A representation formula for maps on supermanifolds

Frédéric Hélein

A representation formula for maps on supermanifolds

Frédéric Hélein*

March 17, 2006

Introduction

The theory of supermanifolds, first proposed by Salam and Strathdee [15] as a geometrical framework for understanding the supersymmetry, is now well understood mathematically and can be formulated in roughly two different ways: either by defining a notion of superdifferential structure with "supernumbers" which generalizes the differential structure of $\mathbb{R}^p$ and by gluing together these local models to build a supermanifold. This is the approach proposed by Dewitt [6] and Rogers [14]. Alternatively one can define supermanifolds as ringed spaces, i.e. objects on which the algebra (or the sheaf) of functions is actually a superalgebra (or a sheaf of superalgebras). This point of view was adopted by Berezin [4], Leĭtes [12], Manin [13] and was recently further developped by Deligne and Morgan [8], Freed [10] and Varadarajan [16]. The first approach is influenced by differential geometry, whereas the second one is inspired by algebraic geometry. Of course all these points of view are strongly related, but they may lead to some subtle differences (see Batchelor [3], Bartocci, Bruzzo and Hernández-Ruipérez [2] and Bahraini [1]).

The starting point of this paper was to understand some implications of the theory of supermanifolds according to the second point of view [4, 12, 13, 8, 10, 16], i.e. the one inspired by algebraic geometry. The basic question is to understand $\mathbb{R}^{p|q}$, the space with p ordinary (bosonic) coordinates and q odd (fermionic) coordinates. There is no direct definition nor picture of such a space beside the fact that the algebra of functions on $\mathbb{R}^{p|q}$ should be isomorphic to $\mathcal{C}^\infty(\mathbb{R}^p)[\eta^1,\cdots,\eta^q]$, i.e. the algebra over $\mathcal{C}^\infty(\mathbb{R}^p)$ spanned by q generators $\eta^1,\cdots,\eta^q$ which satisfy the anticommutation relations $\eta^i\eta^j+\eta^j\eta^i=0$. Hence $\mathcal{C}^\infty(\mathbb{R}^p)[\eta^1,\cdots,\eta^q]$ is isomorphic to the set of sections of the flat vector bundle over $\mathbb{R}^p$ whose fiber is the exterior algebra $\Lambda^*\mathbb{R}^q$. To experiment further $\mathbb{R}^{p|q}$ we define what should be maps from open subsets of $\mathbb{R}^{p|q}$ to ordinary manifolds. We adopt the provisional definition of an *open subset* of Ω of $\mathbb{R}^{p|q}$ to be a space on which the algebra of functions is isomorphic to $\mathcal{C}^\infty(|\Omega|)[\eta^1,\cdots,\eta^q]$, where $|\Omega|$ is an open subset of $\mathbb{R}^p$. So we choose such

*helein@math.jussieu.fr Institut de Mathématiques de Jussieu, UMR 7586 Université Denis Diderot – Paris 7, Case 7012, 2 place Jussieu 75251 Paris Cedex 5, France

an open set Ω and a smooth ordinary manifold $\mathcal{N}$ and analyze what should be maps ϕ from Ω to $\mathcal{N}$. Again there is no direct definition of such an object except that by the chain rule it should define a ring morphism ϕ^* from the ring $\mathcal{C}^\infty(\mathcal{N})$ of smooth functions on $\mathcal{N}$ to the ring $\mathcal{C}^\infty(|\Omega|)[\eta^1,\cdots,\eta^q]$. The morphism property means that

$$\forall\lambda,\mu\in\mathbb{R},\forall f,g\in\mathcal{C}^\infty(\mathcal{N}),\quad \phi^*(\lambda f+\mu g)=\lambda\phi^* f+\mu\phi^* g \tag{1}$$

and

$$\forall f,g\in\mathcal{C}^\infty(\mathcal{N}),\quad \phi^*(fg)=(\phi^* f)(\phi^* g). \tag{2}$$

We restrict ourself to *even* morphisms, which means here that we impose to $\phi^* f$ to be in the even part $\mathcal{C}^\infty(|\Omega|)[\eta^1,\cdots,\eta^q]_0$ of $\mathcal{C}^\infty(|\Omega|)[\eta^1,\cdots,\eta^q]$.

In the first section we prove our main result (Theorem 1.1) which shows that, for any even morphism ϕ^*, there exists a smooth map φ from $|\Omega|$ to $\mathcal{N}$ and a family of vector fields $(\Xi_x)_{x\in|\Omega|}$ depending on $x\in|\Omega|$ and tangent to $\mathcal{N}$ and with coefficients in the commutative subalgebra $\mathbb{R}[\eta^1,\cdots,\eta^q]_0$ such that

$$\forall f\in\mathcal{C}^\infty(\mathcal{N}),\quad \phi^* f=(1\times\varphi)^*\left(e^{\Xi}f\right). \tag{3}$$

One may interpret the term e^{Ξ} as an analogue with odd variables of the standard Taylor series representation

$$g(x)=\sum_{k=0}^{\infty}\frac{\partial^k g}{(\partial x)^k}(x_0)\frac{(x-x_0)^k}{k!}=\left(e^{\sum_{i=1}^n(x^i-x_0^i)\frac{\partial}{\partial x^i}}g\right)(x_0),$$

for a function g which is analytic in a neighbourhood of x_0. We also show that the vector field Ξ (which is not unique) can be build as a combination of commuting vector fieds. Then the rest of this paper is devoted to the consequences of this result.

The second section explores in details the structure behind relation (3). First we exploit the fact that one can assume that the vector fields which compose Ξ commute, so that one can integrate them locally. This gives us an alternative description of morphisms. Eventually this study leads us to a factorization result for all even morphisms as follows. First let us denote by $\Lambda_+^{2*}\mathbb{R}^q$ the subspace of all even elements of the exterior algebra $\Lambda^*\mathbb{R}^q$ of positive degree (i.e. $\Lambda_+^{2*}\mathbb{R}^q\simeq\mathbb{R}^{2^{q-1}-1}$). We construct an ideal $\mathcal{I}^q(|\Omega|)$ of the algebra $\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)$ in such a way that, if we consider the quotient algebra $\mathcal{A}^q(|\Omega|):=\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)/\mathcal{I}^q(|\Omega|)$, then there exists a canonical isomorphism $T_\Omega^*:\mathcal{A}^q(|\Omega|)\longrightarrow\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)$. By following the theory of scheme of Grothendieck we associate to $\mathcal{A}^q(|\Omega|)$ its spectrum $\mathrm{Spec}\mathcal{A}^q(|\Omega|)$, a kind of geometric object embedded in $|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q$. Then for any even morphism ϕ^* from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(|\Omega|)[\eta^1,\cdots,\eta^q]$, there exists a smooth map Φ from $|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q$ to $\mathcal{N}$, such that

$$\phi^*=T_\Omega^*\circ\Phi^*_{|\star},$$

where $\forall f \in \mathcal{C}^\infty(\mathcal{N})$, $\Phi^*_{|\star} f = f \circ \Phi \ \mathrm{mod}\mathcal{I}^q(|\Omega|)$. So by dualizing we can think of the map $\Phi_{|\star} : \mathrm{Spec}\mathcal{A}^q(|\Omega|) \longrightarrow \mathcal{N}$ as the restriction of Φ to $\mathrm{Spec}\mathcal{A}^q(|\Omega|)$. Hence we obtain an interpretation of a map on a supermanifold as a function defined on an (almost) ordinary space. This reminds somehow the theory developped by Vladimirov and Volovich [17] who represent a map on a superspace as a function depending on many auxiliary ordinary variables satisfying a system of so-called "Cauchy–Riemann type equations". However their description in terms of ordinary functions satisfying first order equations differs from our point of view.

The last section is devoted to applications of our results for understanding the use of supermanifolds by physicists. First we explain briefly how one can reduced the study of maps between two supermanifolds to the study of maps from a super manifold to an ordinary one, by using charts. Second we recall why it is necessary to incorporate the notion of the functor of point (as illustrated in this framework in [8, 10, 18]) in the definition of a map ϕ between supermanifolds in terms of ring morphisms. Then we address the simple question of computing the pull-back image ϕ^*f of a map f on an ordinary manifold $\mathcal{N}$ by a map ϕ from an open subset of $\mathbb{R}^{p|q}$ to $\mathcal{N}$. For instance consider a superfield $\phi = \varphi + \theta^1\psi_1 + \theta^2\psi_2 + \theta^1\theta^2 F$ from $\mathbb{R}^{3|2}$ (with coordinates $(x^1, x^2, t, \theta^1, \theta^2)$) to $\mathbb{R}$ and look at the Berezin integral

$$I := \int_{\mathbb{R}^{3|2}} d^3x d^2\theta \ \phi^* f,$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function. Such a quantity arises for instance in the action $\int_{\mathbb{R}^{3|2}} d^3x d^2\theta \left(\frac{1}{4}\epsilon^{ab} D_a\phi D_b\phi + \phi^* f\right)$ and then f plays the role of a superpotential. Following Berezin's rules the integral I is equal to the integral over $\mathbb{R}^3$ of the coefficient of $\theta^1\theta^2$ in the development of $\phi^* f$, which is actually

$$\phi^* f = f \circ \varphi + \theta^1(f' \circ \varphi)\psi_1 + \theta^2(f' \circ \varphi)\psi_2 + \theta^1\theta^2[(f' \circ \varphi)F - (f'' \circ \varphi)\psi_1\psi_2], \qquad (4)$$

so that $I = \int_{\mathbb{R}^3} d^3x[(f' \circ \varphi)F - (f'' \circ \varphi)\psi_1\psi_2]$. The development (4) is well-known and can be obtained by several approaches. For instance in [7] or in [10] one computes the coefficient of $\theta^1\theta^2$ in the development of $\phi^* f$ by the rule $\iota^*\left(-\frac{1}{2}(D_1D_2 - D_2D_1)\phi^* f\right)$, where D_1 and D_2 are derivatives with respect to θ^1 and θ^2 respectively and ι is the canonical embedding $\mathbb{R}^3 \hookrightarrow \mathbb{R}^{3|2}$. Here we propose a recipe which, I find, is simple, intuitive, but mathematically safe for performing this computation (this recipe is of course equivalent to the already existing rules !). It consists roughly in the following: we reinterpret the relation $\phi = \varphi + \theta^1\psi_1 + \theta^2\psi_2 + \theta^1\theta^2 F$ as

$$\phi^* = \varphi^* e^{\theta^1\psi_1 + \theta^2\psi_2 + \theta^1\theta^2 F} = \varphi^*(1 + \theta^1\psi_1)(1 + \theta^2\psi_2)(1 + \theta^1\theta^2 F), \qquad (5)$$

where

- ψ_1, ψ_2 and F are first order differential operators which acts on the right, i.e. for instance $\forall f \in \mathcal{C}^\infty(\mathbb{R})$, $\psi_a f = df(\psi_a) = f'\psi_a$ and so $\varphi^*\psi_a f = (f' \circ \varphi)\psi_a$

- ψ_1, ψ_2 and F are $\mathbb{Z}_2$-graded in such a way that ϕ^* is even, i.e. since θ^1 and θ^2 are odd, ψ_1 and ψ_2 are odd and F is even

- all the symbols θ^1, θ^2, ψ_1, ψ_2 and F supercommute.

Let us use the supercommutation rules to developp (5), we obtain: $\forall f \in \mathcal{C}^\infty(\mathbb{R})$,

$$\phi^* f = \varphi^* f + \theta^1\varphi^*\psi_1 f + \theta^2\varphi^*\psi_2 f + \theta^1\theta^2\varphi^* F f - \theta^1\theta^2\varphi^*\psi_1\psi_2 f.$$

Then we let the first order differential operators act and this gives us exactly (4).

All these rules are expounded in details in the third section of this paper. Their justification is precisely based on the results of the first section.

