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ABSTRACT. We give some a priori estimates of type $\sup \times \inf$ on Riemannian manifolds for Yamabe and prescribed curvature type equations. An application of those results is the uniqueness result for $\Delta u + \epsilon u = u^{N-1}$ with ϵ small enough.

INTRODUCTION AND RESULTS.

We are on Riemannian manifold (M, g) of dimension $n \geq 3$. In this paper we denote $\Delta = -\nabla^j(\nabla_j)$ the geometric laplacian and $N = \dfrac{2n}{n-2}$.

The scalar curvature equation is:

$$\frac{4(n-1)}{n-2}\Delta u + R_g u = V u^{N-1},\ u > 0.$$

Where R_g is the scalar curvature and V is a function (prescribed scalar curvature).

When we suppose $V \equiv 1$, the previous equation is the Yamabe equation.

Here we study some properties of Yamabe and prescribed scalar curvature equations. The existence result for the Yamabe equation on compact Riemannian manifolds was proved by T. Aubin and R. Schoen (see for example [Au]).

First, we suppose the manifold (M, g) compact. We have:

Theorem 1. For all $a, b, m > 0$, there exist a positive constant $C = C(a, b, m, M, g)$ such that for every $\epsilon > 0$, for every smooth function V such that $a \leq V_\epsilon(x) \leq b$ and every positive solution u_ϵ of:

$$\Delta u_\epsilon + \epsilon u_\epsilon = V_\epsilon {u_\epsilon}^{N-1}$$

with $\max_M u_\epsilon \geq m$, we have:

$$\epsilon \max_M u_\epsilon \min_M u_\epsilon \geq C.$$

Now, we consider a Riemannian manifold (M, g) of dimension $n \geq 3$ (not necessarily compact) and we work with Yamabe type equation,

$$\Delta u - \lambda u = n(n-2)u^{N-1}.$$

We look for a priori bounds for solutions of the previous equation.

Theorem 2. If $0 < m \leq \lambda + R_g \leq 1/m$ then for every compact K of M, there exist a positive constant $c = c(K, M, m, n, g)$ such that:

$$\sup_K u \times \inf_M u \leq c.$$

Note that there is lot of estimates of those type for prescribed scalar curvature on open set Ω of $\mathbb{R}^n$, see ([B],[B-M], [B-L-S], [C-L 1], [C-L 2], [L 1], [L 2], and [S]).

In dimension 2 Brezis, Li and Shafrir [B-L-S], have proved that $\sup + \inf$ is bounded from above when we suppose the prescribed curvature uniformly lipschitzian. In [S], Shafrir got a result of type $\sup + C \inf$, with L^∞ assumption on prescribed curvature.

In dimensions $n \geq 3$, we can find many results with different assumptions on prescribed curvature, see [B], [L 2], [C-L 2].

Note that an important estimates was proved for Yamabe equation about the product $\sup \times \inf$, in dimensions 3,4 by Li and Zhang [L-Z].

In our work we have no assumption on energy. There is an important work if we suppose the energy bounded, see for example [D-H-R].

<u>Application:</u>

We assume that M is compact and $1/m \geq R_g \geq m > 0$ on M. For small values of λ we can have some upper bounds for the product $\sup \times \inf$ for the following equation:

$$\Delta u_\epsilon + \epsilon u_\epsilon = n(n-2) u_\epsilon^{N-1}.$$

Theorem 3. If $\epsilon \to 0$, then,

$$\sup_M u_\epsilon \times \inf_M u_\epsilon \leq c(n, m, M, g).$$

A consequence of Theorems 1 and 3 is the following corollary:

Corollary. Any sequence $u_i > 0$ solutions of the following equation:

$$\Delta u_i + \epsilon_i u_i = n(n-2) {u_i}^{N-1},$$

converge uniformly to 0 on M when ϵ_i tends to 0.

We have:

Theorem 4. On compact Riemannian manifold (M, g) with $R_g > 0$ every-where, the sequence $u_i > 0$ solutions of the previous equation is such that for i large, $u_i \equiv \left[\dfrac{\epsilon_i}{n(n-2)} \right]^{(n-2)/4}$.

Note that the previous result assert that $\left[\dfrac{\epsilon_i}{n(n-2)} \right]^{(n-2)/4}$ is the only solution of the previous equation for ϵ_i small.

We remark an important result in [B-V,V]; they have a same consequence than in theorem 4 with assumption on Ricci curvature ($Ric \geq \epsilon_i c g$, with $c > 0$). Here we give a condition on scalar curvature to obtain an uniqueness result.

Proof of theorem 1:

We need two lemmata and one proposition. We are going to prove some estimates for the Green function G_ϵ of the operator $\Delta + \epsilon$.

Lemma 1.

For each point $x \in M$ there exist $\epsilon_0 > 0$ and $C(x, M, g) > 0$ such that for every $z \in B(x_0, \epsilon_0)$, and every $\mu \leq \epsilon_0$, every $a, b \in \partial B(z, \mu)$, there exist a curve $\gamma_{a,b}$ of classe C^1 linking a to b which included in $\partial B(z, \mu)$. The length of this curve is $l(\gamma_{a,b}) \leq C(x, M, g)\mu$.

Proof:

Let $x \in M$, we consider a chart (Ω, φ) around x.

We take exponential map on the compact manifold M. According to T. Aubin and E. Hebey see [Au] and [He], there exist $\epsilon > 0$ such that $\exp_x$ is C^∞ function of $B(x, \epsilon) \times B(0, \epsilon)$ into M and for all $z \in B(x, \epsilon)$, $\exp_z$ is a diffeomorphism from $B(0, \epsilon)$ to $B(z, \epsilon)$ with $\exp_z[\partial B(0, \mu)] = \partial B(z, \mu) \subset M$ for $\mu \leq \epsilon/2$. If we take two points a, b of $\partial B(z, \mu)$ ($\mu \leq \epsilon/2$), then $a' = \exp_z^{-1}(a), b' = \exp_z^{-1}(b)$ are two points of $\partial B(0, \mu) \subset \mathbb{R}^n$. On this sphere of center 0 and radius μ, we can link a' to b' by a great circle arc whose length is $\leq 2\pi\mu$. Then, there exist a curve of class C^1 $\delta_{a',b'}$ in $\partial B(0, \mu) \subset \mathbb{R}^n$ such that $l(\delta_{a',b'}) \leq 2\pi\mu$. Now we consider the curve $\gamma_{a,b} = \exp_z(\delta_{a',b'})$, this curve of class C^1, link a to b and it is included in $\partial B(z, \mu) \subset M$. The length of $\gamma_{a,b}$ is giving by the following formula :

$$l(\gamma_{a,b}) = \int_0^1 \sqrt{g_{ij}[\gamma_{a,b}(s)](\frac{d\gamma_{a,b}}{dt})^i(s)(\frac{d\gamma_{a,b}}{dt})^j(s)}ds.$$

where g_{ij} is the local expression of the metric g in the chart (Ω, φ).

