Continuum limit of the Volterra model, separation of variables and non standard realizations of the Virasoro Poisson bracket.

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Abstract

The classical Volterra model, equipped with the Faddeev-Takhtadjan Poisson bracket provides a lattice version of the Virasoro algebra. The Volterra model being integrable, we can express the dynamical variables in terms of the so called separated variables. Taking the continuum limit of these formulae, we obtain the Virasoro generators written as determinants of infinite matrices, the elements of which are constructed with a set of points lying on an infinite genus Riemann surface. The coordinates of these points are separated variables for an infinite set of Poisson commuting quantities including L_0. The scaling limit of the eigenvector can also be calculated explicitly, so that the associated Schroedinger equation is in fact exactly solvable.

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1 Introduction.

The relation between integrable systems and conformal field theory has long been recognized [1, 2]. Although the emphasis has been put rightfully on Baxter Q operator and therefore on Sklyanin's separated variables [3], to the best of our knowledge there is no explicit expressions of the Virasoro generators in terms of these variables.

We make here a first step in this direction by considering the classical version of this problem. Our strategy will be to start with the Volterra model on the lattice [4, 6] equipped with the Faddeev-Takhtadjan [7, 8] Poisson bracket. Since the Volterra model is integrable, we can rewrite everything in terms of separated variables. Now, the Faddeev-Takhtadjan bracket goes directly to the Virasoro Poisson bracket in the continuum limit, and therefore by taking this limit in the separated variables formulae we will obtain the Virasoro generators expressed in terms of separated variables.

This leads to the following rather new type of formula for the Virasoro generators:

$$u(x) = \sum L_n e^{2in\pi x} = p_0^2 + 2\partial_x^2 \log\det\Theta(x) + (L_0 - p_0^2)\delta(x)$$

Here p_0 is the zero mode and Poisson commutes with everything, the term $(L_0 - p_0^2)\delta(x)$ will be explained later and the formula for L_0 is given in eq.(66). The infinite matrix $\Theta(x)$ reads $(k, m \in \{1, \cdots, \infty\})$:

$$\Theta_{km}(x) = \frac{W_k(x)\partial_x E_m(x) - \partial_x W_k(x) E_m(x)}{Z_k^2 - m^2\pi^2}, \quad 0 \le x \le 1 \tag{1}$$

with

$$W_k(x) = \frac{\sin Z_k x}{Z_k} + \mu_k \frac{\sin Z_k(1-x)}{Z_k}, \quad E_m(x) = 2m\pi \sin m\pi x \tag{2}$$

The above formula for $u(x)$ is valid on the interval $0 \le x \le 1$, and should be extended outside this interval by periodicity (in particular the $\delta(x)$ term in a Dirac comb).

The result of this paper is that if the variables Z_k, μ_k, have Poisson bracket [3]

$$\{Z_k, Z_{k'}\} = 0, \quad \{Z_k, \mu_{k'}\} = 2(Z_k - p_0^2 Z_k^{-1})\mu_k \delta_{kk'}, \quad \{\mu_k, \mu_{k'}\} = 0 \tag{3}$$

then $u(x)$ does satisfies the Virasoro Poisson bracket:

$$\{u(x), u(y)\} = 4(u(x) + u(y))\,\delta'(x-y) + 2\delta'''(x-y) \tag{4}$$

Morever, the variables Z_k, μ_k are separated variables for an infinite set of higher commuting quantities, including L_0.

Since the separated variables are also the ones which solve the classical inverse problem, the Schroedinger equation with the potential $u(x)$

$$(-\partial_x^2 - u(x))\psi(x,\Lambda) = \Lambda^2\psi(x,\Lambda))$$

[3]Notice that if we redefine $\Lambda_k = \sqrt{Z_k^2 - p_0^2}$, the Poisson bracket becomes a standard quadratic bracket $\{\Lambda_k, \mu_k\} = 2\Lambda_k\mu_k$. However p_0 will then enter the formula for $\Theta(x)$ and, in this work, we prefer to keep that formula simple at the expense of a slightly more complicated Poisson bracket.

is exactly solvable, meaning that we have explicit formulae for both the potential $u(x)$ and a basis of solutions $\psi(x,\Lambda)$. Constructing the linear combination which is quasi periodic (the so called Bloch waves) introduces an infinite genus Riemann surface. The coefficients in the expression of this curve define a complete set of Poisson commuting Hamiltonians including L_0. The separated variables are points on this curve.

The paper is organized as follows. In the first three sections we recall some known facts about the Volterra model on the lattice. In particular we recall the formulae expressing the dynamical degrees of freedom in terms of the separated variables.

In section 5 we compute the continuum (scaling) limit of the spectral curve. The result is eq.(41). We then show that the Hamiltonians H_m in this formula are in involution. Moreover we show that the scaling limit of the dynamical divisor still belongs to that curve, and hence define separated variables for these Hamiltonians.

In sections 6, 7 and 8, we compute the scaling limit of the eigenvector of the Lax matrix at each point of the spectral curve. The result is rather simple and is given in eq.(50). We then show that the obtained expression does satisfy a second order Schroedinger equation and we compute its potential $u(x)$. Finally, we construct the two quasi periodic solutions of that equation, the Bloch waves, and recover in this way exactly the same spectral curve as the one obtained in section 5.

In section 9, we give conditions under which the determinants of the infinite matrices that appeared in the previous sections exist. We then perform a few checks in a certain perturbative scheme. In section 10 we prepare the ground for the serious calculations coming next.

In sections 11 and 12 we prove that the potential $u(x)$ does satisfy the Virasoro Poisson bracket. An essential use is made of certain quartic relations, proven very much like the Hirota–Sato bilinear identities. These identities should be considered as generalizations for τ-functions of the quartic relations on Riemann's Theta functions.

2 The Volterra model.

In this and the following two sections we recall some well known facts about the Volterra model. The Volterra model, as an integrable system, was introduced in [4]. It is a restricted version of the Toda lattice. We consider a periodic lattice with $N+1$ sites, and on each lattice site we attach a dynamical variable a_i on which we impose the Faddeev-Takhtadjan Poisson bracket [7]:

$$\{a_i, a_j\} = a_i a_j \Big[(4 - a_i - a_j)(\delta_{i,j+1} - \delta_{j,i+1}) - a_{j+1}\delta_{i,j+2} + a_{i+1}\delta_{j,i+2} \Big] \tag{5}$$

This bracket[4] is interesting because taking the continuum limit as

$$a_i \simeq 1 + \Delta^2 u(x), \quad \Delta = \frac{1}{(N+1)}$$

[4]In terms of Toda Hamiltonian structures, it is a linear combination of restrictions of the second and fourth Poisson brackets.

it becomes the Virasoro Poisson bracket eq.(4). For precisely this reason, and in this perspective, the lattice model has been extensively studied both at the classical level [7, 8] and at the quantum level [9, 10, 11, 13, 12]. The present paper is one more contribution this series of works.

The Lax matrix for the Volterra model is defined by:

$$L(\mu) \quad = \quad \begin{pmatrix} 0 & \sqrt{a_1} & 0 & & \cdots & & \mu^{-1}\sqrt{a_{N+1}} \\ \sqrt{a_1} & 0 & \sqrt{a_2} & & \cdots & & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & & \sqrt{a_{i-1}} & 0 & \sqrt{a_i} & & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & & \cdots & & \sqrt{a_{N-1}} & 0 & \sqrt{a_N} \\ \mu\sqrt{a_{N+1}} & & \cdots & & 0 & \sqrt{a_N} & 0 \end{pmatrix} \tag{6}$$

It is well known that $\mathrm{Tr}L^n(\mu)$ are in involution with respect to the Poisson bracket eq.(5). Hence we have an integrable system on the lattice whose continuum limit is directly related to conformal field theory.

The spectral curve Γ is defined as usual:

$$\Gamma \quad : \quad \det(L(\mu) - \lambda) = 0 \tag{7}$$

Expanding the determinant we see that it is of the form:

$$\mu + \mu^{-1} - t(\lambda) = 0 \tag{8}$$

where $t(\lambda)$ is polynomial of degree $N+1$.

$$t(\lambda) = \mathcal{A}^{-1}\lambda^{N+1} - \mathcal{A}^{-1}\left(\sum_i a_i\right)\lambda^{N-1} + \cdots \tag{9}$$

where

$$\mathcal{A} = \sqrt{a_1 a_2 \cdots a_{N+1}}$$

Assuming $N = 2n$ *even*, $t(\lambda)$ is an *odd* polynomial, $t(-\lambda) = -t(\lambda)$, and has exactly $n+1$ independent coefficients. However, in that case, there is one Casimir function $K = t(2)$:

$$\{t(2), a_i\} = 0, \quad \forall i$$

The dimension of phase space is $N = 2n$ and we have exactly n commuting quantities. The genus of the curve Γ is $g = N$.

At each point (λ, μ) of the spectral curve, we can attach an eigenvector $\Psi(\lambda, \mu) = (\psi_i(x))$, $i = 1, \cdots, N+1$, corresponding to the eigenvalue λ of $L(\mu)$. Explicitly, the equation $(L(\mu) - \lambda)\Psi = 0$ reads

$$\begin{aligned} \sqrt{a_1}\psi_2 + \mu^{-1}\sqrt{a_{N+1}}\psi_{N+1} &= \lambda\psi_1 \\ \sqrt{a_{i-1}}\psi_{i-1} + \sqrt{a_i}\psi_{i+1} &= \lambda\psi_i \\ \mu\sqrt{a_{N+1}}\psi_1 + \sqrt{a_N}\psi_N &= \lambda\psi_{N+1} \end{aligned} \tag{10}$$

We extend the definition of the coefficients a_i by periodicity $a_{i+N+1} = a_i$, and introduce a second order difference operator $\mathcal{D}$:

$$\left(\mathcal{D}\Psi\right)_i \equiv \sqrt{a_{i-1}}\psi_{i-1} + \sqrt{a_i}\psi_{i+1}$$

This operator is a discrete version of a Schroedinger operator with periodic potential. Eqs.(10) are then equivalent to:

$$\left(\mathcal{D}\Psi\right)_i = \lambda\psi_i, \quad \text{with} \quad \psi_{i+N+1} = \mu\psi_i \tag{11}$$

Therefore, the eigenvector Ψ is a Bloch wave for the difference operator $\mathcal{D}$ with a Bloch momentum μ.

In the continuum limit, eq.(11) becomes the Schroedinger equation.

$$(-\partial_x^2 - u(x))\psi(x) = \Lambda^2\psi(x), \quad \lambda \simeq 2 - \Delta^2\Lambda^2$$

3 The free case.

Since in the continuum limit $a_i \to 1$, it is useful to first recall some formulae in the trivial case $a_i = 1$. They will be generalized to the full case in the next section. To introduce the zero mode from the start, we consider the slightly more general case $a_i = a$:

$$\begin{aligned} \sqrt{a}(\psi_2 + \mu^{-1}\psi_{N+1}) &= \lambda\psi_1 \\ \sqrt{a}(\psi_{i-1} + \psi_{i+1}) &= \lambda\psi_i \\ \sqrt{a}(\mu\psi_1 + \psi_N) &= \lambda\psi_{N+1} \end{aligned}$$

The solution of the bulk equations is $\psi_i = \alpha x_+^i + \beta x_-^i$ where $x_\pm$ are solutions of the equation

$$x^2 - zx + 1 = 0, \quad x_\pm(\lambda) = \frac{1}{2}(z \pm \sqrt{z^2 - 4}), \quad z = \frac{\lambda}{\sqrt{a}}$$

Imposing the two boundary equations, we get

$$\begin{aligned} (x_+^{N+1} - \mu)\,\alpha + (x_-^{N+1} - \mu)\,\beta &= 0 \\ (x_+^{N+1} - \mu)x_+\,\alpha + (x_-^{N+1} - \mu)x_-\,\beta &= 0 \end{aligned}$$

The compatibility of this system yields the spectral curve

$$\mu + \mu^{-1} = x_+^{N+1} + x_-^{N+1} \equiv t(\lambda) \tag{12}$$

We now impose that the curve passes through the point $(\lambda = 2, \mu = \mu_0)$ where μ_0 is related to the value of the Casimir function by

$$K = \mu_0 + \mu_0^{-1} \tag{13}$$

Setting

$$x_\pm(2) = \frac{1}{\sqrt{a}} \pm \sqrt{\frac{1}{a} - 1} = e^{\pm\alpha}, \quad \mu_0 = e^{ip_0}$$

eq.(12) gives $\alpha = i\frac{p_0}{N+1}$. Hence the constant a is related to the value of the zero mode p_0 by:

$$\sqrt{a} = \frac{1}{\cos\frac{p_0}{N+1}}$$

The components of the eigenvector, properly normalized, are meromorphic functions on the spectral curve eq.(12). Choosing the normalization $\psi_{N+1} = \mu$, they read

$$\psi_i(\lambda, \mu) = \frac{P_{N+1-i}(z) + \mu P_i(z)}{P_{N+1}(z)}, \quad z = \frac{\lambda}{\sqrt{a}} \tag{14}$$

We have introduced the polynomials of degree $j-1$:

$$P_j(z) = \frac{x_+^j - x_-^j}{x_+ - x_-}, \quad P_j(z) = z^{j-1} + O(z^{j-3}) \tag{15}$$

The first few polynomials are

$$P_0 = 0, \quad P_1 = 1, \quad P_2 = z, \quad P_3 = z^2 - 1, \quad P_4 = z^3 - 2z$$

They are essentially the Tchebitchev polynomials of the second kind.

As we will see, eq.(14) is the general form of the meromorphic function $\psi_i(\lambda, \mu)$ even when $a_i \neq a$. In particular, in order to take the continuum limit, the poles of the eigenvector[5] will have to be close to the roots of the equation $P_{N+1}(\frac{\lambda}{\sqrt{a}}) = 0$, that is

$$\lambda_k^{(0)} = 2\sqrt{a}\cos\frac{Z_k^{(0)}}{N+1}, \quad Z_k^{(0)} = k\pi \quad k = 1, \cdots, N \tag{16}$$

For these special values $\lambda_k^{(0)}$, we have $x_\pm = e^{\pm i\frac{k\pi}{N+1}}$ and eq.(12) gives $\mu_k^{(0)} = (-1)^k$. The set of points $(\lambda_k^{(0)}, \mu_k^{(0)}), k = 1, \cdots, N$, will be called the free configuration and will play an important role below.

It is simple to take the continuum limit in this free case. We set

$$\lambda \simeq 2 - \Delta^2\Lambda^2, \quad z \simeq 2 - \Delta^2 Z^2, \quad \psi_{i\pm1} = \psi(x \pm \Delta), \quad \Delta = \frac{1}{N+1}$$

where we have introduced the variable

$$Z = \sqrt{\Lambda^2 + p_0^2} \tag{17}$$

The eigenvector equation becomes the Schroedinger equation

$$-\psi''(x) - p_0^2\psi(x) = \Lambda^2\psi(x), \quad x = \Delta j \tag{18}$$

We also have $x_\pm = 1 \pm i\Delta Z$ and the equation of the spectral curve reads:

$$\mu + \mu^{-1} = (1 + i\Delta Z)^{1/\Delta} + (1 - i\Delta Z)^{-1/\Delta}$$

[5]In fact in this simple case the eigenvector has no poles at finite distance because they are compensated by zeroes in the numerator. This degeneracy is lifted as soon as the a_i are not all equal.

In the limit $\Delta \to 0$, it becomes

$$\mu + \mu^{-1} = 2\cos Z \tag{19}$$

Similarly, the eigenvector becomes a Baker-Akhiezer function:

$$\psi(x) = \frac{\sin Z(1-x) + \mu \sin Zx}{\sin Z} \tag{20}$$

When $\mu = e^{\pm iZ}$ this reduces to $\psi(x) = e^{\pm iZx}$ as it should be. Notice that when μ is kept as a free parameter, the above formula gives two independent solutions of eq.(18), but when μ belongs to the spectral curve eq.(19) one has

$$\psi(x+1) = \mu\psi(x)$$

Eq.(20) presents the two Bloch waves as a single function on the hyperelliptic spectral curve eq.(19).