1 Even maps from $\mathbb{R}^{p|q}$ to a manifold $\mathcal{N}$

Our first task will be to study even morphisms ϕ^* from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(|\Omega|)[\eta^1,\cdots,\eta^q]$, i.e. maps between these two superalgebras which satisfy (1) and (2). Let us first precise the sense of *even*. If $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ are two $\mathbb{Z}_2$-graded rings with unity, a ring morphism $\phi : B \longrightarrow A$ is sayed to be *even* is it respects the grading, i.e. $\forall b \in B_\alpha$, $\phi(b) \in A_\alpha$ for $\alpha = 0, 1$. In the case at hand $B = \mathcal{C}^\infty(\mathcal{N})$ is purely even, i.e. $B_1 = \{0\}$, and so ϕ^* is even if and only if it maps $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(|\Omega|)[\eta^1,\cdots,\eta^q]_0$, the even part of $\mathcal{C}^\infty(|\Omega|)[\eta^1,\cdots,\eta^q]$. We then say that ϕ is an *even* map from Ω to $\mathcal{N}$. In the following we shall denote by $\mathcal{C}^\infty(|\Omega|)[\eta^1,\cdots,\eta^q]$ and $\mathcal{C}^\infty(|\Omega|)[\eta^1,\cdots,\eta^q]_0$ respectively by $\mathcal{C}^\infty(\Omega)$ and $\mathcal{C}^\infty(\Omega)_0$ and we shall denote by $\mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}), \mathcal{C}^\infty(\Omega)_0)$ the set of even morphisms from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(\Omega)$.

We observe that because of the hypothesis (1) any such morphism is given by a finite family $(a_{i_1\cdots i_{2k}})$ of linear functionals on $\mathcal{C}^\infty(\mathcal{N})$ with values in $\mathcal{C}^\infty(|\Omega|)$, where $(i_1,\cdots i_{2k}) \in [\![1,q]\!]^{2k}$ and $0 \le k \le [q/2]$ ($[q/2]$ is the integer part of $q/2$), by the relation

$$\phi^* f = \sum_{k=0}^{[q/2]} \sum_{1\le i_1<\cdots<i_k\le q} a_{i_1\cdots i_{2k}}(f)\eta^{i_1}\cdots\eta^{i_{2k}} = a_\emptyset(f) + \sum_{1\le i_1<i_2\le q} a_{i_1 i_2}(f)\eta^{i_1}\eta^{i_2} + \cdots$$

Here we will assume that the $a_{i_1\cdots i_{2k}}$'s are skew symmetric in $(i_1,\cdots i_{2k})$. At this point it is useful to introduce the following notations: For any positive integer k we let $\mathbb{I}^q(k) := \{(i_1,\cdots i_k) \in [\![1,q]\!]^k | i_1 < \cdots < i_k\}$, we denote by $I = (i_1,\cdots i_k)$ an element of $\mathbb{I}^q(k)$ and we then write $\eta^I := \eta^{i_1}\cdots\eta^{i_k}$. It will be also useful to use the convention $\mathbb{I}^q(0) = \{\emptyset\}$. We let $\mathbb{I}^q := \cup_{k=0}^q \mathbb{I}^q(k)$, $\mathbb{I}^q_0 := \cup_{k=0}^{[q/2]} \mathbb{I}^q(2k)$, $\mathbb{I}^q_1 := \cup_{k=0}^{[(q-1)/2]} \mathbb{I}^q(2k+1)$ and $\mathbb{I}^q_2 := \cup_{k=1}^{[q/2]} \mathbb{I}^q(2k)$. Hence the preceding relation can be written

$$\phi^* f = \sum_{k=0}^{[q/2]} \sum_{I\in\mathbb{I}^q(2k)} a_I(f)\eta^I = \sum_{I\in\mathbb{I}^q_0} a_I(f)\eta^I \tag{6}$$

or

$$\forall x \in |\Omega|, \quad (\phi^* f)(x) = \sum_{I \in \mathbb{I}_0^q} a_I(f)(x)\eta^I.$$

Construction of morphisms

We start by providing a construction of morphisms satisfying (1) and (2). We note $\pi : |\Omega| \times \mathcal{N} \longrightarrow \mathcal{N}$ the canonical projection map and consider the vector bundle $\pi^* T\mathcal{N}$: the fiber over each point $(x, q) \in |\Omega| \times \mathcal{N}$ is the tangent space $T_q\mathcal{N}$. For any $I \in \mathbb{I}_2^q$, we choose a smooth section ξ_I of $\pi^* T\mathcal{N}$ over $|\Omega| \times \mathcal{N}$ and we consider the $\mathbb{R}[\eta^1, \cdots, \eta^q]_0$-valued vector field

$$\Xi := \sum_{I \in \mathbb{I}_2^q} \xi_I \eta^I.$$

Alternatively Ξ can be seen as a smooth family $(\Xi_x)_{x \in |\Omega|}$ of smooth tangent vector fields on $\mathcal{N}$ with coefficients in $\mathbb{R}[\eta^1, \cdots, \eta^q]_0$. So each Ξ_x defines a first order differential operator which acts on the algebra $\mathcal{C}^\infty(\mathcal{N}) \otimes \mathbb{R}[\eta^1, \cdots, \eta^q]_0$, i.e. the set of smooth functions on $\mathcal{N}$ with values in $\mathbb{R}[\eta^1, \cdots, \eta^q]_0$, by the relation

$$\forall f \in \mathcal{C}^\infty(\mathcal{N}) \otimes \mathbb{R}[\eta^1, \cdots, \eta^q]_0, \quad \Xi_x f = \sum_{I \in \mathbb{I}_2^q} ((\xi_I)_x \cdot f)\eta^I.$$

Here we do not need to worry about the position of η^I since it is an even monomial. We now define (letting $\Xi^0 = 1$)

$$e^\Xi := \sum_{n=0}^{\infty} \frac{\Xi^n}{n!} = \sum_{n=0}^{[q/2]} \frac{\Xi^n}{n!},$$

which can be considered again as a smooth family parametrized by $x \in |\Omega|$ of differential operators of order at most $[q/2]$ acting on $\mathcal{C}^\infty(\mathcal{N}) \otimes \mathbb{R}[\eta^1, \cdots, \eta^q]_0$. Now we choose a smooth map $\varphi : |\Omega| \longrightarrow \mathcal{N}$ and we consider the map

$$\begin{array}{rccc} 1 \times \varphi : & |\Omega| & \longrightarrow & |\Omega| \times \mathcal{N} \\ & x & \longmapsto & (x, \varphi(x)) \end{array}$$

which parametrizes the graph of φ. Lastly we construct the following linear operator on $\mathcal{C}^\infty(\mathcal{N}) \subset \mathcal{C}^\infty(\mathcal{N}) \otimes \mathbb{R}[\eta^1, \cdots, \eta^q]_0$:

$$\mathcal{C}^\infty(\mathcal{N}) \ni f \longmapsto (1 \times \varphi)^* \left(e^\Xi f\right) \in \mathcal{C}^\infty(\Omega),$$

where

$$\forall x \in |\Omega|, \quad (1 \times \varphi)^* \left(e^\Xi f\right)(x) := \left(e^{\Xi_x} f\right)(\varphi(x)) = \sum_{n=0}^{[q/2]} \left(\frac{(\Xi)_x^n}{n!} f\right)(\varphi(x)).$$

We observe that actually, for any $x \in |\Omega|$, we only need to define Ξ_x on a neighbourhood of $\varphi(x)$ in $\mathcal{N}$, i.e. it suffices to define the section Ξ on a neighbourhood of the graph of φ in $|\Omega| \times \mathcal{N}$ (or even on their Taylor expansion in q at order $[q/2]$ around $\varphi(x)$).

Lemma 1.1 *The map* $f \longmapsto (1\times\varphi)^*\left(e^{\Xi}f\right)$ *is a morphism from* $\mathcal{C}^\infty(\mathcal{N})$ *to* $\mathcal{C}^\infty(\Omega)_0$*, i.e. satisfies assumptions (1) and (2).*

Proof — Property (1) is obvious, so we just need to prove (2). We first remark that, for any $x \in |\Omega|$, Ξ_x satisfies the Leibniz rule:

$$\forall f,g \in \mathcal{C}^\infty(\mathcal{N})\otimes\mathbb{R}[\eta^1,\cdots,\eta^q]_0, \quad \Xi_x(fg) = (\Xi_x f)g + f(\Xi_x g),$$

which immediately implies by recursion that

$$\forall f,g \in \mathcal{C}^\infty(\mathcal{N})\otimes\mathbb{R}[\eta^1,\cdots,\eta^q]_0, \forall n\in\mathbb{N}, \quad \Xi_x^n(fg) = \sum_{j=1}^{n}\frac{n!}{(n-j)!j!}(\Xi_x^{n-j}f)(\Xi_x^j g). \qquad (7)$$

We deduce easily that

$$\forall x \in |\Omega|, \forall f,g \in \mathcal{C}^\infty(\mathcal{N})\otimes\mathbb{R}[\eta^1,\cdots,\eta^q]_0, \quad e^{\Xi_x}(fg) = \left(e^{\Xi_x}f\right)\left(e^{\Xi_x}g\right), \qquad (8)$$

by developping both sides and using (7). Now relation (8) is true in particular for functions $f,g \in \mathcal{C}^\infty(\mathcal{N})$ and if we evaluate this identity at the point $\varphi(x)\in\mathcal{N}$ we immediately conclude that $f \longmapsto (1\times\varphi)^*\left(e^{\Xi}f\right)$ satisfies (2). ■

The following result says that actually all morphisms are of the previous type.

Theorem 1.1 *Let* $\phi^* : \mathcal{C}^\infty(\mathcal{N}) \longrightarrow \mathcal{C}^\infty(\Omega)_0$ *be a morphism. Then there exists a smooth map* $\varphi : |\Omega| \longrightarrow \mathcal{N}$ *and a smooth family* $(\xi_I)_{I\in\mathbb{I}_2^q}$ *of sections of* $\pi^*T\mathcal{N}$ *defined on a neighbourhood of the graph of* φ *in* $|\Omega|\times\mathcal{N}$*, such that if* $\Xi := \sum_{I\in\mathbb{I}_2^q}\xi_I\eta^I$*, then*

$$\forall f \in \mathcal{C}^\infty(\mathcal{N}), \quad \phi^*f = (1\times\varphi)^*\left(e^{\Xi}f\right). \qquad (9)$$

Proof — Let $\phi^* : \mathcal{C}^\infty(\mathcal{N}) \longrightarrow \mathcal{C}^\infty(\Omega)_0$ which satisfies (1) and (2). We denote by a_I the functionals involved in the identity (6). We also introduce the following notation: for any $N\in\mathbb{N}$, $\mathcal{O}(\eta^{(N)})$ will represent a quantity of the form

$$\mathcal{O}(\eta^{(N)}) = \sum_{n=N}^{\infty}\sum_{I\in\mathbb{I}^q(n)} c_I\eta^I,$$

where the coefficients c_I's may be real constants or functions. The result will follow by proving by recursion on $n\in\mathbb{N}^*$ the following property:

- (P_n)*: There exists a smooth map* $\varphi : |\Omega| \longrightarrow \mathcal{N}$ *and there exists a family of vector fields* $(\xi_I)_I$*, where* $I\in\mathbb{I}^q(2k)$ *and* $1\le k\le n$*, defined on a neighbourhood of the graph of* φ *in* $|\Omega|\times\mathcal{N}$*, such that if*

  $$\Xi_n := \sum_{k=1}^{n}\sum_{I\in\mathbb{I}^q(2k)}\xi_I\eta^I,$$

  then

  $$\forall f \in \mathcal{C}^\infty(\mathcal{N}), \quad \phi^*f = (1\times\varphi)^*\left(e^{\Xi_n}f\right) + \mathcal{O}\left(\eta^{(2n+1)}\right).$$

Proof of (P_1) — For start from relation (2) and we expand both sides by using (6): we first obtain by identifying the terms of degree 0 in the η^i's:

$$\forall x \in |\Omega|, \forall f,g \in \mathcal{C}^\infty(\mathcal{N}), \quad a_\emptyset(fg)(x) = (a_\emptyset(f)(x))\,(a_\emptyset(g)(x))\,,$$

which implies that, for any $x \in |\Omega|$, there exists some value $\varphi(x) \in \mathcal{N}$ such that

$$\forall x \in |\Omega|, \forall f \in \mathcal{C}^\infty(\mathcal{N}), \quad a_\emptyset(f)(x) = f(\varphi(x)).$$

