

AN OBSERVER AND REGULATOR FOR LINEAR SYSTEMS
WITH PURE AND DISTRIBUTED DELAYS

A THESIS

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SUMMARY

A representation theory based on convolution operations was developed for a large class of linear systems containing pure and distributed delays in state and control by E. W. Kamen [12]. Using this framework, Kamen [12, 13] has also studied the state feedback problem. However, in order to utilize state feedback, it is often necessary to reconstruct missing state-variable information. In this work we consider the design of observers (state-estimators) using Kamen's convolution operator framework. In particular, conditions for coefficient assignability of the error dynamics will be developed by using duality. The observer construction will then be used in the design of input/output regulators.

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CHAPTER I

INTRODUCTION

We will be looking at the class of systems given by a first-order delay differential equation of the form

$$\frac{dx(t)}{dt} = F_0 x(t) + \sum_i F_i x(t-a_i) + g_0 u(t) + \sum_i g_i u(t-b_i) \quad (1.1)$$

where a_i and b_i are positive real numbers. F_i (resp. g_i) are $n \times n$ (resp. $n \times 1$) matrices over the reals R , $x(t) \in R^n$ is the "instantaneous state", and $u(t) \in R$ is the input or control. Systems of this type occur in many applications (see [7] for examples). A good deal of work has been carried out on systems of the form (1.1) especially in recent years (see survey [15]). Essentially all this previous work is based on operators defined on Hilbert or Banach spaces. This (functional analytical) approach has produced many results but as a result of the infinite dimensionality of the underlying spaces, it is necessary to use approximation methods in carrying out computations.

The observer problem for (1.1) has been studied also in terms of the functional-analytical framework (e.g., [4, 5]). However very few results exist on the construction of observers for (1.1), due to the infinite dimensionality of the Hilbert (or Banach) space setting. In addition, not

much work exists on the design of input/output regulators for systems with time delays. Recent work by Bhat [3] does involve the application of the functional analytical setting to the study of regulators, although in general his results are not algorithmic.

Recently, Kamen [11] has developed an operator framework for the study of systems given by (1.1). Kamen's approach is based on viewing (1.1) as a vector differential equation defined over a ring of delay operators. This algebraic structure makes it possible to approach the computation of complete solutions by using an operational calculus. The "computability" of the algebraic framework is primarily a result of representing operators by (finite) matrices defined over a convolution ring. Kamen has also studied state-feedback in terms of the convolution operator framework [12, 13]. In [12, 13] conditions for spectrum assignability are given in terms of module generation conditions.

In this work we will use the "finiteness" properties of Kamen's operator theory to develop a theory for the design of observers and regulators. We shall develop a duality theory which allows us to carry over the state-feedback construction of Kamen [12, 13]. The use of duality here is very similar to the manner in which duality is used in the theory of finite-dimensional systems.

CHAPTER II

PRELIMINARIES

Introduction

In this section we present Kamen's [12, 13] algebraic theory for the class of systems given by the first order delay differential equation (1.1).

System Definition

We begin by constructing the commutative rings, in terms of which we will define our concept of a system. In this development we follow closely that of Kamen [12, 13].

Let R denote the field of real numbers. Let δ_a denote the Dirac distribution concentrated at the point a .

Let D denote the set of sums $\sum_{i=1}^q c_i \delta_{a_i}$, $a_i \geq 0$, $a_i \in R$, q finite. This set D is a commutative ring with the obvious addition operation and with convolution defined by

$$\left(\sum_i c_i \delta_{a_i}\right) * \left(\sum_i d_i \delta_{b_i}\right) = \sum_j \sum_i c_j d_i \delta_{a_j + b_i} \quad (2.1)$$