Another example, important to us, will be the Dirac comb.

$$[-\partial^2 - H_0\delta(x)]\psi(x) = \Lambda^2\psi(x), \quad \delta(x+1) = \delta(x)$$

On each interval $x_j = j < x < x_{j+1} = j+1$, one has

$$\psi(x) = \alpha_j e^{i\Lambda x} + \beta_j e^{-i\Lambda x}, \quad x_j < x < x_{j+1}$$

The Bloch condition

$$\psi(x + x_j) = \mu^j \psi(x), \quad 0 < x < 1$$

gives $\alpha_j = (\mu e^{-i\Lambda})^j \alpha_0$, $\beta_j = (\mu e^{i\Lambda})^j \beta_0$. The continuity of $\psi(x)$ gives $\alpha_0 = -\frac{(\mu - e^{-i\Lambda})}{(\mu - e^{i\Lambda})}\beta_0$, while the gap equation on the first derivative, $-\psi'(x_j+0) + \psi'(x_j-0) - H_0\psi(x_j) = 0$, gives the spectral curve

$$\mu + \mu^{-1} = 2\cos\Lambda - H_0\frac{\sin\Lambda}{\Lambda} \tag{21}$$

The Bloch wave itself is $\psi(x) = \mu^j \psi_{vac}(x - x_j)$, $x_j < x < x_{j+1}$, where $\psi_{vac}(x)$ is given by eq.(20) (with $p_0 = 0$) but Λ, μ now belonging to the curve eq.(21).

4 Separated variables.

In this section, we generalize the previous analysis when $a_i \neq a$ and express the dynamical variables of the Volterra model in terms of the separated variables. Equivalent formulae where already obtained a long time ago in [4, 5]. A quantum version of this construction for the closed Toda chain can be found in [19].

We have to reconstruct the eigenvectors Ψ of $L(\mu)$. Let us set $\Psi = (\psi_i)$, $i = 1, \cdots, N+1$. We normalize the last component $\psi_{N+1} = \mu$. Notice that due to eq.(8), μ does not vanish for finite λ. The components ψ_i are meromorphic functions on the spectral curve and are uniquely characterized by their poles and behavior at infinity which we now describe.

We will call $P^+(\lambda = \infty, \mu = \infty)$ and $P^-(\lambda = \infty, \mu = 0)$ the two points above $\lambda = \infty$. In the neighbourhood of $P^\pm$, the local parameter is λ^{-1} and we have by direct expansion of eq.(8):

$$P^+ \ : \ \mu = \mathcal{A}^{-1}\lambda^{N+1}\Big(1 + O(\lambda^{-2})\Big) \tag{22}$$

$$P^- \ : \ \mu = \mathcal{A}\lambda^{-N-1}\Big(1 + O(\lambda^{-2})\Big) \tag{23}$$

At the points P^+ and P^-, the eigenvector $\Psi(P)$ behaves as:

$$\psi_i(P) = \frac{1}{\sqrt{a_{N+1}a_1a_2\cdots a_{i-1}}}\,\lambda^i\Big(1 + O(\lambda^{-2})\Big), \quad P \sim P^+ \tag{24}$$

$$\psi_i(P) = \sqrt{a_{N+1}a_1a_2\cdots a_{i-1}}\,\lambda^{-i}\Big(1 + O(\lambda^{-2})\Big), \quad P \sim P^- \tag{25}$$

This is easily deduced by inspection of eqs.(10).

From the general results of the classical inverse scattering theory, we expect $g + (N+1) - 1 = 2N$ poles for the eigenvector (see e.g. [6, 15]). From eq.(24), we see that we have a fixed pole of order N at P^+ (on the component ψ_N), and there remains $g = N$ poles at finite distance, the so called *dynamical poles*. But we notice the symmetry property

$$\psi_i(-\lambda, -\mu) = (-1)^i\psi_i(\lambda, \mu)$$

so that the dynamical poles come in pairs

$$\lambda_{N+1-k} = -\lambda_k, \quad \mu_{N+1-k} = -\mu_k$$

and only $(\lambda_k, \mu_k), k = 1 \cdots n$, are independent parameters.

Everything can be expressed in terms of these $2n = N$ quantities $(\lambda_k, \mu_k), k = 1 \cdots n$. In fact, they can be viewed as coordinates on (an open set of) phase space.

First, the commuting Hamiltonians are easy to reconstruct. Indeed the spectral curve is determined by requiring that it passes through the points $(\lambda_k, \mu_k), k = 1 \cdots n$, and through the point $(2, \mu_0)$, where μ_0 is related to the Casimir function as in eq.(13).

The equation of the curve itself can be written as a determinant

$$\det\begin{pmatrix} \lambda & \lambda^3 & \cdots & \lambda^{N+1} & \mu + \mu^{-1} \\ 2 & 2^3 & \cdots & 2^{N+1} & \mu_0 + \mu_0^{-1} \\ \lambda_1 & \lambda_1^3 & \cdots & \lambda_1^{N+1} & \mu_1 + \mu_1^{-1} \\ \vdots & & & & \vdots \\ \lambda_n & \lambda_n^3 & \cdots & \lambda_n^{N+1} & \mu_n + \mu_n^{-1} \end{pmatrix} = 0 \tag{26}$$

Expanding over the first row, we obtain a curve of the form eq.(8), and we can read directly the Hamiltonians as the coefficients of $t(\lambda)$. They appear as functions of the (λ_k, μ_k) and can be shown to Poisson commute (see [16, 17, 18] for a proof and for the quantum generalization of this fact).

Eqs.(24,25) and the data of the N dynamical poles also determine the functions ψ_i uniquely. Being meromorphic functions on a hyperelliptic curve, we can write quite generally

$$\psi_i = \frac{Q^{(i)}(\lambda) + \mu R^{(i)}(\lambda)}{\prod_{k=1}^n (\lambda^2 - \lambda_k^2)} \tag{27}$$

where $Q^{(i)}$ and $R^{(i)}$ are polynomials such that

$$Q^{(i)}(-\lambda) = (-1)^i Q^{(i)}(\lambda), \quad R^{(i)}(-\lambda) = (-1)^{i+1} R^{(i)}(\lambda)$$

Above λ_k, we have two points on the curve: (λ_k, μ_k) and (λ_k, μ_k^{-1}). We want the poles to be at (λ_k, μ_k) only so that the numerator in eq.(27) should vanish at the points (λ_k, μ_k^{-1}). This gives n conditions

$$Q^{(i)}(\lambda_k) + \mu_k^{-1} R^{(i)}(\lambda_k) = 0, \quad k = 1 \cdots n \tag{28}$$

To have a pole of order i at P^+ and a zero of order i at P^- we must choose

$$\text{degree } Q^{(i)} = N - i, \quad \text{degree } R^{(i)} = i - 1$$

Hence, these two polynomials depend altogether on $n+1$ coefficients which are determined by imposing the n conditions eq.(28) and requiring that the normalizations coefficients are inverse to each other at $P^\pm$ as in eqs.(24,25).

It is convenient to use the basis of polynomials $P_j(\lambda)$ given by eq.(15). We will write the formulae for ψ_i in the case i *odd*, the case i *even* is similar.

The polynomial $Q^{(i)}(\lambda)$ can be expanded over

$$Q^{(i)}(\lambda): \quad P_2(\lambda), P_4(\lambda), \cdots P_{N+1-i}$$

and the polynomial $R^{(i)}(\lambda)$ can be expanded over

$$R^{(i)}(\lambda): \quad P_1(\lambda), P_3(\lambda), \cdots P_i(\lambda)$$

Solving the linear system eq.(28), the eigenvector can be written as

$$\psi_i = \frac{K_i}{\prod(\lambda^2 - \lambda_k^2)} \det \begin{pmatrix} \mu P_1(\lambda) & \cdots & \mu P_i(\lambda) & -P_{N+1-i}(\lambda) & \cdots & -P_2(\lambda) \\ P_1(\lambda_1) & \cdots & P_i(\lambda_1) & -\mu_1 P_{N+1-i}(\lambda_1) & \cdots & -\mu_1 P_2(\lambda_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_1(\lambda_k) & \cdots & P_i(\lambda_k) & -\mu_k P_{N+1-i}(\lambda_k) & \cdots & -\mu_k P_2(\lambda_k) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_1(\lambda_n) & \cdots & P_i(\lambda_n) & -\mu_n P_{N+1-i}(\lambda_n) & \cdots & -\mu_n P_2(\lambda_n) \end{pmatrix} \tag{29}$$

where K_i are constants independant of λ, μ. Defining

$$\Theta_i = \begin{pmatrix} P_1(\lambda_1) & \cdots & P_{i-2}(\lambda_1) & -\mu_1 P_{N+1-i}(\lambda_1) & \cdots & -\mu_1 P_2(\lambda_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_1(\lambda_k) & \cdots & P_{i-2}(\lambda_k) & -\mu_k P_{N+1-i}(\lambda_k) & \cdots & -\mu_k P_2(\lambda_k) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_1(\lambda_n) & \cdots & P_{i-2}(\lambda_n) & -\mu_n P_{N+1-i}(\lambda_n) & \cdots & -\mu_n P_2(\lambda_n) \end{pmatrix} \tag{30}$$

we can compute the leading terms in eq.(29) when $\lambda \to \infty$. At $P^{(+)}$ the leading term comes from $\mu P_i(\lambda)$, while at $P^{(-)}$ it comes from $P_{N+1-i}(\lambda)$.

$$
\begin{aligned}
\psi_i &\simeq (-1)^{\frac{i-1}{2}} \mathcal{A}^{-1} K_i \det \Theta_i \, \lambda^i, \quad P^+ \\
\psi_i &\simeq (-1)^{\frac{i-1}{2}} K_i \det \Theta_{i+2} \, \lambda^{-i}, \quad P^-
\end{aligned}
$$

Imposing that the two coefficients of λ^i and λ^{-i} are inverse to each other, we get

$$
K_i^2 = \frac{\mathcal{A}}{\det \Theta_i \det \Theta_{i+2}}
$$

Comparing with eqs.(24, 25), we finally obtain

$$
a_i = \frac{\det \Theta_i \det \Theta_{i+3}}{\det \Theta_{i+1} \det \Theta_{i+2}}, \quad a_N = \frac{\det \Theta_N}{\det \Theta_{N+2}} \mathcal{A}, \quad a_{N+1} = \frac{\det \Theta_3}{\det \Theta_1} \mathcal{A} \tag{31}
$$

Here $\mathcal{A}^{-1}$ is the coefficient of λ^{N+1} in $t(\lambda)$, eq.(9), computed from eq.(26).

We impose the Poisson bracket on the variables λ_k, μ_k

$$
\{\lambda_k, \lambda_{k'}\} = 0, \quad \{\lambda_k, \mu_{k'}\} = -\frac{1}{2}\delta_{kk'} \left(4\lambda_k - \lambda_k^3\right) \mu_k, \quad \{\mu_k, \mu_{k'}\} = 0 \tag{32}
$$

One can then check that the Hamiltonians defined by eq.(26) are all in involution (this is a general result), and that the a_i defined above do satisfy the Faddeev-Takhtadjan Poisson bracket. The fact that the expressions for a_N, a_{N+1} are different from the ones in the bulk is due to the choice of normalization of the eigenvector. However, the Poisson bracket of the a_i is periodic. All this can be proved using techniques similar to the ones in [19].

5 Continuum limit of the spectral curve.

We now take the continuum limit of the spectral curve eq.(26). The result is eq.(41). We set

$$
\lambda = \sqrt{a} z, \quad \lambda_k = \sqrt{a} z_k, \quad \frac{2}{\sqrt{a}} = 2 \cos \frac{p_0}{N+1} = z_0
$$

From these, the scaled variables Λ, Z, Z_k are defined like this:

$$
\lambda = 2 \cos \frac{\Lambda}{N+1}, \quad z = 2 \cos \frac{Z}{N+1}, \quad z_k = 2 \cos \frac{Z_k}{N+1} \tag{33}
$$

Notice that we have $Z = \sqrt{\Lambda^2 + p_0^2}$. In the following, we will refer to the terminology "perturbation theory" when the points (Z_k, μ_k) are small deviations from the free configuration eq.(16). The formulae we will write will make sense in this perturbative setting. This however does not exclude the possibility to have a finite number of points which are large deviation. We will also be interested in the deviation from the zero mode configuration. That is we make the substitution $\sqrt{a_i} \to \sqrt{a}\sqrt{\tilde{a}_i}$ everywhere on the lattice. Alternatively this amounts to using the variable $z = \lambda/\sqrt{a}$.

Using the basis of polynomials $P_j(z)$ defined in eq.(15) instead of the z^j, we can write the spectral curve as (it has the right form and passes through the right points)

$$\det \begin{pmatrix} \mu+\mu^{-1} & P_{N+2}(z) & \cdots & P_4(z) & P_2(z) \\ \mu_0+\mu_0^{-1} & P_{N+2}(z_0) & \cdots & P_4(z_0) & P_2(z_0) \\ \mu_1+\mu_1^{-1} & P_{N+2}(z_1) & \cdots & P_4(z_1) & P_2(z_1) \\ \vdots & & & & \vdots \\ \mu_n+\mu_n^{-1} & P_{N+2}(z_n) & \cdots & P_4(z_n) & P_2(z_n) \end{pmatrix} = 0$$

Without changing the determinant, we can subtract to the first column the linear combination of the next two columns:

$$P_{N+2}(z_k) - P_N(z_k) = 2\cos Z_k$$

The first column becomes

$$\begin{pmatrix} \mu+\mu^{-1}-2\cos Z \\ \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$

where we have set

$$\gamma_k = \mu_k + \mu_k^{-1} - 2\cos Z_k \tag{34}$$

Notice that $\gamma_0 = 0$. The reason for this subtraction is that for the free configuration we also have $\gamma_k^{(0)} = 0$, so that the spectral curve becomes simply $\mu + \mu^{-1} = 2\cos Z$ as it should be. The subtraction gives sense to the spectral curve in perturbation theory. Expanding the determinant over the first row, we can write

$$\mu + \mu^{-1} - 2\cos Z = \sum_{j=1}^{n+1} H_{2j} P_{2j}(z)$$

The H_{2j} are given by

$$H = N^{-1} V$$

where we have defined

$$H = \begin{pmatrix} H_{N+2} \\ H_N \\ \vdots \\ H_2 \end{pmatrix}, \quad N = \begin{pmatrix} P_{N+2}(z_0) & \cdots & P_4(z_0) & P_2(z_0) \\ P_{N+2}(z_1) & \cdots & P_4(z_1) & P_2(z_1) \\ \vdots & & & \vdots \\ P_{N+2}(z_n) & \cdots & P_4(z_n) & P_2(z_n) \end{pmatrix}, \quad V = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$

We will need to treat separately the first row and column in the matrix N. Let us write it as

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where

$$A = P_{N+2}(z_0), \quad B = (\, P_N(z_0) \quad \cdots P_4(z_0) \quad P_2(z_0)\,)$$

$$C = \begin{pmatrix} P_{N+2}(z_1) \\ \vdots \\ P_{N+2}(z_n) \end{pmatrix}, \quad D = \begin{pmatrix} P_N(z_1) & \cdots & P_4(z_1) & P_2(z_1) \\ \vdots & & & \vdots \\ P_N(z_n) & \cdots & P_4(z_n) & P_2(z_n) \end{pmatrix}$$

To zero-th order in perturbation theory, we denote $N = N^{(0)}$ and similarly $A^{(0)}, B^{(0)}, C^{(0)}, D^{(0)}$. To take the continuum limit we have to consider the matrix $NN^{(0)-1}$.