In other words there exists a function $\varphi : |\Omega| \longrightarrow \mathcal{N}$ such that $a_\emptyset(f) = f \circ \varphi$. Since $a_\emptyset(f)$ must be $\mathcal{C}^\infty$ for any smooth f, this implies that $\varphi \in \mathcal{C}^\infty(|\Omega|, \mathcal{N})$. The relations between the terms of degree 2 in (2) are: $\forall x \in |\Omega|$, $\forall f,g \in \mathcal{C}^\infty(\mathcal{N})$,

$$\begin{aligned} \forall I \in \mathbb{I}^q(2), \quad a_I(fg)(x) &= (a_I(f)(x))\,(a_\emptyset(g)(x)) + (a_\emptyset(f)(x))\,(a_I(g)(x)) \\ &= (a_I(f)(x))\,g(\varphi(x)) + f(\varphi(x))\,(a_I(g)(x))\,, \end{aligned}$$

which implies that for any $x \in |\Omega|$, each $a_I(\cdot)(x)$ is a derivation acting on $\mathcal{C}^\infty(\mathcal{N})$, with support $\{\varphi(x)\}$, i.e. $\forall I \in \mathbb{I}^q(2)$ there exist tangent vectors $(\xi_I)_x \in T_{\varphi(x)}\mathcal{N}$ such that

$$\forall f \in \mathcal{C}^\infty(\mathcal{N}), \quad a_I(f)(x) = ((\xi_I)_x \cdot f)\,(\varphi(x)).$$

And since $a_I(f)$ must be smooth for any $f \in \mathcal{C}^\infty(\mathcal{N})$, the vectors $(\xi_I)_x$ should depend smoothly on x, i.e. $x \longmapsto (\xi_I)_x$ is a smooth section of $\varphi^*T\mathcal{N}$. It is then possible (see the Proposition 1.1 below) to extend it to a smooth section of $\pi^*T\mathcal{N}$ on a neighbourhood of the graph of φ. If we now set $(\Xi_1)_x := \sum_{I\in\mathbb{I}^q(2)} (\xi_I)_x\, \eta^I$ we have on the one hand, $\forall x \in |\Omega|$,

$$\forall f \in \mathcal{N}, \quad e^{(\Xi_1)_x} f = f + \sum_{I\in\mathbb{I}^q(2)} ((\xi_I)_x \cdot f)\,\eta^I + \mathcal{O}(\eta^{(3)})$$

and on the other hand $\forall x \in |\Omega|$,

$$(\phi^* f)(x) = f(\varphi(x)) + \sum_{I\in\mathbb{I}^q(2)} ((\xi_I)_x \cdot f)\,(\varphi(x))\eta^I + \mathcal{O}(\eta^{(3)}),$$

from which (P_1) follows.
Proof of $(P_n) \Longrightarrow (P_{n+1})$ — We assume (P_n) so that a map $\varphi \in \mathcal{C}^\infty(|\Omega|, \mathcal{N})$ and a vector field Ξ_n have been constructed. Let us denote by b_I the linear forms on $\mathcal{C}^\infty(\mathcal{N})$ such that

$$(1 \times \varphi)^* \left(e^{\Xi_n} f\right) = \sum_{k=0}^{[q/2]} \sum_{I\in\mathbb{I}^q(2k)} b_I(f)\eta^I. \tag{10}$$

Then property (P_n) is equivalent to

$$\forall k \in [\![0,n]\!], \forall I \in \mathbb{I}^q(2k), \quad a_I = b_I. \tag{11}$$

We use Lemma 1.1: it says us that $f \longmapsto (1\times\varphi)^*\left(e^{\Xi_n}f\right)$ is a morphism, hence $(1\times\varphi)^*\left(e^{\Xi_n}(fg)\right) = \left[(1\times\varphi)^*\left(e^{\Xi_n}f\right)\right]\left[(1\times\varphi)^*\left(e^{\Xi_n}g\right)\right]$, so by using (10):

$$\sum_{k=0}^{n+1}\sum_{I\in\mathbb{I}^q(2k)} b_I(fg)\eta^I = \sum_{k=0}^{n+1}\sum_{j=0}^{k}\sum_{J\in\mathbb{I}^q(2k-2j),K\in\mathbb{I}^q(2j)} b_J(f)b_K(g)\eta^J\eta^K + \mathcal{O}(\eta^{(2n+3)}). \tag{12}$$

But the morphism property (2) for ϕ^* implies also

$$\sum_{k=0}^{n+1}\sum_{I\in\mathbb{I}^q(2k)} a_I(fg)\eta^I = \sum_{k=0}^{n+1}\sum_{j=0}^{k}\sum_{J\in\mathbb{I}^q(2k-2j),K\in\mathbb{I}^q(2j)} a_J(f)a_K(g)\eta^J\eta^K + \mathcal{O}(\eta^{(2n+3)}). \tag{13}$$

We now substract (12) to (13) and use (11): it gives us

$$\sum_{I\in\mathbb{I}^q(2n+2)} \left(a_I(fg)-b_I(fg)\right)\eta^I = \sum_{I\in\mathbb{I}^q(2n+2)} \left[\left(a_I(f)-b_I(f)\right)a_\emptyset(g) + a_\emptyset(f)\left(a_I(g)-b_I(g)\right)\right]\eta^I.$$

Hence if we denote $\delta a_I := a_I - b_I$, we obtain that

$$\forall I\in\mathbb{I}^q(2n+2),\quad \delta a_I(fg) = \delta a_I(f)(g\circ\varphi) + (f\circ\varphi)\delta a_I(g).$$

By the same reasoning as in the proof of (P_1), we conclude that, $\forall I\in\mathbb{I}^q(2n+2)$, there exist smooth sections ξ_I of $\pi^*T\mathcal{N}$ defined on a neighbourhood of the graph of φ, such that

$$\forall x\in|\Omega|, \forall I\in\mathbb{I}^q(2n+2),\quad \delta a_I(f)(x) = \left((\xi_I)_x\cdot f\right)(\varphi(x)).$$

Now let us define

$$\Xi_{n+1} := \Xi_n + \sum_{I\in\mathbb{I}^q(2n+2)} \xi_I\eta^I.$$

Then it turns out that

$$\begin{aligned} e^{\Xi_{n+1}}f &= \sum_{k=0}^{n+1}\frac{\left(\Xi_n + \sum_{I\in\mathbb{I}^q(2k+2)}\xi_I\eta^I\right)^k}{k!}f + \mathcal{O}(\eta^{(2n+3)}) \\ &= \sum_{k=0}^{n+1}\frac{\Xi_n^k}{k!}f + \sum_{I\in\mathbb{I}^q(2n+2)}\xi_I\cdot f\eta^I + \mathcal{O}(\eta^{(2n+3)}) \\ &= e^{\Xi_n}f + \sum_{I\in\mathbb{I}^M(2n+2)}\xi_I\cdot f\eta^I + \mathcal{O}(\eta^{(2n+3)}), \end{aligned}$$

so that

$$(1\times\varphi)^*\left(e^{\Xi_{n+1}}f\right) = \phi^*f + \mathcal{O}(\eta^{(2n+3)}).$$

Hence we deduce (P_{n+1}). ■

Proposition 1.1 *In the preceding result, it is possible to construct smoothly the vector fields ξ_I's in such a way that, $\forall x \in |\Omega|$,*

$$\forall I, J \in \mathbb{I}_2^q, \quad [(\xi_I)_x, (\xi_J)_x] = 0.$$

Proof — Recall that in the previous proof, in order to build Ξ_{n+1} out of Ξ_n, we introduced, for each $I \in \mathbb{I}^q(2n+2)$, an unique smooth section $x \longmapsto (\xi_I)_x$ of $\varphi^* T\mathcal{N}$. We will explain here how to extend each such vector fields defined along the graph of φ to a neighbourhood of the graph of φ in $|\Omega| \times \mathcal{N}$ in order to achieve the claim in the proposition. For that purpose we prove that for some set

$$\mathcal{V} := \{(x, \xi, q) \in \varphi^* T\mathcal{N} \times \mathcal{N} |\ x \in |\Omega|, \xi \in T_{\varphi(x)}\mathcal{N}, q \in V_{\varphi(x)}\},$$

where each $V_{\varphi(x)}$ is a neighbourhood of $\varphi(x)$ in $\mathcal{N}$, there exists a smooth map

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & T\mathcal{N} \\ (x, \xi, q) & \longmapsto & (q, \mathbb{V}(x, \xi, q)) \end{array}$$

such that $\forall (x, \xi) \in \varphi^* T\mathcal{N}$, $\mathbb{V}(x, \xi, \varphi(x)) = \xi$ and $\forall x \in |\Omega|$ fixed, $\forall \xi, \zeta \in T_{\varphi(x)}\mathcal{N}$, $[\mathbb{V}(x, \xi, \cdot), \mathbb{V}(x, \zeta, \cdot)] = 0$, i.e. the vector fields $q \longmapsto \mathbb{V}(x, \xi, q)$ and $q \longmapsto \mathbb{V}(x, \zeta, q)$ commute on $V_{\varphi(x)}$. Then the proposition will follows by extending each vector $(\xi_I)_x \in T_{\varphi(x)}\mathcal{N}$ on $V_{\varphi(x)}$ by $q \longmapsto \mathbb{V}(x, (\xi_I)_x, q)$.

The construction is the following. Let $(U_a)_{a \in A}$ be a covering of $\mathcal{N}$ by open subsets, let $(\chi_a)_{a \in A}$ be a partition of unity and $(y_a)_{a \in A}$ be a family of charts associated with this covering. For any $x \in |\Omega|$, let $A_x := \{a \in A |\ \varphi(x) \in U_a\}$. For any $a \in A_x$ and for any linear isomorphism $\ell : T_{\varphi(x)}\mathcal{N} \longrightarrow \mathbb{R}^n$, where $n = \dim \mathcal{N}$, let $R_{x,\ell,a}$ be the unique linear automorphism of $\mathbb{R}^n$ such that

$$R_{x,\ell,a} \circ dy_{a|\varphi(x)} = \ell.$$

We then set

$$\forall q \in \mathcal{N}, \quad y_{x,\ell}(q) := \sum_{a \in A_x} \chi_a(q) R_{x,\ell,a} \circ y_a(q).$$

We observe that $dy_{x,\ell|\varphi(x)} = \ell$ and hence, by the inverse mapping theorem, there exists an open neighbourhood $V_{\varphi(x)}$ of $\varphi(x)$ in $\mathcal{N}$ such that the restriction of $y_{x,\ell}$ to $V_{\varphi(x)}$ is a diffeomorphism. We then define

$$\forall q \in V_{\varphi(x)}, \quad \mathbb{V}(x, \xi, q) := \left(dy_{x,\ell|q}\right)^{-1} (\ell(\xi)).$$

Because of the obvious relation $y_{x,u\circ\ell} = u \circ y_{x,\ell}$ for all linear automorphism u of $\mathbb{R}^n$, it is clear that the definition of $\mathbb{V}(x, \xi, q)$ does not depend on ℓ (for the same reason $V_{\varphi(x)}$ is also independant of ℓ). Moreover $q \longmapsto \mathbb{V}(x, \xi, q)$ is simply a vector field which is a linear combination with constant coefficients of the vector fields $\left(\frac{\partial}{\partial y^i_{x,\ell}}\right)_{i=1,\cdots,n}$ so that the property $[\mathbb{V}(x, \xi, \cdot), \mathbb{V}(x, \zeta, \cdot)] = 0$ follows. Note also that these vector fields are of course not canonical since they obviously depend on the charts. ■

Remark 1.1 *If we assume furthermore that the image of* φ *is contained in an open subset* U *of* $\mathcal{N}$ *such that there exists a local chart* $y = (y^1, \cdots, y^n) : U \longrightarrow \mathbb{R}^n$, *then it is possible to choose all the vector fields* ξ_I *such that*

$$\forall x \in |\Omega|, \forall I, J \in \mathbb{I}_2^q, \quad (\xi_I)_x \cdot (\xi_J)_x \cdot y = 0. \tag{14}$$

