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Part I

Karsten Henckell[*], Stuart W. Margolis[†], Jean-Eric Pin[‡] and John Rhodes[§]

Abstract

This paper is concerned with the many deep and far reaching consequences of Ash's positive solution of the type II conjecture for finite monoids. After rewieving the statement and history of the problem, we show how it can be used to decide if a finite monoid is in the variety generated by the Malcev product of a given variety and the variety of groups. Many interesting varieties of finite monoids have such a description including the variety generated by inverse monoids, orthodox monoids and solid monoids. A fascinating case is that of block groups. A block group is a monoid such that every element has at most one semigroup inverse. As a consequence of the cover conjecture — also verified by Ash — it follows that block groups are precisely the divisors of power monoids of finite groups. The proof of this last fact uses earlier results of the authors and the deepest tools and results from global semigroup theory. We next give connections with the profinite group topologies on finitely generated free monoids and free groups. In particular, we show that the type II conjecture is equivalent with two other conjectures on the structure of closed sets (one conjecture for the free monoid and another one for the free group). Now Ash's theorem implies that the two topological conjectures are true and independently, a direct proof of the topological conjecture for the free group has been recently obtained by Ribes and Zalesskii. An important consequence is that a rational subset of a finitely generated free group G is closed in the profinite topology if and only if it is a finite union of sets of the form $gH_1H_2\cdots H_n$, where each H_i is a finitely generated subgroup of G. This significantly extends classical results by M. Hall. Finally we return to the roots of this problem and give connections with the complexity theory of finite semigroups. We show that the largest local complexity function in the sense of Rhodes and Tilson is computable.

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1 Introduction

The excitement caused by Chris Ash's proof of the type II conjecture, published in this journal [6], is well justified. The beauty and depth of the proof can be amply seen by reading Ash's article [5] and the full version [6]. The conjecture itself had obtained wide circulation as one of the outstanding open problems in finite semigroup theory of the past few years. This was spurred by many talks by the fourth author (Rhodes) over the years and by its presentation of a set of problems at the Chico conference in 1986 [42], successive survey articles of the other authors [36, 14, 24], and by a sequence of articles of the third author (Pin) [31, 33, 34, 37], who emphasized the topological and language theoretic aspects of the problem. The purpose of this paper is to explain the many deep consequences of Ash's theorem. It is divided into two parts. Part II should appear in the next issue of this journal.

We will assume in this paper that the reader has only a basic background in finite semigroup theory. See [10, 11, 19, 23, 32] for additional background material. For the most part, we follow the notations and terminology of Eilenberg [11]. In particular, if $\varphi : S \to T$ is a function from S into T, we denote by $s\varphi$ (instead of the usual $\varphi(s)$) the image of an element s of S by φ. All monoids except free monoids and free groups are assumed finite. A monoid M *divides* a monoid N if M is a quotient of a submonoid of N. The set of idempotents of a monoid M is denoted $E(M)$. Given a subset P of a monoid M, $\langle P \rangle$ denotes the submonoid of M generated by P. The word "variety" will be used for pseudovariety — that is, a collection of finite monoids closed under division and finite direct product.

We begin with a statement of the problem. Recall that a relational morphism between monoids M and N is a relation $\tau : M \to N$ such that:

(1) $(m\tau)(n\tau) \subset (mn)\tau$ for all $m, n \in M$,

(2) $(m\tau)$ is non-empty for all $m \in M$,

(3) $1 \in 1\tau$

Equivalently, τ is a relation whose graph

$$\mathrm{graph}(\tau) = \{\ (m, n) \mid n \in m\tau\ \}$$

is a submonoid of $M \times N$ that projects onto M. See [54] for an introduction to this notion and the related notion of derived category.

We will be only interested in relational morphisms into groups in this paper. Note that if $\tau : M \to G$ is a relational morphism into a group G, then $1\tau^{-1}$ is a submonoid of M. We define the *kernel* or *type II submonoid* of M, $K(M)$, to be the intersection of the submonoids $1\tau^{-1}$ over all relational morphisms $\tau : M \to G$ into a group. Note that even though all relational morphisms are between finite objects, it is not a priori clear that membership in $K(M)$ is decidable since there are an infinity of possible relational morphisms.

As one would like to decide membership in $K(M)$, this leads to a search for elements of M that are sure to be in $K(M)$. It is fairly easy to prove using standard facts about finite semigroups that the following is true [36, 53, 46].

(1) $E(M)$ is contained in $K(M)$,

(2) Let $m, n \in M$ be such that $mnm = m$. Then $mK(m)n \cup nK(M)m \subset K(M)$.

That is, every idempotent is in $K(M)$ and $K(M)$ is closed under *weak conjugation*: if m is a *weak inverse* of n, that is, if $mnm = m$, then, for every $k \in K(M)$, $mkn \in K(M)$ and $nkm \in K(M)$. Define $D(M)$ to be the the smallest submonoid of M closed under weak conjugation. Then $D(M)$ contains the idempotents of M: indeed, if e is idempotent, then e is an inverse of itself, and thus $1 \in D(M)$ implies $e = e1e \in D(M)$. Note that membership in $D(M)$ is decidable given the multiplication table of M. It follows from (1) and (2) above that $D(M)$ is a submonoid of $K(M)$, and the fourth author (Rhodes) conjectured that $K(M) = D(M)$ — the "type II" conjecture. Ash's result proves that this conjecture is true.

Theorem 1.1 (Ash [5, 6]) *For every finite monoid M, $K(M) = D(M)$.*

Actually, Ash's results imply another related result conjectured in [17]. We define a subset X of a monoid M to be *pointlike* (with respect to groups) if, for all relational morphisms $\tau : M \to G$ into a finite group G, there is a g such that $X \subset g\tau^{-1}$. For $m \in M$ let $m^{(1)} = \{m\}$ and $m^{(-1)} = \{\ x \in M \mid xmx = x\ \}$, the set of weak inverses of m. The *Pointlike (or Cover) Conjecture* [18] states that a subset X of a monoid M is pointlike (with respect to groups) if and only if there are elements $m_1, m_2, \ldots m_n$ of M such that

$$X \subset D(M)m_1^{(\varepsilon_1)}D(M)m_2^{(\varepsilon_2)}D(M) \cdots D(M)m_n^{(\varepsilon_n)}D(M)$$

where $\varepsilon_i \in \{1, -1\}$ for $1 \leq i \leq n$. It follows from Ash's work [5, 6] that the Pointlike Conjecture is true.

Theorem 1.2 (Ash [5, 6]) *A subset X of a monoid M is pointlike (with respect to groups) if and only if there are elements $m_1, m_2, \ldots m_n$ of M such that*

$$X \subset D(M)m_1^{(\varepsilon_1)}D(M)m_2^{(\varepsilon_2)}D(M) \cdots D(M)m_n^{(\varepsilon_n)}D(M)$$

where $\varepsilon_i \in \{1, -1\}$ for $1 \leq i \leq n$.

By analogy, we can define the pointlike sets with respect to any variety $\mathbf{V}$. The first author (Henckell [13]) has proved that the pointlike sets with respect to the variety of aperiodic monoids (that is, group-free monoids) are decidable. See the second part of this paper for questions related to this problem.

By theorem 1.2, it is now clear that it is decidable if a set of elements of M is pointlike (with respect to groups). Before going on to give the history and consequences of these results, perhaps an explanation of terminology is due to the reader. We are sure that it has occurred to a number of people who have heard of this conjecture to wonder what type I semigroups are and perhaps are there type III, type IV semigroups, etc.

Here is a simplified definition. See [47] for a fuller treatment. Let $\mathbf{V}$ be a variety and let M be a monoid. A submonoid N of M is called a type $\mathbf{V}$ submonoid of M if for all relational morphisms $\tau : M \to T$ with $T \in \mathbf{V}$, there is a $t \in T$ such that $N \subset \mathrm{Stab}(t)\tau^{-1}$. Here $\mathrm{Stab}(t) = \{\ s \in T \mid ts = t\ \}$ is the right stabilizer of t. The importance of stabilizers stems from the local structure of Tilson's derived category [54]. This definition was especially of interest for the two varieties singled out by the Krohn-Rhodes decomposition theorem as being of central importance in finite semigroup theory — $\mathbf{A}$, the variety of aperiodic monoids, and the variety $\mathbf{G}$ of finite groups. Submonoids of type $\mathbf{A}$ where called type I and submonoids of type $\mathbf{G}$ were called type II. It is clear that $K(M)$ (and now $D(M)$!) is the unique maximal type $\mathbf{G}$ submonoid of M. There is not a unique maximal type I submonoid of M, but we shall see that it is decidable if a monoid M is a type I submonoid of itself (termed *absolute type I*) and explore the consequences of this later in this paper.

2 Some history

Like many problems that resist immediate solution, a number of partial results and equivalent conditions to the conjecture have appeared or will appear in the literature. In this section we briefly survey these previous results. We will concentrate on the special cases of the conjecture that have been proved and leave the important connections with topology for a later section.

The type II subsemigroup first appeared in [46]. The motivations came from the complexity theory of finite semigroups and monoids. We recall some basic facts. Let $\mathbf{V_0} = \mathbf{A}$ and, for $n \geq 0$, let $\mathbf{V_{n+1}} = \mathbf{A} * \mathbf{G} * \mathbf{V_n}$. Here $\mathbf{V} * \mathbf{W}$ denotes the variety generated by all monoid semidirect products $M * N$ where $M \in \mathbf{V}$ and $N \in \mathbf{W}$. It follows from the Krohn-Rhodes decomposition theorem that every monoid M is in $\mathbf{V_n}$ for some n. The least such n is called the *complexity* of M [52].