Let L_+ denote the space of R -valued, Lebesgue-measurable, locally-integrable functions (defined a.e. on R) with supports bounded on the left (i.e., given $v \in L_+$, there exist a t_0 , which depends on v , such that $v(t) = 0$ for all $t < t_0$). The set L_+ is a commutative ring with pointwise

addition $(u + v)(t) = u(t) + v(t)$, and with convolution

$$(\mu * v)(t) = \int_{-\infty}^{\infty} \mu(t-\tau)v(\tau)d\tau \quad (2.2)$$

where $u, v \in L_+$

Given $z = \sum_i c_i \delta_{a_i} \in D$ and $u \in L_+$ we define the following convolution operation

$$(z * u) = \sum_i c_i u(t - a_i) \in L_+ \quad (2.3)$$

Let L_0 denote the subring of L_+ consisting of all functions having bounded support contained in $[0, \infty)$. The convolution $(v * u)$, with $v \in L_0$ and $u \in L_+$, is that given by (2.2), but the support of v is contained in some interval $[0, a]$, $a > 0$, and then (2.2) can be written as

$$(v * u)(t) = \int_{t-a}^t v(t-s)u(s)ds \quad (2.4)$$

Let J denote the set consisting of all sums $z + v$ where $z \in D$ and $v \in L_0$. J is a commutative ring with convolution

$$(z + v) * (y + w) = z * y + z * w + y * v + v * w \quad (2.5)$$

with $z, y \in D$, $v, w \in L_0$, $z * y$ is convolution in D (2.1), $z * w$ and $y * v$ are defined by (2.3), and $v * w$ is convolution in L_0 (2.2).

Note that the rings J and D contain the identity δ_0 ,

where as L_0 and L_+ do not. Also J and D contain $R\delta_0 = \{a\delta_0 : a \in R\}$. With the obvious addition and multiplication operation $R\delta_0$ is a field isomorphic to the field R . We can thus view J and D as ring extensions of R .

Given $u \in L_+$, $G \in J$, with $G = z + v$, $z = \sum_i c_i \delta_{a_i} \in D$, $v \in L_0$, we define the following convolution operation

$$(G * \mu)(t) = \sum_i c_i \mu(t - a_i) + (v * \mu)(t) \in L_+ \quad (2.6)$$

We are now able to define the notion of a system over a ring.

Definition 2.1

Let N be a fixed subring of J with $R\delta_0 \leq N \subseteq J$. A system Σ over N is a triple (F, G, H) of $n \times n$, $m \times n$, $n \times k$ matrices over N , together with the dynamical equations

$$\begin{aligned} \dot{x}(t) &= (F * x)(t) + (G * u)(t) \\ y(t) &= (H * x)(t) \end{aligned} \quad (2.7)$$

where $u \in L_+$, and x a column vector over L_+ .

The Characteristic Operator

Let p denote the generalized derivative of δ_0 , and p^i denote the i th generalized derivative of δ_0 . Given $\theta \in L_+$, we have $p^i * \theta = i$ th generalized derivative of θ where $p^i * \theta$ is convolution in a space of distributions. Let $J[p]$

denote the set of finite sums $\left\{ \sum_{i=0}^{\alpha} \alpha_i * p^i : \alpha_i \in J \right\}$.

$J[p]$ is a commutative ring with operations

$$\begin{aligned}\sum_i e_i * p^i + \sum_i f_i * p^i &= \sum_i (e_i + f_i) * p^i \\ (\sum_i e_i * p^i) * (\sum_i f_i * p^i) &= \sum_j \sum_i e_j * f_i * p^{j+i}\end{aligned}$$

Now consider a system $\Sigma = (F, G, H)$ over $N, (R \delta_0 \leq N \leq J)$ given by the equation (2.7). Following Kamen [12] in the study of (2.7) we shall consider $(pI - F) = nxn$ matrix over $J[p]$, where I is the nxn identity matrix. Since $J[p]$ is a commutative ring we are able to consider the determinant of $(pI - F)$, denoted by $\det(pI - F)$.

Definition 2.2

The element $\det(pI - F) \in J[p]$ is the characteristic operator of the system given by (2.7). In functional analytical theory [13], the Laplace transform of $\det(pI - F)$ is the characteristic function. The zeros of the characteristic function are the eigenvalues. It is well known [6] that the asymptotic behavior of (2.7) is determined by the location of the eigenvalues in the complex plane. In particular (2.7) is asymptotically stable if and only if all eigenvalues have negative real parts.