We know that there exist a constant $C = C(x, M, g) > 1$ such that:

$$\frac{1}{C}||X||_{\mathbb{R}^n} \leq g_{ij}(z)X^iX^j \leq C||X||_{\mathbb{R}^n} \text{ for all } z \in B(x, \epsilon/2) \text{ and all } X \in \mathbb{R}^n.$$

We have,

$$l(\gamma_{a,b}) = \int_0^1 \sqrt{g_{ij}[\exp_z[\delta_{a',b'}(s)]](\frac{d[\exp_z[\delta_{a',b'}]]}{dt})^i(s)(\frac{d[\exp_z[\delta_{a',b'}]]}{dt})^j(s)}ds,$$

$$l(\gamma_{a,b}) \leq C\int_0^1 ||d\exp_z(\frac{d\delta_{a',b'}}{dt})(s)||_{\mathbb{R}^n}ds,$$

and $u : (z, v) \to \exp_z(v)$ is C^∞ on $B(x, \epsilon) \times B(0, \epsilon)$,

but,

$$||du_{z,v}|| = ||d\exp_z(v)|| \leq C'(x, M, g) \ \forall \ (z, v) \in B(x, \epsilon/2) \times B(0, \epsilon/2) (\text{in the sense of linear form}),$$

Finaly,

$$l(\gamma_{a,b}) \leq \tilde{C}(x, M, g)\int_0^1 ||\frac{d\delta_{a',b'}}{dt}(s)||_{\mathbb{R}^n}ds = \tilde{C}(x, M, g)l(\delta_{a',b'}) \leq 2\pi\tilde{C}(x, M, g)\mu.$$

We need to estimate the singularities of Green functions. Set $G_i = G_{\epsilon_i}$.

Lemma 2.

The function G_i satisfies:

$$G_i(x, y) \leq \frac{C(M, g)}{\epsilon_i[d_g(x, y)]^{n-2}}.$$

where $C(M, g) > 0$ and d_g is the distance on M for the metric g.

Proof:

According to the Appendix of [D-H-R] (see also [Au]), we can write the function G_i :

$$G_i(x,y) = H(x,y) + \Sigma_{j=1}^k \Gamma_{i,k}(x,y) + u_{i,k+1}(x,y),$$

with, $k = [n/2]$ and $u_{i,k+1}$ is solution of $\Delta u_{i,k+1} + \epsilon_i u_{i,k+1} = \Gamma_{i,k+1}$.

According to Giraud (see [Au] and [D-H-R]), we have:

i) $0 \leq H(x,y) \leq \dfrac{C_0(M,g)}{[d_g(x,y)]^{n-2}}$,

ii) $|\Gamma_{i,j}(x,y)| \leq \dfrac{C_j(M,g)}{[d_g(x,y)]^{n-2}}, j = 1, \ldots, k$ and,

iii) $\Gamma_{i,k+1}(x,y) \leq C_{k+1}(M,g)$ and continuous on $M \times M$.

We write $u_{i,k+1}$ by using the Green function G_i, we obtain with iii):

$$u_{i,k+1}(x,y) = \int_M G_i(x,y)\Gamma_{i,k+1}(x,y)dV_g(y) \leq C_{k+1}(M,g)\int_M G_i(c,y)dV_g(y) = \frac{C_{k+1}(M,g)}{\epsilon_i}.$$

If we combine the last inequality and i) et ii), we obtain the result of the lemma.

We have to estimate the Green function from below.

Proposition.

Consider two sequences of points of M, (x_i) et (y_i) such that $x_i \neq y_i$ for all i and $x_i \to x$, $y_i \to y$. Then, there exist a positive constant C depending on x, y, M and g, and a subsequence (i_j) such that:

$$G_{i_j}(x_{i_j}, y_{i_j}) \geq \frac{C}{\epsilon_{i_j}} \qquad \forall\ j.$$

Proof:

We know that $G_i(x_i, .)$ is $C^\infty(M - x_i)$ and satisfies the following equation:

$$\Delta G_i(x_i, .) + \epsilon_i G_i(x_i, .) = 0, \text{ in } M - x_i.$$

Case 1: $y = x$.

Let $R_i = \frac{1}{2} d_g(x_i, y_i) > 0$ and $\Omega_i = M - B(x_i, R_i)$, according to maximum principle, the function $G_i(x_i, .)$ has its maximum on the boundary of Ω_i. Then;

$$\max_{\Omega_i} G_i(x_i, z) = G_i(x_i, z_i),\ d(x_i, z_i) = R_i.$$

Let t_i be a point of M such that $d_g[y_i, B(x_i, R_i)] = d(t_i, y_i)$. We have $t_i \in \partial B(x_i, R_i)$ then $d(x_i, t_i) = R_i$. Because the manifold M is compact, we can find a minimizing curve L_i between y_i and t_i. Let δ_i a curve in $\partial B(x_i, R_i)$ with minimal length linking t_i to z_i. We can choose it like in lemma 1. Then $l(\delta_i) \leq c(x, M, g)R_i$ and if we note $\bar{\delta}_i = \delta_i \cup L_i$, we have $l(\bar{\delta}_i) = l(\delta_i) + l(L_i) \leq R_i[1 + c(x, M, g)]$. The curve $\bar{\delta}_i$ link z_i to y_i, and it is included in Ω_i. Let $r_i = \frac{1}{5}R_i$. We cover the curve $\bar{\delta}_i$ by balls of radii r_i, if we consider N_i the minimal number of those balls, then we have $N_i r_i \leq [c(x, M, g) + 1]R_i$, and $N_i \leq 5[c(x, M, g) + 1]$.

If we work on open set of one chart Ω centered in x, with a small ball around x_i removed, $\tilde{\Omega}_i = \Omega - B(x_i, \frac{1}{100}R_i)$ then, we can apply the theorem 8.20 of [GT] (Harnack inequality) in each ball of the finite covering of $\bar{\delta}_i$ defined previous. In this Harnack inequality the constant which depends on the radius is explicit and equal to $C_0(n)^{(\Lambda/\lambda)+\nu R_i)}$ but here $R_i \to 0$, and the constant do not depend on the radius. We obtain:

$$\sup_{B(z_i,r_i)} G_i(x_i, z) \leq C(x, M, g) \inf_{B(y_i,r_i)} G_i(x_i, z).$$

Then, $G_i(x_i, z) \leq G_i(x_i, z_i) \leq C(x, M, g)G_i(x_i, y_i)$ for all $z \in \Omega_i$.