The idea of type I (resp. type II) submonoids is that if M is in a variety of the form $\mathbf{V} * \mathbf{A}$ (resp. $\mathbf{V} * \mathbf{G}$), then any type I (type II) submonoid of M should be a member of $\mathbf{V}$. In particular, one obtains a *lower bound* to complexity by taking the maximal length of a chain of submonoids alternating type I and type II and containing a non-aperiodic type I submonoid. It was hoped at the time of the publication of [46] that this latter number would in fact give the complexity of an arbitrary monoid. This was in fact true for inverse monoids and completely regular monoids but a counterexample was constructed in [44].

Nonetheless, we will return in the last section of this paper to examine these chains of submonoids which give the largest *local* complexity function as proved in [47]. Furthermore, the paper [46] went on to prove important facts about type I and type II submonoids. In particular, the following result was proved there.

Theorem 2.1 [46] *Let M be a monoid and let m be a regular element of M. Then $m \in K(M)$ if and only if $m \in D(M)$. In particular, if M is a regular monoid, then $K(M) = D(M)$.*

The proof in [46] was long and based on certain renormalizations of the structure matrix of a regular $\mathcal{D}$-class. Tilson gave a much more accessible proof in [53] by directly associating an injective automaton with every regular $\mathcal{D}$-class. This work had major influence on subsequent work on this problem. It was used to show [17] that the type II conjecture could be reduced to the case of block groups — monoids in which every element has at most one semigroup inverse. It also played an important role in Ash's formulation of and proof of the conjecture. We shall see why block groups play such an important role in the theory later in this paper. We will provide more details of these facts in section 5.

Another particular case of interest was the case of monoids with commuting idempotents. Indeed it was shown [27] that the type II conjecture implied that the variety generated by inverse monoids was in fact equal to the variety of monoids with commuting idempotents. This weak form of the type II conjecture was also solved by Ash as a warm up to his future proof of the full conjecture.

Theorem 2.2 (Ash [3, 4]) *Every monoid with commuting idempotents divides an inverse monoid. Equivalently, the variety generated by inverse monoids is equal to the variety of monoids with commuting idempotents.*

The proof of this result handled non-regular elements by appealing to Ramsey's theorem in a non trivial way. In 1986, T.E. Hall gave an illuminating lecture on this proof at the Chico Conference on Semigroups. This stimulated Birget and two of the authors to see how to extend this result to the case of monoids whose idempotents form a submonoid.

Theorem 2.3 [8, 9] *Every monoid whose idempotents form a submonoid divides an orthodox monoid.*

Recall that an orthodox monoid is a regular monoid whose idempotents form a submonoid. Actually more was proved. By combining the results of [4] and the methods introduced in [53], the following result was obtained:

Theorem 2.4 [8, 9] *Let M be a monoid such that $D(M)$ is a regular submonoid. Then $D(M) = K(M)$. That is the type II conjecture is true in this case.*

It is fairly easy to prove that if the idempotents of a monoid form a band (resp. generate a completely regular monoid), then $D(M)$ is a band (resp. completely regular monoid). We will clarify the connection between these results in the next section of this paper.

An other interesting special case was proved in [16]:

Theorem 2.5 [16] *Let M be a $\mathcal{J}$-trivial monoid. Then $D(M) = K(M)$.*

The interesting part of this proof is that it directly constructs a relational morphism from a $\mathcal{J}$-trivial monoid onto a finite group that proves $D(M) = K(M)$. The proof is very different than Ash's and it is unknown at the present time if this proof technique can be extended to the general case.

3 Malcev products and semidirect products with groups

As indicated in the previous section, the motivation for the definition of $K(M)$ came from a desire to decompose M into a semidirect product $N * G$ where N is "simpler" than M and G is a finite group. The paper [46] was written before the influential notion of variety of finite monoids and languages had been formulated by Eilenberg and Schützenberger [11, 32]. It is within this context that we can make these statements precise.

Let $\mathbf{V}$ and $\mathbf{W}$ be varieties. We have defined $\mathbf{V}*\mathbf{W}$ to be the variety generated by monoid semidirect products of members of $\mathbf{V}$ and $\mathbf{W}$. There is a related variety that we now define. Let

$$\begin{aligned}\mathbf{V} \circledM \mathbf{W} = \{\ M \mid & \text{ There is a relational morphism } \tau : M \to N \text{ with} \\ & N \in \mathbf{W} \text{ and such that } e\tau^{-1} \in \mathbf{V} \text{ for all idempotents } e \in N\ \}\end{aligned}$$

The variety $\mathbf{V} \circledM \mathbf{W}$ is called the *Malcev product* of $\mathbf{V}$ and $\mathbf{W}$. We will be interested in when a variety decomposes in the form $\mathbf{V} * \mathbf{G}$ or $\mathbf{V} \circledM \mathbf{G}$. We first list some preliminary observations.

Theorem 3.1 *Let $\mathbf{V}$ be any variety. Then $\mathbf{V} * \mathbf{G} \subset \mathbf{V} \circledM \mathbf{G}$.*

Proof. Let $M \in \mathbf{V} * \mathbf{G}$. Then M divides a monoid of the form $N * K$ where $N \in \mathbf{V}$ and $K \in \mathbf{G}$. Let $\pi : N * K \to K$ be the projection. Then $1\pi^{-1}$ is isomorphic to $N \in \mathbf{V}$. Therefore $N * K$ is in $\mathbf{V} \circledM \mathbf{G}$ and so is M since M divides $N * K$. $\square$

We now give the connection between Malcev product with the variety of groups and $K(M)$. We first have the following result that is proved by a compactness argument [36, 46].

Theorem 3.2 *Let M be a monoid. Then there is a group G and a relational morphism $\tau : M \to G$ such that $1\tau^{-1} = K(M)$.*

Proof. Since M is finite, there are only a finite number of sets of the form $1\tau^{-1}$, where τ is a relational morphism from M onto a finite group G. Therefore one can select a finite set of relational morphisms $\tau_i : M \to G_i$ $(1 \le i \le n)$ such that every $1\tau^{-1}$ is equal to one of the $1\tau_i^{-1}$. Set $G = G_1 \times G_2 \times \cdots \times G_n$ and define a relational morphism $\tau : M \to G$ by setting $m\tau = m\tau_1 \times m\tau_2 \times \cdots \times m\tau_n$. Then

$$1\tau^{-1} = \bigcap_{1 \le i \le n} 1\tau_i^{-1} = K(M) \qquad \square$$

A slight improvement of the previous proof leads to the following stronger result:

Theorem 3.3 *Let M be a monoid. Then there is a group G and a relational morphism $\tau : M \to G$ such that*

(1) $1\tau^{-1} = K(M)$,

(2) *A subset P of M is a pointlike subset of M if and only if there exists $g \in G$ such that P is a subset of $g\tau^{-1}$.*

A relational morphism $\tau : M \to G$ satisfying the conditions (1) and (2) of theorem 3.3 is called *universal* for M (with respect to groups).

Theorem 3.4 *Let M be a monoid and let $\mathbf{V}$ be a variety. The following conditions are equivalent*

(1) $M \in \mathbf{V} \circledM \mathbf{G}$,

(2) *there exists a relational morphism $\tau : M \to G$, universal for M, such that $1\tau^{-1} \in \mathbf{V}$,*

(3) $K(M) \in \mathbf{V}$

Proof. (1) implies (3). Let $M \in \mathbf{V} ⓜ \mathbf{G}$. Then there is a relational morphism $\tau : M \to G$ onto a group G such that $1\tau^{-1} \in \mathbf{V}$. Now $K(M)$ is a submonoid of $1\tau^{-1}$ by definition and thus $K(M) \in \mathbf{V}$.
(3) implies (2). Assume that $K(M) \in \mathbf{V}$. By theorem 3.3 there exists a relational morphism $\tau : M \to G$, universal for M, such that $1\tau^{-1} = K(M)$. Thus $1\tau^{-1} \in \mathbf{V}$.
(2) implies (1). If there exists a relational morphism $\tau : M \to G$ such that $1\tau^{-1} \in \mathbf{V}$, then $M \in \mathbf{V} ⓜ \mathbf{G}$ by definition. □

We have the following very important corollary to Ash's theorem. Recall that a variety $\mathbf{V}$ is called *decidable* if there is an algorithm that decides whether a given monoid is a member of $\mathbf{V}$ or not.

Theorem 3.5 *Let $\mathbf{V}$ be a decidable variety. Then $\mathbf{V} ⓜ \mathbf{G}$ is a decidable variety.*

Proof. By theorem 3.4 and theorem 1.1, $M \in \mathbf{V} ⓜ \mathbf{G}$ if and only if $D(M) \in \mathbf{V}$. Since $\mathbf{V}$ is decidable by hypothesis and membership in $D(M)$ is decidable given the multiplication table of M, the result follows. □

The importance of theorem 3.5 becomes apparent when contrasted with the fact that the collection of decidable varieties is not closed under join, semidirect product or Malcev product [1].

In general, $\mathbf{V} * \mathbf{G}$ is a proper subvariety of $\mathbf{V} ⓜ \mathbf{G}$. For example, an unpublished example of the fourth author proves that the variety $(\mathbf{A} * \mathbf{G}) ⓜ \mathbf{G}$ contains monoids of complexity n for any $n \geq 0$. These examples will appear in the second part of this paper. On the other hand, $(\mathbf{A} * \mathbf{G}) * \mathbf{G} = \mathbf{A} * \mathbf{G}$ is contained in $\mathbf{V_1}$, the variety of monoids of complexity less than or equal to one.

The question of when equality holds is a special case of a question that has attracted great attention over the past few years. It has to do with the notion and application of the derived category of a morphism to the decomposition theory of monoids. See [54] and [43] or the survey article [24]. As the details would take us too far afield, we list the following theorems for the readers familiar with the notion of division of categories and of a local variety in the sense of [54].