Indeed in this case the proof of Proposition 1.1 is much simpler, since we do not need to use a partition of unity in order to build $\mathbb{V}$. *We just set* $\mathcal{V} := \{(x, \xi, q) \in \varphi^*T\mathcal{N} \times \mathcal{N} | \ x \in |\Omega|, \xi \in T_{\varphi(x)}\mathcal{N}, q \in U\}$ *and define* $\mathbb{V}$ *by* $\mathbb{V}(x, \xi, q) := (dy_{|q})^{-1} \circ dy_{|\varphi(x)}(\xi)$. *Then for each* $(x, \xi) \in \varphi^*T\mathcal{N}$ *fixed, the vector field* $q \longmapsto \mathbb{V}(x, \xi, q)$ *has constant coordinates in the variables* y^α. *Hence (14) follows.*

Remark 1.2 *We can write an alternative formula for* e^Ξ *by developping this exponential: in each term of the form* $\left(\sum_I \xi_I \eta^I\right)^n$ *we can see that each monomial which appears contains at most one time any operator* ξ_I, *so we obtain*

$$e^\Xi = \sum_{I \in \mathbb{I}_0^q} \eta^I \left(\sum_{n \geq 0} \frac{1}{n!} \sum_{I_1, \cdots, I_n \in \mathbb{I}_0^q} \epsilon_I^{I_1 \cdots I_n} \xi_{I_1} \cdots \xi_{I_n} \right), \tag{15}$$

with the convention that the $\mathbb{I}_0^q(0) = \emptyset$ *contribution is the identity. Here we have introduced the notation* $\epsilon_I^{I_1 \cdots I_n}$: *first all the* $\epsilon_\emptyset^{I_1 \cdots I_n}$*'s vanish except for* $\epsilon_\emptyset^\emptyset = 1$, *so that* $e^\Xi = 1 \text{ mod}[\eta^1, \cdots, \eta^q]$. *Second, for* $k \geq 1$, *if* $I_1 = (i_{1,1}, \cdots, i_{1,2k_1})$, $\cdots$, $I_n = (i_{n,1}, \cdots, i_{n,2k_n})$ *and* $I = (i_1, \cdots, i_{2k})$, *we write that* $I_1 \sqcup \cdots \sqcup I_n = I$ *if and only if* $k_1 + \cdots + k_n = k$, $\{i_{1,1}, \cdots, i_{1,2k_1}, \cdots, i_{n,1}, \cdots, i_{n,2k_n}\} = \{i_1, \cdots, i_{2k}\}$ *and* $\forall j$, $I_j \neq \emptyset$ *(i.e.* $k_j > 0$*). Then*

- *if* $I_1 \sqcup \cdots \sqcup I_n \neq I$, $\epsilon_I^{I_1 \cdots I_n} = 0$
- *if* $I_1 \sqcup \cdots \sqcup I_n = I$, $\epsilon_I^{I_1 \cdots I_n}$ *is the signature of the permutation* $(i_{1,1}, \cdots, i_{1,2k_1}, \cdots, i_{n,1}, \cdots, i_{n,k_{2n}}) \longmapsto (i_1, \cdots, i_{2k})$.

The preceding expression of e^Ξ *can be recovered by another way: since all the operators* $\eta^I \xi_I$ *commute, we have*

$$e^\Xi = e^{\sum_{I \in \mathbb{I}_2^q} \eta^I \xi_I} = \prod_{I \in \mathbb{I}_2^q} e^{\eta^I \xi_I} = \prod_{I \in \mathbb{I}_2^q} (1 + \eta^I \xi_I),$$

which gives also the same result by a straightforward development.

2 A factorization of the morphism ϕ^*

2.1 Integrating the vector fields ξ_I's

In the same spirit as a tangent vector at a point q to a manifold $\mathcal{N}$ can be seen as the time derivative of a smooth curve which reaches q we can describe the η^I-components of the morphism ϕ^* as higher order approximations of a smooth map from some vector space with values in $\mathcal{N}$. Indeed let $\phi^* \in \mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}), \mathcal{C}^\infty(\Omega)_0)$: then by the preceding result ϕ^* is characterized by a map $\varphi \in \mathcal{C}^\infty(|\Omega|, \mathcal{N})$ and $2^{q-1}-1$ vector fields[1] ξ_I tangent to $\mathcal{N}$ defined on a neighbourhood of the graph of φ in $|\Omega| \times \mathcal{N}$. By proposition 1.1 these vector fields can moreover be chosen so that they pairwise commute when $x \in |\Omega|$ is fixed. So, for any $x \in |\Omega|$ we can integrate simultaneously all vector fields $(\xi_I)_x$ in order to construct a map

$$\Phi(x, \cdot) : U_x(\Lambda_+^{2*}\mathbb{R}^q) \longrightarrow \mathcal{N},$$

where $\Lambda_+^{2*}\mathbb{R}^q \simeq \mathbb{R}^{2^{q-1}-1}$ is the subspace of even elements of positive degree of the exterior algebra $\Lambda^*\mathbb{R}^q$ and $U_x(\Lambda_+^{2*}\mathbb{R}^q)$ is a neighbourhood of 0 in $\Lambda_+^{2*}\mathbb{R}^q$, such that

$$\Phi(x, 0) = \varphi(x) \tag{16}$$

and, denoting by $\left(\mathfrak{s}^I\right)_{I \in \mathbb{I}_2^q}$ the linear coordinates on $\Lambda_+^{2*}\mathbb{R}^q$,

$$\frac{\partial \Phi}{\partial \mathfrak{s}^I}(x, \mathfrak{s}) = \xi_I(\Phi(x, \mathfrak{s})) \quad \forall \mathfrak{s} \in U_x(\Lambda_+^{2*}\mathbb{R}^q), \forall I \in \mathbb{I}_2^q. \tag{17}$$

We hence obtain a map Φ from a neighbourhood of $|\Omega| \times \{0\}$ in $|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q$ to $\mathcal{N}$. By using a cut-off function argument we can extend this map to an application $\Phi : |\Omega| \times \Lambda_+^{2*}\mathbb{R}^q \longmapsto \mathcal{N}$. Lastly we introduce the $\mathbb{R}[\eta^1, \cdots, \eta^q]$-valued vector field on $|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q$

$$\vartheta := \sum_{I \in \mathbb{I}_2^q} \eta^I \frac{\partial}{\partial \mathfrak{s}^i},$$

so that by (17) $\Phi_* \vartheta = \Xi = \sum_{I \in \mathbb{I}_2^q} \eta^I \xi_I$. Then relation (9) implies

$$\forall f \in \mathcal{C}^\infty(\mathcal{N}), \forall x \in |\Omega|, \quad \phi^* f(x) = \left(e^\vartheta (f \circ \Phi)\right)(x, 0)$$

or by letting $\iota : |\Omega| \longrightarrow |\Omega| \times \Lambda_+^{2*}\mathbb{R}^q$, $x \longmapsto (x, 0)$ to be the canonical injection,

$$\phi^* f = \iota^* \left(e^\vartheta (f \circ \Phi)\right). \tag{18}$$

Alternatively by using (15) we have

$$\forall x \in |\Omega|, \quad \phi^* f(x) = \sum_{I \in \mathbb{I}_0^q} \eta^I \left(\sum_{k \geq 0} \frac{1}{k!} \sum_{I_1, \cdots, I_k \in \mathbb{I}_0^q} \epsilon_I^{I_1 \cdots I_k} \frac{\partial^k (f \circ \Phi)}{\partial \mathfrak{s}^{I_1} \cdots \partial \mathfrak{s}^{I_k}}(x, 0) \right). \tag{19}$$

It is useful to introduce the differential operators $\mathcal{D}_\emptyset := 1$ and

$$\mathcal{D}_I := \sum_{k \geq 0} \frac{1}{k!} \sum_{I_1, \cdots, I_k \in \mathbb{I}_0^q} \epsilon_I^{I_1 \cdots I_k} \frac{\partial^k}{\partial \mathfrak{s}^{I_1} \cdots \partial \mathfrak{s}^{I_k}},$$

[1] note that $\mathrm{card}\mathbb{I}^q(2k) = \frac{q!}{(q-2k)!(2k)!}$ and $\sum_{k=0}^{[q/2]} \frac{q!}{(q-2k)!(2k)!} = 2^{q-1}$

so that $\phi^* f(x) = \sum_{I\in\mathbb{I}_0^q} \eta^I \mathcal{D}_I(f\circ\Phi)(x,0)$. Conversely to any map smooth map $\Phi \in \mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q,\mathcal{N})$ we can associate a unique morphism $\phi^* \in \mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}),\mathbb{R}[\eta^1,\cdots,\eta^q]_0)$ defined by (18) or (19). This defines an application

$$\begin{array}{ccc} \mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q,\mathcal{N}) & \longrightarrow & \mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}),\mathcal{C}^\infty(\Omega)_0) \\ \Phi & \longmapsto & \Phi_{|\circ}^*, \end{array}$$

where $\forall f\in\mathcal{C}^\infty(\mathcal{N})$, $\Phi_{|\circ}^* f = \iota^*\left(e^{\vartheta}(f\circ\Phi)\right)$. It is clear from the previous discussion that this application is onto. It is however certainly not injective, since $\Phi_{|\circ}^*$ depends only on the $[q/2]$-th order Taylor expansion of Φ at 0. This will be precised in the following.

2.2 Expressions using local coordinates on the target manifold

Assume that we have local coordinates on $\mathcal{N}$: we let U to be an open subset of $\mathcal{N}$ and we consider a chart $y=(y^1,\cdots,y^n): U\longrightarrow V\subset\mathbb{R}^n$. Then any function $f: U\longrightarrow\mathbb{R}$ can be represented by an unique function $F: V\longrightarrow\mathbb{R}$ such that $f = F\circ y$. For any $y_0\in V\subset\mathbb{R}^n$ let $P_{F,y_0}^{[q/2]}$ be the $[q/2]$-th order Taylor expansion of F at y_0 and $R_{F,y_0}^{[q/2]}$ be the rest, so that we have the decomposition $F(y) = P_{F,y_0}^{[q/2]}(y) + R_{F,y_0}^{[q/2]}(y)$. The expressions for $P_{F,y_0}^{[q/2]}$ and $R_{F,y_0}^{[q/2]}$ are:

$$\forall y\in\mathbb{R}^n,\quad P_{F,y_0}^{[q/2]}(y) = \sum_{r\in\mathbb{N}^n,|r|\le[q/2]} \frac{\partial^r F}{(\partial y)^r}(y_0)\frac{(y-y_0)^r}{r!}$$

and

$$\forall y\in V,\quad R_{F,y_0}^{[q/2]}(y) = \sum_{r\in\mathbb{N}^n,|r|=[q/2]+1} (y-y_0)^r R_{F,y_0,r}(y),$$

where, if $r=(r_1,\cdots,r_n)\in\mathbb{N}^n$, $|r| := r_1+\cdots+r_n$, $(y)^r := (y^1)^{r_1}\cdots(y^n)^{r_n}$ and $\frac{\partial^r F}{(\partial y)^r} := \frac{\partial^{|r|}F}{(\partial y^1)^{r_1}\cdots(\partial y^n)^{r_n}}$, assuming that V is star-shaped around y_0,

$$R_{F,y_0,r}(y) := \frac{[q/2]+1}{r!}\int_0^1 (1-t)^{[q/2]}\frac{\partial^r F}{(\partial y)^r}(y_0+t(y-y_0))dt.$$

Proposition 2.1 *Let $y:\mathcal{N}\supset U\longrightarrow V\in\mathbb{R}^n$ be a local chart and $\phi^*:\mathcal{C}^\infty(U)\longrightarrow\mathcal{C}^\infty(\Omega)_0$ be a morphism. For any $f\in\mathcal{C}^\infty(U)$ let $F\in\mathcal{C}^\infty(V)$ such that $f=F\circ y$. Then $\forall x_0\in|\Omega|$,*

$$(\phi^* f)(x_0) = \sum_{r\in\mathbb{N}^n,|r|\le[q/2]} \frac{\partial^r F}{(\partial y)^r}(y_0)\frac{(\phi^* y - y_0)^r}{r!}, \tag{20}$$

where y_0 is the unique point in $\mathbb{R}^n$ such that $y\circ\phi(x_0)-y_0$ has nilpotent components.