Now we write:

$$\frac{1}{\epsilon_i} = \int_M G_i(x_i,z)dV_g(z) = \int_{\Omega_i} G_i(x_i,z)dV_g(z) + \int_{B(x_i,2R_i)} G_i(x_i,z)dV_g(z),$$

but,

$$\int_{\Omega_i} G_i(x_i,z)dV_g(z) \leq |\Omega_i| \sup_{\Omega_i} G_i(x_i,z) \leq |\Omega_i| C(x,M,g) G_i(x_i,y_i),$$

we take $A_i = \int_{B(x_i,2R_i)} G_i(x_i,z)dV_g(z)$, we have,

$$A_i = \int_{B(0,2R_i)} G_i[x_i, \exp_{x_i}(v)]\sqrt{|g|}du = \int_0^{2R_i}\int_{\mathbb{S}_{n-1}} t^{n-1}\sqrt{|g|}G_i[x_i,\exp_{x_i}(t\theta)]dtd\theta,$$

if we use the lemma 8 in Hebey-Vaugon (see [H-V]), we obtain $\sqrt{|g|} \leq c(M,g)$. But $R_i \to 0$, then $d_g[x_i, \exp_{x_i}(t\theta)] = t$ (the geodesic are minimizing). We use the lemma 2 and we find:

$$\int_{B(x_i,2R_i)} G_i(x_i,z)dV_g(z) \leq \frac{c'(M,g)(R_i)^2}{\epsilon_i}.$$

Finaly:

$$G_i(x_i,y_i) \leq \frac{1-c'(M,g)(R_i)^2}{|\Omega_i|C(x,M,g)\epsilon_i} \geq \frac{C'(x,M,g)}{\epsilon_i}.$$

<u>Case 2</u>: $x \neq y$.

We write,

$$\frac{1}{\epsilon} = \int_M G_i(x_i,z)dV_g(z) = \int_{M-B(x_i,\delta)} G_i(x_i,z)dV_g(z) + \int_{B(x_i,\delta)} G_i(x_i,z)dV_g(z).$$

We take $0 < \delta \leq \dfrac{inj_g(M)}{2}$, where $inj_g(M)$ is the injectivity radius of the compact manifold M. We use the exponential map and we have:

$$\int_{B(x_i,\delta)} G_i(x_i,z)dV_g(z) = \int_{B(0,\delta)} G_i[x_i,\exp_{x_i}(v)]\sqrt{|g|}dv = \int_0^\delta t^{n-1}\int_{\mathbb{S}_{n-1}} G_i[x_i,\exp_{x_i}(t\theta)]\sqrt{|}g|dtd\theta,$$

If we use the lemma 8 in Hebey-Vaugon (see [H-V]), we obtain $|g| \leq c(M,g)$. Using the fact $t \to \exp_{x_i}(t\theta)$ is minimizing for $t \leq \delta < inj_g(M)$ and the lemma 2, we obtain:

$$\int_{B(x_i,\delta)} G_i(x_i,z)dV_g(z) \leq \frac{C'(M,g)\delta^2}{\epsilon_i}.$$

Then,

$$\int_{M-B(x_i,\delta)} G_i(x_i,z)dV_g(z) \geq \frac{1-C'(M,g)\delta^2}{\epsilon_i},$$

we can choose $0 < \delta < \dfrac{1}{\sqrt{C'(M,g)}}$.

Between x and y, we work like in the first case. We take $0 < \delta < \dfrac{d_g(x,y)}{2}$, for each i and we consider the maximum of $G_i(x_i,.)$ in $\Omega_i = M - B(x_i,\delta)$. By maximum principle $G_i(x_i,z_i) = \max_{\Omega_i} G_i(x,z) = \max_{\partial B(x_i,\delta)} G_i(x_i,z)$. After passing to a subsequence, we can suppose that $z_i \to z$.

We have $0 < \delta = d(x_i,z_i) \to d(x,z)$. We choose $\delta > 0$ such that the ball of center x and radius 2δ is included in open chart centred in x. (we can choose the exponential map in x and use the lemma 1).

Let t be the point of $B(x,\delta)$ such that $d(y,t) = d[y, B(x,\delta)]$, t depend on x and y. We consider a minimizing curve L_1 between t and y. The manifold is compact and $\delta << inj_g(M)$, then, in each point u of L_1, $[B(u,\delta/2), exp_u]$ is a local chart. We cover the curve L_1 by a finite number of balls of radii $\delta/10$. We apply the Harnack inequality between those balls for the functions $G_i(x_i,.)$. We infer that:

$$\sup_{B(t,\delta/10)} G_i(x_i,s) \leq C(x,y,M,g) \inf_{B(y_i,\delta/10)} G_i(x_i,s) \leq C(x,y,M,g) G_i(x_i,y_i).$$

Now we want to know what happens between the balls $B(t,\delta/10)$ and $B(z,\delta/10)$. The ball $B(x,2\delta)$ is open chart set centered in x. We choose a curve L_2 between z and t like in the first case. This curve must stay in $\partial B(x,\delta)$ and its length $l \leq C_1(x,y,M,g)\delta$, then, we can have a covring of this curve by a minimal number N of balls od radii $\delta/10$, in fact $N \leq C_2(x,y,M,g)$ (like in the first case). Those balls are included in the open chart set centered in x which we choose as in the begining. Then, the operator $\Delta + \epsilon_i$ has those coefficients depending only on the open chart set centred in x and not depending on z, we can apply the Harnack inequality (theorem 8.20 of [GT]) in this open set without $B(x,\delta/100)$, for the functions $G_i(x_i,.)$. Finaly, we obtain the same conclusion than in the case 1, there exist $C = C(x,y,M,g) > 0$ such:

$$G_i(x_i,s) \leq C G_i(x_i,y_i) \ \forall \ s \in \Omega_i = M - B(x_i,\delta) \ \forall \ i \geq i_0.$$

The rest of the proof is the same than in the case 1.

Proof of Theorem 1.