Theorem 3.6 *A monoid M belongs to $\mathbf{V} * \mathbf{G}$ if and only if there is a relational morphism $\tau : M \to G$ onto a finite group G such that the derived category $D(\tau)$ divides a member of $\mathbf{V}$.*

Thus membership in $\mathbf{V} * \mathbf{G}$ is a "global" question in that it depends not just on the inverse image of the identity, but on the global structure of the derived category of certain relational morphisms. When being able to determine if a category divides a member of a variety only depends on the local structure of the category — that is on the loop monoids being members of the variety — then we can replace Malcev product by semidirect product. Intuitively a variety is "local" if this latter condition holds. Again the reader is urged to read [54, 44, 24] for more details.

Theorem 3.7 *Let $\mathbf{V}$ be a local variety. Then $\mathbf{V} * \mathbf{G} = \mathbf{V} ⓜ \mathbf{G}$.*

Many important varieties are known to be local. The following result combines the work of Simon, Thérien and Jones-Szendrei [11, 50, 20].

Theorem 3.8 *Any variety of bands is local. The variety* $\mathbf{CR}$ *of completely regular monoids is local.*

Corollary 3.9 *Let* $\mathbf{V}$ *be any variety of bands or the variety of completely regular monoids. Then* $\mathbf{V} * \mathbf{G} = \mathbf{V} Ⓜ \mathbf{G}$ *is decidable.*

Proof. The equality $\mathbf{V} * \mathbf{G} = \mathbf{V} Ⓜ \mathbf{G}$ follows from Theorems 3.7 and 3.8. Now the variety $\mathbf{CR}$ is clearly decidable and since any variety of bands is defined by a finite number of identities, varieties of bands are also decidable. Therefore the result follows from theorem 3.5. □

In fact, it was long known that many varieties decompose as a semidirect product of a variety of bands and the variety of groups. One of the impetuses for the introduction of the derived category into decomposition theory and the notion of local and global membership of finite categories in a variety was a desire to show that the semidirect product could be replaced by the Malcev product in these cases. Recall that a regular monoid M is *orthodox* if $E(M)$ is a submonoid of M.

Theorem 3.10 *Let* $\mathbf{V}$ *be a variety of bands and let* M *be a monoid. The following conditions are equivalent:*

(1) $M \in \mathbf{V} * \mathbf{G}$,
(2) $M \in \mathbf{V} Ⓜ \mathbf{G}$,
(3) $E(M) \in \mathbf{V}$,
(4) *M divides an orthodox monoid N such that* $E(N) \in \mathbf{V}$.

Proof. It is easy to prove that of M is a monoid such that $E(M)$ is a submonoid, then $E(M) = D(M)$. The equivalence of (1) and (2) follows from Corollary 3.9. The equivalence of (2) and (3) follows from theorem 3.4. Furthermore, let $\mathbf{W}$ be the variety generated by orthodox monoids such that $E(M) \in \mathbf{V}$. It is easy to show that if $M \in \mathbf{V}$, then $M * G \in \mathbf{W}$ for any group G. Also, well known results about orthodox monoids [8, 9] show that if M is an orthodox monoid with $E(M) \in \mathbf{V}$, then $M \in \mathbf{V} Ⓜ \mathbf{G}$. We have then that $\mathbf{V} * \mathbf{G} \subset \mathbf{W} \subset \mathbf{V} Ⓜ \mathbf{G}$ and equality follows from the equivalence of (1) and (2). Thus $\mathbf{V} * \mathbf{G} = \mathbf{W}$ and (1) is equivalent with (4). □

Notice that Theorems 2.2 and 2.3 are special cases of this result. Recall also that a regular monoid M is *solid* if the monoid $\langle E(M) \rangle$ generated by the idempotents of M is completely regular. The following theorem is proved analogously to Theorem 3.10.

Theorem 3.11 [8, 9] *Let M be a monoid. The following conditions are equivalent:*

(1) $M \in \mathbf{CR} * \mathbf{G}$,
(2) $M \in \mathbf{CR} Ⓜ \mathbf{G}$,
(3) $\langle E(M) \rangle \in \mathbf{CR}$,

(4) *M divides a solid monoid.*

Theorem 3.7 is only a sufficient condition for $\mathbf{V} * \mathbf{G}$ to be equal to $\mathbf{V} \textcircled{\scriptsize M} \mathbf{G}$. The case of the variety $\mathbf{J}$ of $\mathcal{J}$-trivial monoids is fascinating. Recall that a *block group* is a monoid M such that every element has at most one semigroup inverse. By classical results it follows that M is a block-group if and only if every $\mathcal{R}$-class and every $\mathcal{L}$-class has at most one idempotent. Thus in the eggbox picture of M, the maximal blocks containing idempotents are groups [10].
Note that a regular monoid is a block group if and only if it is inverse. It is easy to see that the collection $\mathbf{BG}$ of all block groups is a variety.

An interesting class of block groups comes from the well known fact that if G is a group, then the monoid $\mathcal{P}(G)$ of all subsets of G under the usual multiplication (also called the *power group* of G) is a block group [30]. If $\mathbf{PG}$ denotes the variety generated by all power groups, then we have $\mathbf{PG} \subset \mathbf{BG}$. Denote by $\mathbf{EJ}$ the variety of monoids such that $\langle E(M) \rangle \in \mathbf{J}$. The following summarizes some of the work that appeared in [28].

Theorem 3.12 *The following formulae hold:* $\mathbf{PG} = \mathbf{J} * \mathbf{G} \subset \mathbf{BG} = \mathbf{J} \textcircled{\scriptsize M} \mathbf{G} = \mathbf{EJ}$.

The proof uses a number of techniques, some purely algebraic, and some using results from the theory of languages. For example, it is easy to show that if G is a group, then $\mathcal{P}(G)$ divides $\mathcal{P}_1(G) * G$, where $\mathcal{P}_1(G)$ is the submonoid of $\mathcal{P}(G)$ consisting of all subsets of G containing the identity and where G acts on $\mathcal{P}_1(G)$ by conjugation. Furthermore, $\mathcal{P}_1(G)$ is easily seen to be $\mathcal{J}$-trivial and this shows that $\mathbf{PG} \subset \mathbf{J} * \mathbf{G}$. The other inclusion involves a detailed study of the languages accepted by power groups. By using the wreath product principle of Straubing [32, 35], which gives a description of the languages accepted by wreath products of monoids in terms of those accepted by the factors, one shows that every language accepted by a wreath product of a group by a $\mathcal{J}$-trivial monoid is also accepted by a power group. By Eilenberg's theorem [11], the opposite inclusion also holds. This method of proof also pioneered by Straubing and Thérien, is a very powerful method for proving results like these.

A very deep result of Knast [22] shows that $\mathbf{J}$ is not a local variety. In the framework of global semigroup theory, we state the theorem of Knast using the notions of Tilson [54]. Recall that the *exponent* of a monoid M is the smallest integer ω such that m^ω is idempotent for every $m \in M$. Similarly, the *exponent of a finite category C* is the smallest integer ω such that m^ω is idempotent for every loop $m \in C$. It is also equal to the l.c.m. of the exponents of all loop monoids of C.

Theorem 3.13 (Knast) *A finite category C divides a $\mathcal{J}$-trivial monoid if and only if it satisfies the following path identity, where ω denotes the exponent of C:*

$$(ab)^\omega ad(cd)^\omega = (ab)^\omega (cd)^\omega$$

for every subgraph of C of the form

There are categories such that every loop monoid of the category is $\mathcal{J}$-trivial, but the category itself does not divide any member of $\mathbf{J}$. This is what led two of the authors to conjecture in [28] that $\mathbf{PG}$ was a proper subvariety of

BG. However the conjecture turned out to be false, as shown by the two other authors in [18] (when combined with Ash's theorem).

Theorem 3.14 *If the Pointlike Conjecture is true, then* $\mathbf{PG} = \mathbf{BG}$.

Proof. This theorem is proved by combining the results of theorem 3.12 with Knast's theorem 3.13. By theorem 3.12, $\mathbf{BG} = \mathbf{PG}$ if and only if $\mathbf{J} \,ⓜ\, \mathbf{G} = \mathbf{J} * \mathbf{G}$. Let $M \in \mathbf{J} \,ⓜ\, \mathbf{G}$. Then, by theorem 3.4, there exists a relational morphism $\tau : M \to G$ onto a group G, universal for M, such that $1\tau^{-1} \in \mathbf{J}$. Let C_τ be the derived category of τ: the objects of C_τ are the elements of G and the arrows are the triples $(h, (m, g), hg)$ such that $(m, g) \in \mathrm{graph}(\tau)$ — or equivalently $g \in m\tau$. Composition is given by the formula

$$(h, (m, g), hg)(hg, (m', g'), hgg') = (h, (mm', gg'), hgg')$$

pictured below Note that for all $g \in G$, $C(g, g) = 1\tau^{-1}$, because $(g, (m, 1)g)$ is an arrow if and only if $m \in 1\tau^{-1}$. In other words, the category C_τ is *locally* in $\mathbf{J}$, and we have to show, by Theorem 3.6, that it is *globally* in $\mathbf{J}$. To verify Knast's identity, consider a subgraph of C_τ of the form shown in Figure 2. Then we have $\{a, c\} \subset g\tau^{-1}$ and $\{b, d\} \subset g^{-1}\tau^{-1}$. Set $\bar{a} = (ab)^\omega a$, $\bar{b} = b(ab)^{2\omega-1}$, $\bar{c} = (cd)^\omega c$, $\bar{d} = d(cd)^{2\omega-1}$, $e = \bar{a}\bar{b} = (ab)^\omega$ and $f = \bar{c}\bar{d} = (cd)^\omega$. The following lemma can be verified by a straightforward calculation.