State Feedback

Given a system $\Sigma = (F, G, H)$ over $N = D$, we can consider state feedback by setting

$$u = K*x + r \tag{2.8}$$

where $K \in J^{m \times n}$, $x(t) \in L_+^n$, r is an external input. Note we are allowing feedback element K to contain distributed delays. Combining (2.7) and (2.8) yields the closed loop equation

$$\dot{x}(t) = (F - G^*K)x + G^*r$$

We will characterize the state feedback by considering the characteristic operator of the closed-loop system, given by

$$\det(pI - F + G^*K) \quad (2.9)$$

Since K is over J we have

$$\det(pI - F + G^*K) = p^n + \sum_{i=0}^{n-1} \gamma_i^* p^i, \quad \gamma_i^* \in J$$

In the remainder of this paper we will consider the single input-single output case ($G = g$, $H = h$). We will let Σ denote an arbitrary matrix. For the single input case the feedback will be an n element column vector (k).

Given $\Sigma = (F, g, h)$ we write

$$\det(pI - F) = p^n + \sum_{i=0}^{n-1} \alpha_i^* p^i, \quad \alpha_i \in D$$

and define

$$\bar{F} = \begin{bmatrix} 0 & \delta_0 & 0 & \cdots & 0 \\ \cdot & 0 & \delta_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & \delta_0 \\ -\alpha_0 & -\alpha_1 & \cdots & \cdots & -\alpha_n & -1 \end{bmatrix} \quad \bar{g} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \delta_0 \end{bmatrix}$$

Definition 2.3

The controllability matrix of a pair (F, g) is the $n \times n$ matrix

$$C = (g, F^*g, \dots, F^{n-1}g)$$

It follows for the pair (\bar{F}, \bar{g})

$$\bar{C} = (\bar{g}, \bar{F}^*\bar{g}, \dots, \bar{F}^{n-1}\bar{g})$$

Definition 2.4

The pair (F, g) is J -equivalent (resp. Q -equivalent) to the pair (\bar{F}, \bar{g}) if $\det(C)$ is a unit in J (resp. $\det(C) \neq 0$). J -equivalent implies

$$\bar{F} = PFP^{-1}, \quad \bar{g} = Pg, \quad P = \bar{C}C^{-1} \in J^{n \times n}$$

Q -equivalent implies

$$\bar{F} = PFP^{-1}, \quad \bar{g} = Pg, \quad P = \bar{C}C^{-1} \in Q^{n \times n}$$

where Q is the quotient ring of J given as $D^{-1}J = (\theta/\lambda; \theta \in J, \lambda \in D)$.

Definition 2.5

For (F, g) and (\bar{F}, \bar{g}) J - equivalent (resp. Q - equivalent), the system $\Sigma = (\bar{F}, \bar{g}, \bar{h})$ where $\bar{h} = hP^{-1}$ is called the control canonical J form (resp. Q form) of the system $\Sigma = (F, g, h)$.

Definition 2.6

Given a system $\Sigma = (F, g, -)$ over D the characteristic operator $\det(pI - F + g*k)$ is Coefficient Assignable with respect to J if for any e_0, e_1, \dots, e_{n-1} belonging to J there exist an n -element row vector k over J such that

$$\det(pI - F + g*k) = p^n + \sum_{i=0}^{n-1} e_i * p^i$$

By choosing the e_i to be equal to $c_i \delta_0$, $c_i \in R$, we see that coefficient assignability implies that the eigenvalues of the closed loop system can be made finite in number and can be placed anywhere in the complex plane. Thus coefficient assignability implies eigenvalue placement. Following Kamen [12, 13] we have the following sufficient condition for coefficient assignability.

Theorem 2.2

The characteristic operator $\det(pI - F + g*k)$ is coefficient assignable with respect to D if $\det(c)$ is an invertible element of D .

Proof

$$\text{Write } \det(pI - F) = p^n + \sum_{i=0}^{n-1} \alpha_i * p^i, \alpha_i \in D$$

and let

$$\sigma(p) = p^n + \sum_{i=0}^{n-1} \gamma_i * p^i, \gamma_i \in J$$

Then from the results of Kamen [12, 13] we have

$$\det(pI - F + g*k) = \sigma(p)$$

if

$$k = (\gamma_0 - \alpha_0, \gamma_1 - \alpha_1 \cdots \gamma_{n-1} - \alpha_{n-1}) * \bar{C} * C^{-1}$$

The above condition is rather severe for all but the finite dimensional case. In particular an inverse will only exist for $\det(C) = a\delta_0$ with $a \neq 0, a \in \mathbb{R}$. As seen from the results of Kamen [12], if we allow the feedback k to be over J , then the condition in theorem (2.2) is not necessary.