We write u_i by using the Green function G_i, then:

$$\min_M u_i = u_i(x_i) = \int_M G_i(x_i,y) V_i(y) u_i(y)^{N-1} dV_g(y),$$

then,

$$\sup_M u_i \times \inf_M u_i \geq \int_M G_i(x_i,y) V_i(y) u_i(y)^N dV_g(y) \geq a \min_M G_i(x_i,.) \int_M u_i(y)^N dV_g(y).$$

Let $G_i(x_i,y_i) = \min_M G_i(x_i,.)$, after passing to a subsequence, we can assume that $x_i \to x$ and $y_i \to y$. By using the previous proposition, we can suppose that there exist a positive constant $c = c(x,y,M,g)$ such that:

$$G_i(x_i,y_i) \geq \frac{c}{\epsilon_i}.$$

Then,

$$\int_M [u_i(y)]^N dV_g(y) \leq \epsilon_i \sup_M u_i \inf_M u_i.$$

We know argue by contradiction and assume that $\epsilon_i \sup_M u_i \times \inf_M u_i$ tends to 0. We know (see a previous paper when we use the Moser iterate scheme, see [B1]), that (after passing to a subsequence) for q large:

$$||u_i||_{L^q(M)} \to 0.$$

Assume that G the Green function of the laplacian, we can write:

$$u_i(x) = \frac{1}{Vol(M)} \int_M u_i + \int_M G(x,y)[V_i(y) u_i(y)^{N-1} - \epsilon_i u_i(y)] dV_g(y),$$

and if we use Holder inequality, we obtain:

$$\sup_M u_i \to 0.$$

But, this is a contradiction with $\sup_M u_i \geq m > 0$.

Proof of the theorems 2,3,4.

Part I: The metric in polar coordinates.

Let (M,g) a Riemannian manifold. We note $g_{x,ij}$ the local expression of the metric g in the exponential map centred in x.

We are concerning by the polar coordinates expression of the metric. Using Gauss lemma, we can write:

$$g = ds^2 = dt^2 + g_{ij}^k(r,\theta)d\theta^i d\theta^j = dt^2 + r^2\tilde{g}_{ij}^k(r,\theta)d\theta^i d\theta^j = g_{x,ij}dx^i dx^j,$$

in a polar chart with origin x", $]0,\epsilon_0[\times U^k$, with (U^k,ψ) a chart of $\mathbb{S}_{n-1}$. We can write the element volume:

$$dV_g = r^{n-1}\sqrt{|\tilde{g}^k|}drd\theta^1\dots d\theta^{n-1} = \sqrt{[det(g_{x,ij})]}dx^1\dots dx^n,$$

then,

$$dV_g = r^{n-1}\sqrt{[det(g_{x,ij})]}[\exp_x(r\theta)]\alpha^k(\theta)drd\theta^1\dots d\theta^{n-1},$$

where, α^k is such that, $d\sigma_{\mathbb{S}_{n-1}} = \alpha^k(\theta)d\theta^1\dots d\theta^{n-1}$. (Riemannian volume element of the sphere in the chart (U^k,ψ)).

Then,

$$\sqrt{|\tilde{g}^k|} = \alpha^k(\theta)\sqrt{[det(g_{x,ij})]}.$$

Clearly, we have the following proposition:

Proposition 1: Let $x_0\in M$, there exist $\epsilon_1>0$ and if we reduce U^k, we have:

$$|\partial_r\tilde{g}_{ij}^k(x,r,\theta)| + |\partial_r\partial_{\theta^m}\tilde{g}_{ij}^k(x,r,\theta)| \le Cr,\ \forall\ x\in B(x_0,\epsilon_1)\ \forall\ r\in[0,\epsilon_1],\ \forall\ \theta\in U^k.$$

and,

$$|\partial_r|\tilde{g}^k|(x,r,\theta)| + \partial_r\partial_{\theta^m}|\tilde{g}^k|(x,r,\theta) \le Cr,\ \forall\ x\in B(x_0,\epsilon_1)\ \forall\ r\in[0,\epsilon_1],\ \forall\ \theta\in U^k.$$

Remark:

$\partial_r[\log\sqrt{|\tilde{g}^k|}]$ is a local function of θ, and the restriction of the global function on the sphere $\mathbb{S}_{n-1}$, $\partial_r[\log\sqrt{det(g_{x,ij})}]$. We will note, $J(x,r,\theta) = \sqrt{det(g_{x,ij})}$.

Part II: The laplacian in polar coordinates

Let's write the laplacian in $[0,\epsilon_1]\times U^k$,

$$-\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \partial_r[\log\sqrt{|\tilde{g}^k|}]\partial_r + \frac{1}{r^2\sqrt{|\tilde{g}^k|}}\partial_{\theta^i}(\tilde{g}^{\theta^i\theta^j}\sqrt{|\tilde{g}^k|}\partial_{\theta^j}).$$

We have,

$$-\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \partial_r\log J(x,r,\theta)\partial_r + \frac{1}{r^2\sqrt{|\tilde{g}^k|}}\partial_{\theta^i}(\tilde{g}^{\theta^i\theta^j}\sqrt{|\tilde{g}^k|}\partial_{\theta^j}).$$

We write the laplacian (radial and angular decomposition),

$$-\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \partial_r[\log J(x,r,\theta)]\partial_r - \Delta_{S_r(x)},$$

where $\Delta_{S_r(x)}$ is the laplacian on the sphere $S_r(x)$.

We set $L_\theta(x,r)(...) = r^2\Delta_{S_r(x)}(...)[\exp_x(r\theta)]$, clearly, this operator is a laplacian on $\mathbb{S}_{n-1}$ for particular metric. We write,

$$L_\theta(x,r) = \Delta_{g_{x,r,\mathbb{S}_{n-1}}},$$

and,

$$\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \partial_r[J(x,r,\theta)]\partial_r - \frac{1}{r^2}L_\theta(x,r).$$

If, u is function on M, then, $\bar u(r,\theta) = u[\exp_x(r\theta)]$ is the corresponding function in polar coordinates centred in x. We have,

$$-\Delta u = \partial_{rr}\bar u + \frac{n-1}{r}\partial_r\bar u + \partial_r[J(x,r,\theta)]\partial_r\bar u - \frac{1}{r^2}L_\theta(x,r)\bar u.$$

Part III: "Blow-up" and "Moving-plane" methods

The "blow-up" technic

Let, $(u_i)_i$ a sequence of functions on M such that,

$$\Delta u_i - \lambda u_i = n(n-2){u_i}^{N-1},\ u_i > 0,\ N = \frac{2n}{n-2}, \qquad (E)$$

We argue by contradiction and we suppose that $\sup \times \inf$ is not bounded.