Lemma 3.15 *The following properties hold:*

(1) *e and f are idempotent,*

(2) *$ea = e\bar{a} = \bar{a}$ and $fc = f\bar{c} = \bar{c}$.*

(3) *$\bar{b} = \bar{a}^{-1}$ and $\bar{d} = \bar{c}^{-1}$ in the block group M.*

Here $\bar{a}^{-1}$ and $\bar{c}^{-1}$ denote the unique semigroup inverses of $\bar{a}$ and $\bar{c}$ respectively. Now $e \mathrel{\mathcal{R}} \bar{a}$, $f \mathrel{\mathcal{R}} \bar{c}$, and we have the picture of two (perhaps not distinct) $\mathcal{J}$-classes. By Lemma 3.15, we have $(ab)^\omega(cd)^\omega = ef$ and $(ab)^\omega ad(cd)^\omega = \bar{a}\bar{d} = \bar{a}\bar{c}^{-1}$ and thus proving Knast's identity reduces to showing the equality

$$ef = \bar{a}\bar{c}^{-1}$$

Since τ is universal for M, the inclusion $\{a, c\} \subset g\tau^{-1}$ shows that $\{a, c\}$ is a pointlike subset of M. By the Pointlike Conjecture, there exist elements $d_0, \ldots, d_k, d'_0, \ldots, d'_k$ of $D(M)$, elements $m_1, \ldots, m_k$ of M and integers $\varepsilon_1, \ldots, \varepsilon_k \in \{-1, 1\}$ such that

$$a \in d_0 m_1^{(\varepsilon_1)} d_1 m_2^{(\varepsilon_2)} \cdots \; d_{k-1} m_k^{(\varepsilon_k)} d_k \qquad (1)$$

$$c \in d'_0 m_1^{(\varepsilon_1)} d'_1 m_2^{(\varepsilon_2)} \cdots \; d'_{k-1} m_k^{(\varepsilon_k)} d'_k \qquad (2)$$

where, as before, $m^{(1)} = \{m\}$ and $m^{(-1)} = \{x \in M \mid xmx = x\}$. We need the following lemma, which follows from the work of Tilson [53].

Lemma 3.16 *Let M be a block group and let R be a regular $\mathcal{R}$-class of M.*

(1) *For every $r \in R$, for every $m \in M$ such that $rm \in R$, there exists $\bar{m} \in m^{(-1)}$ such that $rm\bar{m} = r$.*

(2) *For every $r \in R$, for every $m \in M$ and for every $\bar{m} \in m^{(-1)}$ such that $r\bar{m} \in R$, one has $r\bar{m}m = r$.*

(3) *For every $m \in M$, the action of m on R defined, for every $r \in R$, by*

$$r \cdot m = \begin{cases} rm & \text{if } rm \in R \\ \text{undefined} & \text{otherwise} \end{cases}$$

is partially one-to-one and is a partial identity if $m \in K(M)$.

Of course a dual lemma holds by considering $\mathcal{L}$-classes instead of $\mathcal{R}$-classes.

Proof. (1) Let $r, rm \in R$. Then since $r \mathrel{\mathcal{R}} rm$, there exists $m' \in M$ such that $rmm' = r$. Set $\bar{m} = (m'm)^{2\omega-1}m'$. Then $\bar{m} \in m^{-1}$ since

$$\bar{m}m\bar{m} = (m'm)^{2\omega-1}m'm(m'm)^{2\omega-1}m' = (m'm)^{4\omega-1}m' = (m'm)^{2\omega-1}m' = \bar{m}$$

Furthermore, $rm\bar{m} = rm(m'm)^{2\omega-1}m' = r(mm')^{2\omega} = r$.

(3) Suppose now that $r_1m = r_2m = r$ for some $r, r_1, r_2 \in R$. Since R is regular, it is contained in a regular $\mathcal{J}$-class J and the $\mathcal{L}$-class of r contains an idempotent e. Therefore $re = r$, that is

$$r_1(me) = r_2(me) = r$$

By the first part of the lemma, there exist weak inverses $\bar{m}_1$, $\bar{m}_2$ of me such that

$$r_1(me)\bar{m}_1 = r_1 \qquad \text{and} \qquad r_2(me)\bar{m}_2 = r_2$$

It is clear that if $n \in m^{(-1)}$, then $n \leq_{\mathcal{J}} m$. It follows that $\bar{m}_1, \bar{m}_2 \in J$ and since $\bar{m}_1$ and $\bar{m}_2$ are weak inverses of $me \in J$, they are in fact inverses of me. But since M is a block group, every element has a unique inverse, and thus $\bar{m}_1 = \bar{m}_2$. It follows that

$$r_1 = r_1(me)\bar{m}_1 = r_2(me)\bar{m}_1 = r_2(me)\bar{m}_2 = r_2$$

and thus the action of m on R is partially one-to-one.

Let S_R be the symmetric group on R and let $\tau : M \to S_R$ be the relation that associates to any $m \in M$ the set $m\tau$ of all permutations on R that extends the partial permutation on R defined by m. It is easy to see that τ is a relational morphism. In particular, if $m \in K(M)$, then $m \in 1\tau^{-1}$ by definition and thus m is a partial identity.

(2) If $r\bar{m} \in R$ and if $\bar{m} \in m^{(-1)}$, then $\bar{m}m\bar{m} = \bar{m}$, and thus $r\bar{m}m \in R$. Now $\bar{m}m$ is idempotent, and thus belongs to $K(M)$, and by (3), induces a partial identity on R. In particular $r\bar{m}m = r$. □

Let R be the $\mathcal{R}$-class of f. By (2), there exists a sequence $m_i^{[\varepsilon_i]} \in m_i^{(\varepsilon_i)}$ $(1 \leq i \leq k)$ such that

$$\bar{c} = fc = fd'_0m_1^{[\varepsilon_1]}d'_1 \ \cdots \ d'_{k-1}m_k^{[\varepsilon_k]}d'_k \in R$$

In particular, all the elements f, fd'_0, $fd'_0m_1^{[\varepsilon_1]}$, $fd'_0m_1^{[\varepsilon_1]}d'_1, \ldots$ are elements of R. But since $d'_0, \ldots, d'_k \in D(M) \subset K(M)$, part (3) of lemma 3.16 shows that the actions of these elements on R are partial identities. Therefore

$$\bar{c} = fm_1^{[\varepsilon_1]} \ \cdots \ m_k^{[\varepsilon_k]} \in fm_1^{(\varepsilon_1)} \ \cdots \ m_k^{(\varepsilon_k)} \tag{3}$$

and similarly,

$$\bar{a} \in e m_1^{(\varepsilon_1)} \cdots m_k^{(\varepsilon_k)} \tag{4}$$

We next compute the inverse of $\bar{c}$ in M. Set, for $0 \leq i \leq k$, $r_i = f m_1^{[\varepsilon_1]} \cdots m_i^{[\varepsilon_i]}$ and

$$m_i^{[-\varepsilon_i]} = \begin{cases} m_i & \text{if } \varepsilon_i = -1 \\ \text{an element } m_i^{[-1]} \text{ of } m_i^{(-\varepsilon_i)} \text{ such that } r_{i-1} m_i m_i^{[-1]} = r_{i-1} & \text{if } \varepsilon_i = 1 \end{cases} \tag{5}$$

Of course, we are using the first part of lemma 3.16 to define $m_i^{[-1]}$. Now we have

$$r_{i-1} m_i^{[\varepsilon_i]} m_i^{[-\varepsilon_i]} = r_{i-1} \qquad (1 \leq i \leq k)$$

This follows from the definition of $m_i^{[-1]}$ if $\varepsilon_i = 1$ and from the second part of lemma 3.16 if $\varepsilon_i = -1$. It follows by induction

$$f m_1^{[\varepsilon_1]} \cdots m_k^{[\varepsilon_k]} m_k^{[-\varepsilon_k]} \cdots m_1^{[-\varepsilon_1]} f = f$$

and thus

$$\bar{c}^{-1} = m_k^{[-\varepsilon_k]} \cdots m_1^{[-\varepsilon_1]} f \tag{6}$$

Furthermore, by (4), we have

$$\bar{a} = e {m'}_1^{[\varepsilon_1]} \cdots {m'}_k^{[\varepsilon_k]} \tag{7}$$

for some elements ${m'}_i^{[\varepsilon_i]}$ such that ${m'}_i^{[\varepsilon_i]} = m_i$ if $\varepsilon_i = 1$ and ${m'}_i^{[\varepsilon_i]} \in m_i^{(-1)}$ if $\varepsilon_i = -1$. Set, for $1 \leq i \leq k$,

$$s_i = e {m'}_1^{[\varepsilon_1]} \cdots {m'}_i^{[\varepsilon_i]} m_i^{[-\varepsilon_i]} \cdots m_1^{[-\varepsilon_1]} f$$

We claim that $s_i = s_{i-1}$ for all $i \geq 1$. Indeed, if $\varepsilon_i = -1$, we have

$$e {m'}_1^{[\varepsilon_1]} \cdots {m'}_{i-1}^{[\varepsilon_{i-1}]} {m'}_i^{[-1]} {m'}_i = e {m'}_1^{[\varepsilon_1]} \cdots {m'}_{i-1}^{[\varepsilon_{i-1}]}$$

by part (2) of lemma 3.16 and, if $\varepsilon_i = 1$, then

$${m'}_i m_i^{[-1]} m_{i-1}^{[\varepsilon_{i-1}]} \cdots m_1^{[-\varepsilon_1]} f = m_{i-1}^{[\varepsilon_{i-1}]} \cdots m_1^{[-\varepsilon_1]} f$$

by the dual version of the same lemma. Therefore, starting from (4) and (6), we obtain by induction

$$\bar{a}\bar{c}^{-1} = e {m'}_1^{[\varepsilon_1]} \cdots {m'}_k^{[\varepsilon_k]} m_k^{[-\varepsilon_k]} \cdots m_1^{[-\varepsilon_1]} f = \cdots = e {m'}_1^{[\varepsilon_1]} m_1^{[-\varepsilon_1]} f = ef$$

Therefore M satisfies Knast's identity and thus $M \in \mathbf{J} * \mathbf{G}$. $\square$

Of course, since we know by theorem 1.2 that the pointlike conjecture is true, we have the following theorem.