Lemma 1 *In the continuum limit, we have*

$$NN^{(0)-1} = \begin{pmatrix} 1 & 0 \\ \frac{\sin Z_k}{Z_k}\frac{p_0}{\sin p_0} & \frac{\sin Z_k}{Z_k}\left\{\frac{1}{Z_k^2-m^2\pi^2} - \frac{1}{p_0^2-m^2\pi^2}\right\} 2(-1)^m m^2\pi^2 \end{pmatrix}, \quad k,m = 1,\cdots,\infty \quad (35)$$

Proof. Since $P_{N+2}(z) = zP_{N+1}(z) - P_N(z)$ and $P_{N+1}(z_k^{(0)}) = 0$, we have

$$C_k^{(0)} = -D_{k1}^{(0)} \Longrightarrow (D^{(0)-1}C^{(0)})_k = -\delta_{k1}$$

so that

$$N^{(0)-1} = \frac{1}{A+B_1}\begin{pmatrix} 1 & -BD^{(0)-1} \\ F & (A+B_1)D^{(0)-1} - F\otimes BD^{(0)-1} \end{pmatrix}$$

where F is the column vector with components $F_k = \delta_{k,1}$, $k = 1,\cdots,n$. It follows that

$$NN^{(0)-1} = \begin{pmatrix} 1 & 0 \\ \frac{1}{A+B_1}(C+DF) & DD^{(0)-1} - \frac{1}{A+B_1}(C+DF)\otimes BD^{(0)-1} \end{pmatrix}$$

Noticing that

$$A + B_1 = P_{N+2}(z_0) + P_N(z_0)$$
$$(C+DF)_k = P_{N+2}(z_k) + P_N(z_k)$$

we get in the continuum limit

$$\frac{1}{A+B_1}(C+DF)_k \to \frac{\sin Z_k}{Z_k}\frac{p_0}{\sin p_0}$$

The main trick to proceed is an explicit formula for the inverse of the matrix $D^{(0)}$. It is not difficult to check that

$$(D^{(0)-1})_{jk} = \frac{4}{N+1}\sin^2\frac{k\pi}{N+1}\; P_{2j}(z_k^{(0)})$$

With this, we find

$$\begin{aligned}(BD^{(0)-1})_m &= \frac{4}{N+1}\sin^2\frac{m\pi}{N+1}\sum_{j=1}^{n} P_{2j}(z_0)P_{2j}(z_m^{(0)}) \\ &= \frac{\sin^2\frac{m\pi}{N+1}}{\sin\frac{p_0}{N+1}\sin\frac{m\pi}{N+1}}\frac{4}{N+1}\sum_{j=1}^{n}\sin\frac{2jp_0}{N+1}\sin\frac{2jm\pi}{N+1} \to 2\frac{m\pi}{p_0}\int_0^1 dx\sin p_0x\sin m\pi x\end{aligned}$$

and the last integral is easily evaluated with the result

$$(BD^{(0)-1})_m \to (-1)^m\frac{2m^2\pi^2\sin p_0}{p_0(p_0^2-m^2\pi^2)}$$

Similarly we compute

$$(DD^{(0)-1})_{km} = \frac{\sin Z_k}{Z_k}\frac{(-1)^m 2m^2\pi^2}{Z_k^2-m^2\pi^2}$$

Gathering all this we get eq.(35). ∎

We now introduce the important infinite matrix

$$M_{km} = \frac{1}{Z_k^2 - m^2\pi^2}, \quad k, m = 1, \cdots, \infty \tag{36}$$

and the important vector $|\eta\rangle$

$$|\eta\rangle = M^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{37}$$

With these notations we can compute the inverse of the matrix $NN^{(0)-1}$:

Lemma 2

$$(NN^{(0)-1})^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{p_0}{\sin p_0}\left(\frac{1}{1-\langle\chi(p_0)|\eta\rangle}\right)\frac{(-1)^{m+1}}{2m^2\pi^2}\eta_m & \frac{(-1)^m}{2m^2\pi^2}\left\{\left(1+\frac{|\eta\rangle\langle\chi(p_0)|}{1-\langle\chi(p_0)|\eta\rangle}\right)M^{-1}\right\}_{mk}\frac{Z_k}{\sin Z_k} \end{pmatrix}$$

where we have defined the vector $\langle\chi(p_0)|_m = \frac{1}{p_0^2-\pi^2m^2}$.

Proof. With the above notations we can write

$$NN^{(0)-1} = \begin{pmatrix} 1 & 0 \\ \frac{\sin Z_k}{Z_k}\frac{p_0}{\sin p_0} & \frac{\sin Z_k}{Z_k}M_{kl}\left(\delta_{lm} - \eta_l\chi_m(p_0)\right)2(-1)^m m^2\pi^2 \end{pmatrix}$$

Letting

$$NN^{(0)-1} = \begin{pmatrix} 1 & 0 \\ Y & X \end{pmatrix} \Longrightarrow (NN^{(0)-1})^{-1} = \begin{pmatrix} 1 & 0 \\ -X^{-1}Y & X^{-1} \end{pmatrix}$$

we find

$$(X^{-1})_{mk} = \frac{(-1)^m}{2m^2\pi^2}\left\{\left(1+\frac{|\eta\rangle\langle\chi(p_0)|}{1-\langle\chi(p_0)|\eta\rangle}\right)M^{-1}\right\}_{mk}\frac{Z_k}{\sin Z_k} \tag{38}$$

and

$$(-X^{-1}Y)_m = \frac{p_0}{\sin p_0}\left(\frac{1}{1-\langle\chi(p_0)|\eta\rangle}\right)\frac{(-1)^{m+1}}{2m^2\pi^2}\eta_m$$

∎

Let us return to the formula for the spectral curve. We assume that the conditions explained in Section 9 are satisfied so that the infinite sums we will manipulate are convergent. Denote

$$\eta(Z) = 1 - \sum_{m=1}^{\infty}\frac{\eta_m}{Z^2 - m^2\pi^2} \tag{39}$$

and

$$|\Gamma\rangle_m = \sum_k M^{-1}_{mk}\frac{Z_k}{\sin Z_k}\gamma_k \tag{40}$$

These quantities enter the expression of the continuum limit of the spectral curve.

Proposition 1 *In the continuum limit, the equation of the spectral curve becomes:*

$$\mu + \mu^{-1} = 2\cos Z + \frac{\sin Z}{Z}\left(-H_0 + \sum_m \frac{H_m}{Z^2 - m^2\pi^2}\right) \tag{41}$$

where the conserved quantities H_m can be taken as

$$H_m = \Gamma_m + \frac{1}{\eta(p_0)}\langle\chi(p_0)|\Gamma\rangle\eta_m, \quad H_0 = \sum_m \frac{H_m}{p_0^2 - \pi^2 m^2} = \frac{1}{\eta(p_0)}\sum_m \frac{\Gamma_m}{p_0^2 - \pi^2 m^2} \tag{42}$$

Proof. We have

$$\mu + \mu^{-1} - 2\cos Z = \sum_{i=0}^{n} P_{N+2-2i}(z) H_{N+2-2i} = \sum_{i=0}^{n} P_{N+2-2i}(z)(N^{-1}V)_i$$

we insert $1 = N^{(0)-1}N^{(0)}$ into the above expression:

$$\mu + \mu^{-1} - 2\cos Z = \sum_{i,j,k} P_{N+2-2i}(z)(N^{(0)-1})_{ik}(N^{(0)})_{kj}(N^{-1}V)_j \tag{43}$$

Hence, we need to compute

$$\begin{aligned}\sum_i P_{N+2-2i}(z)(N^{(0)-1})_{im} \quad &= \\ &\left(\frac{P_{N+2}(z)+P_N(z)}{P_{N+2}(z_0)+P_N(z_0)}, -\frac{P_{N+2}(z)+P_N(z)}{P_{N+2}(z_0)+P_N(z_0)}BD^{(0)-1} + \sum_i P_{N+2-2i}(z)(D^{(0)-1})_{im}\right)\end{aligned}$$

whose limit $N \to \infty$ is easy to take

$$\sum_i P_{N+2-2i}(z)(N^{(0)-1})_{im} \quad \to \quad \frac{\sin Z}{Z}\left(\frac{p_0}{\sin p_0}, 2(-1)^m m^2\pi^2\left\{\frac{1}{Z^2 - m^2\pi^2} - \frac{1}{p_0^2 - m^2\pi^2}\right\}\right)$$

We can now take the limit $N \to \infty$ in eq.(43)

$$\mu + \mu^{-1} - 2\cos Z \quad = \quad \frac{\sin Z}{Z}\left(\frac{p_0}{\sin p_0}\widetilde{H}_0 + \sum_{m=1}^{\infty} 2(-1)^m m^2\pi^2\left\{\frac{1}{Z^2 - m^2\pi^2} - \frac{1}{p_0^2 - m^2\pi^2}\right\}\widetilde{H}_m\right)$$

where

$$\widetilde{H}_m = (NN^{(0)-1})^{-1}\begin{pmatrix}\gamma_0\\ \gamma_1\\ \vdots\\ \gamma_n\end{pmatrix} = \begin{pmatrix} 0 \\ X^{-1}\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_n\end{pmatrix}\end{pmatrix}$$

where X^{-1} is given in eq.(38) and we remembered that $\gamma_0 = 0$. Since $1 - \langle\chi(p_0)|\eta\rangle = \eta(p_0)$ the equation of the spectral curve finally becomes

$$\mu + \mu^{-1} - 2\cos Z \quad = \quad \frac{\sin Z}{Z}\left(\sum_m\left(\frac{1}{Z^2 - m^2\pi^2} - \frac{1}{p_0^2 - m^2\pi^2}\right)\left(\Gamma_m + \frac{1}{\eta(p_0)}\eta_m\langle\chi(p_0)|\Gamma\rangle\right)\right)$$

■

Another useful expression of this result is:

$$\mu + \mu^{-1} = 2\cos Z + \frac{\sin Z}{Z}\left(\sum_m \frac{\Gamma_m}{Z^2 - m^2\pi^2} - \frac{\eta(Z)}{\eta(p_0)}\frac{\Gamma_m}{p_0^2 - m^2\pi^2}\right) \tag{44}$$

The next proposition performs a few consistency checks.

Proposition 2 *The points $(Z = p_0, \mu_0^{\pm 1})$, and $(Z = Z_k, \mu_k^{\pm 1})$, all belong to the curve eq.(44).*

Proof. When $Z = p_0$, we find $\mu + \mu^{-1} = 2\cos p_0$, hence the curve passes through the point $\Lambda = 0, \mu_0^{\pm 1}$, as it should be. When $Z = Z_k$, recalling that $\eta(Z_k) = 0$, we find

$$\begin{aligned}
\mu + \mu^{-1} &= 2\cos Z_k + \frac{\sin Z_k}{Z_k}\sum_m \frac{1}{Z_k^2 - m^2\pi^2} M_{ml}^{-1}\frac{Z_l}{\sin Z_l}(\mu_l + \mu_l^{-1} - 2\cos Z_l) \\
&= 2\cos Z_k + \frac{\sin Z_k}{Z_k}\sum_m M_{km}M_{ml}^{-1}\frac{Z_l}{\sin Z_l}(\mu_l + \mu_l^{-1} - 2\cos Z_l) = \mu_k + \mu_k^{-1}
\end{aligned}$$

Hence the curve passes through the points $Z_k, \mu_k^{\pm 1}$. ■

We now show that the H_m in eq.(41) all Poisson commute. We need the following result:

Lemma 3 *One has*

$$\begin{aligned}
\{\Gamma_n, \Gamma_m\} &= 0 &(45)\\
\{\Gamma_n, \eta_m\} &= \{\Gamma_m, \eta_n\} &(46)\\
\{\eta_n, \eta_m\} &= 0 &(47)
\end{aligned}$$

Proof. Recall the definitions eqs.(37,40) of η_m and Γ_m.

$$\eta_m = M_{mk}^{-1}|1\rangle_k, \quad \Gamma_m = M_{mk}^{-1}|\tilde{\gamma}\rangle_k, \quad \tilde{\gamma}_k = \frac{Z_k}{\sin Z_k}\gamma_k$$

where M_{mk}^{-1} is the inverse of the matrix defined in eq.(36). The relation eq.(47) is obvious because the η_m depend only on the Z_k. Consider the second relation eq.(46):

$$\begin{aligned}
\{\eta_n, \Gamma_m\} &= \{M_{nk}^{-1}1_k, M_{ml}^{-1}\tilde{\gamma}_l\} = M_{m,l}^{-1}\{M_{nk}^{-1}, \tilde{\gamma}_l\}1_k \\
&= -M_{m,l}^{-1}M_{n,k'}^{-1}\{M_{k'p}, \tilde{\gamma}_l\}M_{pk}^{-1}|1\rangle_k = -M_{m,l}^{-1}M_{n,l}^{-1}\{M_{lp}, \tilde{\gamma}_l\}\eta_p
\end{aligned}$$

where in the last step we used that $\{M_{k'p}, \tilde{\gamma}_l\} = 0$ if $k' \neq l$. The result is obviously symmetric in m and n. Finally the first statement, eq.(45), is simple. One has

$$\{\Gamma_m, \Gamma_n\} = \{M_{mk}^{-1}\tilde{\gamma}_k, M_{nl}^{-1}\tilde{\gamma}_l\} = -M_{mr}^{-1}M_{nl}^{-1}\Big[\{M_{rs}, \tilde{\gamma}_l\} - \{M_{ls}, \tilde{\gamma}_r\}\Big]M_{sk}^{-1}\tilde{\gamma}_k$$

but because of the structure of M, we have $\{M_{rs}, \tilde{\gamma}_l\} = 0$ if $r \neq l$ and for $r = l$ the term in the square bracket obviously vanishes. This is a special case of a general theorem [17, 18]. ∎

We are now ready to prove

Proposition 3 *The quantities H_0, H_m, Poisson commute*

$$\{H_0, H_n\} = 0, \quad \{H_n, H_m\} = 0$$

Proof. Using eq.(42), one has

$$\{H_n, H_m\} = \eta_n(\{C, \Gamma_m\} + C\{C, \eta_m\}) - \eta_m(\{C, \Gamma_n\} + C\{C, \eta_n\})$$

where we denoted

$$C = \frac{1}{\eta(p_0)} \langle \chi(p_0) | \Gamma \rangle = \frac{1}{\eta(p_0)} \sum_l \frac{\Gamma_l}{p_0^2 - \pi^2 l^2}$$

One has

$$\{\Gamma_m, C\} = \frac{1}{\eta(p_0)} C \sum_l \frac{\{\Gamma_m, \eta_l\}}{p_0^2 - \pi^2 l^2}, \quad \{\eta_m, C\} = \frac{1}{\eta(p_0)} \sum_l \frac{\{\eta_m, \Gamma_l\}}{p_0^2 - \pi^2 l^2}$$

hence

$$\{C, \Gamma_m\} + C\{C, \eta_m\} = -\frac{1}{\eta(p_0)} C \sum_l \frac{\{\Gamma_m, \eta_l\} + \{\eta_m, \Gamma_l\}}{p_0^2 - \pi^2 l^2} = 0$$

∎

All this means that (Z_k, μ_k) are separated coordinates for the Hamiltonians H_m.

6 Continuum limit of the eigenvector.

Having found the continuum limit of the spectral curve, we now consider the limit of the eigenvector. Again, the continuum limit can be computed, the result being eq.(50).

As seen from eq.(29), the eigenvector can be written as (for i *odd*)

$$\psi_i = \frac{\sqrt{\mathcal{A}}}{\prod(z^2 - z_j^2)} \frac{\det N_i}{\sqrt{\det \Theta_i \det \Theta_{i+2}}}$$

where

$$N_i = \begin{pmatrix} \mu P_i(z) + P_{N+1-i}(z) & \mu P_1(z) & \cdots & -P_{N+1-i}(z) & \cdots & -P_2(z) \\ P_i(z_1) + \mu_1 P_{N+1-i}(z_1) & P_1(z_1) & \cdots & -\mu_1 P_{N+1-i}(z_1) & \cdots & -\mu_1 P_2(z_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_i(z_k) + \mu_j P_{N+1-i}(z_k) & P_1(z_k) & \cdots & -\mu_k P_{N+1-i}(z_k) & \cdots & -\mu_k P_2(z_k) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_i(z_n) + \mu_n P_{N+1-i}(z_n) & P_1(z_n) & \cdots & -\mu_n P_{N+1-i}(z_n) & \cdots & -\mu_n P_2(z_n) \end{pmatrix} \tag{48}$$

Compared to eq.(29), we have subtracted the i-th column to the first one for the same reason than in the previous section. Also we have used the variable z, z_k instead of λ, λ_k. Let us decompose the matrix N_i in blocs particularizing the first row and first column:

$$N_i = \begin{pmatrix} U_i & V_i \\ W_i & \Theta_i \end{pmatrix}$$

where

$$U_i \equiv \mu P_i(z) + P_{N+1-i}(z)$$
$$(W_i)_k \equiv \mu_k P_i(z_k) + P_{N+1-i}(z_k)$$
$$(V_i)_j = \mu P_j(z)\theta(i-j) - P_{N+1-j}(z)\theta(j-i), \quad i,j \;\; odd$$

To order zero in perturbation, we have $-(-1)^k P_j(z_k^{(0)}) = P_{N+1-j}(z_k^{(0)})$ so that

$$N_i^{(0)} = \begin{pmatrix} U_i & V_i \\ 0 & \Theta_i^{(0)} \end{pmatrix}$$

where $\Theta_i^{(0)}$ is the matrix eq.(30) evaluated on the free configuration. It is in fact independent of i and we will denote it by $\Theta^{(0)}$. The appearance of zero in the lower left corner was the reason for the subtraction in eq.(48) and makes things better behaved in perturbation.