Proof — The morphism property implies that

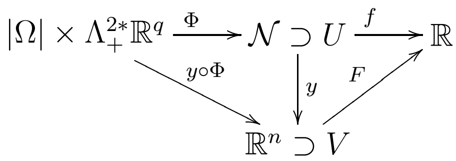
$$\phi^*(F\circ y)=\phi^*\left(P^{[q/2]}_{F,y_0}(y)\right)+\sum_{r\in\mathbb{N}^n,|r|=[q/2]+1}\phi^*\left((y-y_0)^r\right)\phi^*\left(R_{F,y_0,r}\circ y\right). \qquad (21)$$

But still by using the morphism property we have $\phi^*\left(P(y)\right)=P\left(\phi^*y\right)$ for any polynomial P in n real variables. Hence

$$\phi^*f=\phi^*(F\circ y)=P^{[q/2]}_{F,y_0}\left(\phi^*y\right)+\sum_{r\in\mathbb{N}^n,|r|=[q/2]+1}\left(\phi^*y-y_0\right)^r\phi^*\left(R_{F,y_0,r}\circ y\right).$$

In particular when we evaluate this last identity at the point x_0 we get (20) because $\left(\phi^*y-y_0\right)^r(x_0)=0$ for $|r|=[q/2]+1$. ■

Now let $\Phi:|\Omega|\times\Lambda^{2*}_+\mathbb{R}^q\longrightarrow U\subset\mathcal{N}$, then we have the diagram:



Corollary 2.1 *Let $y:U\longrightarrow\mathbb{R}^n$ be a local chart on $\mathcal{N}$ and let $\Phi,\widetilde{\Phi}\in\mathcal{C}^\infty(|\Omega|\times\Lambda^{2*}_+\mathbb{R}^q,\mathcal{N})$ such that $\iota^*\Phi=\iota^*\widetilde{\Phi}=:\varphi$. Then*

$$\Phi^*_{|\circ}=\widetilde{\Phi}^*_{|\circ} \qquad (22)$$

if and only if

$$\forall\alpha,\forall I\in\mathbb{I}^q_0,\forall x\in|\Omega|,\quad \mathcal{D}_I(y^\alpha\circ\Phi)(x,0)=\mathcal{D}_I(y^\alpha\circ\widetilde{\Phi})(x,0). \qquad (23)$$

Proof — Since $\Phi^*_{|\circ}f=\iota^*\sum_{I\in\mathbb{I}^q_0}\eta^I\mathcal{D}_I(f\circ\Phi)$ condition (23) just means that $\forall\alpha$, $\Phi^*_{|\circ}y^\alpha=\widetilde{\Phi}^*_{|\circ}y^\alpha$ and hence is a trivial consequence of (22). Conversely if (23) is true then we recover (22) by applying (20) for $\phi^*=\Phi^*_{|\circ}$ and $\phi^*=\widetilde{\Phi}^*_{|\circ}$ and with $y_0=\varphi(x_0)$. ■

It is natural to define the following equivalence relation in $\mathcal{C}^\infty(|\Omega|\times\Lambda^{2*}_+\mathbb{R}^q,\mathcal{N})$: for any $\Phi,\widetilde{\Phi}\in\mathcal{C}^\infty(|\Omega|\times\Lambda^{2*}_+\mathbb{R}^q,\mathcal{N})$

$$\Phi\sim\widetilde{\Phi}\quad\Longleftrightarrow\quad\Phi^*_{|\circ}=\widetilde{\Phi}^*_{|\circ}.$$

Then clearly morphisms in $\mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}),\mathcal{C}^\infty(\Omega)_0)$ are in one to one correspondence with equivalence classes in $\mathcal{C}^\infty(|\Omega|\times\Lambda^{2*}_+\mathbb{R}^q,\mathcal{N})/\sim$. This gives us a direct geometric picture (which we shall discuss below) of a map $\phi:\mathbb{R}^{p|q}\supset\Omega\longrightarrow\mathcal{N}$ (thought as dual to a morphism ϕ^* in $\mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}),\mathcal{C}^\infty(\Omega)_0)$): it can be identified with a class of maps in $\mathcal{C}^\infty(|\Omega|\times\Lambda^{2*}_+\mathbb{R}^q,\mathcal{N})/\sim$, i.e. a map into $\mathcal{N}$ surrounded by a family of infinitesimal deformations inside $\mathcal{N}$.

2.3 The chain rule for the operators $\mathcal{D}_I$

We exploit relation (20) again but we use a different expression for the Taylor polynomial

$$P^{[q/2]}_{F,y_0}(y) = \sum_{k=0}^{[q/2]} \frac{1}{k!} \sum_{\alpha_1,\cdots,\alpha_k=1}^{n} \frac{\partial^k F}{\partial y^{\alpha_1}\cdots\partial y^{\alpha_k}}(y_0)(y^{\alpha_1}-y_0^{\alpha_1})\cdots(y^{\alpha_k}-y_0^{\alpha_k}).$$

Hence by (20)

$$\Phi^*_{|\circ} f(x_0) = \sum_{k=0}^{[q/2]} \frac{1}{k!} \sum_{\alpha_1,\cdots,\alpha_k=1}^{n} \frac{\partial^k F}{\partial y^{\alpha_1}\cdots\partial y^{\alpha_k}}(y_0) \prod_{\ell=1}^{k} \left(\Phi^*_{|\circ} y^{\alpha_\ell} - y_0^{\alpha_\ell}\right). \tag{24}$$

But since

$$\Phi^*_{|\circ} y^\alpha(x_0) - y_0^\alpha = \sum_{I\in\mathbb{I}^q_2} \eta^I \mathcal{D}_I(y^\alpha\circ\Phi)(x_0,0),$$

we deduce by a substitution

$$\Phi^*_{|\circ} f(x_0) = F(y_0) + \sum_{k=1}^{[q/2]} \frac{1}{k!} \sum_{I;I_1,\cdots,I_k\in\mathbb{I}^q_0} \eta^I \epsilon_I^{I_1\cdots I_k} \sum_{\alpha_1,\cdots,\alpha_k=1}^{n} \frac{\partial^k F}{\partial y^{\alpha_1}\cdots\partial y^{\alpha_k}}(y_0) \prod_{\ell=1}^{k} \mathcal{D}_{I_\ell}(y^{\alpha_\ell}\circ\Phi)(x_0,0).$$

But on the other hand we have

$$\Phi^*_{|\circ} f(x_0) = f\circ\varphi(x_0) + \sum_{I\in\mathbb{I}^q_2} \eta^I \mathcal{D}_I(f\circ\Phi)(x_0,0) = F(y_0) + \sum_{I\in\mathbb{I}^q_2} \eta^I \mathcal{D}_I(F\circ y\circ\Phi)(x_0,0).$$

These two relations give us by an identification an expression for each $\mathcal{D}_I(F\circ y\circ\Phi)(x_0,0)$ in terms of $\mathcal{D}_I(y^\alpha\circ\Phi)(x_0,0)$. By setting $Y^\alpha := y^\alpha\circ\Phi$ it can be formulated as follows

Proposition 2.2 *For any map $Y\in\mathcal{C}^\infty(|\Omega|\times\Lambda^{2*}_+\mathbb{R}^q,\mathbb{R}^n)$, for any $x_0\in|\Omega|$, for any open neighbourhood V of $y_0 := Y(x_0,0)$ in $\mathbb{R}^n$ and for any map $F\in\mathcal{C}^\infty(V)$, we have $\forall I\in\mathbb{I}^q_2$,*

$$\mathcal{D}_I(F\circ Y)(x_0,0) = \sum_{k\geq 0} \frac{1}{k!} \sum_{I_1,\cdots,I_k\in\mathbb{I}^q_0} \epsilon_I^{I_1\cdots I_k} \sum_{\alpha_1,\cdots,\alpha_k=1}^{n} \frac{\partial^k F}{\partial y^{\alpha_1}\cdots\partial y^{\alpha_k}}(y_0) \prod_{\ell=1}^{k} \mathcal{D}_{I_\ell} Y^{\alpha_\ell}(x_0,0). \tag{25}$$

An application

We use a specialization of the identity (25) by choosing $\mathbb{R}^n = \Lambda^{2*}_+\mathbb{R}^q$, and by substituting to Y a smooth map $S:\Lambda^{2*}_+\mathbb{R}^q \longrightarrow \Lambda^{2*}_+\mathbb{R}^q$ such that $S(0)=0$. We hence get

$$\mathcal{D}_I(F\circ S)(0) = \sum_{p\geq 0} \frac{1}{p!} \sum_{I_1,\cdots,I_p\in\mathbb{I}^q_0} \epsilon_I^{I_1\cdots I_p} \sum_{J_1,\cdots,J_p\in\mathbb{I}^q_0} \frac{\partial^p F}{\partial \mathfrak{s}^{J_1}\cdots\partial\mathfrak{s}^{J_p}}(0) \left(\mathcal{D}_{I_1}S^{J_1}\cdots\mathcal{D}_{I_p}S^{J_p}\right)(0).$$

In the special case where $\mathcal{D}_I S^J(0) = \delta_I^J$ this simplifies to

$$\mathcal{D}_I(F\circ S)(0) = \sum_{p\geq 0}\frac{1}{p!}\sum_{I_1,\cdots,I_p\in\mathbb{I}_0^q}\epsilon_I^{I_1\cdots I_p}\frac{\partial^p F}{\partial\mathfrak{s}^{I_1}\cdots\partial\mathfrak{s}^{I_p}}(0) = \mathcal{D}_I F(0). \tag{26}$$

We conclude that if $S:\Lambda_+^{2*}\mathbb{R}^q \longrightarrow \Lambda_+^{2*}\mathbb{R}^q$ is a smooth diffeomorphism such that $S(0)=0$ and $\mathcal{D}_I S^J(0) = \delta_I^J$, then $V\sim V\circ S$. Hence if we define

$$\mathcal{T}_q := \{\text{diffeomorphisms } S:\Lambda_+^{2*}\mathbb{R}^q \longrightarrow \Lambda_+^{2*}\mathbb{R}^q | S(0)=0, \mathcal{D}_I S^J(0) = \delta_I^J\}$$

then we remark that $\mathcal{T}_q$ is a group for the composition law (another consequence of (26)) and we see that the morphism $\Phi_{|\circ}^*$ is characterized by the behaviour of Φ modulo the action of $\mathcal{T}_q$ hence by duality we can identify a map $T:\mathbb{R}^{0|q}\longrightarrow\mathcal{N}$ with a class of maps from $\Lambda_+^{2*}\mathbb{R}^q$ to $\mathcal{N}$ modulo the action of $\mathcal{T}_q$ on $\Lambda_+^{2*}\mathbb{R}^q$.

2.4 Leibniz identities for the operators $\mathcal{D}_I$

The operators $\mathcal{D}_I$ satisfy nice Leibniz type identities:

Proposition 2.3 *For any pair of functions $a,b\in\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)$ and for any $I\in\mathbb{I}_0^q$,*

$$\mathcal{D}_I(ab) = \sum_{I_1,I_2\in\mathbb{I}_0^q}\epsilon_I^{I_1I_2}\left(\mathcal{D}_{I_1}a\right)\left(\mathcal{D}_{I_2}b\right), \tag{27}$$

where in the summation we allow $(I_1,I_2)=(\emptyset,I)$ or $(I,\emptyset)$.