We assume that:

$\forall\ c, R > 0\ \exists\ u_{c,R}$ solution of (E) such that:

$$R^{n-2}\sup_{B(x_0,R)} u_{c,R} \times \inf_M u_{c,R} \geq c. \qquad (H)$$

Proposition 2:

There exist a sequence of points $(y_i)_i$, $y_i \to x_0$ and two sequences of positive real number $(l_i)_i, (L_i)_i$, $l_i \to 0$, $L_i \to +\infty$, such that if we consider $v_i(y) = \dfrac{u_i[\exp_{y_i}(y)]}{u_i(y_i)}$, we have:

$$i) \qquad 0 < v_i(y) \leq \beta_i \leq 2^{(n-2)/2},\ \beta_i \to 1.$$

$$ii) \qquad v_i(y) \to \left(\frac{1}{1+|y|^2}\right)^{(n-2)/2},\ \text{uniformly on every compact set of } \mathbb{R}^n.$$

$$iii) \qquad l_i^{(n-2)/2}[u_i(y_i)] \times \inf_M u_i \to +\infty$$

Proof:

We use the hypothesis (H). We can take two sequences $R_i > 0, R_i \to 0$ and $c_i \to +\infty$, such that,

$$R_i^{(n-2)}\sup_{B(x_0,R_i)} u_i \times \inf_M u_i \geq c_i \to +\infty.$$

Let, $x_i \in B(x_0,R_i)$, such that $\sup_{B(x_0,R_i)} u_i = u_i(x_i)$ and $s_i(x) = [R_i - d(x,x_i)]^{(n-2)/2}u_i(x), x \in B(x_i,R_i)$. Then, $x_i \to x_0$.

We have,

$$\max_{B(x_i,R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = {R_i}^{(n-2)/2}u_i(x_i) \geq \sqrt{c_i} \to +\infty.$$

Set :

$$l_i = R_i - d(y_i,x_i),\ \bar u_i(y) = u_i[\exp_{y_i}(y)],\ v_i(z) = \frac{u_i[\exp_{y_i}\left(z/[u_i(y_i)]^{2/(n-2)}\right)]}{u_i(y_i)}.$$

Clearly, $y_i \to x_0$. We obtain:

$$L_i = \frac{l_i}{(c_i)^{1/2(n-2)}}[u_i(y_i)]^{2/(n-2)} = \frac{[s_i(y_i)]^{2/(n-2)}}{c_i^{1/2(n-2)}} \geq \frac{c_i^{1/(n-2)}}{c_i^{1/2(n-2)}} = c_i^{1/2(n-2)} \to +\infty.$$

If $|z| \leq L_i$, then $y = \exp_{y_i}[z/[u_i(y_i)]^{2/(n-2)}] \in B(y_i, \delta_i l_i)$ with $\delta_i = \dfrac{1}{(c_i)^{1/2(n-2)}}$ and $d(y, y_i) < R_i - d(y_i, x_i)$, thus, $d(y, x_i) < R_i$ and, $s_i(y) \leq s_i(y_i)$, we can write,

$$u_i(y)[R_i - d(y, y_i)]^{(n-2)/2} \leq u_i(y_i)(l_i)^{(n-2)/2}.$$

But, $d(y, y_i) \leq \delta_i l_i$, $R_i > l_i$ and $R_i - d(y, y_i) \geq R_i - \delta_i l_i > l_i - \delta_i l_i = l_i(1-\delta_i)$, we obtain,

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \left[\frac{l_i}{l_i(1-\delta_i)}\right]^{(n-2)/2} \leq 2^{(n-2)/2}.$$

We set, $\beta_i = \left(\dfrac{1}{1-\delta_i}\right)^{(n-2)/2}$, clearly $\beta_i \to 1$.

The function v_i is solution of:

$$-g^{jk}[\exp_{y_i}(y)]\partial_{jk} v_i - \partial_k \left[g^{jk}\sqrt{|g|}\right][\exp_{y_i}(y)]\partial_j v_i + \frac{R_g[\exp_{y_i}(y)]}{[u_i(y_i)]^{4/(n-2)}} v_i = n(n-2){v_i}^{N-1},$$

By elliptic estimates and Ascoli, Ladyzenskaya theorems, $(v_i)_i$ converge uniformely on each compact to the function v solution on $\mathbb{R}^n$ of,

$$\Delta v = n(n-2)v^{N-1},\ v(0) = 1,\ 0 \leq v \leq 1 \leq 2^{(n-2)/2},$$

By using maximum principle, we have $v > 0$ on $\mathbb{R}^n$, the result of Caffarelli-Gidas-Spruck (see [C-G-S]) give, $v(y) = \left(\dfrac{1}{1+|y|^2}\right)^{(n-2)/2}$. We have the same properties for v_i in the previous paper [B2].

<u>Polar coordinates and "moving-plane" method</u>

Let,

$$w_i(t, \theta) = e^{(n-2)/2}\bar{u}_i(e^t, \theta) = e^{(n-2)t/2} u_i o \exp_{y_i}(e^t\theta),\ \text{ et } a(y_i, t, \theta) = \log J(y_i, e^t, \theta).$$

<u>Lemma 1:</u>

The function w_i is solution of:

$$-\partial_{tt} w_i - \partial_t a \partial_t w_i - L_\theta(y_i, e^t) + c w_i = n(n-2)w_i^{N-1},$$

with,

$$c = c(y_i, t, \theta) = \left(\frac{n-2}{2}\right)^2 + \frac{n-2}{2}\partial_t a - \lambda e^{2t},$$

<u>Proof:</u>

We write:

$$\partial_t w_i = e^{nt/2}\partial_r \bar{u}_i + \frac{n-2}{2} w_i,\ \partial_{tt} w_i = e^{(n+2)t/2}\left[\partial_{rr}\bar{u}_i + \frac{n-1}{e^t}\partial_r \bar{u}_i\right] + \left(\frac{n-2}{2}\right)^2 w_i.$$

$$\partial_t a = e^t \partial_r \log J(y_i, e^t, \theta), \partial_t a \partial_t w_i = e^{(n+2)t/2}\left[\partial_r \log J \partial_r \bar{u}_i\right] + \frac{n-2}{2}\partial_t a w_i.$$

the lemma is proved.

Now we have, $\partial_t a = \dfrac{\partial_t b_1}{b_1}$, $b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0$,

We can write,

$$-\frac{1}{\sqrt{b_1}}\partial_{tt}(\sqrt{b_1} w_i) - L_\theta(y_i, e^t) w_i + [c(t) + b_1^{-1/2} b_2(t, \theta)] w_i = n(n-2){w_i}^{N-1},$$

where, $b_2(t,\theta) = \partial_{tt}(\sqrt{b_1}) = \dfrac{1}{2\sqrt{b_1}}\partial_{tt} b_1 - \dfrac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2$.