The matrix $N_i^{(0)}$ being bloc triangular we can compute its inverse:

$$N_i^{(0)-1} = \begin{pmatrix} U_i^{-1} & -U_i^{-1} V_i \Theta^{(0)-1} \\ 0 & \Theta^{(0)-1} \end{pmatrix}$$

so that

$$N_i N_i^{(0)-1} = \begin{pmatrix} 1 & 0 \\ U_i^{-1} W_i & \Theta_i \Theta^{(0)-1} - U_i^{-1} W_i \otimes V_i \Theta^{(0)-1} \end{pmatrix}$$

Returning to the formula for ψ_i, we multiply all the matrices by $\Theta^{(0)-1}$. The factors $\det \Theta^{(0)-1}$ cancel between the numerator and denominator. We arrive at

$$\psi_i = \frac{\sqrt{\mathcal{A}}\;\, U_i}{\prod_k z^2 - z_k^2} \;\; \frac{\det\left(\Theta_i \Theta^{(0)-1} - U_i^{-1} W_i \otimes V_i \Theta^{(0)-1}\right)}{\sqrt{\det\left(\Theta_i \Theta^{(0)-1}\right) \det\left(\Theta_{i+2} \Theta^{(0)-1}\right)}}$$

We want to take the scaling limit of this expression. Again, the main trick is an explicit formula for the inverse of $\Theta^{(0)}$. It is not difficult to check that

$$(\Theta^{(0)-1})_{jk} = \frac{4}{N+1} \sin^2 \frac{k\pi}{N+1} \; P_j(z_k^{(0)})$$

Let us compute $\Theta_i \Theta^{(0)-1}$. Using the parametrization eq.(33) we find (recall that i is assumed to be odd)

$$(\Theta_i \Theta^{(0)-1})_{km} = \frac{4}{N+1} \frac{\sin \frac{m\pi}{N+1}}{\sin \frac{Z_k}{N+1}} \left\{ \sum_{j=1,\text{odd}}^{i-2} \sin \frac{jZ_k}{N+1} \sin \frac{jm\pi}{N+1} \right.$$
$$\left. -\mu_k \sum_{j=i,\text{odd}}^{N-1} \sin \frac{(N+1-j)Z_k}{N+1} \sin \frac{jm\pi}{N+1} \right\}$$

Defining $\Theta(x)$ as the scaling limit of $\Theta_i \Theta^{(0)-1}$, we find (there is a factor 1/2 because the sum is over j odd only)

$$(\Theta(x))_{km} = 2\frac{m\pi}{Z_k} \left\{ \int_0^x dy \sin Z_k y \sin m\pi y - \mu_k \int_x^1 dy \sin Z_k(1-y) \sin m\pi y \right\}$$

Similarly, we define $U(x, \Lambda, \mu)$ and $W_k(x)$ by

$$U_i = (N+1)U(x, \Lambda, \mu), \quad (W_i)_k = (N+1)W_k(x)$$

We find

$$U(x, \Lambda, \mu) = \mu \frac{\sin xZ}{Z} + \frac{\sin(1-x)Z}{Z}, \quad Z = \sqrt{\Lambda^2 + p_0^2}$$

and

$$W_k(x) = \frac{\sin Z_k x}{Z_k} + \mu_k \frac{\sin Z_k(1-x)}{Z_k}$$

Finally, we have (again there is a factor 1/2 because the sum is over j odd only)

$$(V_i \Theta^{(0)-1})_m = V_m(x, Z, \mu)$$

where

$$V_m(x, \Lambda, \mu) = 2m\pi \left\{ \int_0^x dy\ \mu \frac{\sin yZ}{Z} \sin m\pi y - \int_x^1 \frac{\sin(1-y)Z}{Z} \sin m\pi y \right\} \tag{49}$$

Putting everything together, we arrive at (up to a factor[6] independent of x)

Proposition 4

$$\psi(x, \Lambda, \mu) = U(x, \Lambda, \mu) - \langle V(x, \Lambda, \mu)|\Theta^{-1}(x)|W(x)\rangle \tag{50}$$

where we denoted by $\langle V(x, \Lambda, \mu)|$ the row vector with components $V_m(x, \Lambda, \mu)$ and by $|W(x)\rangle$ the column vector with components $W_k(x)$.

[6] In this factor we will include in particular $\frac{1}{\prod_k (1 - Z_k^2/Z^2)}$ which produces the poles at $Z^2 = Z_k^2$. This is important for the analyticity properties of $\psi(x)$ but plays little role for the considerations of this paper.

It is easy to show that the infinite sums involved in this formula converge under the conditions eq.(65) of Section 9.

Eq.(50) is the generalization of eq.(20). Here, the Λ and μ dependence is entirely contained in the function $U(x,\Lambda,\mu)$ and the vector $V(x,\Lambda,\mu)$. For the moment they are free complex parameters.

We now want to specialize to $\Lambda = 0, \mu_0 = e^{\pm ip_0}$. We have

$$U(x,\Lambda,\mu)|_{0,e^{\pm ip_0}} = \frac{\sin p_0}{p_0} e^{\pm ip_0 x}, \quad V_m(x,\Lambda,\mu)|_{0,e^{\pm ip_0}} = U(x,\Lambda,\mu)|_{0,e^{\pm ip_0}} \widetilde{V}_m^{(\pm)}(x,p_0)$$

where

$$\widetilde{V}_m^{(\pm)}(x,p_0) = -m\pi \left[\frac{e^{im\pi x}}{m\pi \pm p_0} + \frac{e^{-im\pi x}}{m\pi \mp p_0} \right]$$

Hence, up to a constant

$$\psi^{(\pm)}(x,p_0) = e^{\pm ip_0 x} \Big[1 - \langle \widetilde{V}^{(\pm)}(x,p_0)|\Theta^{-1}(x)|W(x)\rangle \Big]$$

These are the primary fields of CFT. Their logarithmic derivatives are the free fields of the Coulomb gaz representation. Notice that we have two such fields playing a completely symmetrical role: we go from one to the other by changing $p_0 \to -p_0$. This circumstance was recognized and used with great profit in [14]. The separated variables make this symmetry explicit and built in.

7 Schroedinger equation.

Having found a formula for the wave function $\psi(x,\Lambda,\mu)$, the next question is to find the potential in the Schroedinger equation that $\psi(x,\Lambda,\mu)$ is expected to satisfy. At this point it is simpler to forget the lattice model and work directly with eq.(50).

Let us denote by $|E(x)\rangle$ the vector with components

$$E_m(x) = 2m\pi \sin m\pi x$$

Calculating explicitly the integrals in eq.(49), we find (' denotes the derivative with respect to x)

$$V_m(x,\Lambda,\mu) = \frac{U(x,\Lambda,\mu)E'_m(x) - U'(x,\Lambda,\mu)E_m(x)}{Z^2 - m^2\pi^2}, \quad Z^2 = \Lambda^2 + p_0^2 \tag{51}$$

and similarly for $\Theta(x)$, we find

$$\Theta_{km}(x) = \frac{W_k(x)E'_m(x) - W'_k(x)E_m(x)}{Z_k^2 - m^2\pi^2} \tag{52}$$

Notice the important formulae

$$\langle V'(x,\Lambda,\mu)| = U(x,\Lambda,\mu)\langle E(x)|, \quad \Theta'(x) = |W(x)\rangle\langle E(x)|$$

The derivative of the matrix $\Theta(x)$ is a rank one projector. The matrix $\Theta(x)$ has a form familiar in the theory of integrable systems, and we know that it leads to non linear differential equations. Indeed, let us define the vector $|\Phi(x)\rangle$ by

$$|\Phi(x)\rangle = \Theta^{-1}(x)|W(x)\rangle \tag{53}$$

and let K be the diagonal matrix

$$K_{mm'} = m\pi\, \delta_{mm'}$$

Proposition 5 *The vector $|\Phi(x)\rangle$ satisfies the set of coupled non linear second order differential equations*

$$|\Phi''\rangle + 2(\langle\Phi|E\rangle)'|\Phi\rangle + K^2|\Phi\rangle = 0 \tag{54}$$

where

$$\langle\Phi|E\rangle(x) = \sum_m \Phi_m(x)E_m(x)$$

<u>Proof</u>. By derivation, and using the formula for Θ', we get

$$\begin{aligned} |W\rangle &= \Theta|\Phi\rangle \\ |W'\rangle &= \Theta'|\Phi\rangle + \Theta|\Phi'\rangle = \langle\Phi|E\rangle|W\rangle + \Theta|\Phi'\rangle \\ |W''\rangle &= \langle\Phi|E\rangle|W'\rangle + \Theta|\Phi''\rangle + (\langle\Phi|E\rangle' + \langle\Phi'|E\rangle)|W\rangle \end{aligned}$$

But we also have

$$\mathcal{Z}^2\Theta - \Theta K^2 = |W\rangle\langle E'| - |W'\rangle\langle E| \tag{55}$$

where we defined the diagonal matrix $\mathcal{Z}$

$$\mathcal{Z}_{kk'} = Z_k\delta_{kk'}$$

Applying this identity to $|\Phi\rangle$, we get

$$\mathcal{Z}^2\Theta|\Phi\rangle - \Theta K^2|\Phi\rangle = \langle E'|\Phi\rangle|W\rangle - \langle E|\Phi\rangle|W'\rangle$$

But

$$\mathcal{Z}^2\Theta|\Phi\rangle = \mathcal{Z}^2|W\rangle = -|W''\rangle$$

and therefore

$$|W''\rangle = \langle E|\Phi\rangle|W'\rangle - \langle E'|\Phi\rangle|W\rangle - \Theta K^2|\Phi\rangle$$

Comparing these two expressions of $|W''\rangle$, we find

$$\Theta|\Phi''\rangle + (\langle\Phi|E\rangle' + \langle\Phi'|E\rangle)|W\rangle = -\langle E'|\Phi\rangle|W\rangle - \Theta K^2|\Phi\rangle$$

Multiplying by Θ^{-1} yields eq.(54). ∎

We are now ready to find the Schroedinger equation satisfied by ψ.

Proposition 6 *The function $\psi(x,\Lambda,\mu)$ defined by eq.(50) satisfies the linear second order differential equation*

$$-\psi''(x,\Lambda,\mu) - [p_0^2 + 2(\langle E|\Phi\rangle)']\,\psi(x,\Lambda,\mu) = \Lambda^2\psi(x,\Lambda,\mu) \tag{56}$$

Proof. We have

$$\psi(x,\Lambda,\mu) = U(x,\Lambda,\mu) - \langle V(x,\Lambda,\mu)|\Theta^{-1}|W(x)\rangle = U(x,\Lambda,\mu) - \langle V(x,\Lambda,\mu)|\Phi(x)\rangle$$

Using the formula for $\langle V(x,\Lambda,\mu)|$, we get

$$\psi(x,\Lambda,\mu) = U\left(1 - \langle E'|\frac{1}{Z^2-K^2}|\Phi\rangle\right) + U'\langle E|\frac{1}{Z^2-K^2}|\Phi\rangle$$

Next, remembering that $U''(x,\Lambda,\mu) = -Z^2U(x,\Lambda,\mu)$, $|E''(x)\rangle = -K^2|E(x)\rangle$, we obtain

$$\psi'(x,\Lambda,\mu) = -U\left(\langle E|\Phi\rangle + \langle E'|\frac{1}{Z^2-K^2}|\Phi'\rangle\right) + U'\left(1 + \langle E|\frac{1}{Z^2-K^2}|\Phi'\rangle\right)$$

and

$$\begin{aligned}\psi''(x,\Lambda,\mu) \;\; &= \;\; -U\left(Z^2 + \langle E|\Phi'\rangle + (\langle E|\Phi\rangle)' + \langle E'|\frac{1}{Z^2-K^2}|\Phi''\rangle\right) \\ &\quad + U'\left(-\langle E|\Phi\rangle + \langle E|\frac{1}{Z^2-K^2}|\Phi''\rangle\right)\end{aligned}$$

Using now the equation for $|\Phi''\rangle$, we get eq.(56). ∎

The potential $T(x) = 2(\langle E|\Phi\rangle)'$ can also be written directly in terms of Θ. In fact, we have

$$\partial_x^2 \log\det\Theta = \partial_x \;\mathrm{Tr}\;\Theta^{-1}\Theta' = \partial_x\;\langle E|\Theta^{-1}W\rangle = \partial_x\langle E|\Phi\rangle$$

hence

$$T(x) = 2\partial_x\langle E|\Phi\rangle = 2\partial_x^2\log\det\Theta(x)$$

The Schroedinger equation therefore also reads

$$\psi''(x,\Lambda,\mu) + [p_0^2 + 2\,\partial_x^2\log\det\Theta]\;\psi(x,\Lambda,\mu) = -\Lambda^2\psi(x,\Lambda,\mu) \tag{57}$$

In this formula both the potential and the function $\psi(x,\Lambda,\mu)$ are known. The potential therefore belongs to the class of exactly solvable potentials. It is strongly reminiscent of the formula for finite zones potentials [20, 21, 22, 23]. It can probably also be obtained by an infinite sequence of Darboux transformations [24].

The parameter μ which enters the function $U(x,\Lambda,\mu)$ and the vector $\langle V(x,\Lambda,\mu)|$ was, up to now, a free parameter. Eq.(50) therefore provides two linearly independent solutions of eq.(56). We now introduce the spectral curve by imposing the quasiperiodicity of $\psi(x,\Lambda,\mu)$.

8 Bloch Waves and Spectral Curve.

So far, $\psi(x, \Lambda, \mu)$ was defined on the interval $[0, 1]$. We extends its definition by imposing

$$\psi(x + 1, \Lambda, \mu) = \mu\psi(x, \Lambda, \mu)$$

This extension is continuous as we now show.

Proposition 7

$$\psi(1, \Lambda, \mu) - \mu\psi(0, \Lambda, \mu) = 0$$

Proof. This follows immediately from

$$W_k(1) = \mu_k^{-1} W_k(0), \quad \Theta_{km}(1) = \mu_k^{-1}\Theta_{km}(0)(-1)^m \quad U(1, \Lambda, \mu) = \mu U(0, \Lambda, \mu)$$

∎

It is worth computing explicitly $\psi(0, \Lambda, \mu)$. In terms of the matrix M introduced in eq.(36), we have

$$\Theta_{km}(0) = W_k(0)M_{km}E'_m(0), \quad U(0, \Lambda, \mu) = \frac{\sin Z}{Z}, \quad V_m(0, \Lambda, \mu) = U(0, \Lambda, \mu)\frac{1}{Z^2 - m^2\pi^2}E'_m(0)$$

It follows that

$$\psi(0, \Lambda, \mu) = \frac{\sin Z}{Z}\left(1 - \sum_m \frac{\eta_m}{Z^2 - m^2\pi^2}\right) = \frac{\sin Z}{Z}\eta(Z) \tag{58}$$

where η_m and $\eta(Z)$ are defined in eqs.(37, 39). Notice that when $Z^2 = Z_k^2$, we have $\psi(0, \Lambda, \mu) = 0$, by definition of $\eta(Z)$.

We now turn to the derivative of $\psi(x, \Lambda, \mu)$.

Lemma 4 *One has*

$$\psi'(1, \Lambda, \mu) - \mu\psi'(0, \Lambda, \mu) = \mu\, \widetilde{\Gamma}(\Lambda, \mu)$$

$$\widetilde{\Gamma}(\Lambda, \mu) = \mu + \mu^{-1} - 2\cos Z - \frac{\sin Z}{Z}\sum_{m=1}^{\infty}\frac{E'_m(0)\Phi'_m(0) - E'_m(1)\Phi'_m(1)}{Z^2 - m^2\pi^2} \tag{59}$$

Proof. We have

$$U(1, \Lambda, \mu) = \mu\frac{\sin Z}{Z}, \quad U(0, \Lambda, \mu) = \frac{\sin Z}{Z}$$

and

$$U'(1, \Lambda, \mu) = \mu\cos Z - 1, \quad U'(0, \Lambda, \mu) = \mu - \cos Z$$

Using $E_k(1) = E_k(0) = 0$, we get

$$\psi'(1, \Lambda, \mu) = -U(1, \Lambda, \mu)\langle E'(1)|\frac{1}{Z^2 - K^2}|\Phi'(1)\rangle + U'(1, \Lambda, \mu)$$

$$\psi'(0,\Lambda,\mu) = -U(0,\Lambda,\mu)\langle E'(0)|\frac{1}{Z^2-K^2}|\Phi'(0)\rangle + U'(0,\Lambda,\mu)$$

From this the result follows. ■

At this point it is tempting to identify the spectral curve as $\widetilde{\Gamma}(\Lambda,\mu) = 0$. However, this cannot be correct because the point $\Lambda = 0, \mu = \mu_0$ does not belong to it.