Proof — By applying relation (8) for $\vartheta := \sum_I \eta^I\frac{\partial}{\partial\mathfrak{s}^I}$ we obtain

$$\forall a,b\in\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q),\quad e^\vartheta(ab) = \left(e^\vartheta a\right)\left(e^\vartheta b\right). \tag{28}$$

And by using $e^\vartheta a = \sum_{I\in\mathbb{I}_0^q}\eta^I\left(\mathcal{D}_I a\right)$ to developp this relation we obtain (27). ■

A straightforward consequence of Proposition 2.3 is that the set

$$\mathcal{I}^q(|\Omega|) := \{f\in\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)|\forall I\in\mathbb{I}_0^q, \iota^*\left(\mathcal{D}_I f\right)=0\}$$

is an ideal of the commutative algebra $\left(\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q),+,\cdot\right)$. Hence the quotient $\mathcal{A}^q(|\Omega|) := \mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)/\mathcal{I}^q(|\Omega|)$ is an algebra over $\mathbb{R}$. We will recover that this algebra is isomorphic to $\mathcal{C}^\infty(|\Omega|)[\eta^1,\cdots,\eta^q]_0$. First we may also write $\mathcal{A}^q(|\Omega|)\simeq\mathcal{C}_{pol}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)/\mathcal{I}_{pol}^q(|\Omega|)$, where $\mathcal{C}_{pol}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)$ is the subalgebra of smooth functions on $|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q$ which have a polynomial dependence in the variables $\mathfrak{s}^I$ and $\mathcal{I}_{pol}^q(|\Omega|) = \mathcal{C}_{pol}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)\cap\mathcal{I}^q(|\Omega|)$. And any function $f\in\mathcal{C}_{pol}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)$ can be written

$$f(x,\mathfrak{s}) = \sum_{n=0}^\infty\frac{1}{n!}\sum_{I_1,\cdots,I_n\in\mathbb{I}_0^q}\frac{\partial^n f}{\partial\mathfrak{s}^{I_1}\cdots\partial\mathfrak{s}^{I_n}}(x,0)\mathfrak{s}^{I_1}\cdots\mathfrak{s}^{I_n}.$$

Now $f \in \mathcal{I}^q_{pol}(|\Omega|)$ if and only if, $\forall I \in \mathbb{I}^q_2$,

$$\forall x \in |\Omega|, \quad \frac{\partial f}{\partial \mathfrak{s}^I}(x,0) = -\sum_{n=2}^{\infty} \frac{1}{n!} \sum_{I_1,\cdots,I_n \in \mathbb{I}^q_0} \epsilon_I^{I_1\cdots I_n} \frac{\partial^n f}{\partial \mathfrak{s}^{I_1}\cdots \partial \mathfrak{s}^{I_n}}(x,0).$$

Hence for such a function

$$f(x,\mathfrak{s}) = \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{I_1,\cdots,I_n \in \mathbb{I}^q_0} \frac{\partial^n f}{\partial \mathfrak{s}^{I_1}\cdots \partial \mathfrak{s}^{I_n}}(x,0) \left[\mathfrak{s}^{I_1}\cdots \mathfrak{s}^{I_n} - \sum_{I \in \mathbb{I}^q_0} \epsilon_I^{I_1\cdots I_n} \mathfrak{s}^I \right].$$

So $\mathcal{I}^q_{pol}(|\Omega|)$ is the ideal spanned by the family

$$\left(\mathfrak{s}^{I_1}\cdots \mathfrak{s}^{I_n} - \sum_{I \in \mathbb{I}^q_0} \epsilon_I^{I_1\cdots I_n} \mathfrak{s}^I \right)_{n\geq 2,\ I_1,\cdots,I_n \in \mathbb{I}^q_0}.$$

Hence it is clear that the linear application from $\mathrm{Span}_{\mathcal{C}^\infty(|\Omega|)}(\mathfrak{s}^I)$ to $\mathrm{Span}_{\mathcal{C}^\infty(|\Omega|)}(\eta^I)$ which maps $\mathfrak{s}^I$ to η^I can be extended in an unique way into an algebra *isomorphism* from $\mathcal{A}^q(|\Omega|)$ to $\mathcal{C}^\infty(|\Omega|) \otimes \mathbb{R}[\eta^1,\cdots,\eta^q]_0$. Moreover this isomorphism is nothing but

$$\begin{array}{rccc} \iota^* \circ e^\vartheta : & \mathcal{A}^q(|\Omega|) & \longrightarrow & \mathcal{C}^\infty(|\Omega|) \otimes \mathbb{R}[\eta^1,\cdots,\eta^q]_0 \\ & f & \longmapsto & \iota^* \circ \left(e^\vartheta f\right) \end{array}$$

2.5 An alternative description using schemes

Let us start by assuming that $p = 0$ for simpliclity. The "geometry" of $\mathbb{R}^{0|q}$ appears to be related with another "geometric" object living in a neighbourhood of 0 in $\Lambda_+^{2*}\mathbb{R}^q$ and such that the ring of functions on it is isomorphic to the algebra $\mathcal{A}^q := \mathcal{A}^q(\{0\})$ that we just constructed. It turns out that this object can be described accurately by using Grothendieck's theory of schemes. We refer to [9] for a complete and comprehensive presentation of this theory and recall here only notions which may be relevant for us. To any commutative ring R we can associate an (affine) scheme which is called the *spectrum* of R and is denoted by $\mathrm{Spec}R$. It consists in three data: a set of points, a topology (the Zariski topology) and a sheaf of regular functions on it. The set of points is simply the set of prime ideals of R. In the case at hand where $R = \mathcal{A}^q$ the prime ideals are of the form[2]

$$\mathfrak{A} = \left(\sum_{I \in \mathbb{I}^q_2} \alpha_{1,I}\mathfrak{s}^I, \cdots, \sum_{I \in \mathbb{I}^q_2} \alpha_{p,I}\mathfrak{s}^I \right),$$

where $p \in \mathbb{N}$ and the $\alpha_{j,I}$ are real parameters so that, $\forall f,g \in R$, if $fg \in \mathfrak{A}$ then either $f \in \mathfrak{A}$ or $g \in \mathfrak{A}$. The "point" which corresponds to such an ideal is the "generic point" living in the vector subspace defined by $\sum_{I\in\mathbb{I}^q_2} \alpha_{1,I}\mathfrak{s}^I = \cdots = \sum_{I\in\mathbb{I}^q_2} \alpha_{p,I}\mathfrak{s}^I = 0$. Note that by dualizing the canonical ring morphism $\mathcal{C}^\infty_{pol}(\Lambda_+^{2*}\mathbb{R}^q) \longrightarrow \mathcal{A}^q$ we can view $\mathrm{Spec}\mathcal{A}^q$ as embedded in $\Lambda_+^{2*}\mathbb{R}^q$.

[2]here if $a_1,\cdots,a_p \in R$, we denote by $(a_1,\cdots,a_p)$ the ideal $\{a_1f_1+\cdots+a_pf_p | f_1,\cdots,f_p \in R\}$

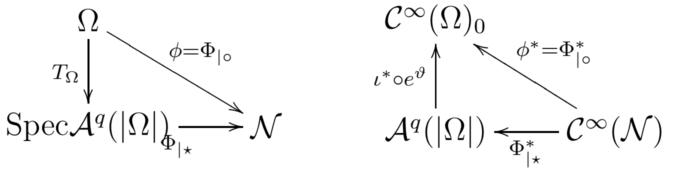
Example 2.1 *For all* $q \in \mathbb{N}$, *set* $\mathcal{A}^q_{(2)} := \{\sum_{1 \le i \le j < q} \alpha_{ij} \mathfrak{s}^{ij}\}$. *Then for any* $1 \le p \le \frac{q(q-1)}{2}$ *if* $f_1, \cdots, f_p$ *are* p *linearly independants vectors of* $\mathcal{A}^q_{(2)}$, *then* $(f_1, \cdots, f_p)$ *is a prime ideal of* $\mathcal{A}^q$ *(and for* $p = \frac{q(q-1)}{2}$ *it is the maximal ideal, see below). For* $q \le 4$ *there are no other prime ideals. However for* $q \ge 5$ *other instances of prime ideal exist like* $(\mathfrak{s}^{1234} + \mathfrak{s}^{15})$ *for* $q = 5$.

So in general the concept of a "point" of a scheme is different from the usual one, except if the point is a maximal ideal. For $R = \mathcal{A}^q$ there is only one maximal ideal[3] which is $(\mathfrak{s}^I)_{I \in \mathbb{I}^q(2)}$: it corresponds to the point $0 \in \Lambda^{2*}_+ \mathbb{R}^q$. This point is also the unique closed point for the Zariski topology, all the other ones are open[4].

For $p \ge 1$, similarly we can associate to any open subset Ω of $\mathbb{R}^{p|q}$ the scheme associated with $\mathcal{A}^q(|\Omega|)$, and we can picture its spectrum $\mathrm{Spec}\mathcal{A}^q(|\Omega|)$ as an object embedded in $|\Omega| \times \Lambda^{2*}_+ \mathbb{R}^q$. Then we can interpret our results as follows: first for any morphism $\phi^* : \mathcal{C}^\infty(\mathcal{N}) \longrightarrow \mathcal{C}^\infty(\Omega)_0$ we have found that there exists a family of maps $\Phi : |\Omega| \times \Lambda^{2*}_+ \mathbb{R}^q \longrightarrow \mathcal{N}$ (a class of maps modulo $\sim$) such that $\Phi^*_{|\circ} = \phi^*$. We can simply denote by $\Phi_{|\circ} = \phi$ this relation. Second through the algebra isomorphism $\iota^* \circ e^\vartheta : \mathcal{A}^q(|\Omega|) \longrightarrow \mathcal{C}^\infty(\Omega)_0$ constructed in the previous section, we can decompose $\Phi^*_{|\circ} = \iota^* \circ e^\vartheta \circ \Phi^*_{|\star}$, where

$$\left(\Phi^*_{|\star} f\right)(x, \mathfrak{s}) := \sum_{I \in \mathbb{I}^q_0} \mathfrak{s}^I \mathcal{D}_I (f \circ \Phi)(x, 0) = \left(e^{\sum_{I \in \mathbb{I}^q_2} \mathfrak{s}^I \frac{\partial}{\partial \mathfrak{s}^I}} f \circ \Phi\right)(x, 0) = (f \circ \Phi)(x, \mathfrak{s}) \mod \mathcal{I}^q(|\Omega|).$$

Hence $\Phi_{|\star}$ can be thought as a restriction of Φ to $\mathrm{Spec}\mathcal{A}^q(|\Omega|)$. Moreover if we denote by T_Ω the isomorphism from $\mathrm{Spec}\mathcal{C}^\infty(\Omega)_0$ to $\mathrm{Spec}\mathcal{A}^q(|\Omega|)$ which is dual of $\iota^* \circ e^\vartheta$ we can dualize the relation $\Phi^*_{|\circ} = \iota^* \circ e^\vartheta \circ \Phi^*_{|\star}$ as $\phi = \Phi_\circ = \Phi_{|\star} \circ T_\Omega$. All that can be summarized in the following diagrams:



[3]rings with an unique maximal ideal are *called local rings*

[4]Then R can be interpreted as the ring of functions on the points of $\mathrm{Spec}R$: to each prime ideal $\mathfrak{A}$ of R we associate the *residue field* $R/\mathfrak{A}$ and each $f \in R$ has an image $[f \mod \mathfrak{A}]$ in $R/\mathfrak{A}$ through the canonical projection, so each $f \in R$ is identified with the "map"

$$\begin{array}{rccc} f: & \mathrm{Spec}R & \longrightarrow & \text{residue fields} \\ & \mathfrak{A} & \longmapsto & [f \mod \mathfrak{A}]. \end{array}$$

Here we can interpret $[f \mod \mathfrak{A}]$ as being isomorphic to the set of functions on the zero set of all functions contained in $\mathfrak{A}$. A more refined description of functions on $\mathrm{Spec}R$ is given by the construction of a sheaf $\mathcal{O}_{\mathrm{Spec}R}$ on the topological space $\mathrm{Spec}R$ such that the ring of global sections of $\mathcal{O}_{\mathrm{Spec}R}$ is R (see [9]).