Again, from the results of Kamen [12], we have the following necessary and sufficient condition for coefficient assignability.

Theorem 2.3

Given a system $\Sigma = (F, g, -)$ over D the $\det(pI - F + g*k)$ is coefficient assignable with respect to J if and only if there exist a $n \times n$ matrix W over J such that

$$W^*(\overline{pI - F}) * g = \begin{bmatrix} \delta_0 \\ p \\ \cdot \\ \cdot \\ p^{n-1} \end{bmatrix} \quad (2.11)$$

where $(\overline{pI - F})$ denotes the adjoint of $(pI - F)$. Using this result we can derive the following constructive procedure for eigenvalue placement. For desired

$$\det(pI - F + g * k) = \sigma(p) = p^n + \sum_{i=0}^{n-1} a_i p^i, \quad a_i \in \mathbb{R}$$

and we have

$$\det(pI - F) = p^n + \sum_{i=0}^{n-1} \alpha_i * p^i$$

Then the feedback vector is given by

$$k = (a_0 - \alpha_0, a_1 - \alpha_1, \dots, a_{n-1} - \alpha_{n-1}) * W$$

Input Feedback

Recently Kamen [13] has studied the feedback control problem for the case in which g in the system equation (2.7) is of the form $g * \delta_a$, $g * \in D^n$. It is easily shown that for this case, condition (2.11) cannot be satisfied. However Kamen [13] has shown that through the use of both state and input feedback, it may still be possible to achieve coefficient assignability. We can achieve this structure

by setting

$$u = -v*u - k*x + r \quad (2.12)$$

with $v \in L_0$, $k \in J^n$ combining (2.7) and (2.12) we obtain the following closed loop equation

$$(\delta_0 + v)*\dot{x} = (\delta_0 + v)*F*x - g*k*x + g*r \quad (2.13)$$

We then have the following result from Kamen [13].

Theorem 2.4

The characteristic operator $\det(pI - F + (\delta_0 + v)^{-1} * g * k)$ is coefficient assignable up to a multiplicative factor if there exists a $n \times (n + 1)$ matrix $W = (W_1, W_2, \dots, W_{n+1})$ with $W_i \in J^n$ for $i = 1, 2, \dots, n$ and $W_{n+1} \in L_0^n$ such that

$$W^* \begin{bmatrix} \overline{(pI - F)} * g \\ \det(pI - F) \end{bmatrix} = \begin{bmatrix} \delta_0 \\ p \\ \vdots \\ p^{n-1} \end{bmatrix}$$

We will not go into the proof here as it does not pertain to the following work.

CHAPTER III

OBSERVER

In designing the state-feedback controller we assumed that the entire state vector was available. Often some components of the state vector are not available or are delayed. Thus we need a suitable estimate of the unavailable states. In this section we consider the design of an observer whose inputs are the inputs and outputs of the given system, and whose output is the estimated value of the state vector. The general equation for observers were first presented for finite dimensional systems by Luenberger [14] and Bass [2].

We will use the operator structure of Kamen, making use of duality construction. Conditions are given for the existence of observers with arbitrary eigenvalues for the error dynamics. The conditions can be checked via matrix operations defined over the convolution ring, resulting in a constructive procedure for the design of observers.

For the observer problem we will consider systems with single input and single output. $\Sigma = (F, g, h)$.

Definition 3.1

Given a system $\Sigma = (F, g, h)$ as defined (def. 2.1),

an observer is the system given by

$$\begin{aligned}\dot{\hat{x}}(t) &= (F\hat{x})(t) + (g*u)(t) + \ell*(y - h*\hat{x})(t) \\ \hat{y}(t) &= \hat{x}(t)\end{aligned}\tag{3.1}$$

where F, g, h are over D , $u \in L_+^n$, $\ell \in J^n$ and \hat{x} is the state estimate.

We let \tilde{x} denote the error given by

$$\tilde{x}(t) = x(t) - \hat{x}(t)\tag{3.2}$$

By combining (3.1) and (3.2) with the system equations (2.7) we have the "error dynamical equation" given by

$$\dot{\tilde{x}}(t) = (F - \ell*h)*\tilde{x}(t)\tag{3.3}$$

Definition 3.2

The characteristic operator of the error dynamics is

$$\det(pI - F + \ell*h)\tag{3.4}$$

Note that if $\det(pI - F + \ell*h)$ is coefficient assignable then we can place the eigenvalues of the error dynamics anywhere in the left half of the complex plane.