We have to change the Schroedinger equation. The only possible modification is at the edges. We consider therefore the equation

$$\psi''(x,\Lambda,\mu) + \Big[p_0^2 + 2\langle E\Phi\rangle' + H_0\delta(x)\Big]\psi(x,\Lambda,\mu) = -\Lambda^2\psi(x,\Lambda,\mu) \tag{60}$$

The bulk formula for $\psi(x,\Lambda,\mu)$ does not change. The continuity of $\psi(x,\Lambda,\mu)$ at $x=1$ still holds, but the derivative now has a discontinuity

$$\int_{1-}^{1+} dx\ \psi'' + H_0\psi(1) = 0$$

Using

$$\psi(1,\Lambda,\mu) = \mu\ \eta(Z)\frac{\sin Z}{Z}$$

the Bloch condition becomes

$$\psi'(1,\Lambda,\mu) - \mu\psi'(0,\Lambda,\mu) - H_0\mu\ \eta(Z) = 0$$

that is

$$\mu + \mu^{-1} = 2\cos Z + \frac{\sin Z}{Z}\left(\sum_m \frac{\Gamma_m}{Z^2 - m^2\pi^2} - H_0\eta(Z)\right)$$

where we have set

$$\Gamma_m = E'_m(0)\Phi'_m(0) - E'_m(1)\Phi'_m(1) \tag{61}$$

We now determine the coefficient H_0 by requiring that the curve passes through the points $p_0, \mu_0^{\pm1}$. We find

$$H_0 = \frac{1}{\eta(p_0)}\sum_m \frac{\Gamma_m}{p_0^2 - m^2\pi^2}$$

Hence the curve takes the form

$$\mu + \mu^{-1} = 2\cos Z + \frac{\sin Z}{Z}\left(\sum_m \frac{\Gamma_m}{Z^2 - m^2\pi^2} - \frac{\eta(Z)}{\eta(Z_0)}\frac{\Gamma_m}{p_0^2 - m^2\pi^2}\right)$$

In order to compare with eq.(44), we must compute Γ_m. We have

$$W_k(0) = \mu_k\frac{\sin Z_k}{Z_k}, \quad W_k(1) = \frac{\sin Z_k}{Z_k}, \quad W'_k(0) = 1 - \mu_k\cos Z_k, \quad W'_k(1) = \cos Z_k - \mu_k$$

and

$$\Theta_{km}(0) = \frac{W_k(0)E'_m(0)}{Z_k^2 - \pi^2 m^2} = W_k(0)M_{km}E'_m(0), \quad \Theta_{km}(1) = \frac{W_k(1)E'_m(1)}{Z_k^2 - \pi^2 m^2} = W_k(1)M_{km}E'_m(1)$$

where M is the matrix introduced in eq.(36). Since $|\Phi'(0)\rangle = \Theta^{-1}(0)|W'(0)\rangle$ and $|\Phi'(1)\rangle = \Theta^{-1}(1)|W'(1)\rangle$, one has

$$E'_m(0)\Phi'_m(0) = \sum_k M^{-1}_{mk}\,(\mu_k^{-1} - \cos Z_k)\frac{Z_k}{\sin Z_k}, \quad E'_m(1)\Phi'_m(1) = \sum_k M^{-1}_{mk}\,(\cos Z_k - \mu_k)\frac{Z_k}{\sin Z_k}$$

hence

$$\Gamma_m = E'_m(0)\Phi'_m(0) - E'_m(1)\Phi'_m(1) = \sum_k M^{-1}_{mk}\,\frac{Z_k}{\sin Z_k}\gamma_k$$

where we recall that $\gamma_k = \mu_k + \mu_k^{-1} - 2\cos Z_k$. This is exactly eq.(40) and shows that we have recovered precisely the spectral curve eq.(44).

Finally, let us close this section by proving the

Proposition 8 *The function $\psi(x,\Lambda,\mu)$ has no pole at the point $Z = Z_k, \mu_k^{-1}$*

Proof. Here we restaure the factor $\frac{1}{\prod_k(1-Z_k^2/Z^2)}$.

$$U(x,\Lambda,\mu)|_{Z=Z_k,\mu_k^{-1}} = \mu_k^{-1}W_k(x), \Longrightarrow V_{k'}(x,\Lambda,\mu)|_{Z=Z_k,\mu_k^{-1}} = \mu_k^{-1}\Theta_{kk'}(x)$$

$$\psi(x,\Lambda,\mu)|_{Z=Z_k,\mu_k^{-1}} \simeq \frac{1}{\prod_k(1-Z_k^2/Z^2)}\ \mu_k^{-1}\Big[W_k(x) - \Theta_{kk'}(x)\Theta^{-1}_{k'k''}(x)W_{k''}(x)\Big] = \text{regular}$$

∎

This shows that the same property for the eigenvector on the lattice has been preserved when taking the continuum limit.

9 Perturbation theory.

In the previous sections, we have manipulated determinants of infinite matrices quite freely. It is necessary now to investigate the conditions for the existence of the determinant $\det\Theta(x)$. We recall the free configuration eq.(16). In the scaled variables it reads

$$Z_k^{(0)} = k\pi, \quad \mu_k^{(0)} = (-1)^k$$

By construction, when $(Z_k,\mu_k) = (Z_k^{(0)},\mu_k^{(0)})$, we have $\Theta(x) = \text{Id}$, so that $\det\Theta(x) = 1$. Clearly for $\det\Theta(x)$ to exist, we have to assume $(Z_k,\mu_k) \to (Z_k^{(0)},\mu_k^{(0)})$ when $k \to \infty$. Hence we set

$$Z_k = k\pi + \delta Z_k, \quad \mu_k = (-1)^k(1+\delta\mu_k) \tag{62}$$

It is not difficult to see that to leading order in $(\delta Z_k, \delta\mu_k)$, we have

$$W_k(x) = \frac{1}{k\pi}(\delta Z_k \cos k\pi x - \delta\mu_k \sin k\pi x)$$

and this implies

$$\begin{aligned}\Theta_{k,m}(x) &= \frac{2m\pi}{k(Z_k^2 - \pi^2 m^2)}\Big\{\delta Z_k(m\cos k\pi x\cos m\pi x + k\sin k\pi x\sin m\pi x) \\ &\qquad +\delta\mu_k(-m\sin k\pi x\cos m\pi x + k\cos k\pi x\sin m\pi x)\Big\}\end{aligned} \tag{63}$$

Notice that when $m = k$ this formula gives that to leading order $\Theta_{k,k}(x) = 1$, as it should be.

A first consequence of these formulae is that if $\delta Z_k = 0, \delta\mu_k = 0$, beyond a certain index $k = k_{max}$, then for $k > k_{max}$ we have $W_k(x) = 0$, $\Theta_{k,k}(x) = 1$, and $\Theta_{k,m}(x) = 0, \forall m \neq k$. As a result only the first block of size $k_{max} \times k_{max}$ of the matrix $\Theta(x)$ plays a role and all the constructions of the previous sections reduce to finite size matrices and vectors.

If however we want to retain an infinite number of modes in order to keep the field theoretical character of the model, one has to say something about the rate at which δZ_k and $\delta\mu_k$ tend to zero when $k \to \infty$. Disregarding a finite number of possibly large $\delta Z_k, \delta\mu_k$ which play no role in these convergence questions, we may assume that $\Theta(x)$ is given by eq.(63). As we have seen, it is of the form

$$\Theta(x) = \mathrm{Id} + \widetilde{\Theta}(x)$$

where $\widetilde{\Theta}(x)$ is small. In fact, bounding the trigonometric functions by 1, we have

$$|\widetilde{\Theta}_{k,m}(x)| \leq c\frac{m}{k|k-m|}(|\delta Z_k| + |\delta\mu_k|), \quad m \neq k$$

Since $|k - m| \geq 1$ when $k \neq m$, we may write as well

$$|\widetilde{\Theta}_{k,m}(x)| \leq c\frac{m}{k}(|\delta Z_k| + |\delta\mu_k|), \quad m \neq k$$

It is not difficult to see that this formula is valid also for $m = k$ (we have to adapt the constant c). It follows that

$$\begin{aligned}\log\det\Theta(x) &= \mathrm{Tr}\log(1 + \widetilde{\Theta}(x)) \\ &\leq \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}|\widetilde{\Theta}(x)|^n \leq \sum_{n=1}^{\infty}\frac{c^n}{n}\left(\sum_k\Big(|\delta Z_k| + |\delta\mu_k|\Big)\right)^n\end{aligned} \tag{64}$$

Hence a sufficient condition for the existence of the determinant is that the series $\sum_k \Big(|\delta Z_k| + |\delta\mu_k|\Big)$ converges. This is achieved if

$$|\delta Z_k| + |\delta\mu_k| < \frac{c'}{k^{1+\epsilon}}, \quad \epsilon > 0 \tag{65}$$

One can then adjust the constant c' such that the series in eq.(64) converges. The condition eq.(65) ensures that $\log\det\Theta(x)$ exists. To build the potential $u(x)$ however, this function has to be twice differentiable in x and this may require stronger conditions on ϵ.

Now that we have found an expression for the potential $u(x)$ in terms of a countable set of variables Z_k, μ_k, we would like to check the Virasoro Poisson bracket directly. Recall that

$$u(x) = p_0^2 + T(x) + H_0\delta(x), \quad T(x) = 2\partial_x\langle E|\Phi\rangle = \sum_n T_n e^{2i\pi x}$$

Notice first that $T(x)$ has no Fourier component T_0:

$$T_0 = \int_0^1 dxT(x) = 2\int_0^1 dx(\langle E|\Phi\rangle)' = 2(\ \langle E(1)|\Phi(1)\rangle - \langle E(0)|\Phi(0)\rangle\ ) = 0$$

where we used that $E_m(0) = E_m(1) = 0$. The Fourier expansion of the potential $u(x)$ reads

$$u(x) = \sum_n L_n e^{2i\pi nx} = p_0^2 + \sum_n (T_n + H_0)e^{2i\pi nx}$$

we must therefore identify

$$L_0 = p_0^2 + H_0 = p_0^2 + \frac{1}{\eta(Z_0)}\sum_m \frac{\Gamma_m}{p_0^2 - m^2\pi^2} \tag{66}$$

and

$$L_n = T_n + H_0, \quad \Longrightarrow T_n = L_n - L_0 + p_0^2, \quad n \neq 0$$

If $u(x)$ has Poisson bracket eq.(4), the algebra of the L_n reads

$$\{L_n, L_m\} = 8i\pi(n-m)L_{n+m} - 16i\pi^3 n^3 \delta_{n+m,0}$$

The Poisson algebra for the T_n is then closed:

$$\{T_n, T_m\} = 8i\pi(n-m)T_{n+m} - 8i\pi n T_n + 8i\pi m T_m - 16i(\pi^3 n^3 - \pi p_0^2 n)\delta_{n+m,0}$$

or, in a form that will be useful later

$$\{T(x), T(y)\} = 2\delta'''(x-y) + 4(2p_0^2 + T(x) + T(y))\delta'(x-y) - 4T'(x)\delta(y) + 4T'(y)\delta(x) \tag{67}$$

In this section, we consider the situation where all the variables (Z_k, μ_k) are close to the free configuration as in eq.(62), and we perform a perturbation theory in $\delta Z_k, \delta\mu_k$. We have seen that to lowest order $\Theta^{(0)}(x) = \mathrm{Id}$ by construction. So, we can write the expansion

$$\begin{aligned} \Theta(x) &= \mathrm{Id} + \Theta^{(1)}(x) + \Theta^{(2)}(x) + \cdots \\ |W(x)\rangle &= |W^{(1)}(x)\rangle + |W^{(2)}(x)\rangle + \cdots \end{aligned}$$

where we have taken into account that $|W^{(0)}(x)\rangle = 0$. It follows that

$$|\Phi(x)\rangle = |\Phi^{(1)}(x)\rangle + |\Phi^{(2)}(x)\rangle + \cdots$$

where

$$|\Phi^{(1)}\rangle = |W^{(1)}\rangle, \quad |\Phi^{(2)}\rangle = |W^{(2)}\rangle - \Theta^{(1)}|W^{(1)}\rangle, \cdots$$

To lowest order, we find easily

$$T^{(1)}(x) = 2\langle E|\Phi^{(1)}\rangle' = 4\sum_k k\pi(\delta Z_k \cos 2k\pi x - \delta\mu_k \sin 2k\pi x)$$

This shows in particular that δZ_k and $\delta \mu_k$ are just the Fourier components of the potential in this first approximation. We see here clearly that for $T^{(1)}(x)$ to exist as a function (and not just as a distribution), we need $\epsilon > 1$ in eq.(65). The Poisson bracket eq.(3) becomes to leading order

$$\{\delta Z_k, \delta \mu_k\} = 2k\pi \left(1 - \frac{p_0^2}{k^2\pi^2}\right)$$

To define modes independent of the zero mode p_0 we introduce

$$a_k = \alpha_k(\delta\mu_k - i\delta Z_k), \quad a_k^\dagger = \bar{\alpha}_k(\delta\mu_k + i\delta Z_k)$$

where the coefficients α_k, $\bar{\alpha}_k$ satisfy[7]

$$\alpha_k \bar{\alpha}_k = \frac{1}{4\pi} \frac{k^2\pi^2}{p_0^2 - k^2\pi^2}$$

With this choice, one has

$$\{a_k, a_k^\dagger\} = ik$$

we can then rewrite

$$T^{(1)}(x) = 2i \sum_k k\pi \left(\frac{a_k}{\alpha_k} e^{2ik\pi x} - \frac{a_k^\dagger}{\bar{\alpha}_k} e^{-2ik\pi x} \right)$$

It is now straightforward to compute the Poisson bracket

$$\{T^{(1)}(x), T^{(1)}(y)\} \;=\; 2\delta'''(x-y) + 8p_0^2 \delta'(x-y)$$

This is the correct result for the Virasoro Poisson bracket in this approximation. Notice that the term $\delta'''(x-y)$ is exact already at this level. Higher order terms cannot contribute to it.

Next, we look at the conserved quantities. The leading terms in the expansions of η_m and H_m are easy to find:

$$\eta_m \simeq 2m\pi\, \delta Z_m, \quad \Gamma_m \simeq 2m^2\pi^2(\delta\mu_m^2 + \delta Z_m^2)$$

To see it, consider the defining relations of η_m

$$\sum_m \frac{\eta_m}{Z_k^2 - \pi^2 m^2} = 1, \forall k$$

When Z_k is given by eq.(62), the dominant term in the above sum is $m = k$. The equation becomes $\eta_k/(2k\pi\delta Z_k) = 1$. The same argument starting with the equation of the spectral curve, eq.(41), taken at the point Z_k, μ_k which belongs to it, yields the formula for H_m. Remark that the condition eq.(65) ensures that the sums in the definition of the function $\eta(Z)$, eq.(39), or in the definition of the spectral curve, eq.(41), are convergent.

Written with the oscillators $a_m, a_m^\dagger$, we find

$$H_m = 8\pi(p_0^2 - \pi^2 m^2) a_m^\dagger a_m, \quad H_0 = 8\pi \sum_m a_m^\dagger a_m$$

[7]It is known [14] that the poles at $p_0^2 = k^2\pi^2$ are classical remnants of the zeroes of the Kac determinant.

It is clear that the H_m are in involution at this order. As we see, in first approximation, the dynamical system reduces to a set of decoupled harmonic oscillators. The generator L_0 is given by

$$L_0 = p_0^2 + 8\pi \sum_m a_m^\dagger a_m$$

It is easy to verify that

$$\{L_0, T^{(1)}(x)\}_0 = -4\partial_x T^{(1)}(x)$$

These perturbative arguments are good indications that $u(x)$ indeed satisfies the Virasoro Poisson bracket. Clearly we will not go very far in perturbation and we now look for a more formal proof of this fact. For that purpose, we need some preparation.

10 Some identities.

Before computing Poisson brackets to check the Virasoro algebra, we collect a number of useful identities. We start with a formula for the inverse matrix $\Theta^{-1}(x)$. It has the same form as $\Theta(x)$.

Proposition 9 *Let us define*

$$\langle F| = \langle E|\Theta^{-1} \tag{68}$$

Then, we can write

$$\Theta^{-1}_{mk} = \frac{\Phi_m F'_k - \Phi'_m F_k}{Z_k^2 - \pi^2 m^2} \tag{69}$$

Proof. Multiplying eq.(55) on both sides by Θ^{-1}, we get

$$\Theta^{-1}\mathcal{Z}^2 - K^2\Theta^{-1} = \Theta^{-1}|W\rangle\langle E'|\Theta^{-1} - \Theta^{-1}|W'\rangle\langle E|\Theta^{-1}$$

and so

$$\Theta^{-1}_{mk} = \frac{(\Theta^{-1}|W'\rangle)_m(\langle E|\Theta^{-1})_k - (\Theta^{-1}|W\rangle)_m(\langle E'|\Theta^{-1})_k}{\pi^2 m^2 - Z_k^2}$$

But

$$\Theta^{-1}|W'\rangle = |\Phi'\rangle + \langle E|\Phi\rangle|\Phi\rangle, \quad \langle E'|\Theta^{-1} = \langle F'| + \langle E|\Phi\rangle\langle F|$$

Plugging into the above formula, we obtain eq.(69). ∎

Proposition 10 *The vector $\langle F|$ satisfies a set of differential equations.*

$$\langle F''| + 2(\langle E|\Phi\rangle)'\langle F| + \langle F|\mathcal{Z}^2 = 0 \tag{70}$$

Proof. The proof is the same as for $|\Phi\rangle$.