Theorem 3.17 *The following equalities hold:* $\mathbf{PG} = \mathbf{BG} = \mathbf{J} * \mathbf{G} = \mathbf{J} \,Ⓜ\, \mathbf{G}$. *In particular, it is decidable whether a monoid divides a power group.*

At the present time, this seems to be one of the most difficult results in finite semigroup theory in that it requires both Ash's theorem and Knast's theorem. That is, given a block group M, how does one find a finite group H such that M divides $\mathcal{P}(H)$? First one must trace through the (purely algebraic) proof that $\mathbf{BG} = \mathbf{J} ⓜ \mathbf{G}$ to construct a group G_1 and a relational morphism $\tau : M \to G_1$ such that $1\tau^{-1}$ is $\mathcal{J}$-trivial. Then one must use theorem 3.14 and Ash's proof of theorem 1.2 to find a group G_2 and a $\mathcal{J}$-trivial monoid N such that M divides $N * G_2$ for some action of G_2 on N. Then we use the language theoretic approach of [26, 28] to construct a group G_3 such that $N * G_2$ divides $\mathcal{P}(G_3)$ and we can take $H = G_3$.

4 Local Complexity of Finite Semigroups

In this section we return to the origins of the type II problem and give the connections with the complexity of finite semigroups. We have given the definition of the complexity function $c : \mathbf{Sgp} \to \mathbb{N}$ in section 1. Here $\mathbf{Sgp}$ is the variety of all finite semigroups. The following theorem characterizing the complexity function appeared in [47].

Theorem 4.1 *The complexity function $c : \mathbf{Sgp} \to \mathbb{N}$ is the largest function, in the pointwise sense, satisfying the following axioms:*

(1) *$Sc = 0$ for every aperiodic semigroup.*

(2) *$Sc \leq 1$ for every finite group.*

(3) *If S divides T, then $Sc \leq Tc$.*

(4) *$(S \times T)c \leq \max\{Sc, Tc\}$.*

(5) *$(S \circ T)c \leq Sc + Tc$.*

Here "$\circ$" denotes wreath product. A function $\ell : \mathbf{Sgp} \to \mathbb{N}$ is called a *local complexity function* if it satisfies Axioms 1-5 and in addition:

(1) $S\ell = \max\{(eSe)\ell \mid e = e^2 \in S\}$.

For a time it was thought that the complexity function c was local. An example in [44] constructed a semigroup S such that $Sc = 2$, but $\max\{(eSe)c \mid e = e^2 \in S\} = 1$. In the second part of this paper, a sequence of semigroups S_n $(n > 0)$ is constructed such that:

(1) $S_n c = n$.

(2) $K(S_n) \in \mathbf{A} * \mathbf{G}$. In particular, $K(S_n)c \leq 1$.

(3) $S_n \in (\mathbf{A} * \mathbf{G}) ⓜ \mathbf{G}$. (See theorem 3.4).

Thus the complexity of a semigroup S can differ arbitrarily from the complexity of its type II subsemigroup $K(S)$. Also, note that $(\mathbf{A} * \mathbf{G}) * \mathbf{G} = \mathbf{A} * \mathbf{G}$ is contained in $\mathbf{V}_1$, but (3) shows that $(\mathbf{A} * \mathbf{G}) ⓜ \mathbf{G}$ contains semigroups of arbitrary complexity. Letting $\mathbf{V} = \mathbf{A} * \mathbf{G}$, we see that in general $\mathbf{V} * \mathbf{G}$ can be a "small" proper subvariety of $\mathbf{V} ⓜ \mathbf{G}$.

Furthermore, consider a relation $\tau_n : S_n \to G_n$ onto some finite group G_n, such that the inverse image of the identity is $K(S_n)$. Let D_n be the derived semigroup of τ_n as defined in [52]. It follows easily from the basic properties of the derived semigroup that eD_ne is contained in $K(S_n)^0$ for all idempotents $e \in D_n$. On the other hand the Derived Semigroup Theorem [52] shows that S_n divides the wreath product $D_n \circ G_n$. It follows that $D_nc \geq n - 1$ and thus, the complexity of a semigroup can differ arbitrarily from that of any of its local submonoids. This shows that the complexity function is very global.

On the other hand, the main Theorem of [47] implies that the lower bound to complexity considered in [46] is the largest local complexity function. We review the definition here. Let S be a semigroup and let

$$S = U_0 \geq T_1 \geq U_1 \leq \ \ldots \ \geq T_n \geq U_n$$

be a descending sequence of subsemigroups such that each T_i, $(1 \leq i \leq n)$, is a nonaperiodic absolute type I semigroup and each U_i is a type II subsemigroup of T_i for $1 \leq i \leq n$. We say that the above chain is an alternating series for S of length n.

Recall that an absolute type I semigroup is a semigroup T such that, for every relational morphism $\tau : T \to A$ where A is an aperiodic semigroup, there is a $t \in A$ such that $T \subset \mathrm{Stab}(t)\tau^{-1}$. That is, T is a type I subsemigroup of itself.

Let S be a semigroup. Define $S\ell$ to be the length of the longest alternating series of S. $S\ell$ is called the *local complexity* of S. The following is the main Theorem of [47].

Theorem 4.2 *The local complexity function $\ell : \mathbf{Sgp} \to \mathbb{N}$ is the largest function in the pointwise sense satisfying Axioms 1-6.*

At the time that [47] was published it was not known whether being either an absolute type I semigroup or a type II semigroup was decidable. Ash's Theorem takes care of type II. The main result of this section shows that there is an easy criterion to decide if a semigroup is absolute type I as well. We first need the following hitherto unpublished result of the fourth author. Let U_2 be the semigroup consisting of two right zeroes $\{a, b\}$ and an identity.

Theorem 4.3 *A semigroup T is absolute type I if and only if for every relational morphism $\tau : T \to U_2$, $T \subset \{1, a\}\tau^{-1}$ or $T \subset \{1, b\}\tau^{-1}$.*

Note that the maximal right stabilizers in U_2 are $\{1, a\}$ and $\{1, b\}$, so that Theorem 4.3 says that we need only check that T behaves like a type I semigroup with respect to relational morphisms into U_2. Since there are only a finite number of relational morphisms between any two finite semigroups, we have the following important corollary.

Corollary 4.4 *There is an algorithm to decide if a finite semigroup is an absolute type I semigroup.*

Before commencing with the proof of theorem 4.3, we recall that $\mathbf{A}$, the variety of aperiodic semigroups is the smallest collection $\mathbf{A}$ of semigroups containing the trivial semigroup, closed under division, direct product and such that if $T \in \mathbf{A}$, then so is $T \circ U_2$. This is an immediate corollary of the Krohn-Rhodes Theorem and is a useful tool for inductive proofs involving $\mathbf{A}$.

It will be useful in the proof of theorem 4.3 to have the following concept as well. A semigroup S is R_1 if every $\mathcal{R}$-class of S has at most one idempotent. The collection $\mathbf{R}_1$ of all R_1 semigroups forms a variety of finite semigroups. The next lemma ties up our definition with that of [45] and gives a proof of 4.3.

Lemma 4.5 *The following conditions are equivalent for a semigroup S:*

(1) *For every relational morphism $\tau : S \to T$ where T is aperiodic, there is an R_1 subsemigroup $R \subset T$ such that $S \subset R\tau^{-1}$.*

(2) *S is an absolute type I semigroup.*

(3) *For every relational morphism $\tau : S \to U_2$, $S \subset \{1, a\}\tau^{-1}$ or $S \subset \{1, b\}\tau^{-1}$.*

Proof. Let R be an aperiodic R_1 semigroup. Let e be an idempotent in the minimal ideal of R. It is clear that eR is just the $\mathcal{R}$-class of e. Since $R \in \mathbf{R}_1$, the only idempotent in eR is e and since eR is an aperiodic simple semigroup, it follows that $eR = \{e\}$ and thus $R \subset \mathrm{Stab}(e)$. This gives immediately that (1) implies (2). (2) implies (3) is obvious.

(3) implies (1). We will use the inductive scheme for aperiodic semigroups outlined above. Let S be a finite semigroup and let $A_S = \{T \in \mathbf{A} \mid$ for all relational morphisms $\tau : S \to T$, there is an R_1 subsemigroup R of T such that $S \subset R\tau^{-1}\}$. Then the trivial semigroup is in A_S. Now assume that $T \in A_S$ and that T' divides T. Then there is an injective relational morphism (i.e. the inverse of a surjective partial function) $\iota : T' \to T$. Let $\tau : S \to T'$ be a relational morphism. Then $\tau\iota : S \to T$ is a relational morphism and thus there is an R_1 subsemigroup R of T such that $S \subset R(\tau\iota)^{-1}$. Therefore, $S \subset R'\tau^{-1}$ where $R' = R\iota^{-1}$. Since ι is an injective relational morphism it follows that R' divides R and is also an R_1 semigroup. This proves that $T' \in A_S$. A similar type proof shows that A_S is closed under direct product as well.

Now assume that $T \in A_S$ and let $\tau : S \to T \circ U_2$ be a relational morphism. Let $\pi : T \circ U_2 \to U_2$ be the projection morphism. By the assumption (3) it follows without loss of generality, that $S \subset \{1, a\}(\tau\pi)^{-1}$. Notice that $\{1, a\}$ is isomorphic to the semigroup U_1 consisting of an identity and a zero, and thus we can consider $\tau : S \to T \circ (\{1, a, b\}, U_1)$. Thus every element of this wreath product is of the form (f, x) where $x \in \{1, a\}$ and $f : \{1, a, b\} \to T$.