We will now develop conditions for coefficient assignability of the characteristic operator of the error dynamics. Our approach is based on the construction of a dual system. The dual is defined by extending the setup over a field Kalman [8] to equations over rings.

Definition 3.3

Given a system $\Sigma = (F, g, h)$ the Dual of Σ is the system $\Sigma' = (F', h', g')$ with equations

$$\begin{aligned}\dot{z}(t) &= (F' * z)(t) + (h' * u)(t) \\ y(t) &= (g' * z)(t)\end{aligned}$$

where F', h', g' are over D , $u \in L_+$ and where F' indicates the transpose of F .

Theorem 3.1

The characteristic operator of the error dynamics is coefficient assignable if and only if the characteristic operator of the dual system with state feedback, given by $\det(pI - F' + h' * k)$, is coefficient assignable.

Proof of 3.1

We let $k = \ell'$ and it follows directly that

$$\det(pI - F + \ell * h) = \det(pI - F' + h' * k)$$

Corollary 1

Let F and h be over D and ℓ is over J then the $\det(pI - F + \ell * h)$ is coefficient assignable if the element $(h', F' * h', \dots, F'^{n-1} * h')$ is a unit in J .

From theorem 3.1 we have equivalence between the characteristic operators of the error dynamics of the given system and the feedback control case of the dual. Therefore through duality and theorem 2.2 we have the above sufficient condition, with construction of ℓ as given in Theorem 2.2.

But as stated in the preliminaries this condition is rarely met for delay systems. Constructing the dual of theorem (2.3) we have the following necessary and sufficient condition for coefficient assignability.

Corollary 2

The characteristic operator of the error dynamics, $\det(pI - F + \ell^*h)$, is coefficient assignable with respect to J if and only if there exist a $n \times n$ matrix W over J such that

$$h^*(\overline{pI - F})^*W = \begin{bmatrix} \delta_0 \\ p \\ \vdots \\ p^{n-1} \end{bmatrix} \quad (3.5)$$

Using (3.5) we can apply the constructive procedure for designing feedback in theorem 2.3 to design observers. This procedure is illustrated in the following example.

Example

We will construct a coefficient assignable observer for the system by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_1(t - a) + x_2(t) + \mu(t) \\ y(t) &= x_2(t) \end{aligned}$$

Which has system matrices

$$F = \begin{bmatrix} 0 & \delta_0 \\ \delta_a & \delta_0 \end{bmatrix} \quad g = \begin{bmatrix} 0 \\ \delta_0 \end{bmatrix} \quad h = [0, \delta_0]$$

and we have matrices of the Dual system

$$F' = \begin{bmatrix} 0 & \delta_a \\ \delta_0 & \delta_0 \end{bmatrix} \quad h' = \begin{bmatrix} 0 \\ \delta_0 \end{bmatrix} \quad g' = [0, \delta_0]$$

We first check the sufficient condition

$$C' = (h', F' * h') = \begin{bmatrix} 0 & \delta_a \\ \delta_0 & \delta_0 \end{bmatrix}$$

$$(\det(C'))^{-1} = -\delta_{-a} \notin J$$

Since the sufficient condition is not met we check the necessary condition by checking if there is a nxn matrix W such that

$$W * (\overline{pI - F'}) * h' = \begin{pmatrix} \delta_0 \\ p \end{pmatrix}$$

We have a nonunique W given by

$$W = \begin{bmatrix} \delta_0 & \lambda \\ 0 & \delta_0 \end{bmatrix} \quad \text{where } \lambda(t) = \begin{cases} 1, & 0 \leq t \leq a \\ 0, & \text{otherwise} \end{cases}$$