3 Supermanifolds

The previous and provisional definition of $\mathbb{R}^{p|q}$ can be recast in the more sophisticated language of ringed space, then functions on such superspaces can be seen as sections of sheaves of superalgebras. Let us recall the definition of a supermanifold according to [12], [13], [8], [16]. First one defines the space $\mathbb{R}^{p|q}$ to be the topological space $\mathbb{R}^p$ endowed with the sheaf of real superalgebras $\mathcal{O}_{\mathbb{R}^{p|q}}$ whose sections are smooth functions on open subsets of $\mathbb{R}^p$, with values in $\mathbb{R}[\theta^1,\cdots,\theta^q]$, where $\theta^1,\cdots,\theta^q$ are odd variables. So for any open subset $|\Omega|$ of $\mathbb{R}^p$ the superalgebra $\Gamma(|\Omega|,\mathcal{O}_\Omega)$ of sections of $\mathcal{O}_{\mathbb{R}^{p|q}}$ over $|\Omega|$ is spanned over $\mathcal{C}^\infty(\Omega)$ by $\theta^1,\cdots,\theta^q$: $\forall f\in\Gamma(|\Omega|,\mathcal{O}_\Omega)$, $f=\sum_{I\in\mathbb{I}^q} f_I\theta^I$, where $f_I\in\mathcal{C}^\infty(|\Omega|)$, $\forall I\in\mathbb{I}^q$. The open subsets of $\mathcal{M}$ are then the objects $\Omega=(|\Omega|,\mathcal{O}_\Omega)$, where $|\Omega|$ is an open subset of $\mathbb{R}^p$. If Ω and Ω' are two such open subsets then a *morphism* $\varphi:\Omega\longrightarrow\Omega'$ is given by a continuous map $|\varphi|:|\Omega|\longrightarrow|\Omega'|$ and an even morphism φ^* of sheaves of superalbegras from $|\varphi|^*\mathcal{O}_{\Omega'}$ to $\mathcal{O}_\Omega$[5] (this implies in particular that $|\varphi|$ should be smooth). If furthermore $|\varphi|$ is a *homeomorphism* and φ^* is an isomorphism of sheaves we then say that φ is an *isomorphism.*

A supermanifold $\mathcal{M}$ of dimension $p|q$ is a topological space $|\mathcal{M}|$ endowed with a sheaf $\mathcal{O}_\mathcal{M}$ of real superalgebras which is *locally isomorphic* to $\mathbb{R}^{p|q}$. An *open subset* U of $\mathcal{M}$ is an open subset $|U|$ of $|\mathcal{M}|$ endowed with the sheaf of superalgebras $\mathcal{O}_U$ which is the restriction of $\mathcal{O}_\mathcal{M}$ over $|U|$. By saying *locally isomorphic* we mean that for any point $m\in|\mathcal{M}|$ there is an open subset U of $\mathcal{M}$ such that $m\in|U|$, an open subset V of $\mathbb{R}^{p|q}$ and a isomorphism of sheaves X from U to V. There is however a difference with $\mathbb{R}^{p|q}$: the sheaf $\mathcal{O}_{|U|}$ of smooth real valued functions on $|U|$ is not embedded in a canonical way in $\mathcal{O}_U$[6]. But it may be identified with $\mathcal{O}_U/\mathcal{J}$, where $\mathcal{J}$ is the nilpotent ideal $(\theta^1,\cdots,\theta^q)$[7]. Then the isomorphism $X:U\longrightarrow V$ plays the role of a local chart and the pull-back image of the canonical coordinates $x^1,\cdots,x^p,\theta^1,\cdots,\theta^q$ by X are the analogues of local coordinates.

3.1 Maps from an open subset of $\mathbb{R}^{p|q}$ to a supermanifold

Let $\mathcal{N}$ be a supermanifold of dimension $n|m$, U be an open subset of $\mathcal{N}$ and $Y:U\longrightarrow V\subset\mathbb{R}^{n|m}$ be a local chart (i.e. a sheaf isomorphism). Let $y^1,\cdots,y^n,\psi^1,\cdots,\psi^m$ be the canonical coordinates on $\mathbb{R}^{n|m}$. By abusing notations we write also $y^\alpha:\simeq Y^*y^\alpha$ and $\psi^j:\simeq Y^*\psi^j$. Then any section f of $\mathcal{O}_\mathcal{N}$ over U decomposes as

$$f=\sum_{J\in\mathbb{I}_0^m} F_J(y^1,\cdots,y^n)\psi^J,$$

[5]then, when restricted to the subsheaf $\mathcal{O}_{|\Omega'|}$ of smooth functions on $|\Omega'|$, φ^* it corresponds to the usual pull-back operation on functions by $|\varphi|$

[6]i.e. by dualizing there is no canonical fibration $\mathcal{M}\longrightarrow|\mathcal{M}|$

[7]i.e. by dualizing the projection map $\mathcal{O}_\mathcal{M}\longrightarrow\mathcal{O}_\mathcal{M}/\mathcal{J}$, there is a canonical embbeding $|\mathcal{M}|\hookrightarrow\mathcal{M}$

where $\forall J \in \mathbb{I}_0^m$, $F_J \in \mathcal{C}^\infty(|V|)$ and $\forall J = (j_1, \cdots, j_k)$, $\psi^J := \psi^{j_1} \cdots \psi^{j_k}$.

Now let Ω be an open subset of $\mathbb{R}^{p|q}$ and ϕ be a map from Ω to U, i.e. by dualizing an even morphism ϕ^* of superalgebra from $\mathcal{C}^\infty(U)$ to $\mathcal{C}^\infty(\Omega)$. Then the morphism property of ϕ^* implies that

$$\phi^* f = \sum_{J \in \mathbb{I}_0^m} \phi^* \left(F_J \circ (y^1, \cdots, y^n) \right) \chi^J,$$

where $\forall j \in [\![1, m]\!]$, $\chi^j := \phi^* \psi^j$, $\forall J = (j_1, \cdots, j_k)$, $\chi^J := \chi^{j_1} \cdots \chi^{j_k}$ and each $\phi^* \left(F_J \circ (y^1, \cdots, y^n) \right)$ can be expressed in terms of $(\phi^* y^1, \cdots, \phi^* y^n)$ by using Proposition 2.1. Hence *$\phi^* f$ can be computed as soon as we know $(\phi^* y^1, \cdots, \phi^* y^n)$ and $(\phi^* \psi^1, \cdots, \phi^* \psi^m)$*. This generalizes Proposition 2.1.

3.2 The use of the functor of point

When we study supersymmetric differential equations, a brutal application of the previous definitions suffers from incoherences. These are largely discussed in [10]. An instance is the superspace formulation of supergeodesics on an Euclidean sphere S^n. Let us view S^n as a submanifold of $\mathbb{R}^{n+1}$ and we consider the "supertime" $\mathbb{R}^{1|1}$ with coordinates t, θ. Then we look at maps $\phi : \mathbb{R}^{1|1} \longrightarrow S^n$ (i.e. morphisms ϕ^* from $\mathcal{C}^\infty(S^n)$ to $\mathcal{C}^\infty(\mathbb{R}^{1|1})$) which are solutions of

$$D \frac{\partial \phi}{\partial t} + \left\langle D\phi, \frac{\partial \phi}{\partial t} \right\rangle \phi = 0,$$

where $D := \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial t}$. This means that the image of any coordinate function y^α on $\mathbb{R}^{n+1} \supset S^n$ by $D \frac{\partial \phi^*}{\partial t} + \left\langle D\phi^*, \frac{\partial \phi^*}{\partial t} \right\rangle \phi^*$ vanishes. Set $\phi^* y = \varphi + \theta \psi$, where $\varphi \in \mathcal{C}^\infty(\mathbb{R}, S^n)$ and ψ is a section of $\varphi^* TS^n$. A first problem is that ψ should be *odd*: this is the usual requirement made by physicists and in our context it is imposed by the fact that ϕ^* should be an even morphism, because θ is odd. This could be cared by introducing a further (dumb) odd variable, say η, and by letting $\psi = \eta v$, where v is an ordinary section of $\varphi^* TS^n$. But then the next problem is that the preceding equation is equivalent to the system

$$\frac{\partial^2 \varphi}{(\partial t)^2} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \varphi = - \left\langle \psi, \frac{\partial \varphi}{\partial t} \right\rangle \psi \quad \text{and} \quad \frac{\partial \psi}{\partial t} + \left\langle \psi, \frac{\partial \varphi}{\partial t} \right\rangle \varphi = 0.$$

And we see that the right hand side of the first equation contains two times ψ, hence $\eta\eta$, which vanishes. So we should build ψ out of a linear combination of at least two dumb odd variables, say η^1 and η^2. But then we see that φ cannot be an ordinary map into S^n, still because of the first equation. Note that all these difficulties are absent in the differential geometric point of view used in [6, 14] for defining supermanifolds.

An alternative solution is proposed in [8] and [18] (see also [16]): it relies on Grothendieck's notion of *functor of points* in algebraic geometry. We will adopt that point of view in the following. For any $L \in \mathbb{N}$ we set $B := \mathbb{R}^{0|L}$. The starting point is to see a map ϕ from a supermanifold $\mathcal{M}$ of dimension $p|q$ into a supermanifold $\mathcal{N}$ of dimension $n|m$ as *a functor* *from* $\mathcal{C}^\infty(B)$ *to even morphisms* $\phi^* : \mathcal{C}^\infty(\mathcal{N}) \longrightarrow \mathcal{C}^\infty(\mathcal{M}\times B)$. So we need to understand morphisms ϕ^* from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(\mathcal{M}\times B)$: from a technical point of view nothing is new and it suffices to apply all the previous results. For simplicity we restrict ourself to the case where the target manifold $\mathcal{N}$ is an ordinary manifold and the source domain Ω is an open subset of $\mathbb{R}^{p|q}$.

3.3 Our final representation of a map from an open subset of $\mathbb{R}^{p|q}$ to an ordinary manifold

It is convenient to note $(x^1,\cdots,x^p)$, $(\theta^1,\cdots,\theta^q)$ respectively the even and the odd local coordinates on Ω and $(\eta^1,\cdots,\eta^L)$ the odd coordinates on B. Hence for any open subset Ω of $\mathcal{M}$, $\mathcal{C}^\infty(\Omega\times B)\simeq \mathcal{C}^\infty(|\Omega|)[\theta^1,\cdots,\theta^q,\eta^1,\cdots,\eta^L]$. Furthermore we note $\mathbb{A}^q(0)=\{\emptyset\}$ and for any $k\in\mathbb{N}^*$, $\mathbb{A}^q(k) := \{(a_1,\cdots a_k)\in [\![1,q]\!]^k | a_1<\cdots<a_k\}$. We denote by $A=(a_1,\cdots a_k)$ an element of $\mathbb{A}^q(k)$ and we then write $\theta^A := \theta^{a_1}\cdots\theta^{a_k}$. And we let $\mathbb{A}^q := \cup_{k=0}^q \mathbb{A}^q(k)$, $\mathbb{A}^q_0 := \cup_{k=0}^{[q/2]}\mathbb{A}^q(2k)$, $\mathbb{A}^q_1 := \cup_{k=0}^{[(q-1)/2]}\mathbb{A}^q(2k+1)$, $\mathbb{A}^q_2 := \cup_{k=1}^{[q/2]}\mathbb{A}^q(2k)$ and $\mathbb{A}^q_+ := \mathbb{A}^q_1\cup\mathbb{A}^q_2$. Lastly we set $\mathbb{AI} := \{AI | A\in\mathbb{A}^q, I\in\mathbb{I}^L\}$ and, defining the degree of AI to be the some of the degrees of A and I, we define similarly $\mathbb{AI}(j)$, $\mathbb{AI}_0$, $\mathbb{AI}_1$ and $\mathbb{AI}_2$. Hence any (even) function $f\in\mathcal{C}^\infty(\Omega\times B)$ (where Ω is an open subset of $\mathbb{R}^{p|q}$) can be decomposed as $f=\sum_{AI\in\mathbb{AI}_0}\theta^A\eta^I f_{AI}$, where $f_{AI}\in\mathcal{C}^\infty(|\Omega|)$, $\forall AI\in\mathbb{AI}_0$.