Let,

$$\tilde{w}_i = \sqrt{b_1} w_i,$$

Lemma 2:

The function $\tilde{w}_i$ is solution of:

$$-\partial_{tt}\tilde{w}_i + \Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i).\nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2} b_2 - c_2)\tilde{w}_i =$$

$$= n(n-2)\left(\frac{1}{b_1}\right)^{(N-2)/2} \tilde{w}_i^{N-1},$$

where, $c_2 = [\dfrac{1}{\sqrt{b_1}}\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2]$.

Proof:

We have:

$$-\partial_{tt}\tilde{w}_i - \sqrt{b_1}\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}} w_i + (c + b_2)\tilde{w}_i = n(n-2)\left(\frac{1}{b_1}\right)^{(N-2)/2} \tilde{w}_i^{N-1},$$

But,

$$\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}(\sqrt{b_1} w_i) = \sqrt{b_1}\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}} w_i - 2\nabla_\theta w_i.\nabla_\theta \sqrt{b_1} + w_i \Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}(\sqrt{b_1}),$$

and,

$$\nabla_\theta(\sqrt{b_1} w_i) = w_i \nabla_\theta \sqrt{b_1} + \sqrt{b_1}\nabla_\theta w_i,$$

we deduce than,

$$\sqrt{b_1}\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}} w_i = \Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i).\nabla_\theta \log(\sqrt{b_1}) - c_2\tilde{w}_i,$$

with $c_2 = [\dfrac{1}{\sqrt{b_1}}\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2]$. The lemma is proved.

The "moving-plane" method:

Let ξ_i a real number, and suppose $\xi_i \leq t$. We set $t^{\xi_i} = 2\xi_i - t$ and $\tilde{w}_i^{\xi_i}(t,\theta) = \tilde{w}_i(t^{\xi_i},\theta)$.

We have,

$$-\partial_{tt}\tilde{w}_i^{\xi_i} + \Delta_{g_{y_i,e^{t^{\xi_i}}\mathbb{S}_{n-1}}}(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i^{\xi_i}).\nabla_\theta \log(\sqrt{b_1})\tilde{w}_i^{\xi_i} + [c(t^{\xi_i}) + b_1^{-1/2}(t^{\xi_i},.)b_2(t^{\xi_i}) - c_2^{\xi_i}]\tilde{w}_i^{\xi_i} =$$

$$= n(n-2)\left(\frac{1}{b_1^{\xi_i}}\right)^{(N-2)/2} (\tilde{w}_i^{\xi_i})^{N-1}.$$

By using the same arguments than in [B2], we have:

Proposition 3:

We have:

$$1)\ \tilde{w}_i(\lambda_i,\theta) - \tilde{w}_i(\lambda_i + 4,\theta) \geq \tilde{k} > 0,\ \forall\ \theta \in \mathbb{S}_{n-1}.$$

For all $\beta > 0$, there exist $c_\beta > 0$ such that:

$$2)\ \frac{1}{c_\beta} e^{(n-2)t/2} \leq \tilde{w}_i(\lambda_i + t,\theta) \leq c_\beta e^{(n-2)t/2},\ \forall\ t \leq \beta,\ \forall\ \theta \in \mathbb{S}_{n-1}.$$

We set,

$$\bar{Z}_i = -\partial_{tt}(...) + \Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}(...) + 2\nabla_\theta(...).\nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)(...)$$

Remark: In the operator $\bar{Z}_i$, by using the proposition 3, the coeficient $c + b_1^{-1/2}b_2 - c_2$ satisfies:

$$c + b_1^{-1/2}b_2 - c_2 \geq k' > 0, \text{ pour } t << 0,$$

it is fundamental if we want to apply Hopf maximum principle.

<u>Goal:</u>

Like in [B2], we have elliptic second order operator. Here it is $\bar{Z}_i$, the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$\bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) \leq 0, \text{ if } \tilde{w}_i^{\xi_i} - \tilde{w}_i \leq 0.$$

We write:

$$\bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) = (\Delta_{g_{y_i,e^{t^{\xi_i}},\mathbb{S}_{n-1}}} - \Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}})(\tilde{w}_i^{\xi_i}) +$$

$$+2(\nabla_{\theta,e^{t^{\xi_i}}} - \nabla_{\theta,e^t})(w_i^{\xi_i}).\nabla_{\theta,e^{t^{\xi_i}}} \log(\sqrt{b_1^{\xi_i}}) + 2\nabla_{\theta,e^t}(\tilde{w}_i^{\xi_i}).\nabla_{\theta,e^{t^{\xi_i}}}[\log(\sqrt{b_1^{\xi_i}}) - \log\sqrt{b_1}] +$$

$$+2\nabla_{\theta,e^t} w_i^{\xi_i}.(\nabla_{\theta,e^{t^{\xi_i}}} - \nabla_{\theta,e^t})\log\sqrt{b_1} - [(c + b_1^{-1/2}b_2 - c_2)^{\xi_i} - (c + b_1^{-1/2}b_2 - c_2)]\tilde{w}_i^{\xi_i} +$$

$$+n(n-2)\left(\frac{1}{b_1^{\xi_i}}\right)^{(N-2)/2}(\tilde{w}_i^{\xi_i})^{N-1} - n(n-2)\left(\frac{1}{b_1}\right)^{(N-2)/2}\tilde{w}_i^{N-1}. \qquad (***1)$$

Clearly, we have:

<u>Lemma 3 :</u>

$$b_1(y_i,t,\theta) = 1 - \frac{1}{3}Ricci_{y_i}(\theta,\theta)e^{2t} + \ldots,$$

$$R_g(e^t\theta) = R_g(y_i) + < \nabla R_g(y_i)|\theta > e^t + \ldots.$$

According to proposition 1 and lemma 3,

<u>Propostion 4 :</u>

$$\bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) \leq {b_1}^{(2-N)/2}[(\tilde{w}_i^{\xi_i})^{N-1} - \tilde{w}_i^{N-1}] +$$

$$+C|e^{2t} - e^{2t^{\xi_i}}|\left[|\nabla_\theta \tilde{w}_i^{\xi_i}| + |\nabla_\theta^2(\tilde{w}_i^{\xi_i})| + |Ricci_{y_i}|[\tilde{w}_i^{\xi_i} + (\tilde{w}_i^{\xi_i})^{N-1}] + |R_g(y_i)|\tilde{w}_i^{\xi_i}\right] + C'|e^{3t^{\xi_i}} - e^{3t}|.$$