We then construct the feedback of the dual system such that

$$\det(pI - F' + h' * k) = (p + \delta_0)(p + 2\delta_0) = p^2 + 3p + 2\delta_0$$

$$a_0 = 2\delta_0$$

$$a_1 = 3\delta_0$$

$$\det(pI - F') = p^2 - p - \delta_0$$

$$\alpha_0 = -\delta_a$$

$$\alpha_1 = -\delta_0$$

$$k = (a_0 - \alpha_0, a_1 - \alpha_1)^*W$$

$$k = (2\delta_0 + \delta_a, 2\lambda + \lambda(t - a) + 4\delta_0)$$

From duality we have observer vector

$$l = k' = \begin{bmatrix} 2\delta_0 + \delta_a \\ 2\lambda + \lambda(t - a) + 4\delta_0 \end{bmatrix}$$

Let us consider the case in which, h is of the form $h^* \delta_a$, $h^* \in D^n$. In this case the condition given in Corollary 2 cannot be met. However we can "dualize" the state and input feedback construction in Chapter II.

Definition 3.4

An observer of a system with output feedback is given by equations

$$\begin{aligned} \dot{\hat{x}}(t) &= (F^* \hat{x})(t) + (g^* u)(t) + (l^* z)(t) \\ z(t) &= (\delta_0 + \alpha)^* (y - h^* \hat{x}) \\ \hat{y}(t) &= \hat{x}(t) \end{aligned} \quad (3.7)$$

We let \tilde{x} denote the error given by $\tilde{x}(t) = x(t) - \hat{x}(t)$ and combined with the observer equation (3.9) and the system

equations (2.7) we obtain the error dynamical equation which can be characterized by

$$\det(pI - F + (\delta_o + \alpha)^{-1} * \ell * h) \quad (3.8)$$

Theorem 3.2

The characteristic operator of the error dynamics (3.8) is coefficient assignable if and only if the characteristic operator of the dual system with state and input feedback is coefficient assignable.

Proof

Using duality and equation (2.13) yields the characteristic operator of the dual system with state and input feedback

$$\det(pI - F + (\delta_o + \alpha)^{-1} * \ell * h)$$

if we let

$$k = \ell'$$

it follows that

$$\det(pI - F + (\delta_o + \alpha)^{-1} * \ell * h) = \det(pI - F' + (\delta_o + \alpha)^{-1} * h' * k)$$

We then have the condition for coefficient assignability of Kamen [13] for a system with state and input feedback given by theorem (2.4).

CHAPTER IV

REGULATOR

In this section we will look at the construction of an input/output regulator. We will show that the regulator construction can be "separated" into two parts: the design of a controller and an observer. We then give conditions for coefficient assignability for the regulator's characteristic operator. Finally, use is made of the control canonical form to simplify regulator construction.

We again consider a system $\Sigma = (F, g, h)$ given by

$$\begin{aligned}\dot{x}(t) &= (F*x)(t) + (g*u)(t) \\ y(t) &= (h*x)(t)\end{aligned}\tag{4.1}$$

State feedback was incorporated using

$$u = -k*x + r\tag{4.2}$$

We then developed an observer (3.1) of the form

$$\begin{aligned}\dot{\hat{x}}(t) &= (F*\hat{x})(t) + (g*u)(t) + \ell*(y - h*\hat{x})(t) \\ \hat{y}(t) &= \hat{x}(t)\end{aligned}\tag{4.3}$$

We now use the estimated value of the state in the state feedback. Combining equation (4.2) and (4.3) we obtain

$$\begin{aligned}\dot{\hat{x}}(t) &= (F + \ell^*h)^*x(t) + (g, \ell)^*\begin{pmatrix} u \\ y \end{pmatrix} \\ \hat{y}(t) &= (-k^*\hat{x})(t)\end{aligned}\tag{4.4}$$

where $\mu = \hat{y}(t) + r$

Definition 4.1

Equation (4.4) is a regulator. We now look at the closed loop equation of the system with regulator.

Theorem 4.1

The characteristic operator of the overall system ((F, g, h) and regulator) is equal to

$$\det(pI - F + g^*k)^*\det(pI - F + \ell^*h)$$

Proof

Using an expanded state vector we have

$$\begin{bmatrix} \dot{\hat{x}} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} F & g^*k \\ \ell^*h & F + g^*k - \ell^*h \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} g \\ g \end{bmatrix} r$$

And then using the linear transformation

$$\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix}$$

We obtain the triangular form

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\tilde{x}} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} F + g^*k & -g^*k \\ 0 & F + \ell^*h \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \\ x \end{bmatrix} + \begin{bmatrix} g \\ 0 \end{bmatrix} r$$

which has the characteristic operator

$$\det(pI - F + g*k)*\det(pI - F + \ell*h)$$

Corollary 1

The characteristic operator of the system with regulator is coefficient assignable if and only if

- a) $\det(pI - F + g*k)$ is coefficient assignable
- b) $\det(pI - F + \ell*h)$ is coefficient assignable

This follows directly from theorem 4.1.