Then Theorem 1.1 implies that for any morphism ϕ^* from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(\Omega\times B)$, there exists a smooth map $\varphi\in\mathcal{C}^\infty(|\Omega|,\mathcal{N})$ and a smooth family $(\xi_{AI})_{AI\in\mathbb{AI}_2}$ of sections of $\pi^*T\mathcal{N}$ defined on a neighbourhood of the graph of φ in $|\Omega|\times\mathcal{N}$ such that if $\Xi := \sum_{AI\in\mathbb{AI}_2}\xi_{AI}\theta^A\eta^I$ then $\forall f\in\mathcal{C}^\infty(\mathcal{N})$, $\phi^*f = (1\times f)^*\left(e^\Xi f\right)$. Moreover, thanks to Proposition 1.1, the vector fields $(\xi_{AI})_{AI\in\mathbb{AI}_2}$ can be chosen in order to commute pairwise. We decompose Ξ as

$$\Xi = \sum_{A\in\mathbb{A}^q}\theta^A\Xi_A = \Xi_\emptyset + \sum_{a\in\mathbb{A}^q(1)}\theta^a\Xi_a + \sum_{(a_1,a_2)\in\mathbb{A}^q(2)}\theta^{a_1}\theta^{a_2}\Xi_{a_1a_2}+\cdots,$$

where $\forall A\in\mathbb{A}^q_1$, $\Xi_A=\sum_{I\in\mathbb{I}^L_1}\xi_{AI}\eta^I$ and $\forall A\in\mathbb{A}^q_0$, $\Xi_A=\sum_{I\in\mathbb{I}^L_2}\xi_{AI}\eta^I$. In particular $\Xi_\emptyset=\sum_{I\in\mathbb{I}^L_2}\xi_{\emptyset I}\eta^I$ and we see that Ξ_A is odd if A is odd and is even if A is even. Then the relations $[\xi_{AI},\xi_{A'I'}]=0$ implies that the vector fields Ξ_A *supercommute* pairwise, i.e.

$$\forall A\in\mathbb{A}^q(k),\forall A'\in\mathbb{A}^q(k'),\quad \Xi_A\Xi_{A'}-(-1)^{kk'}\Xi_{A'}\Xi_A=0.$$

This is equivalent to the fact that $\forall A,A'\in\mathbb{A}^q$, $[\theta^A\Xi_A,\theta^{A'}\Xi_{A'}]=0$. This last commutation relation implies that

$$e^\Xi = e^{\sum_{A\in\mathbb{A}^q}\theta^A\Xi_A} = e^{\Xi_\emptyset}\prod_{A\in\mathbb{A}^q_+}e^{\theta^A\Xi_A} = e^{\Xi_\emptyset}\prod_{A\in\mathbb{A}^q_+}(1+\theta^A\Xi_A),$$

where we have used $\left(\theta^A\Xi_A\right)^2=0$. Hence

$$\forall f\in\mathcal{C}^\infty(\mathcal{N}),\quad \phi^*f=(1\times\varphi)^*\left(e^{\Xi_\emptyset}\prod_{A\in\mathbb{A}^q_+}(1+\theta^A\Xi_A)f\right). \tag{29}$$

Alternatively one can integrate these vector fields as in the second section of this paper. Let us denote by $\left(\mathfrak{s}^{AI}\right)_{AI\in\mathbb{A}\mathbb{I}_2}$ the coordinates on $\Lambda_+^{2*}\mathbb{R}^{q+L}$ and

$$\vartheta := \sum_{AI\in\mathbb{A}\mathbb{I}_2} \theta^A\eta^I \frac{\partial}{\partial\mathfrak{s}^{AI}} = \sum_{A\in\mathbb{A}^q} \theta^A\vartheta_A,$$

where $\forall A\in\mathbb{A}_1^q$, $\vartheta_A := \sum_{I\in\mathbb{I}_1^L}\eta^I\frac{\partial}{\partial\mathfrak{s}^{AI}}$ and $\forall A\in\mathbb{A}_0^q$, $\Xi_A = \sum_{I\in\mathbb{I}_2^L}\eta^I\frac{\partial}{\partial\mathfrak{s}^{AI}}$. Then there exists a smooth map Φ from a neighbourhood of $|\Omega|\times\{0\}$ in $|\Omega|\times\Lambda_+^{2*}\mathbb{R}^{q+L}$ to $\mathcal{N}$ such that

$$\forall f\in\mathcal{C}^\infty(\mathcal{N}),\quad \phi^*f = \iota^*\left(e^{\vartheta_\emptyset}\prod_{A\in\mathbb{A}_+^q}(1+\theta^A\vartheta_A)(f\circ\Phi)\right). \tag{30}$$

3.4 Forgetting the ugly notations

We now propose some abuses and adaptations of notation to lighten all this description. But we try to keep the important property that each Ξ_A is vector field[8] defined along the graph of φ (even if it has coefficients in a Grassmann algebra). First of all we simply write $\varphi^* :\simeq (1\times\varphi)^*$. Second the operator $e^{\Xi_\emptyset}$ has no direct geometrical signification and his presence there is only necessary to "thicken" φ^*, so that we can absorb it by a redefinition of φ^*:

$$\varphi^* :\simeq \varphi^* e^{\Xi_\emptyset} :\simeq (1\times\varphi)^* e^{\Xi_\emptyset}.$$

We can hence rewrite (29) as

$$\forall f\in\mathcal{C}^\infty(\mathcal{N}),\quad \phi^*f = \varphi^*\left(\prod_{A\in\mathbb{A}_+^q}(1+\theta^A\Xi_A)\right)f. \tag{31}$$

For example if $q=2$, we have (keeping in mind the fact that Ξ_1 and Ξ_2 are odd whereas Ξ_{12} is even):

$$\begin{aligned}\phi^*f &= \varphi^*(1+\theta^1\Xi_1)(1+\theta^2\Xi_2)(1+\theta^1\theta^2\Xi_{12})f\\ &= \varphi^*\left(1+\theta^1\Xi_1+\theta^2\Xi_2+\theta^1\theta^2(\Xi_{12}-\Xi_1\Xi_2)\right)f.\end{aligned} \tag{32}$$

Similarly relation (30) can be written

$$\forall f\in\mathcal{C}^\infty(\mathcal{N}),\quad \phi^*f = \left(\prod_{A\in\mathbb{A}_+^q}(1+\theta^A\vartheta_A)\right)\Phi^*f. \tag{33}$$

Use of a local chart on the target manifold

The use of relations (31) is particularly convenient if we assume that the image of $\phi:\Omega\longrightarrow\mathcal{N}$ is contained in an open subset $U\subset\mathcal{N}$ on which there is a chart $y:U\longrightarrow\mathbb{R}^n$.

[8] i.e. a *first order* differential operator

Indeed Remark 1.1 tells us that we can choose the vector fields (ξ_I) in such a way that $\xi_I\xi_J y = 0$ (see (14)). This implies that $\Xi_A\Xi_{A'}y = 0$, $\forall A, A' \in \mathbb{A}^q$. Now what physicists denote "ϕ" or "$(\phi^\alpha)_\alpha$" is just $\phi^* y$ or $(\phi^* y^\alpha)_\alpha$ and then when they write the decomposition

$$"\phi = \varphi + \sum_{A\in\mathbb{A}^q} \theta^A \psi_A", \tag{34}$$

it implies by using (31) that

$$\varphi + \sum_{A\in\mathbb{A}^q_+} \theta^A\psi_A = \phi^* y = \varphi^* \left(\prod_{A\in\mathbb{A}^q_+} (1 + \theta^A \Xi_A) \right) y.$$

But since $\Xi_A\Xi_{A'}y = 0$ the development of the right hand side of this identity is particularly simple. We deduce

$$\varphi + \sum_{A\in\mathbb{A}^q_+} \theta^A\psi_A = \varphi^* y + \sum_{A\in\mathbb{A}^q_+} \theta^A \varphi^* \Xi_A y.$$

Hence $\forall A \in \mathbb{A}^q_+$, $\psi_A = \varphi^*\Xi_A y$. Our last abus of notation is to let $\psi_A \simeq \Xi_A$. So we reinterpret (34) as

$$\phi^* = \varphi^* \prod_{A\in\mathbb{A}^q_+} (1 + \theta^A\psi_A),$$

where the rules to manipulate such an expression are

- each ψ_A acts as a first order differential operator to its right
- two different ψ_A, $\psi_{A'}$ supercommute pairwise and with the θ^A's

An example of application

Assume that we find in the physics litterature a map "ϕ" from $\mathbb{R}^{p|2}$ to $\mathbb{R}$ which has the expression

$$"\phi = \varphi + \theta^1\psi_1 + \theta^2\psi_2 + \theta^1\theta^2 F" \tag{35}$$

and we want to compute $\phi^* f \simeq f \circ \phi$, where $f \in \mathcal{C}^\infty(\mathbb{R})$. Then we reinterpret (35) as

$$\phi^* = \varphi^*(1 + \theta^1\psi_1)(1 + \theta^2\psi_2)(1 + \theta^1\theta^2 F).$$

Then

$$\begin{aligned}
\phi^* f &= \varphi^*(1 + \theta^1\psi_1)(1 + \theta^2\psi_2)(1 + \theta^1\theta^2 F) f \\
&= \varphi^* f + \theta^1\varphi^*\psi_1 f + \theta^2\varphi^*\psi_2 f + \theta^1\theta^2\varphi^* F f - \theta^1\theta^2\varphi^*\psi_1\psi_2 f \\
&= f\circ\varphi + \theta^1(f'\circ\varphi)\psi_1 + \theta^2(f'\circ\varphi)\psi_2 + \theta^1\theta^2[(f'\circ\varphi)F - (f''\circ\varphi)\psi_1\psi_2].
\end{aligned}$$

References

[1] A. Bahraini, *Supersymétrie et géométrie complexe*, thèse de Doctorat, Université Paris 7, 2004.

[2] C. Bartocci, U. Bruzzo, D. Hernández-Ruipérez, *The geometry of supermanifolds*, Kluwer Academic Publishers Group, Dordrecht, 1991.

[3] M. Batchelor, *Two approaches to supermanifolds*, Trans. Amer. Math. Soc. **258**, N. 1 (1980), 257–270.

[4] F.A. Berezin, *Introduction to superanalysis*, D. Reidel Publishing Companyn, 1987.

[5] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan. D. Morrison, E. Witten, ed., *Quantum fields and strings: a course for mathematicians*, Volume 1, , AMS, 1999.

[6] B. Dewitt, *Supermanifolds*, Cambridge University Press, 1984.

[7] P. Deligne, D. Freed, *Supersolutions*, in [5].

[8] P. Deligne, J. Morgan, *Notes on supersymmetry*, in [5], 41–96.

[9] D. Eisenbud, J. Harris, *The geometry of schemes*, Graduate text in Maths. Springer.

[10] D. Freed, *Five lectures on supersymmetry*, AMS, 1999.

[11] P.G.O. Freund, *Introduction to supersymmetry*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1986.

[12] D.A. Leĭtes, *Introduction to the theory of supermanifolds*, Uspekhi Mat. Nauk 35 (1980), no. 1, 3–57; Russian Math. Surveys, 35 no. 1 (1980), 1–64.

[13] Y.I. Manin, *Gauge field theory and complex geometry*, Grundlehren der Mathematischen Wissenschaften 289, Springer-Verlag, 1988.

[14] A. Rogers, *A global theory of supermanifolds*, J. Math. Phys. 21 (1980), 1352–65; *Super Lies groups: global topology and local structures*, J. Math. Phys. 22(1981), 939–45.

[15] A. Salam, J. Strathdee, Nucl. Phys. B, 51 (1974).

[16] V.S. Varadarajan, *Supersymmetry for mathematicians: an introduction*, Courant Lecture Notes, AMS 2004.

[17] V.S. Vladimirov, I.V. Volovich, *Superanalysis. I. differential calculus*, Teoret. Mat. Fiz. 59, no. 1 (1984), 3–27.

[18] Fall Problems posed by E. Witten, solutions by P. Deligne, D. Freed, L. Jeffrey and S. Wu, in [5].