a simple inflection of γ. Then, for every fixed $p \in \mathbb{Z}$, Σ_p has generically a simple inflection at the corresponding point $\varphi_p(\xi_0)$.*

Proof. Generically Σ_p is a smooth curve with isolated singularities, having also isolated simple inflections. Fix $\xi_0 \in I$, such that $\tau_p(\xi_0) \in I_p$ and let B_P be a Riemannian ball centered at $P := \varphi_p(\xi_0)$ of radius r arbitrary small.

The geodesic Γ_{ξ_0} divides the boundary of ∂B_P into two disjoint sectors. If Σ_p is convex at P, the tangent geodesics arriving from γ are entering in the ball B_P crossing one of the two sectors of the boundary for $0 < \xi - \xi_0 \ll 1$ and crossing the other sector for $0 < \xi_0 - \xi \ll 1$. As we have seen in the proof of Proposition 1, this is impossible when γ has a simple inflection at ξ_0. Thus Σ_p is not convex at $\varphi_p(\xi_0)$ if γ has a simple inflection at ξ_0.

The same reasoning allows us to exclude the possibility that Σ_p has a semicubic cusp at $\varphi_p(\xi_0)$. Hence Σ_p has an inflection at $\varphi_p(\xi_0)$. This inflection is generically simple. □

In some degenerate cases the point of Σ_P corresponding to a simple inflection can have degenerate inflections of odd order, like $t \mapsto (t, t^5)$ in the Euclidian plane, but also more complicated singularities, like $t \mapsto (t^2, t^4 + t^5)$.

Proposition 6. *The generic singularities of geodesic envelopes of regular curves are semicubic cusps, transversal self-intersections and (local) second order self-tangencies. These singularities are stable under small enough deformations of the curve γ and of the ambien metric (the required smallness of the deformations depending on the order of the tangential caustics contening the singularity).*

The stability concernes only local 2-self-tangencies: in the non-local case, two branches of the envelope may have any kind of tangency, which disappear under small deformations of the initial curve (splitting into transversal self-intersections).

Proof. This follows from Propositions 4 and 5. □

Remark. The stability of local 2-self-tangencies of geodesic envelopes can also be deduced from the stability of these singularities for envelopes of tangential families (see [6]). Indeed, denoting by U a connected component of $I_p \setminus X_p$, the family of geodesics $\{\Gamma_\xi, \xi \in U\}$ is a tangential family, whose support is the corresponding regular component of Σ_p.

We prove now Theorem 2. Consider the manifold PT^*M of all the contact elements on M (a contact element is a pair (x, h) of a point $x \in M$ and a homogeneous hyperplane $h \subset T_xM$). The *projectivized cotangent bundle* PT^*M of M has a natural contact structure.

Each regular curve on M can be lifted to a Legendrian curve of PT^*M: at each point of the curve we associate the contact element formed by the point and the tangent line (on the affine tangent plane) to the curve at this point.

By this construction we can associate a surface (called Legendrian graph, see [7]) to the family of the tangent geodesics Γ_ξ of our curve γ. This surface is the union of the Legendrian lifts of the tangent geodesics. The envelope of the family is the (Legendrian) apparent contour of this Legendrian graph under the natural Legendrian fibration $PT^*M \to M$, $(x, h) \mapsto x$.

Under the hypothesis of Theorem 2, the contour generator (i.e. the critical set of this projection) has some regular closed components, projecting on the caustics Σ_p. Since these components are Legendrian curves, the caustic Σ_p can be viewed as a closed fronts. A deformation of the initial curve and of the ambient metric induces a deformation of these Legendrian curves (among Legendrian curves). By Proposition 4, generically (with respect of the curve γ) the only singularities of such a front are semicubic cusps and transversal self-intersections. It is proved in [2] that such a front is stable under small enough deformations of the Legendrian curve generating it. Theorem 2 is proved.

Remark. In the preceding propositions and in Theorem 2 the order of the caustic is fixed. I do not know whether the same statements hold for all the tangential caustics simultaneously.

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