Since a is the zero of U_1, it is easy to see that the function

$$\eta : T \circ (\{1, a, b\}, U_1) \to T$$

such that $(f, x)\eta = af$ is a morphism. Since we are assuming that $T \in A_S$, there is an R_1 subsemigroup R of T such that $S \subset R\eta^{-1}\tau^{-1}$. Let $R' = R\eta^{-1}$. Then R' is the disjoint union of a subsemigroup $N = R' \cap \{(f, 1) \mid f : \{1, a, b\} \to T\}$ and an ideal $I = R' \cap \{(f, a) \mid f : \{1, a, b\} \to T\}$. It is easy to see that the Rees quotient $R'/I = N \cup \{0\}$ divides $N \times U_1$ and that N divides $T^{\{1,a,b\}}$, so by the above and the easy fact that $U_1 \in A_S$, $R'/I \in A_S$. Let $\rho : R' \to R'/I$ be the Rees morphism. Then $\theta = \big(\tau \cap (S \times R')\big)\rho : S \to R'/I$ is a relational morphism. Thus there is an R_1 subsemigroup T of R'/I such that $S \subset T\theta^{-1}$. All this allows to assume without loss of generality, that N is an R_1 subsemigroup of R'.

It suffices then to prove that I is an R_1 semigroup. Suppose that (f, a) and (g, a) are $\mathcal{R}$-equivalent idempotents of R. Then $(f, a)\eta = (g, a)\eta$, since η is a morphism and R is an R_1 semigroup. That is, $af = ag$. Thus,

$$(f, a) = (g, a)(f, a) = (g + af, a) = (g + ag, a) = (g, a)$$

using the fact that we are dealing with $\mathcal{R}$-equivalent idempotents. This proves that I is an R_1 semigroup and we are done.

Theorem 4.6 *The local complexity function $\ell : \mathbf{Sgp} \to \mathbb{N}$ is computable.*

The rest of this section is devoted to providing a more satisfying descrition of absolute type I semigroups. It confirms a conjecture first published in [45]. We first gather some simple facts about absolute type I semigroups.

Lemma 4.7

(1) *Let $\varphi : S \to T$ be a surjective functional morphism. If S is absolute type I, then so is T.*

(2) *If S is absolute type I, then so are S^1 and S^0.*

Proof. Let $\varphi : S \to T$ be a surjective functional morphism and let $\tau : T \to A$ be a relational morphism onto an aperiodic semigroup A. Then $\varphi\tau : S \to A$ is a relational morphism. Since S is absolute type I there is a $t \in T$, such that $S \subset A_t\tau^{-1}\varphi^{-1}$ where $A_t = \mathrm{Stab}(t)$. Therefore, $T = S\varphi \subset A_t\tau^{-1}\varphi^{-1}\varphi \subset A_t\tau^{-1}$, since φ is a function. Therefore T is absolute type I and this proves (1). The proof of (2) is easy and is left to the reader. $\square$

Theorem 4.8 *A semigroup S is an absolute type I semigroup if and only if S is generated by the union of a chain of $\mathcal{L}$-classes, $L_1 >_{\mathcal{L}} L_2 >_{\mathcal{L}} \cdots >_{\mathcal{L}} L_n$.*

Proof. First assume that S is generated by the union of a chain of $\mathcal{L}$-classes, $L_1 >_{\mathcal{L}} L_2 >_{\mathcal{L}} \cdots >_{\mathcal{L}} L_n$. Let $\tau : S \to U_2$ be a relational morphism. It is easy to see that L_n is contained in the inverse image of an $\mathcal{L}$-class of U_2. But $\mathcal{L}$-classes of U_2 are singletons. If L_n is contained in the inverse image of 1, so are all the other L_i for $i < n$, since they are all $\mathcal{L}$-above L_n. Thus S is contained in the inverse image of $\{1\}$. If L_n is contained in the inverse image of $\{a\}$, then again it is easy to see that the chain condition implies that all the L_i are contained in $\{1, a\}\tau^{-1}$. Thus in this case, S is contained in the inverse image of $\{1, a\}$. Similarly, if L_n is contained in the inverse image of $\{b\}$, then S is contained in the inverse image of $\{1, b\}$. Theorem 4.3 implies that S is absolute type I.

Conversely, assume that S is absolute type I. Without loss of generality we may assume that S is a monoid with 0 by 4.7 (2). We induct on the number k of non-zero $\mathcal{J}$-classes of S. If $k = 1$, then S is either a group or a group with 0 and the result is clear.

Assume that every monoid with 0 with less than k non-zero $\mathcal{J}$-classes that is absolute type I is generated by a chain of $\mathcal{L}$-classes. Let S have k non-zero $\mathcal{J}$-classes, where $k \geq 2$. Let $J \neq 0$ be a (0)-minimal $\mathcal{J}$-class. Let $I = SJS$ be the ideal generated by J and let $T = S/I$ be the Rees quotient. By Lemma 4.7 (1), T is absolute type I. Since T has $k - 1$ non-zero $\mathcal{J}$-classes, the inductive hypothesis implies that T is generated by the union of a chain of $\mathcal{L}$-classes, $L_1 >_{\mathcal{L}} \cdots >_{\mathcal{L}} L_r$ in the $\mathcal{L}$-order of T. Without loss of generality, we can assume that $L_r \neq 0$.

Let K be the submonoid of S generated by the union of the L_i $(i = 1, \ldots, r)$. Let Y be the set of $\mathcal{L}$-classes of J that are not contained in K. If Y is empty, then $K = S$ and we are done, so we assume that $Y \neq \emptyset$. Define a relation $\geq$ on Y by $L \geq L'$ if and only if there is a $k \in K$ such that $Lk = L'$. Clearly $\geq$ is reflexive and transitive. Let $\sim$ denote the associated equivalence relation and let $\geq$ also denote the associated partial order on $Y/{\sim}$. Let $[L]$ denote the equivalence class of L.

We claim that the poset $(Y/{\sim}, \geq)$ has a unique maximal element. Otherwise there are two $\geq$ maximal classes $[L]$ and $[L']$ that are incomparable. Define $\tau : S \to U_2$ to be the relational morphism generated by $s\tau = \{1\}$ if $s \in S \setminus ([L] \cup [L'])$, $s\tau = \{a\}$ if $s \in \cup[L]$ and $s\tau = \{b\}$ if $s \in \cup[L']$. It is easy to see that $1 \in s\tau$ if and only if s is in K or s belongs to an $\mathcal{L}$-class that is not equivalent to L or L'. Since L and L' are not contained in K, it follows that there is an $s \in L$ and $s' \in L'$ such that neither s nor s' is τ related to 1. Now $a \in s\tau$. Clearly $b \in s\tau$ if and only if $s = xty$ where t is in an $\mathcal{L}$-class equivalent to L' and $1 \in y\tau$. Since J is (0)-minimal, it follows that $xt \; \mathcal{L} \; t$. Since $1 \in y\tau$, either y is in K or in an $\mathcal{L}$-class not equivalent to L. If $y \in K$, then $s = xty \; \mathcal{L} \; ty$ and thus $L \leq L'$ contradicting the assumption that L and L' are incomparable. On the other hand, if y is in an $\mathcal{L}$-class not equivalent to L, then $s = xty \; \mathcal{L} \; y$ (by (0)-minimality of J again) and this too is a contradiction. Therefore, $s\tau = \{a\}$. Similarly, $s'\tau = \{b\}$. Therefore, S is not absolute type I and this is a contradiction.

Let L be a representative of the unique maximal $\sim$ class. Clearly S is generated by $K \cup L$ and thus by $\cup_{1 \leq i \leq r} L_i \cup L$. It suffices then to prove that we can find an $\mathcal{L}$-class L' equivalent to L and such that $L \leq_{\mathcal{L}} L_r$.

Consider then the relational morphism $\eta : S \to U_2$ generated by $L_i\eta = \{1\}$ for $i = 1, \ldots, r-1$, $L_r\eta = \{a\}$ and $L\eta = \{b\}$. Since S is absolute type I, and L_r is strictly $\mathcal{J}$-above L, it follows that S is contained in $\{1, a\}\eta^{-1}$. As above, we can find an $s \in L$ such that s is not in K. Therefore, $a \in s\eta$. This means that we can factor $s = xty$ where $t \in L_r$ and $y \in 1\eta^{-1}$. But $1\eta^{-1} \subset K$ by construction. Now if xt is not in J, it follows that xt is in K and thus $s = xty \in K$, a contradiction. Therefore xt is in J. Clearly the $\mathcal{L}$-class L' of xt is $\mathcal{L}$-below L_r and satisfies $L' \geq L$. Therefore, $L' \sim L$ by maximality and we are done. $\square$

5 Topology

In a sequence of papers [31, 37, 34], the third author gives an interpretation of the type II conjecture in terms of a conjecture on the structure of closed rational sets in the profinite group topology on a free monoid. In [29], it is shown that the type II conjecture is in fact equivalent to this topological conjecture. Related work appeared in [17]. It is proved in [38] that a conjecture on the structure of rational closed subsets in the profinite topology on the free group, generalizing a classical result of M. Hall [12], implies the topological conjecture on the free monoid and thus the type II conjecture. We show here that all these conjectures are equivalent. In particular, Ash's theorem implies that the two topological conjectures are true and leads to a complete and decidable description of the rational closed sets in both the free monoid and the free group. As the authors were preparing this article, a direct proof of the topological conjecture on the free group has been obtained by Ribes and Zalesskii [48], giving in turn a new proof of Ash's theorem! The proof of Ribes and Zaleski uses profinite trees acting on groups, and thus seems at first sight very far from the proof of Ash. However, a more careful study reveals some interesting connections between the two proofs and it would be interesting to combine the two techniques.