<u>Proof:</u>

We use proposition 1, we have:

$$a(y_i,t,\theta) = \log J(y_i,e^t,\theta) = \log b_1, |\partial_t b_1(t)| + |\partial_{tt} b_1(t)| + |\partial_{tt} a(t)| \leq Ce^{2t},$$

and,

$$|\partial_{\theta_j} b_1| + |\partial_{\theta_j,\theta_k} b_1| + \partial_{t,\theta_j} b_1| + |\partial_{t,\theta_j,\theta_k} b_1| \leq Ce^{2t},$$

then,

$$|\partial_t b_1(t^{\xi_i}) - \partial_t b_1(t)| \leq C'|e^{2t} - e^{2t^{\xi_i}}|, \text{ on }]-\infty, \log\epsilon_1] \times \mathbb{S}_{n-1}, \forall\, x \in B(x_0,\epsilon_1)$$

Locally,

$$\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}} = L_\theta(y_i,e^t) = -\frac{1}{\sqrt{|\tilde g^k(e^t,\theta)|}}\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^t,\theta)\sqrt{|\tilde g^k(e^t,\theta)|}\partial_{\theta^j}].$$

Thus, in $[0,\epsilon_1]\times U^k$, we have,

$$A_i = \left[\left[\frac{1}{\sqrt{|\tilde g^k|}}\partial_{\theta^l}(\tilde g^{\theta^l\theta^j}\sqrt{|\tilde g^k|}\partial_{\theta^j})\right]^{\xi_i} - \frac{1}{\sqrt{|\tilde g^k|}}\partial_{\theta^l}(\tilde g^{\theta^l\theta^j}\sqrt{|\tilde g^k|}\partial_{\theta^j})\right](\tilde w_i^{\xi_i})$$

then, $A_i = B_i + D_i$ with,

$$B_i = \left[\tilde g^{\theta^l\theta^j}(e^{t^{\xi_i}},\theta) - \tilde g^{\theta^l\theta^j}(e^t,\theta)\right]\partial_{\theta^l\theta^j}\tilde w_i^{\xi_i}(t,\theta),$$

and,

$$D_i = \left[\frac{1}{\sqrt{|\tilde g^k|}(e^{t^{\xi_i}},\theta)}\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^{t^{\xi_i}},\theta)\sqrt{|\tilde g^k|}(e^{t^{\xi_i}},\theta)] - \frac{1}{\sqrt{|\tilde g^k|}(e^t,\theta)}\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^t,\theta)\sqrt{|\tilde g^k|}(e^t,\theta)]\right]\partial_{\theta^j}\tilde w_i^{\xi_i}(t,\theta),$$

we deduce,

$$A_i \le C_k|e^{2t} - e^{2t^{\xi_i}}|\left[|\nabla_\theta \tilde w_i^{\xi_i}| + |\nabla_\theta^2(\tilde w_i^{\xi_i})|\right],$$

If we take $C = \max\{C_i, 1\le i\le q\}$ and if w use $(***1)$, we obtain proposition 4.
We have,

$$c(y_i,t,\theta) = \left(\frac{n-2}{2}\right)^2 + \frac{n-2}{2}\partial_t a + R_g e^{2t}, \qquad (\alpha_1)$$

$$b_2(t,\theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}}\partial_{tt}b_1 - \frac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2, \qquad (\alpha_2)$$

$$c_2 = [\frac{1}{\sqrt{b_1}}\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2], \qquad (\alpha_3)$$

Then,

$$\partial_t c(y_i,t,\theta) = \frac{(n-2)}{2}\partial_{tt}a + 2e^{2t}R_g(e^t\theta) + e^{3t} < \nabla R_g(e^t\theta)|\theta >,$$

by proposition 1,

$$|\partial_t c_2| + |\partial_t b_1| + |\partial_t b_2| + |\partial_t c| \le K_1 e^{2t},$$

<u>The case: $0 < m < \lambda + R_g < \frac{1}{m}$ for the equation $\Delta u - \lambda u = n(n-2)u^{N-1}$</u>

Let x_0 a point of M, we consider a conformal change of metric $\tilde g = \varphi^{4/(n-2)}g$ such that, $\tilde{Ricci}(x_0) = 0$. See for example [Au] (also Lee and Parker [L,P]).

We are concerning by the following equation,

$$\Delta_g u - \lambda u = n(n-2)u^{N-1},$$

the conformal change of metric give when we set $v = u/\varphi$,

$$\Delta_{\tilde g} v + \tilde R_{\tilde g} v = n(n-2)v^{N-1} + (\lambda + \tilde R_g)\varphi^{N-2}v.$$

The notation $\tilde R$ is for $\frac{n-2}{4(n-1)}R$ and $R = R_g$ or $R = R_{\tilde g}$.

Our calculus for the metric $\tilde g$ are the same that for the metric g. But we have some new properties:

$$\sqrt{det(\tilde g_{y_i,jk})} = 1 - \frac{1}{3}\tilde{Ricci}(y_i)(\theta,\theta)r^2 + ..., \text{ and } \tilde R_{\tilde g}(y_i) \to 0,\ \tilde{Ricci}(y_i) \to 0.$$

If we see the coeficient in the term $e^{2t^{\xi_i}} - e^{2t}$, we can say that all those terms are tending to 0, see proposition 4. Only the term $(\lambda + \tilde{R}_g)(e^{2t^{\xi_i}} - e^{2t}) \leq m(e^{2t^{\xi}} - e^{2t})$ ($m > 0$), is the biggest.

In fact, the increment of the local expression of the metric $\tilde{g}_{jk}^{\xi_i} - \tilde{g}_{jk}$, have terms of type $\partial_{\theta_j} \tilde{w}_i^{\xi_i}$ et $\partial_{\theta_j,\theta_k} \tilde{w}_i^{\xi_i}$ but we know by proposition 2 that those terms tend to 0 because the limit function is radial and do not depend on the angles.

We apply proposition 3. We take $t_i = \log \sqrt{l_i}$ with l_i like in proposition 2. The fact $\sqrt{l_i}[u_i(y_i)]^{2/(n-2)} \to +\infty$ (see proposition 2), implies $t_i = \log \sqrt{l_i} > \dfrac{2}{n-2} \log u_i(y_i) + 2 = \lambda_i + 2$. Finaly, we can work on $]-\infty, t_i]$.