From this we easily deduce

$$(\Theta^{-1})' = -|\Phi\rangle\langle F|$$

which can also be proved using the similar property of Θ. Let us define

$$A_k(x) = \sum_m \frac{E_m(x)\Phi_m(x)}{Z_k^2 - m^2\pi^2}, \quad B_k(x) = \sum_m \frac{E'_m(x)\Phi_m(x)}{Z_k^2 - m^2\pi^2}$$

$$C_k(x) = \sum_m \frac{E_m(x)\Phi_m(x)}{(Z_k^2 - m^2\pi^2)^2}, \quad D_k(x) = \sum_m \frac{E'_m(x)\Phi_m(x)}{(Z_k^2 - m^2\pi^2)^2}$$

Proposition 11 *We have the identity*

$$(1 - B_k)W_k + A_k W'_k = 0 \tag{71}$$

<u>Proof</u>. This is just a rewriting of $|W\rangle = \Theta|\Phi\rangle$ using eq.(52) for $\Theta(x)$. ■

Proposition 12 *The following two identities hold*

$$(1 - B_k + A'_k)F_k - A_k F'_k = 0 \tag{72}$$

$$(B'_k + Z_k^2 A_k)F_k + (1 - B_k)F'_k = 0 \tag{73}$$

<u>Proof</u>. The first identity is a rewriting of $\langle F| = \langle E|\Theta^{-1}$ using eq.(69) for $\Theta^{-1}(x)$. The second identity is just a rewriting of $\langle F'| = \langle E'|\Theta^{-1} - \langle E|\Phi\rangle\langle F|$. ■

The above two identities form a linear system for F_k and F'_k. Its compatibility implies the following:

Proposition 13

$$(1 - B_k)^2 + A_k B'_k - A'_k B_k + A'_k + Z_k^2 A_k^2 = 0 \tag{74}$$

Let us define

$$\widetilde{F}_k = W'_k(1 - B_k) + W''_k A_k \tag{75}$$

An important consequence of eqs.(72,73) is

Proposition 14 *The functions $F_k(x)$ and $\widetilde{F}_k(x)$ are proportional.*

$$F_k(x) = -\frac{\zeta_k}{\mu_k \gamma_k} \widetilde{F}_k(x) \tag{76}$$

The proportionality coefficient is written in this specific way for later convenience. The quantity γ_k is the one defined in eq.(34).

Proof. Let us compute the Wronskian

$$
\begin{aligned}
Wr(F_k,\widetilde{F}_k) &= F_k\Big(W_k''(1-B_k)-W_k'B_k'+W_k'''A_k+W_k''A_k'\Big)-F_k'\Big(W_k'(1-B_k)+W_k''A_k\Big)\\
&= -\Lambda_k^2 W_k\Big((1-B_k+A_k')F_k-A_kF_k'\Big)-W_k'\Big((B_k'+Z_k^2A_k)F_k+(1-B_k)F_k'\Big)=0
\end{aligned}
$$

■

Proposition 15 *The following quantities are constants independent of x*

$$
\eta_m=\Phi_mE_m'-\Phi_m'E_m-<E\Phi>\Phi_mE_m-\sum_{p\neq m}\frac{1}{\pi^2(p^2-m^2)}(\Phi_m\Phi_p'-\Phi_m'\Phi_p)(E_m'E_p-E_mE_p') \quad (77)
$$

The quantities η_m defined in eq.(77) are in fact the same as the ones introduced in eq.(37).

Proof. To prove the first statement, just take the derivative with respect to x and use eq.(54). To prove the second statement, use the η_m defined in eq.(77) to rewrite eq.(74) as

$$
\sum_m\frac{\eta_m}{(Z_k^2-m^2\pi^2)}=1,\quad \forall\, k \quad (78)
$$

which is exactly the same as eq.(37). ■

A straightforward consequence is the "trace" formula that will be useful later:

$$
\langle E\Phi'\rangle-\langle E'\Phi\rangle+\langle E\Phi\rangle^2=-\sum_m\eta_m \quad (79)
$$

We now compute the coefficients ζ_k appearing in eq.(76)

Proposition 16 *The coefficients ζ_k in eq.(76) are determined by the set of equations*

$$
\sum_k\frac{\zeta_k}{Z_k^2-m^2\pi^2}=1,\quad \forall\, m \quad (80)
$$

These equations are dual to eq.(78).

Proof. Start with

$$
\langle F|W\rangle=\langle E|\Theta^{-1}|W\rangle=\langle E|\Phi\rangle
$$

Using eq.(71, 75,76), we have

$$
F_kW_k=-\frac{\zeta_k}{\mu_k\gamma_k}(-W_k'^2+W_k''W_k)A_k=\zeta_kA_k \quad (81)
$$

hence

$$
\langle F|W\rangle=\sum_k\zeta_kA_k=\sum_m\left(\sum_k\frac{\zeta_k}{Z_k^2-m^2\pi^2}\right)E_m\Phi_m=\langle E|\Phi\rangle=\sum_mE_m\Phi_m
$$

Since this has to hold for all x, the only possibility is eq.(80). ■

We collect below a few more identities of the type of eq.(81) that will be important later

Proposition 17

$$\begin{aligned} F_k W_k &= \zeta_k \, A_k && (82)\\ F_k W_k' &= -\zeta_k \, (1 - B_k) && (83)\\ F_k' W_k &= \zeta_k \, (1 - B_k + A_k') && (84)\\ F_k' W_k' &= \zeta_k \, (B_k' + Z_k^2 A_k) && (85) \end{aligned}$$

Next, we relate the η_m and ζ_k

Proposition 18 *The following relation holds:*

$$\sum_m \frac{\eta_m}{(Z_k^2 - m^2\pi^2)^2} = \frac{1}{\zeta_k}$$

Proof. We start from

$$\sum_m \Theta_{km} \Theta^{-1}_{mk'} = \delta_{kk'}$$

When $k = k'$ this gives

$$W_k F_k' D_k - W_k F_k (D_k' - A_k + Z_k^2 C_k) - W_k' F_k' C_k + W_k' F_k (C_k' - D_k) = 1$$

or using eqs.(82-85)

$$\zeta_k \Big\{ 2D_k(1 - B_k) + D_k A_k' - D_k' A_k + A_k^2 - 2Z_k^2 C_k A_k - C_k B_k' - C_k'(1 - B_k) \Big\} = 1$$

Expanding this formula using $(p \neq m)$

$$\begin{aligned} \frac{1}{(Z_k^2 - \pi^2 m^2)^2 (Z_k^2 - \pi^2 p^2)} &= -\frac{1}{\pi^2 (p^2 - m^2)} \frac{1}{(Z_k^2 - \pi^2 m^2)^2} \\ &\quad -\frac{1}{\pi^4 (p^2 - m^2)^2} \left\{ \frac{1}{(Z_k^2 - \pi^2 m^2)} - \frac{1}{(Z_k^2 - \pi^2 p^2)} \right\} \end{aligned}$$

we get the result with η_m represented by eq.(77). ∎

An immediate and important consequence is an expansion of the function $\eta(Z)$ near $Z^2 = Z_k^2$

$$\eta(Z) = (Z^2 - Z_k^2) \frac{1}{\zeta_k} + \cdots \tag{86}$$

Returning to the formula for $\psi(x, \Lambda, \mu)$, we can write it as

$$\psi(x, \Lambda, \mu) = \frac{\mu - e^{-iZ}}{2iZ} w(x, Z) - \frac{\mu - e^{iZ}}{2iZ} w^*(x, Z)$$

where we define

$$w(x,Z) = e^{iZx}\left(1-\langle E'|\frac{1}{Z^2-K^2}|\Phi\rangle+\langle E|\frac{iZ}{Z^2-K^2}|\Phi\rangle\right) \quad (87)$$

$$w^*(x,Z) = e^{-iZx}\left(1-\langle E'|\frac{1}{Z^2-K^2}|\Phi\rangle-\langle E|\frac{iZ}{Z^2-K^2}|\Phi\rangle\right) \quad (88)$$

The functions $w(x,Z)$ and $w^*(x,Z)$ are defined on the Riemann sphere with a puncture at ∞. One can compute the Wronskian

$$w'w^* - w^{*\prime}w = 2iZ\eta(Z)$$

This Wronskian vanishes precisely when $Z^2 = Z_k^2$. Hence, when $Z = Z_k$, $w(x,Z_k)$ becomes proportional to $w^*(x,Z_k)$. Indeed we have

$$\begin{aligned} w(x,Z_k) &= e^{iZ_kx}(1-B_k+iZ_kA_k) \\ w^*(x,Z_k) &= e^{-iZ_kx}(1-B_k-iZ_kA_k) \end{aligned}$$

Then eq.(71) can be rewritten as

$$w(x,Z_k) = \frac{\alpha_k^{(+)}}{\alpha_k^{(-)}}w^*(x,Z_k) \quad (89)$$

where

$$\alpha_k^{(\pm)} = 1-\mu_k e^{\pm iZ_k}, \quad \alpha_k^{(+)}\alpha_k^{(-)} = \mu_k\gamma_k \quad (90)$$

The function $\widetilde{F}_k(x)$ introduced in eq.(75) is solution of eq.(70). It is not difficult to see that the other solution of this equation is

$$\widetilde{G}_k = A_kW_k' + 2Z_k(W_kD_k - W_k'C_k) + x\widetilde{F}_k$$

In terms of $w(x,Z), w^*(x,Z)$ defined in eqs.(87,88), we have

$$\widetilde{F}_k = \frac{1}{2}(\alpha_k^{(-)}w|_{Z_k}+\alpha_k^{(+)}w^*|_{Z_k}) = \alpha_k^{(-)}w|_{Z_k} = \alpha_k^{(+)}w^*|_{Z_k} \quad (91)$$

$$\widetilde{G}_k = -\frac{i}{2}(\alpha_k^{(-)}\partial_Z w|_{Z_k} - \alpha_k^{(+)}\partial_Z w^*|_{Z_k}) \quad (92)$$

This will play an important role below.

11 Virasoro algebra.

We are now ready to compute the Poisson bracket $\{T(x),T(y)\}$. The result is precisely the algebra of the $T_n = L_n - L_0$, eq.(67).

Proposition 19 *Let* $T(x) = 2\partial_x\langle E(x)|\Phi(x)\rangle$, *and* $X(x), Y(x)$ *be two arbitrary test functions. Then we have*

$$\begin{aligned}\left\{\int_0^1 dxX(x)T(x), \int_0^1 dyY(y)T(y)\right\} &= -\int_0^1 dx(X'''Y - XY''') - 4\int_0^1 dx(X'Y - XY')(p_0^2 + T) \\ &+4\int_0^1 dxX(x)\delta(x)\int_0^1 dyY(y)T'(y) - 4\int_0^1 dxX(x)T'(x)\int_0^1 dyY(y)\delta(y) \qquad (93)\end{aligned}$$

The proof is rather long and we will split it into several lemmas. Since $|\Phi\rangle = \Theta^{-1}|W\rangle$, we have

$$\{|\Phi\rangle_1, |\Phi\rangle_2\} = \Theta_1^{-1}\Theta_2^{-1}\Big[\{\Theta_1, \Theta_2\}|\Phi_1\rangle|\Phi_2\rangle - \{\Theta_1, W_2\}|\Phi_1\rangle - \{W_1, \Theta_2\}|\Phi_2\rangle + \{W_1, W_2\}\Big]$$

where the index $1, 2$ refers to the customary tensor notation. In this expression, all Poisson brackets can be computed explicitly. Using the fact that rows with different indices in Θ and W Poisson commute and using only eq.(71), we arrive at

$$\{\Phi_m(x), \Phi_n(y)\} = -2\sum_k \Theta_{mk}^{-1}(x)\Theta_{nk}^{-1}(y)\frac{\sin Z_k}{Z_k\gamma_k}\left(1 - \frac{p_0^2}{Z_k^2}\right)\left[\widetilde{F}_k(x)\widetilde{G}_k(y) - \widetilde{F}_k(y)\widetilde{G}_k(x)\right] \qquad (94)$$

where γ_k are defined in eq.(34). Multypliying by $E_m(x)E_n(y)$ and remembering that $\langle E|\Theta^{-1} = \langle F|$ we get

$$\{\langle E(x)|\Phi(x)\rangle, \langle E(y)|\Phi(y)\rangle\} = 2\sum_k \frac{\zeta_k^2 \sin Z_k}{Z_k\gamma_k^3\mu_k^2}\left(1 - \frac{p_0^2}{Z_k^2}\right) \times \Big(\mathcal{A}_k(y)\mathcal{B}_k(x) - \mathcal{A}_k(x)\mathcal{B}_k(y)\Big) \qquad (95)$$

where

$$\mathcal{A}_k(x) = \widetilde{F}_k^2(x), \quad \mathcal{B}_k(x) = \widetilde{F}_k(x)\widetilde{G}_k(x)$$

Using eqs.(87,88), we can write

$$\begin{aligned}\mathcal{A}_k(x) &= \mu_k\gamma_k\, w(x, Z)w^*(x, Z)|_{Z=Z_k} \\ \mathcal{B}_k(x) &= \frac{\mu_k\gamma_k}{2i}(w^*(x, Z)\partial_Z w(x, Z) - w(x, Z)\partial_Z w^*(x, Z))|_{Z=Z_k}\end{aligned}$$

The strategy to evaluate the right hand side of eq.(95) is to rewrite it as a sum over the residues of certain poles of a function on the Riemann Z-sphere. This sum can then be transformed as a sum over the residues of the other poles (there will be none in our case) plus a integral over a small circle at infinity surrounding an essential singularity. This last integral can then be evaluated using the known asymptotics of the function. Let us define

$$a(Z) = \frac{Z}{2i\sin Z}e^{-iZ}, \quad b(Z) = -\frac{Z}{2i\sin Z}\cos Z, \quad c(Z) = \frac{Z}{2i\sin Z}e^{iZ} \qquad (96)$$

and introduce the functions

$$\begin{aligned}\Omega_1 &= w(x, Z)w^*(y, Z) - w^*(x, Z)w(y, Z) \\ \Omega_2 &= a(Z)w(x, Z)w(y, Z) \\ &\quad +b(Z)(w(x, Z)w^*(y, Z) + w^*(x, Z)w(y, Z)) \\ &\quad +c(Z)w^*(x, Z)w^*(y, Z)\end{aligned}$$

These functions are defined on the Riemann Z-sphere have poles at the points $\pm m\pi$ and have an essential singularity at infinity. Let

$$\Omega = \Omega_1 \Omega_2$$

and recall the definition of $\eta(Z)$ eq.(39).