Corollary 1 shows that in considering the coefficient assignability of the overall system we are able to look at the observer and state feedback systems separately. For this reason theorem 4.1 is often known as the separation theorem.

Corollary 2

The characteristic operator of the overall system is coefficient assignable if and only if

- a) there exists a nxn matrix W such that

$$W*(pI - F)*g = \begin{bmatrix} \delta_0 \\ p \\ \vdots \\ p^{n-1} \end{bmatrix}$$

- b) there exists a nxn matrix T such that

$$h*(pI - F)*T = \begin{bmatrix} \delta_0 \\ p \\ \vdots \\ p^{n-1} \end{bmatrix}$$

From Corollary 1 we have that we need both controller and observer to be coefficient assignable. Here we restate the necessary and sufficient conditions for each as stated in theorem (2.3) and theorem (3.1) Corollary 2.

We now look at the possibility of using an equivalence transformation to simplify either state feedback design or the observer design. We know that the condition given in theorem (2.2), that $(\det C)^{-1}$ exists over D , is rarely met. In particular an inverse will only exist for the $\det C = a\delta_0$ $a \neq 0$, $a \in R$. By transforming the system to control canonical form we obtain alternate conditions.

Corollary 3

The characteristic operator of the overall system is coefficient assignable if and only if

- a) $\det C \neq 0$
- b) there exists a $n \times n$ matrix T such that

$$T^*(\overline{pI - F}) * \overline{C}^{-1} C^* h = (\delta_0, p, \dots, p^{n-1})$$

Recall the control canonical form (def. 2.5). A transformation P to the control canonical form is

$$P = \overline{C} * C^{-1}$$

For regulator construction we are free to use any input output equivalent system representation. A Q -equivalent system (def. 2.4) will be input/output equivalent. We now only need to have $C \neq 0$ for C^{-1} to exist over Q . For this

control canonical form $(\bar{F}, \bar{g}, \bar{h})$ we have easily constructed state feedback as given in proof (theorem 2.2). We now develop the condition for coefficient assignability of the observer for this transformed system. We have from corollary 2 of theorem 3.3 and the system $(\bar{F}, \bar{g}, \bar{h})$ that the observer is coefficient assignable if and only if there exists a T such that

$$h^*(pI - \bar{F})^*T = \begin{bmatrix} \delta_0 \\ p \\ \vdots \\ p^{n-1} - 1 \end{bmatrix}$$

and then using

$$\bar{h} = h^*p^{-1} = h^*(\bar{C}^{-1}*C)$$

and

$$\bar{F} = pFP^{-1} = \bar{C}C^{-1}FC^{-1}C$$

we have the condition as given.

Corollary 4

The characteristic operator of the overall system is coefficient assignable if

- a) $\det C \neq 0$
- b) $(h', F'h', \dots, F'^{n-1}h')$ is a unit in J

Part a is met as in the previous corollary by transforming the system to canonical form. Part b is then the sufficient condition as stated in corollary 1 of theorem 3.1. We know

that this condition will rarely be met for delay systems.

Remark

The conditions in corollary 2 holding does not imply that the conditions in corollary 3 will hold. It is possible to have a system for which the conditions for the existence of a regulator (theorem 4.1, cor. 2) are able to be met, but when the transformation is made to control canonical form, there does not exist a matrix T to meet the condition (theorem 4.1, cor. 3), in particular when $\bar{C}^{-1} * C * h$ is of the form $(\bar{C}^{-1} * C * h)' * \delta_a$ where $(\bar{C}^{-1} * C * h)' \in D$. This is easily seen by example. The use of output feedback as described in Chapter III may make the problem solvable, but this is left for future investigation.

Conjecture

If $\det C = 0$ cannot construct both feedback and observer.