In this section, we only concentrate on the connections between profinite group topologies on free monoids and free groups and the type II conjecture.

For more details on these topologies, see [37].

Let A be a finite set and let A^* and $FG(A)$ denote the free monoid and the free group on A respectively. The rational subsets of a monoid M form the smallest class $\mathcal{R}$ of subsets of M such that

(a) every finite subset of M belongs to $\mathcal{R}$,

(b) if S and T are in $\mathcal{R}$, then so are ST and $S \cup T$,

(c) if $S \in \mathcal{R}$, then so is the submonoid S^* of M generated by S.

If $M = A^*$, the free monoid on a finite set A, a well-known theorem of Kleene states that the rational subsets are exactly the recognizable subsets. In particular, the rational sets form a boolean algebra under union and complement. If $M = FG(A)$, the rational subsets also form a boolean algebra (a non-trivial result), which strictly contains the boolean algebra of recognizable subsets. The rational subsets of the free monoid and of the free group are related as follows. Let $\tilde{A} = A \cup A^{-1}$, where $A \cap A^{-1} = \emptyset$ and let $K \subset \tilde{A}^*$ be the set of group reduced words. We have the canonical injection $\iota : FG(A) \to \tilde{A}^*$ and the canonical maps $\pi : \tilde{A}^* \to FG(A)$ and $\delta : \tilde{A}^* \to K$, where $w\delta$ is the unique reduced word v with $v\pi = w\pi$. Then a theorem of Benois [7] states a subset S of $FG(A)$ is rational if and only if the subset $S\delta$ of $\tilde{A}^*$ is rational.

The profinite group topology on A^* $(FG(A))$ is the smallest topology such that every monoid (group) morphism from A^* $(FG(A))$ onto a finite group G is continuous. This topology was first considered for the free group by M. Hall [12] and by Reutenauer for the free monoid [40, 41]. It is also connected to the study of implicit operations [2, 39].

It is clear that a basis for the topology on A^* is the set of recognizable languages whose syntactic monoid is a finite group and that a basis for the topology on $FG(A)$ is the set of cosets of subgroups of $FG(A)$ of finite index. If X is a subset of the free monoid or the free group, let $\bar{X}$ denote the closure of X in the respective topology. The next theorem gives the first connection between the profinite group topology and the type II conjecture.

Theorem 5.1 [37] *Let M be a finite monoid, represented as a morphic image $\varphi : A^* \to M$ of a finitely generated free monoid. Let $m \in M$. Then $m \in K(M)$ if and only if $1 \in \overline{m\varphi^{-1}}$.*

Thus the question of whether an element m of M is in $K(M)$ is decidable, the so called weak form of the type II conjecture, is equivalent to seeing if the empty word is in the closure, in the profinite group topology on A^*, of the set of words that represent m. The third author refined this to find a conjecture on closed rational sets in the free monoid that is equivalent to the type II conjecture itself. We start with the following observation.

The topologies on the free monoid and on the free group can also be defined by the following metric. Define $d(u, u) = 0$ and if $u \neq v$, then $d(u, v) = 2^{-n}$ where n is the order of the smallest group that separates u and v. Since it is well-known that the free monoid and the free group are residually finite with respect to groups — that is, for every pair of distinct elements there is a monoid (respectively group) morphism onto a finite group that separates them — it is easy to prove that the function d is an ulta-metric, is compatible with multiplication and turns A^* $(FG(A))$ into a topological monoid (group). Here is an interesting limit for this topology. It holds for both cases.

Theorem 5.2 [40] *For all* $x, u, y \in A^*$ *(*$FG(A)$*),* $\lim_{n\to\infty} xu^{n!}y = xy$

In essence one need just observe that $u^{n!} = 1$ in every finite group of order at most n and then use the metric properties listed above. As a closed set contains the limit of any converging sequence, we have the following corollary.

Corollary 5.3 *Let* X *be a closed set. If* $xu^{n!}y \in X$ *for all* $n \geq 0$*, then* $xy \in X$*.*

The third author conjectured that the converse of this corollary holds if X is a recognizable set. More precisely:

Conjecture 5.1 [37] Let L be a recognizable subset of A^*. Then L is closed if and only if for all $x, u, y \in A^*$, if $\{xu^ny \mid n > 0\} \subset L$, then $xy \in L$.

In [34], it is proved that this conjecture implies that $K(M) = D(M)$ for every finite monoid M, that is the topological conjecture 5.1 implies the type II conjecture. In [29] it is proved that the converse is true. The proofs of these equivalences are non-trivial.

In [38] an easier to state conjecture on rational closed sets in the free group is made. First recall the following result of M. Hall. See [12, 49] for proofs.

Theorem 5.4 (Hall) *Let* H *be a finitely generated subgroup of* $FG(A)$*. Then* H *is closed in the profinite topology.*

The conjecture of Pin and Reutenauer just states that the product of finitely generated subgroups of $FG(A)$ is closed:

Conjecture 5.2 (Pin and Reutenauer [38]) Let $H_1, \ldots, H_n$ be finitely generated subgroups of $FG(A)$. Then $H_1H_2\cdots H_n$ is closed in the profinite topology.

The main theorem of this section shows that the two topological conjectures are equivalent to the type II conjecture and therefore are true.

Theorem 5.5 *The following statements are equivalent:*

(a) $K(M) = D(M)$ *for every finite monoid* M*.*

(b) *The topological conjecture for free monoids is true.*

(c) *The topological conjecture for free groups is true.*

Proof. By the remarks above, by the results of [29, 34, 38], we need only prove that (b) implies (c). We use freely the notations $\tilde{A}$, K and δ introduced above. Consider the profinite topologies on $\tilde{A}^*$ and $FG(A)$ and the relative topology on $K \subset \tilde{A}^*$. We have the canonical maps $\pi : \tilde{A}^* \to FG(A)$ and $\delta : \tilde{A}^* \to K$. It is well known that $\beta = \delta^{-1}\pi : K \to FG(A)$ is a bijection. The following is an immediate corollary of [37], theorem 4.11(c).

Lemma 5.6 $\beta : K \to FG(A)$ *is a homeomorphism.*

Let $H_1, \ldots, H_n$ be finitely generated subgroups of $FG(A)$. Let $\mathcal{A}_i = (Q_i, q_i, q_i)$ for $i = 1, \ldots, n$ be the finite state inverse automaton with initial-terminal state q_i such that $|\mathcal{A}_i| \cap K = H_i\beta^{-1}$. That is, $\mathcal{A}_i$ recognizes H_i when considered as an automaton over $FG(A)$. The only difference with an automaton over A^* is that to any edge (q, a, q') is associated an edge $(q', \bar{a}, q)$. In

other words, one can read edges backwards by inversing their labels. Assume $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Consider the automaton $\mathcal{B}_0 = (Q, q_1, q_n)$ pictured below: More formally, the state set of $\mathcal{B}_0$ is $Q = \cup_{1 \leq i \leq n} Q_i$, the edges of $\mathcal{B}_0$ are the union of the edges of the $\mathcal{A}_i$ together with $\{(q_i, 1, q_{i+1}) \mid 1 \leq i \leq n\}$. Note that even though every edge in $\mathcal{A}_i$ has an inverse edge, none of the edges reading the identity can be read backwards.
Clearly, considered as an automaton over $FG(A)$, $\mathcal{A}$ accepts $L = H_1 H_2 \cdots H_n$. However, there may be some reduced words representing elements of L, that are not accepted by $\mathcal{A}$.

Conjecture 5.3 Let $H_1 = \langle aba^{-1} \rangle$ and $H_2 = \langle a \rangle$. Then $ab \in H_1 H_2$, but ab considered as an element of $\tilde{A}^*$ is not accepted by $\mathcal{A}$. The equivalent word $aba^{-1}a$ is accepted.

To rectify this situation we add an edge labelled 1 between any states $p \neq q$ in $\mathcal{B}_0$ such that there is a path labelled $aa^{-1} : p \to q$ for some $a \in \tilde{A}$. Note that since $p \neq q$, if $p \in Q_i$, then $q \in Q_j$ for some $j > i$ since each $\mathcal{A}_i$ is an inverse automaton. We obtain an automaton $\mathcal{B}_1$. One can continue this process of mimicking aa^{-1} pairs with identity arrows to obtain $\mathcal{B}_2, \ldots, \mathcal{B}_k$. It is clear that this process halts after a finite number of steps and we obtain an automaton $\mathcal{B} = (Q, q_1, q_n)$ with the following properties:

(1) considered as an automaton over $FG(A)$, $\mathcal{B}$ accepts $L = H_1 H_2 \cdots H_n$,

(2) considered as an automaton over $\tilde{A}^*$, $|\mathcal{B}| \cap K = L\beta^{-1}$. That is, every reduced word representing an element of L is accepted by $\mathcal{B}$.