We define ξ_i by:

$$\xi_i = \sup\{\lambda \leq \lambda_i + 2,\ \tilde{w}_i(2\lambda - t, \theta) - \tilde{w}_i(t, \theta) \leq 0 \text{ on } [\lambda, t_i] \times \mathbb{S}_{n-1}\}.$$

If we use proposition 4 and the similar technics that in [B2] we can deduce by Hopf maximum principle,

$$\max_{\mathbb{S}_{n-1}} \tilde{w}_i(t_i, \theta) \leq \min_{\mathbb{S}_{n-1}} \tilde{w}_i(2\xi_i - t_i, \theta),$$

which implies,

$$l_i^{(n-2)/2} u_i(y_i) \times \min_M u_i \leq c.$$

It is in contradiction with proposition 2.

Then we have,

$$\sup_K u \times \inf_M u \leq c = c(K, M, m, g, n).$$

Application:

Let M a Riemannian manifold of dimension $n \geq 3$, and consider a sequence of functions u_i such that:

$$\Delta u_i + \epsilon_i u_i = n(n-2) u_i^{N-1},\ \ \epsilon_i \to 0$$

If, the scalar curvature $R_g \geq m > 0$ on M, then, applying the previous result with $\lambda = -\epsilon_i$, we obtain:

$$\sup_M u_i \times \inf_M u_i \leq c,\ \forall\ i,$$

Proof of the theorem 4:

Without loss of generality we suppose,

$$\Delta u_i + \epsilon_i u_i = u_i^{N-1},\ \text{ et } \max_M u_i \to 0.$$

Lemma 1: There exist a positive constant, c such that:

$$\sup_M u_i \leq c \inf_M u_i,\ \forall i.$$

Proof of lemma 1:

Suppose by contradiction:

$$\limsup_{i \to +\infty} \frac{\sup_M u_i}{\inf_M u_i} = +\infty,$$

After passing to a subsequence, we can assume: $\dfrac{\sup_M u_i}{\inf_M u_i} \to +\infty$.

We have, $\sup_M u_i = u_i(y_i)$ et $\inf_M u_i = u_i(x_i)$. We also suppose, $x_i \to x$ et $y_i \to y$.

Let L be a minimizing curve between x and y, take $\delta > 0$ such that $\delta < inj_g(M)$, with $inj_g(M)$ the injectivity radius of the compact manifold M.

For all $a \in L$, $[B(a,\delta), (\exp_a)^{-1}]$ is a local chart around a, but L is compact. We can cover this curve by a finite number of balls centred in a points of L and of radius $\delta/5$. Let $a_1, \dots, a_k$ those points, with, $a_1 = x$ and $a_k = y$.

In each ball $B(a_j, \delta)$, u_i is solution of, $\Delta u_i + (\epsilon_i - u_i^{N-2})u_i = 0$, we use the fact $\sup_M u_i \to 0$ and we apply the Harnack inequality of [G-T] (see theorem 8.20), we obtain:

$$\sup_{B(a_j,\delta/5)} u_i \leq C_j \inf_{B(a_j,\delta/5)} u_i, \; j = 1, \dots, k.$$

We deduce:

$$\sup_{B(y,\delta/5)} u_i \leq C_k C_{k-1} \dots . C_1 \inf_{B(x,\delta/5)} u_i,$$

In other words:

$$\sup_M u_i \leq C_k . \dots . C_1 \inf_M u_i.$$

It's in contradiction with our hypothesis.

<u>**Lemma 2:**</u> There exist two constants, $k_1, k_2 > 0$ such that:

$$k_1 {\epsilon_i}^{(n-2)/4} \leq u_i(x) \leq k_2 {\epsilon_i}^{(n-2)/4}, \; \forall \; x \in M, \; \forall \; i.$$

<u>**Proof of lemma 2:**</u>

Let G_i the Green function of the operator $\Delta + \epsilon_i$, this equation satisfies:

$$\int_M G_i(x,y) dV_g(y) = \frac{1}{\epsilon_i}, \; \forall \; x \in M.$$

We write:

$$\inf_M u_i = u_i(x_i) = \int_M G_i(x_i,y) u_i^{N-1}(y) dV_g(y) \geq (\inf_M u_i)^{N-1} \int_M G_i(x_i,y) dV_g(y) = \frac{(\inf_M u_i)^{N-1}}{\epsilon_i},$$

thus,

$$\inf_M u_i \leq {\epsilon_i}^{(n-2)/4}.$$

We the same idea we can prove, $\sup_M u_i \geq {\epsilon_i}^{(n-2)/4}$. We deduce lemma 2 from lemma 1 and the two last inequalities.

<u>**Lemma 3:**</u> There exist a rank i_0 such that, $u_i \equiv {\epsilon_i}^{(n-2)/4}$. for $i \geq i_0$.

<u>**Proof of lemma 3:**</u>

Let, $w_i = \dfrac{u_i}{{\epsilon_i}^{(n-2)/4}}$. This function is solution of:

$$\Delta w_i = \epsilon_i (w_i^{N-1} - w_i) = \epsilon_i w_i (w_i^{N-2} - 1). \qquad (*)$$

<u>**Case 1:** $N-2 > 1$ **(**$3 < n < 6$**),**</u>

To simplify our computations we suppose that $N-2$ is an integer.

According to binomial formula, $w_i^{N-2} - 1 = (w_i - 1)(1 + w_i + \dots)$, we multiply $(*)$ by $w_i - 1$ and we integrate, we obtain:

$$\int_M |\nabla w_i|^2 \leq C \epsilon_i \int_M |w_i - 1|^2,$$

Suppose that we have infinity i, such that $w_i \not\equiv 1$, then we can consider the following functions: $z_i = \dfrac{w_i - 1}{||w_i - 1||_2}$.

z_i verifiy, $||z_i||_2 = 1, ||\nabla z_i||_2^2 \leq C\epsilon_i \to 0$, thus, $z_i \to 1$ in $L^2(M)$ and in particular, $\int_M z_i w_i(1 + w_i + ...) \to C' \neq 0$ (by using lemma 2). But, if we integrate $(*)$, we find $\int_M z_i w_i(1 + w_i + ...) = 0$, it's a contradiction.

Thus, there exist a rank such that $w_i \equiv 1$ after this rank.

Case 2: $0 < N - 2 < 1$ ($n > 7$):

To simplify our computations, we suppose that $1/(N - 2)$ is an integer.

Now we take $w_i^{N-2} - 1$ and we write $w_i - 1 = (w_i^{N-2})^{1/(N-2)} - 1$, by using the binomial formula and the same ideas than in the previous case we obtain our result.

ACKNOWLEDGEMENT.

This work was done when the author was in Greece at Patras. The author is grateful to Professor Athanase Cotsiolis, the Department of Mathematics of Patras University and the IKY Foundation for hospitality and the very good conditions of work.

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