Example

We will construct a coefficient assignable input/output regulator for the system given by

$$\begin{aligned}\dot{x}_1(t) &= x_2(t - a) \\ \dot{x}_2(t) &= x_1(t - a) + x_2(t) + \mu(t) \\ y(t) &= x_2(t)\end{aligned}$$

We have system matrices

$$F = \begin{bmatrix} 0 & \delta_a \\ \delta_a & \delta_o \end{bmatrix} \quad g = \begin{bmatrix} 0 \\ \delta_o \end{bmatrix} \quad h = [0, \delta_o]$$

We check the sufficient condition (theorem 2.2) for coefficient assignable feedback.

$$C = [g, F^*g] = \begin{bmatrix} 0 & \delta_a \\ \delta_o & \delta_o \end{bmatrix}, \quad (\det(C))^{-1} = -\delta_{-a}^{\notin J}$$

Then using corollary 2 theorem 4.1 since the $\det C = 0$ we look at the possibility of simplifying by constructing

$$\det(pI - F) = p^2 - p - \delta_{2a} \quad \alpha_1 = -\delta_o, \alpha_2 = -\delta_{2a}$$

the

$$F = \begin{bmatrix} 0 & \delta_o \\ 2a & \delta_o \end{bmatrix} \quad g = \begin{bmatrix} 0 \\ \delta_o \end{bmatrix}$$

$$\bar{C} = [\bar{g}, \bar{F}^*\bar{g}] = \begin{bmatrix} 0 & \delta_o \\ \delta_o & -\delta_o \end{bmatrix} \quad (\det(\bar{C}))^{-1} = \delta_o \in J$$

Therefore we can transform the system into canonical form,
We then construct the feedback k as in theorem 2.2 such
that

$$\det(pI - \bar{F} + \bar{g}^* \bar{k}) = (p + \delta_0)(p + 2\delta_0) = p^2 + 3p + 2\delta_0$$

$$\beta_1 = 3\delta_0 \quad \beta_2 = 2\delta_0$$

$$k \equiv (\beta_2 - \alpha_2, \beta_1 - \alpha_1) = (2\delta_0 + \delta_{2a}, 4\delta_0)$$

We then look at the observer for this transformed system, by first checking the sufficient condition. We have

$$\bar{h}' = \begin{bmatrix} 0 \\ \delta_0 \end{bmatrix} \quad \bar{F}' = \begin{bmatrix} 0 & \delta_{2a} \\ \delta_0 & \delta_0 \end{bmatrix}$$

$$\bar{C}' = [\bar{h}', \bar{F}'^* \bar{h}'] = \begin{bmatrix} 0 & \delta_{2a} \\ \delta_0 & \delta_0 \end{bmatrix}$$

$$(\det \bar{C}')^{-1} = -\delta_{-2a} \notin J$$

Since this condition is not met we check the necessary condition given in theorem 3.5. By finding a nxn matrix W such that

$$W^* \overline{(pI - \bar{F}')}^* \bar{h}' = \begin{pmatrix} \delta_0 \\ p \end{pmatrix}$$

There is a nonunique W given by

$$W = \begin{bmatrix} \delta_0 & \lambda \\ 0 & \delta_0 \end{bmatrix}, \quad \lambda(t) = \begin{cases} 1 & 0 \leq t \leq 2a \\ 0 & \text{otherwise} \end{cases}$$

We can then construct the feedback for the dual of the transformed system using the procedure in theorem 2.3 such that

$$\det(pI - \bar{F}' + \bar{h}' * \bar{k}') = (p + \delta_0)(p + 2\delta_0) = p^2 + 3p + 2\delta_0$$

$$a_1 = 3\delta_0$$

$$a_0 = 2\delta_0$$

recall

$$\alpha_0 = \delta_0$$

$$\alpha_1 = \delta_{2a}$$

then

$$\bar{k}' = (a_0 - \alpha_0, a_1 - \alpha_1) * w$$

$$\bar{k}' = (2\delta_0 + \delta_{2a}, 2\lambda + \lambda(t-2a) + 4\delta_0)$$

which yields observer vector of the transformed system.

$$l = k' = \begin{bmatrix} 2\delta_0 + \delta_{2a} \\ 2\lambda + \lambda(t - 2a) + 4\delta_0 \end{bmatrix}$$

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