The topological conjecture for $FG(A)$ states that L is a closed set. By lemma 5.6, it suffices to prove that $L\beta^{-1} = |\mathcal{B}| \cap K$ is closed in K and thus that $|\mathcal{B}|$ is closed in $\tilde{A}^*$ by definition of the relative topology. Thus it suffices to show that if $xu^+y \subset |\mathcal{B}|$, then $xy \in |\mathcal{B}|$, by the topological conjecture for $\tilde{A}^*$. Let $k = |Q|$. If $xu^+y \subset |\mathcal{B}|$, then in particular, $xu^{k!}y \in |\mathcal{B}|$. Thus there is a path $p : q_1 \to q_n$ such that p reads $xu^{k!}y$. By the pigeon-hole principle, there is some state s, and a factorization of $p = abc$ such that $a : q_1 \to s$, $b : s \to s$ and $c : s \to q_n$, such that b reads u^r for some $r \leq k$. By the construction of $\mathcal{B}$, no edge in b reads 1 and if s is a state of $\mathcal{A}_i$ then b is actually a path in $\mathcal{A}_i$. Since $\mathcal{A}_i$ is inverse, there is also a path $b^{-1} : s \to s$ reading u^{-r}. It follows that the path $ab(b^{-1})^{\frac{k!}{r}}c : q_1 \to q_n$ reads a word of the form $xu^i(u^{-1})^j u^l y$ with $i + l = j$. This word reduces to xy and by the construction of $\mathcal{B}$, $xy \in |\mathcal{B}|$ as desired. $\square$

As a consequence of Ash's proof of the type II conjecture, or as a consequence of the result of Ribes and Zalesskii, the topological conjectures now become theorems which we record below.

Theorem 5.7 *Let L be a rational subset of A^*. Then L is closed if and only if L satisfies the implication $xu^+y \subset L \Rightarrow xy \in L$.*

Theorem 5.8 *Let $H_1, \ldots, H_n$ be finitely generated subgroups of $FG(A)$. Then the set $H_1 H_2 \cdots H_n$ is closed in the profinite topology.*

It is shown in [38] that the truth of the topological conjecture for the free group implies a number of results on the structure of rational closed sets in the free monoid and the free group. As these results now become theorems, we repeat them here and leave the details to [38]. Let $\mathcal{F}$ be the smallest class of subsets of the free group such that

(a) The empty set and every singleton $\{g\}$, $g \in FG(A)$, is in $\mathcal{F}$,

(b) If $X, Y \in \mathcal{F}$, then so are XY and $X \cup Y$,

(c) if $X \in \mathcal{F}$, then $\langle X \rangle$, the group generated by X, is in $\mathcal{F}$.

$\mathcal{F}$ is a proper subset of the set of rational subsets of $FG(A)$. Here is an alternative characterization of members of $\mathcal{F}$.

Proposition 5.9 (Proposition 2.2 of [38]) *$\mathcal{F}$ is the class of all subsets of $FG(A)$ that are finite unions of sets of the form $gH_1H_2 \cdots H_n$ where $g \in FG(A)$ and H_1, $\ldots$, H_n are finitely generated subgroups of $FG(A)$. Furthermore $\mathcal{F}$ is precisely the collection of closed rational subsets of $FG(A)$ in the profinite topology.*

The following theorem gives an algorithm to compute the closure of a rational set $L \subset FG(A)$.

Theorem 5.10 (Theorem 2.4 of [38]) *Let $L \subset FG(A)$ be rational. Then the closure $\bar{L}$ of L in the profinite topology is rational. Furthermore $\bar{L}$ can be computed by the following rules where X and Y are rational subsets of the free group:*

(1) $\bar{X} = X$ *if X is finite.*

(2) $\overline{X \cup Y} = \bar{X} \cup \bar{Y}$.

(3) $\overline{XY} = \bar{X}\bar{Y}$.

(4) $\overline{X^*} = \langle X \rangle = (X \cup X^{-1})^*$.

Given a rational expression for L, one can clearly construct a rational expression for $\bar{L}$. The next theorem follows from well-known results about rational sets in free groups.

Lemma 5.11 *Let $L, L' \in Rat(FG(A))$ be given by rational expressions. Then it is decidable whether or not $L = L'$.*

Proof. Let $L \in Rat(FG(A))$ be given by a rational expression. Then by using the theorem of Benois [7], one can effectively construct a finite state automaton $\mathcal{B}(L)$ over $\tilde{A}^*$ such that $|\mathcal{B}(L)| = L\beta^{-1}$, the set of reduced words representing L. But $L = L'$ if and only if $L\beta^{-1} = L'\beta^{-1}$ and the result follows since equality is decidable for languages specified by finite state automata. □

Corollary 5.12 *Let $L \in Rat(FG(A))$ be given by a rational expression. Then it is decidable whether or not L is closed in the profinite topology.*

Proof. L is closed if and only if $L = \bar{L}$. □

Similarly, we can use the truth of the conjecture for the free monoid to show that closed sets have nice properties.

Let $L \subset A^*$. Let $F^*(L)$ be the smallest set K containing L and closed under the implication $xu^+y \subset K \Rightarrow xy \in K$. It is clear that $F^*(L)$ is well defined. Furthermore, it is shown in [37] that if $L \in Rat(A^*)$ is given by a rational expression (or by a finite automaton), then $F^*(L) \in Rat(A^*)$ and a rational expression (resp. a finite automaton) for $F^*(L)$ can be effectively computed.

Theorem 5.13 *Let L be a rational subset of A^*. Then $\bar{L} = F^*(L)$. Furthermore, $\bar{L}$ is rational, effectively constructible from L and its syntactic monoid is a block group.*

Proof. It follows immediately from theorem 7.6 of [37]. $\square$

Corollary 5.14 *It is decidable whether a rational subset of A^* is closed or not in the profinite topology.*

A simple characterization can also be given in terms of syntactic monoids. Let L be a subset of A^*. Then the *syntactic congruence* of L is the equivalence $\sim_L$ on A^* defined by

$$u \sim_L v \text{ if and only if, for every } x, y \in A^* \quad (xuy \in L \Longleftrightarrow xvy \in L).$$

The quotient $A^*/{\sim_L}$ is the *syntactic monoid* of L and the natural morphism $\eta : A^* \to M(L)$ is called the *syntactic morphism*. Finally, the subset $P = L\eta$ of $M(L)$ is called the *syntactic image* of L. It is a well-known fact that L is rational if and only if $M(L)$ is a finite monoid.

Theorem 5.15 *Let $L \subset A^*$ be rational and let M be its syntactic monoid. Then L is closed if and only if its syntactic image P satisfies the following property:*
for every $s, t \in M$ and for every $e \in E(M)$, $set \in P$ implies $st \in P$.

Here is another way to compute the closure of a rational subset of A^*. It too follows from the truth of the conjecture on the free group. A subset $L \subset A^*$ is *simple* if $L = L_0^* w_1 L_1^* w_2 \cdots w_k L_k^*$ where the L_i's are rational subsets of A^* for $i = 0, \ldots, k$ and $w_i \in A^*$ for $i = 1, \ldots, k$. It is an easy consequence of Kleene's theorem that a subset L of A^* is rational if and only if it is a finite union of simple sets. Furthermore, given a rational expression or finite state automaton representing L, we can effectively write L as a finite union of simple sets [37].

Theorem 5.16 (Theorem 7.8 of [37]) *Let $L = L_0^* w_1 L_1^* w_2 \cdots w_k L_k^*$ be simple. Then $\bar{L} = \langle L_0 \rangle w_1 \langle L_1 \rangle \cdots w_k \langle L_k \rangle \cap A^*$.*

Since closure commutes with union, this allows to effectively compute the closure of a rational set.

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${}^*G$		$\cdots$		
	${}^*G$	$\cdots$		
$\vdots$	$\vdots$	$\ddots$		
				${}^*G$

Figure 1: A regular $\mathcal{D}$-class of a block group.

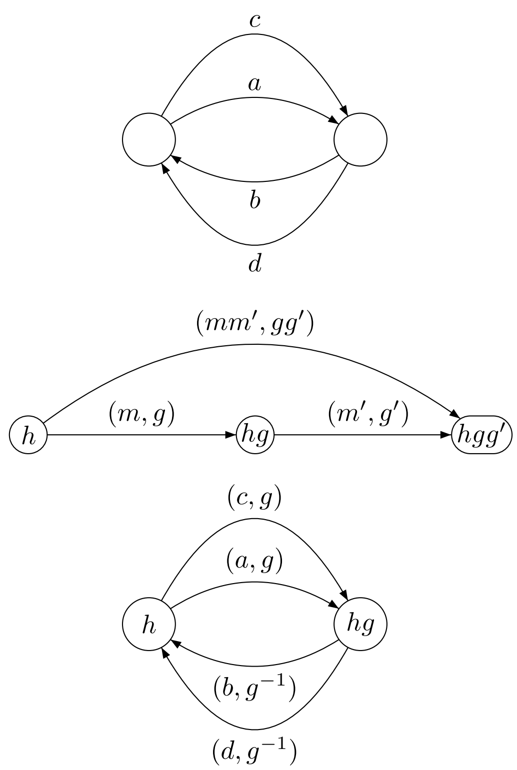
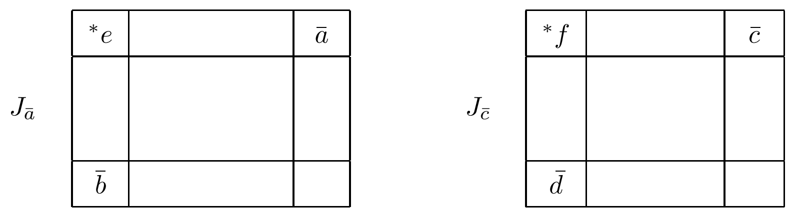


Figure 2: A subgraph of C_τ.



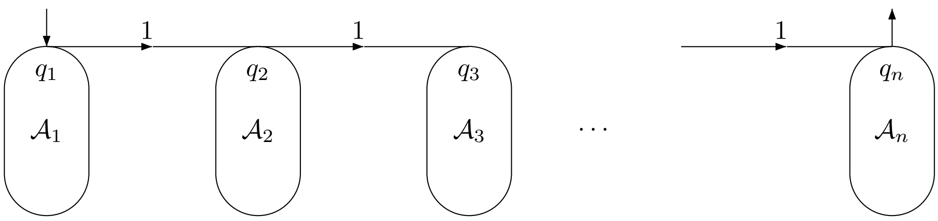


Figure 3: The automaton $\mathcal{B}_0$.