Lemma 5

$$\sum_{\pm Z_k} \mathrm{Res} \frac{\Omega}{\eta^2(Z)} \left(1 - \frac{p_0^2}{Z^2}\right) = -2 \sum_k \frac{\zeta_k^2 \sin Z_k}{Z_k \gamma_k^3 \mu_k^2} \left(1 - \frac{p_0^2}{Z_k^2}\right) \times \Big(\mathcal{A}_k(y)\mathcal{B}_k(x) - \mathcal{A}_k(x)\mathcal{B}_k(y)\Big) \tag{97}$$

Proof. The factor $\eta^2(Z)$ introduces double poles at $\pm Z_k$ because $\eta(Z_k) = 0$. However using eq.(89), we see immediately that $\Omega_1|_{\pm Z_k} = 0$, so that the poles are in fact simple. Remembering eq.(86), we have

$$\sum_{\pm Z_k} \mathrm{Res} \frac{\Omega}{\eta^2(Z)} \left(1 - \frac{p_0^2}{Z^2}\right) = \sum_k \frac{\zeta_k^2}{4Z_k^2} \left(1 - \frac{p_0^2}{Z_k^2}\right) \Big(\partial_Z \Omega|_{Z_k} + \partial_Z \Omega|_{-Z_k}\Big)$$

We need to compute $\partial_Z \Omega|_{\pm Z_k} = \partial_Z \Omega_1|_{\pm Z_k} \Omega_2|_{\pm Z_k}$. Evaluating at Z_k gives

$$\partial_Z \Omega|_{Z_k} = \frac{2i}{\mu_k^2 \gamma_k^2} \left(a(Z_k)\frac{\alpha_k^+}{\alpha_k^-} + 2b(Z_k) + c(Z_k)\frac{\alpha_k^-}{\alpha_k^+}\right) \Big(\mathcal{A}_k(y)\mathcal{B}_k(x) - \mathcal{A}_k(x)\mathcal{B}_k(y)\Big)$$

while using $w(-Z_k) = w^*(Z_k)$, $\partial_Z w|_{-Z_k} = -\partial_Z w^*|_{Z_k}$, we also have

$$\partial_Z \Omega|_{-Z_k} = \frac{2i}{\mu_k^2 \gamma_k^2} \left(c(-Z_k)\frac{\alpha_k^+}{\alpha_k^-} + 2b(-Z_k) + a(-Z_k)\frac{\alpha_k^-}{\alpha_k^+}\right) \Big(\mathcal{A}_k(y)\mathcal{B}_k(x) - \mathcal{A}_k(x)\mathcal{B}_k(y)\Big)$$

The result follows from the identities

$$\begin{aligned} a(Z_k)\frac{\alpha_k^+}{\alpha_k^-} + 2b(Z_k) + c(Z_k)\frac{\alpha_k^-}{\alpha_k^+} &= 2i\frac{Z_k \sin Z_k}{\gamma_k} \\ c(-Z_k)\frac{\alpha_k^+}{\alpha_k^-} + 2b(-Z_k) + a(-Z_k)\frac{\alpha_k^-}{\alpha_k^+} &= 2i\frac{Z_k \sin Z_k}{\gamma_k} \end{aligned}$$

■

Next we have to examine the poles at $\pm m\pi$ in the expression

$$\frac{\Omega}{\eta^2(Z)} \left(1 - \frac{p_0^2}{Z^2}\right)$$

We rewrite Ω_1 and Ω_2 as

$$\begin{aligned} \Omega_1 &= (w(x) - w^*(x))w^*(y) - (w(y) - w^*(y))w^*(x) \\ \Omega_2 &= \frac{Z}{4i \sin Z} \times \Big\{(w^*(x) - w(x))(e^{iZ} w^*(y) - e^{-iZ} w(y)) + \\ &\qquad +(w^*(y) - w(y))(e^{iZ} w^*(x) - e^{-iZ} w(x))\Big\} \end{aligned}$$

Recalling the formula eq.(87, 88) for $w(x,Z)$ and $w^*(x,Z)$, we see that when $Z=0$, we have $w=w^*$ so that $\Omega_1 = 0(Z)$ and $\Omega_2 = 0(Z^2)$. Hence we have no pole at $Z=0$. When $Z=\pm\pi m+\epsilon$

$$w = \frac{1}{\epsilon} w_{\pm m}^{(-1)} + w_{\pm m}^{(0)} + \cdots, \quad w^* = \frac{1}{\epsilon} w_{\pm m}^{*(-1)} + w_{\pm m}^{*(0)} + \cdots$$

with

$$w_{\pm m}^{(-1)}(x) = w_{\pm m}^{*(-1)}(x) = \mp \pi m \Phi_m(x)$$

Because the two leading terms are the same, both $w^*(x,Z)-w(x,Z)$ and $e^{iZ}w^*(x,Z)-e^{-iZ}w(x,Z)$ are regular. So Ω_1 and Ω_2 both behaves like $1/\epsilon$. Since $1/\eta^2(Z)$ behaves like ϵ^2, the whole thing is in fact regular.

We come to the conclusion that everything happens at infinity. We want to compute

$$\left\{ \int dx X(x) T(x), \int dy Y(y) T(y) \right\} = 4 \int_0^1 dx X(x) \int_0^1 dy Y(y) \int_{C_\infty} dZ \left(1 - \frac{p_0^2}{Z^2} \right) \frac{1}{\eta^2(Z)} \partial_x \partial_y \Omega \tag{98}$$

Let

$$\begin{aligned} \widetilde{w}(x,Z) &= \eta^{-1/2}(Z)\, w(x,Z) \\ \widetilde{w}^*(x,Z) &= \eta^{-1/2}(Z)\, w^*(x,Z) \end{aligned}$$

The wronskian of $\widetilde{w}(x,Z)$ and $\widetilde{w}^*(x,Z)$ is $2iZ$ and therefore these functions coincide with the Baker–Akhiezer functions which are usually introduced in the pseudo-differential appproach to the KdV hierarchy (see e.g. [15]). At $Z \simeq \infty$, we have

$$\begin{aligned} \widetilde{w}(x,Z) &= e^{iZx} \left(1 - \frac{\omega(x)}{iZ} + \frac{\omega'(x)+\omega^2(x)}{2(iZ)^2} + \cdots \right) \\ \widetilde{w}^*(x,Z) &= e^{-iZx} \left(1 + \frac{\omega(x)}{iZ} + \frac{\omega'(x)+\omega^2(x)}{2(iZ)^2} + \cdots \right) \end{aligned}$$

where we have set

$$\omega(x) = \langle E\Phi \rangle(x)$$

Eq.(79) is needed to verify this formula. Using theses asymptotic forms, we find

$$\begin{aligned} \partial_x \left(\widetilde{w}^2(x,Z) \right) &= \left(\sum_{n=-\infty}^{1} A_n(x)(iZ)^n \right) e^{2iZx} \\ \partial_x \left(\widetilde{w}^{*2}(x,Z) \right) &= \left(\sum_{n=-\infty}^{1} (-1)^n A_n(x)(iZ)^n \right) e^{-2iZx} \\ \partial_x \left(\widetilde{w}(x,Z)\widetilde{w}^*(x,Z) \right) &= \sum_{n=-\infty}^{-1} C_{2n}(x)(iZ)^{2n} \end{aligned}$$

where

$$A_1 = 2, \quad A_0 = -4\omega(x), \quad A_{-1} = 4\omega^2(x), \quad A_{-2} = -\frac{8}{3}\omega^3 - 2\int^x \omega'^2, \quad A_{-3} = \frac{4}{3}\omega^4 + 4\omega \int^x \omega'^2$$

$$C_{-2}(x) = \omega''(x)$$

Consider the term proportional to $b(Z)$ in eq.(98):

$$4\int_{C_\infty}\frac{dZ}{2i\pi}b(Z)\left(1-\frac{p_0^2}{Z^2}\right)\int dxdy(X(x)Y(y)-X(y)Y(x))\partial_x\widetilde{w}^2(x,Z)\partial_y\widetilde{w}^{*2}(y,Z)$$

Lemma 6 *Let us define*

$$I_p(z)=\int_{C_\infty}\frac{dZ}{2i\pi}b(Z)(iZ)^p e^{2iZz}$$

We have

$$I_{-1}(z)=-\frac{1}{2}\delta(z),\quad I_0(z)=-\frac{1}{4}\delta'(z),\quad I_1(z)=-\frac{1}{8}\delta''(z),\quad I_2(z)=-\frac{1}{16}\delta'''(z)$$

$$I_{-n}(z)=-\frac{2^{n-3}}{(n-2)!}\epsilon(z)z^{n-2},\quad n\geq 2$$

Proof. It is clear that

$$\partial_z I_p(z)=2I_{p+1}(z)$$

hence we can determine all the $I_p(z)$ recursively. For $p\geq 0$ this is done by successively differentiating $I_{-1}(z)$ which is easy to calculate

$$I_{-1}(z)=\int_{C_\infty}\frac{dZ}{2i\pi}b(Z)\frac{1}{iZ}e^{2iZz}=\frac{1}{2}\int_{C_\infty}\frac{dZ}{2i\pi}\frac{\cos Z}{\sin Z}e^{2iZz}=-\frac{1}{2}\sum_{n\in Z}e^{2in\pi z}=-\frac{1}{2}\delta(z)$$

For $p\leq -2$ we have to successively integrate $I_{-1}(z)$. For this we need boundary conditions which are provided by

$$\int_{C_\infty}\frac{dZ}{2i\pi}b(Z)Z^{-p}=0,\quad p\geq 2 \tag{99}$$

This is because

$$\begin{aligned}\int_{C_\infty}\frac{dZ}{2i\pi}b(Z)\sum_{p=2}^{\infty}\zeta^p Z^{-p} &= -\frac{\zeta}{2i}\int_{C_\infty}\frac{dZ}{2i\pi}\frac{\cos Z}{\sin Z}\sum_{p=2}^{\infty}\zeta^{p-1}Z^{-p+1}\\ &=-\frac{\zeta^2}{2i}\int_{C_\infty}\frac{dZ}{2i\pi}\frac{\cos Z}{\sin Z}\frac{1}{Z-\zeta}=\frac{\zeta^2}{2i}\left(\cot\zeta-\frac{1}{\zeta}-\sum_{n=0}^{\infty}\frac{2\zeta}{\zeta^2-n^2\pi^2}\right)=0\end{aligned}$$

∎

Denoting

$$F_p(x,y)=\sum_{m+n=p}(-1)^m A_n(x)A_m(y)$$

we get

$$4\int_0^1 dx\int_0^1 dy\,(X(x)Y(y)-X(y)Y(x))\sum_{-\infty}^{p=2}F_p(x,y)(I_p(x-y)+p_0^2 I_{p-2}(x-y))$$

In this expression, we separate the terms with a $\delta(x-y)$ function or its derivative which will lead to local terms (L_b) and the non local terms (NL_b) with are proportional to $\epsilon(x-y)$.

$$
\begin{aligned}
L_b &= 4\int dxdy(X(x)Y(y)-X(y)Y(x))\Big(F_2I_2+F_1I_1+(F_0+p_0^2F_2)I_0+(F_{-1}+p_0^2F_1)I_{-1}\Big) \\
NL_b &= 4\int dxdy(X(x)Y(y)-X(y)Y(x))\sum_{n=0}^{\infty}(F_{-2-n}+p_0^2F_{-n})I_{-n-2}
\end{aligned} \tag{100}
$$

We have

$$
F_2(x,y)=-4, \quad F_1(x,y)=8\,(\omega(x)-\omega(y))
$$

$$
F_0(x,y)=-8\,(\omega(x)-\omega(y))^2\,, \quad F_{-1}(x,y)=\frac{16}{3}(\omega(x)-\omega(y))^3+4\int_y^x \omega'^2
$$

The local terms are

$$
\begin{aligned}
L_b &= 4\int_0^1 dx\int_0^1 dy(X(x)Y(y)-X(y)Y(x))\Big\{\frac{1}{4}\delta'''(x-y)+p_0^2\delta'(x-y)+ \\
&\quad -(\omega(x)-\omega(y))\delta''(x-y)-2(\omega(x)-\omega(y))^2\delta'(x-y)-\frac{1}{2}F_{-1}(x,y)\delta(x-y)\Big\}
\end{aligned}
$$

The last two terms obviously vanish and what remains is

$$
L_b=\int_0^1 dx\left\{-(X'''Y-XY''')-4((X'Y-XY')(p_0^2+2\omega')\right\}
$$

Lemma 7 *The non local term eq.(100) is identically zero.*

Proof. The non local term reads

$$
\begin{aligned}
NL_b &= -2\int_0^1 dx\int_0^1 dy(X(x)Y(y)-X(y)Y(x))\epsilon(x-y) \\
&\qquad \left[\sum_{n=0}^{\infty}\Big(F_{-2-n}(x,y)+p_0^2F_{-n}(x,y)\Big)\frac{2^n}{(n)!}(x-y)^n\right]
\end{aligned}
$$

The first sum is just the coefficient of $(iZ)^{-2}$ in the formal expansion of $\partial\widetilde{w}^2(x,Z)\partial\widetilde{w}^{*2}(y,Z)$ while the second sum is the coefficient of $(iZ)^0$. Hence we have

$$
NL_b=2\int_0^1 dx\int_0^1 dy(X(x)Y(y)-X(y)Y(x))\epsilon(x-y)\int_{C_\infty}\frac{dZ}{2i\pi}(Z-p_0^2Z^{-1})\,\partial\widetilde{w}^2(x,Z)\partial\widetilde{w}^{*2}(y,Z)
$$

The above expression is zero in the following sense. Let us write

$$
\int_{C_\infty}\frac{dZ}{2i\pi}(Z-p_0^2Z^{-1})\,\partial\widetilde{w}^2(x,Z)\partial\widetilde{w}^{*2}(y,Z)=\sum_{i=1}^{\infty}\frac{(y-x)^i}{i!}\int_{C_\infty}\frac{dZ}{2i\pi}(Z-p_0^2Z^{-1})\,\partial\widetilde{w}^2(x,Z)\partial^{i+1}\widetilde{w}^{*2}(x,Z)
$$

We will show that all the integrals around C_∞ in the right hand side are identically zero.

Since the function $\widetilde{w}(x,Z)$ satisfies the Schroedinger equation, its square $\widetilde{w}^2(x,Z)$ satisfies a third order differential equation

$$\mathcal{D}\widetilde{w}^2(x,Z) = -4Z^2\partial\widetilde{w}^2(x,Z), \quad \mathcal{D} = \partial^3 + 8\omega'\partial + 4\omega''$$

Let us introduce a pseudo differential operator Φ such that

$$\widetilde{w}^2(x,Z) = \Phi e^{2iZx}$$

Then

$$\mathcal{D} = \partial\Phi\partial^2\Phi^{-1}$$

Since $\mathcal{D}$ is anti self-adjoint, we also have

$$\mathcal{D} = -\mathcal{D}^* = \Phi^{*-1}\partial^2\Phi^*\partial$$

it follows that $\widetilde{w}^{*2}(x,Z)$ which is solution of

$$(-\mathcal{D}^*)\widetilde{w}^{*2}(x,Z) = -4Z^2\partial\widetilde{w}^{*2}(x,Z)$$

can be written as

$$\widetilde{w}^{*2}(x,Z) = \partial^{-1}\Phi^{*-1}\partial e^{-2iZx}$$

Hence

$$\partial\widetilde{w}^2 = \partial\Phi e^{2iZx}, \quad \partial^{i+1}\widetilde{w}^{*2} = \partial^i\Phi^{*-1}\partial e^{-2iZx}$$

Finally, we have to compute

$$\begin{aligned}\int_{C_\infty}\frac{dZ}{2i\pi}(Z-p_0^2Z^{-1})\,(\partial\widetilde{w}^2(x,Z))\partial^{i+1}\widetilde{w}^{*2}(x,Z) &= \int_{C_\infty}\frac{dZ}{2i\pi}(Z-p_0^2Z^{-1})\,(\partial\Phi e^{2iZx})\partial^i\Phi^{*-1}\partial e^{-2iZx} \\ &= \frac{-1}{2i}\int_{C_\infty}\frac{dZ}{2i\pi}\,(\partial\Phi e^{2iZx})\partial^i\Phi^{*-1}(\partial^2+4p_0^2)e^{-2iZx}\end{aligned}$$

We recall the formula (see e.g.[15])

$$\int_{C_\infty}\frac{dZ}{2i\pi}(De^{iZx})(Fe^{-iZx}) = \mathrm{Res}_\partial(DF^*)$$

where Res_∂ is Adler's residue [25]. So our expression is equal to

$$\mathrm{Res}_\partial(\partial\Phi(\partial^2+4p_0^2)\Phi^{-1}\partial^i) = \mathrm{Res}_\partial((\mathcal{D}+4p_0^2\partial)\partial^i) = 0$$

because $(\mathcal{D}+4p_0^2\partial)\partial^i$ is a differential operator. ∎

Consider next the term proportional to $a(Z)$ in eq.(98).

$$4\int_0^1 dx\int_0^1 dy\Big(X(x)Y(y)-X(y)Y(x)\Big)\int_{C_\infty}\frac{dZ}{2i\pi}a(Z)\left(1-\frac{p_0^2}{Z^2}\right)\partial_x\widetilde{w}^2(x,Z)\partial_y\Big(\widetilde{w}(y,Z)\;\widetilde{w}^*(y,Z)\Big)$$

Lemma 8 *Let us define*

$$J_p(x) = \int_{C_\infty} \frac{dZ}{2i\pi} a(Z)(iZ)^p e^{2iZx}, \quad p = -1, -2, \cdots$$

One has

$$J_{-1}(x) = \frac{1}{2}\delta(x), \quad J_{-p}(x) = 2^{p-2}\frac{x^{p-2}}{(p-2)!}(\epsilon(x) - 1), \quad p \geq 2$$

Proof. One has

$$\partial_x J_p(x) = 2J_{p+1}(x)$$

The calculation of $J_{-1}(x)$ is easy. Next, we need boundary conditions to determine the other J_p by integration

$$\begin{aligned}\sum_{n=2}^{\infty}(i\zeta)^n J_{-n}(0) &= \frac{\zeta^2}{2i}\int_{C_\infty}\frac{dZ}{2i\pi}\frac{e^{-iZ}}{\sin Z}\frac{1}{Z-\zeta} \\ &= \frac{\zeta^2}{2i}\left[-\frac{e^{-i\zeta}}{\sin\zeta} + \sum_{n\in Z}\frac{1}{\zeta - n\pi}\right] = \frac{\zeta^2}{2i}\left[-\frac{e^{-i\zeta}}{\sin\zeta} + \cot\zeta\right] = \frac{\zeta^2}{2}\end{aligned}$$

∎

It follows that all the non local terms containing $J_p(x)$ for $p \leq -2$ vanish when $0 < x < 1$. The $a(Z)$ term is

$$L_a = 4\int_0^1 dx \int_0^1 dy \Big(X(x)Y(y) - X(y)Y(x)\Big) A_1(x)C_{-2}(y)J_{-1}(x)$$

or

$$L_a = 4\int_0^1 dx X(x)\delta(x)\int_0^1 dy Y(y)\omega''(y) - 4\int_0^1 dx Y(x)\delta(x)\int_0^1 dy X(y)\omega''(y)$$

Finally, the term in $c(Z)$ is just equal to the $a(Z)$ one and double it.

Putting everything together, we arrive at eq.(93). In the course of this proof, we have shown the identities

$$\int_{C_\infty}\frac{dZ}{2i\pi}(Z - p_0^2 Z^{-1})\,(\partial\widetilde{w}^2(x,Z))\partial^{i+1}\widetilde{w}^{*2}(x,Z) = 0, \quad \forall i \geq 0$$

These are quartic identities on the coefficients of $\widetilde{w}(x,Z)$. Of course, we also have the quadratic relations of Hirota and Sato that were interpreted by Sato as Plücker relations defining the infinite Grassmannian, allowing to give a precise definition of τ-functions. The above relations are quartic relations on τ-functions analogous to the quartic relations on Riemann Theta functions.

12 Poisson bracket $\{L_0, u(y)\}$.

In the previous section, we have obtained the Poisson bracket for the generators $T_n = L_n - L_0$. We now have to reintroduce L_0 and check that it has the correct Poisson brackets. The candidate for L_0 was given in eq.(66). Let us recall it:

$$L_0 = p_0^2 + \frac{1}{\eta(p_0)}\sum_k \frac{\Gamma_k}{p_0^2 - k^2\pi^2}$$

where

$$\Gamma_k = E'_k(0)\Phi'_k(0) - E'_k(1)\Phi'_k(1))$$

and

$$\eta(Z) = 1 - \sum_m \frac{\eta_m}{Z^2 - m^2\pi^2} = 1 - \sum_m \frac{E'_m(0)\Phi_m(0)}{Z^2 - m^2\pi^2}$$

where we have used eq.(77), evaluated at $x = 0$, to express η_m.

Proposition 20 *We have the following Poisson bracket:*

$$\left\{L_0, \int_0^1 dyY(y)u(y)\right\} = -4\int_0^1 dyY(y)\Big[(L_0 - p_0^2)\delta'(y) + T'(y)\Big] \tag{101}$$

This shows that $\{L_0, \cdot\}$ acts on $u(y)$ as ∂_y, as it should be.

Again, the proof is long and we will split it into several lemmas. We need to compute $\{\Phi_m(x), T(y)\}$. and $\{\Phi'_m(x), T(y)\}$ for $x = 0$ and $x = 1$.

Multiplying eq.(94) by $E_n(y)$ and remembering that $\langle E|\Theta^{-1} = \langle F|$ and $F_k(x) = -\frac{\zeta_k}{\gamma_k\mu_k}\widetilde{F}_k(x)$, we get

$$\begin{aligned}\{\Phi_m(x), \langle E(y)\Phi(y)\rangle\} &= \qquad\qquad (102)\\ &= -2\Phi_m(x)\sum_k \frac{\sin Z_k}{Z_k}\frac{\zeta_k^2(1-p_0^2Z_k^{-2})}{\mu_k^2\gamma_k^3(Z_k^2-\pi^2m^2)}\Big[\widetilde{F}'_k(x)\widetilde{F}_k(x)\widetilde{F}_k(y)\widetilde{G}_k(y) - \widetilde{F}_k^2(y)\widetilde{F}'_k(x)\widetilde{G}_k(x)\Big]\\ &\quad +2\Phi'_m(x)\sum_k \frac{\sin Z_k}{Z_k}\frac{\zeta_k^2(1-p_0^2Z_k^{-2})}{\mu_k^2\gamma_k^3(Z_k^2-\pi^2m^2)}\Big[\widetilde{F}_k^2(x)\widetilde{F}_k(y)\widetilde{G}_k(y) - \widetilde{F}_k^2(y)\widetilde{F}_k(x)\widetilde{G}_k(x)\Big]\end{aligned}$$

As before, this can be expressed in terms of Ω_1 and Ω_2. We find

$$\{\Phi_m(x), \langle E(y)\Phi(y)\rangle\} = -\sum_k \mathrm{Res}_{\pm Z_k}\frac{(1-p_0^2Z^{-2})}{\eta^2(Z)(Z^2-\pi^2m^2)}\Omega_1(\Phi_m(x)\partial_x\Omega_2 - \Phi'_m(x)\Omega_2)$$

This formula is the starting point to begin the computation of $\{\eta(p_0), \langle E(y)\Phi(y)\rangle\}$. Setting $x = 0$ in eq.(102), the term proportional to $\Phi'_m(x)$ vanishes because $F_k(0) = 0$. We are left with ($x = 0$, but we keep it for a while)

$$\{\Phi_m(x), \langle E(y)\Phi(y)\rangle\} = -\Phi_m(x)\sum_k \mathrm{Res}_{\pm Z_k}\frac{(1-p_0^2Z^{-2})}{\eta^2(Z)(Z^2-\pi^2m^2)}\Omega_1\partial_x\Omega_2 \tag{103}$$

Multiplying by $\frac{E'_m(0)}{(p_0^2-m^2\pi^2)}$ and summing over m, in the right hand side appears the sum

$$\begin{aligned}-\sum_m \frac{E'_m(0)\Phi_m(0)}{(p_0^2-m^2\pi^2)(Z_k^2-\pi^2m^2)} &= -\sum_m \frac{\eta_m}{(p_0^2-\pi^2m^2)(Z_k^2-\pi^2m^2)}\\ &= \frac{-1}{Z_k^2-p_0^2}\sum_m\left(\frac{\eta_m}{p_0^2-m^2\pi^2} - \frac{\eta_m}{Z_k^2-\pi^2m^2}\right)\\ &= \frac{1}{Z_k^2-p_0^2}\eta(p_0)\end{aligned}$$

where we used that $\sum_m \frac{\eta_m}{Z_k^2-\pi^2 m^2} = 1, \quad \forall k$. The factor $1/(Z_k^2 - p_0^2)$ cancels with the factor $(1 - p_0^2 Z_k^{-2})$ in eq.(103) and we are left with

$$\{\eta(p_0), \langle E\Phi\rangle(y)\} \quad = \quad -\eta(p_0)\sum_k \mathrm{Res}_{\pm Z_k}\frac{1}{\eta^2(Z)Z^2}\Omega_1\partial_x\Omega_2$$

Finally

$$\left\{\eta(p_0), \int_0^1 dyY(y)T(y)\right\} = 2\eta(p_0)\int_0^1 dyY(y)\int_{C_\infty}\frac{dZ}{2i\pi}\frac{1}{Z^2}\partial_y(\widetilde{\Omega}_1\partial_x\widetilde{\Omega}_2) \tag{104}$$

where we used the asymptotics expressions for $\widetilde{w}$ and $\widetilde{w}^*$ inside Ω_1 and Ω_2, hence removing the $\eta^2(Z)$ factor. The last step consists in evaluating the integral around C_∞. The reult is:

Lemma 9

$$\left\{\eta(p_0), \int_0^1 dyY(y)T(y)\right\} = 4\eta(p_0)\int_0^1 dyY(y)\delta'(y) \tag{105}$$

Proof. Let us consider the integral over C_∞ in eq.(104). The term containing $b(Z)$ can be written as

$$\begin{aligned}L_b \quad = \quad & \int_{C_\infty}\frac{dZ}{2i\pi}\frac{b(Z)}{Z^2}\times\Big\{(\widetilde{w}(x,Z)\widetilde{w}^{*\prime}(x,Z) - \widetilde{w}'(x,Z)\widetilde{w}^*(x,Z))\partial_y(\widetilde{w}(y,Z)\widetilde{w}^*(y,Z)) + \\ & \widetilde{w}(x,Z)\widetilde{w}'(x,Z)\partial_y\widetilde{w}^{*2}(y,Z) - \widetilde{w}^*(x,Z)\widetilde{w}^{*\prime}(x,Z)\partial_y\widetilde{w}^2(y,Z)\Big\}\end{aligned}$$

On the first line, we recognize the wronskian of $\widetilde{w}(x,Z)$ and $\widetilde{w}^*(x,Z)$, which is just equal to $-2iZ$. Since $\partial_y(\widetilde{w}(y,Z)\widetilde{w}^*(y,Z)) = Z^{-2}S_2'(y) + \cdots$ this term vanish by eq.(99). Hence

$$L_b \quad = \quad \frac{1}{2}\int_{C_\infty}\frac{dZ}{2i\pi}\frac{b(Z)}{Z^2}\Big\{\partial_x\widetilde{w}^2(x,Z)\partial_y\widetilde{w}^{*2}(y,Z) - \partial_x\widetilde{w}^{*2}(x,Z)\partial_y\widetilde{w}^2(y,Z)\Big\} \tag{106}$$

$$= \quad -\sum_{p=-2}^{\infty} F_{-p}(x,y)I_{-p-2}(x-y)$$

$$= \quad \delta'(y-x) + \frac{1}{2}\epsilon(x-y)\int_{C_\infty}\frac{dZ}{2i\pi}\frac{1}{Z}\partial_x\widetilde{w}^2(x,Z)\partial_y\widetilde{w}^{*2}(y,Z) \tag{107}$$

The $\epsilon(x-y)$ term is zero because

$$\begin{aligned}\int_{C_\infty}\frac{dZ}{2i\pi}\frac{1}{Z}\partial^{i+1}\widetilde{w}^2(x,Z)\partial\widetilde{w}^{*2}(x,Z) \quad &= \quad \int_{C_\infty}\frac{dZ}{2i\pi}\frac{1}{Z}(\partial^{i+1}\Phi e^{2iZx})(\Phi^{*-1}\partial e^{-2iZx}) \\ &= \quad \int_{C_\infty}\frac{dZ}{2i\pi}(\partial^{i+1}\Phi e^{2iZx})(\Phi^{*-1}e^{-2iZx}) \\ &= \quad \mathrm{Res}_\partial(\partial^{i+1}\Phi\cdot\Phi^{-1}) = 0\end{aligned}$$

Next, we consider the $a(Z)$ term. It reads

$$L_a \quad = \quad \partial_y\int_{C_\infty}\frac{dZ}{2i\pi}\frac{a(Z)}{Z^2}(\widetilde{w}(x,Z)\widetilde{w}^*(y,Z) - \widetilde{w}^*(x,Z)\widetilde{w}(y,Z))\widetilde{w}'(x,Z)\widetilde{w}(y,Z)$$

$$
\begin{aligned}
= & \int_{C_\infty} \frac{dZ}{2i\pi}\frac{a(Z)}{Z^2}\Big(\frac{1}{2}\partial_x \widetilde{w}^2(x,Z)\partial_y(\widetilde{w}(y,Z)\widetilde{w}^*(y,Z)) \\
& -\frac{1}{2}(\widetilde{w}^*(x,Z)\widetilde{w}'(x,Z)+\widetilde{w}^{*'}(x,Z)\widetilde{w}(x,Z))\partial_y\widetilde{w}^2(y,Z) \\
& -\frac{1}{2}(\widetilde{w}^*(x,Z)\widetilde{w}'(x,Z)-\widetilde{w}^{*'}(x,Z)\widetilde{w}(x,Z))\partial_y\widetilde{w}^2(y,Z)\Big)
\end{aligned}
$$

Again, the last term is the wronskian and so

$$
\begin{aligned}
L_a = & \int_{C_\infty} \frac{dZ}{2i\pi}\frac{a(Z)}{Z^2}\Big(-iZ\partial_y(\widetilde{w}^2(y,Z)+\frac{1}{2}\partial_x\widetilde{w}^2(x,Z)\partial_y(\widetilde{w}(y,Z)\widetilde{w}^*(y,Z)) \\
& -\frac{1}{2}\partial_x(\widetilde{w}^*(x,Z)\widetilde{w}(x,Z))\partial_y(\widetilde{w}^2(y,Z))\Big)
\end{aligned}
$$

The first term is

$$
\begin{aligned}
\int_{C_\infty}\frac{dZ}{2i\pi}a(Z)(iZ)^{-p-1}A_{-p}(y)e^{2iZy} &= \sum_{p=-1}^{\infty}A_{-p}(y)J_{-p-1}(y) \\
&= \frac{1}{2}\delta'(y)-\omega(y)\delta(y)+(\epsilon(y)-1)\sum_{p=1}^{\infty}A_{-p}(y)\frac{(2y)^{p-1}}{(p-1)!}
\end{aligned}
$$

The last sum is zero because $0<y<1$. The $\delta(y)$ term vanishes because $\omega(0)=0$. The second term is

$$
\frac{1}{2}\int_{C_\infty}\frac{dZ}{2i\pi}\frac{a(Z)}{Z^2}A_{-p}(x)C_{-q}(y)(iZ)^{-p-q}e^{2iZx} = -\frac{1}{2}(\epsilon(x)-1)\Big(A_1(x)C_{-2}(y)(2x)+\cdots\Big) \tag{108}
$$

This vanishes when $x=0$. The third term is

$$
-\frac{1}{2}\int_{C_\infty}\frac{dZ}{2i\pi}\frac{a(Z)}{Z^2}A_{-p}(y)C_{-q}(x)(iZ)^{-p-q}e^{2iZy} = \frac{1}{2}(\epsilon(y)-1)\Big(A_1(y)C_{-2}(x)(2y)+\cdots\Big)
$$

and this vanishes when $0<y<1$.

Finally, it is easy to see that the $c(Z)$ term is equal to the $a(Z)$ one. Putting everything together, we get eq.(105). ■

The last result we need is the following:

Lemma 10

$$
\left\{\sum_m \frac{E'_m(0)\Phi'_m(0)-E'_m(1)\Phi'_m(1)}{p_0^2-\pi^2m^2}, \int_0^1 dyY(y)T(y)\right\} = -4\eta(p_0)\int_0^1 dyY(y)T'(y) \tag{109}
$$

Proof. Taking the derivative with respect to x of eq.(102) and remembering that $F_k(0)=F_k(1)=0$, the remaining terms are (there is a cancellation in the $\Phi'_m(x)$ term)

$$
\begin{aligned}
\{\Phi'_m(x),\langle E(y)\Phi(y)\rangle\} = & -2\Phi_m(x)\sum_k\frac{\sin Z_k}{Z_k}\frac{\zeta_k^2(1-p_0^2Z_k^{-2})}{\mu_k^2\gamma_k^3(Z_k^2-\pi^2m^2)}\times \\
& \partial_x\Big[\widetilde{F}'_k(x)\widetilde{F}_k(x)\widetilde{F}_k(y)\widetilde{G}_k(y)-\widetilde{F}_k^2(y)\widetilde{F}'_k(x)\widetilde{G}_k(x)\Big]
\end{aligned}
$$

where it is understood that $x = 0$ or $x = 1$. By exactly the same argument as before

$$\{\sum_m \frac{E'_m(x)\Phi'_m(x)}{p_0^2 - \pi^2 m^2}, \int_0^1 dy Y(y) T(y)\} = 2\eta(p_0) \int_0^1 dy Y(y) \partial_y \partial_x \int_{C_\infty} \frac{dZ}{2i\pi} \frac{1}{Z^2 \eta(Z)} \Omega_1 \partial_x \Omega_2$$

Hence, we just have to take the derivative with respect to x of the previous result, before setting $x = 0$ or $x = 1$. At $x = 0$ we get

$$2\eta(p_0) \int_0^1 dy Y(y) \Big[-\delta''(y) + 4\omega''(y) \Big]$$

The $\delta''(y)$ term comes from eq.(107) while the second term comes from eq.(108) doubled by the $c(Z)$ contribution. At $x = 1$ only the periodic $\delta''(y)$ remains. Taking the difference we obtain eq.(109). ∎

Putting everything together, we arrive at eq.(101), and this finishes our proof that $u(x)$ does satisfy the Virasoro Poisson bracket.

13 Conclusion.

We have succeeded to take the continuum limit in the formulae expressing the dynamical variables of the Volterra model in terms of the separated variables. This yields exactly solvable potentials and formulae for the Virasoro generators of a rather unusual type. Still, we were able to check that they have the correct Poisson brackets. Of course the most interesting thing now is to try to quantize this approach. As a first step, a semiclassical analysis along the lines of [26] should be very enlightening. The full quantum theory however may reserve some surprise. The bracket eq.(3) being in fact an ordinary quadratic bracket, it is natural to quantize it with Weyl type commutation relations. This opens up the possibility of phenomena as those advocated in [9, 27].

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