Sup-Inf inequality on manifold of dimension 3

Samy Skander Bahoura

SUP-INF INEQUALITY ON MANIFOLD OF DIMENSION 3

SAMY SKANDER BAHOURA

ABSTRACT. We give an estimate of type $\sup \times \inf$ on Riemannian manifold of Dimension 3 for prescribed curvature equation.

INTRODUCTION AND RESULTS.

We are on Riemannian manifold (M, g) of dimension 3 not necessary compact. In this paper we denote $\Delta = -\nabla^j(\nabla_j)$ the geometric laplacian.

The scalar curvature equation is:

$$8\Delta u + R_g u = V u^{N-1}, \ u > 0. \qquad (E)$$

Where R_g is the scalar curvature and V is a function (prescribed scalar curvature).

We consider three positive real number a, bA and we suppose V lipschitzian:

$$0 < a \leq V(x) \leq b < +\infty \text{ and } ||\nabla V||_{L^\infty(M)} \leq A. \qquad (C)$$

The equation (E) was studied lot of when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example, [B], [CL1], [L1]. In this case we have a $\sup \times \inf$ inequality.

The corresponding equation in two dimensions on open set Ω of $\mathbb{R}^2$, is:

$$\Delta u = V e^u, \qquad (E')$$

The equation (E') was studed lot of and we can find very important result about a priori estimates in [BM], [BLS], [CL2], [L2], and [S].

In the case $V \equiv 1$ and M compact, the equation (E) is Yamabe equation. It was studed lot of, T.Aubin and R.Schoen have proved the existence of solution in this cas, see for example [Au] and [L-P].

When M is a compact Riemannian manifold, it exist some compactness result for equation (E) see [L-Zh]. Li and Zhu [L-Zh], proved that the energy is bounded and if we suppose M not diffeormorfic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem.

Now, if we suppose M Riemannian manifold (not necessarily compact) and $V \equiv 1$, Li and Zhang [L-Z] proved that the product $\sup \times \inf$ is bounded.

Here, we give an equality of type $\sup \times \inf$ for the equation (E) with general conditions (C). We have:

Theorem. For all compact set K of M and all positive numbers a,b,A, it exists a positive constant c, which depends only on, K,a,b,A,M,g such that:

$$(\sup_K u)^{1/3} \times \inf_M u \leq c,$$

for all u solution of (E) with conditions (C).

Note that in our work, we have not assumption on energy or boundary condition if we suppose the manifold M with boundary.

Next, in the proof of the previous theorem, we can replace the scalar curvature by any smooth function f, but here we do the proof with R_g the scalar curvature.

Proof of the theorem.

Part I: The metric and the laplacian in polar coordinates.

Let (M, g) a Riemannian manifold. We note $g_{x,ij}$ the local expression of the metric g in the exponential map centred in x.

We are concerning by the polar coordinates expression of the metric. By using Gauss lemma, we can write:

$$g = ds^2 = dt^2 + g^k_{ij}(r,\theta)d\theta^i d\theta^j = dt^2 + r^2\tilde{g}^k_{ij}(r,\theta)d\theta^i d\theta^j = g_{x,ij}dx^i dx^j,$$

in a polar chart with origin x", $]0,\epsilon_0[\times U^k$, with (U^k,ψ) a chart of $\mathbb{S}_{n-1}$. We can write the element volume:

$$dV_g = r^{n-1}\sqrt{|\tilde{g}^k|}drd\theta^1\dots d\theta^{n-1} = \sqrt{[det(g_{x,ij})]}dx^1\dots dx^n,$$

then,

$$dV_g = r^{n-1}\sqrt{[det(g_{x,ij})]}[\exp_x(r\theta)]\alpha^k(\theta)drd\theta^1\dots d\theta^{n-1},$$

where, α^k is such that, $d\sigma_{\mathbb{S}_{n-1}} = \alpha^k(\theta)d\theta^1\dots d\theta^{n-1}$. (Riemannian volume element of the la sphere in the chart (U^k,ψ)).

Then,

$$\sqrt{|\tilde{g}^k|} = \alpha^k(\theta)\sqrt{[det(g_{x,ij})]},$$

Clearly, we have the following proposition:

Proposition 1: Let $x_0\in M$, there exist $\epsilon_1>0$ and if we reduce U^k, we have:

$$|\partial_r\tilde{g}^k_{ij}(x,r,\theta)| + |\partial_r\partial_{\theta^m}\tilde{g}^k_{ij}(x,r,\theta)| \leq Cr,\ \forall\ x\in B(x_0,\epsilon_1)\ \forall\ r\in[0,\epsilon_1],\ \forall\ \theta\in U^k.$$

and,

$$|\partial_r|\tilde{g}^k|(x,r,\theta)| + \partial_r\partial_{\theta^m}|\tilde{g}^k|(x,r,\theta) \leq Cr,\ \forall\ x\in B(x_0,\epsilon_1)\ \forall\ r\in[0,\epsilon_1],\ \forall\ \theta\in U^k.$$

Remark:

$\partial_r[\log\sqrt{|\tilde{g}^k|}]$ is a local function of θ, and the restriction of the global function on the sphere $\mathbb{S}_{n-1}$, $\partial_r[\log\sqrt{det(g_{x,ij})}]$. We will note, $J(x,r,\theta) = \sqrt{det(g_{x,ij})}$.

Let's write the laplacian in $[0,\epsilon_1]\times U^k$,

$$-\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \partial_r[\log\sqrt{|\tilde{g}^k|}]\partial_r + \frac{1}{r^2\sqrt{|\tilde{g}^k|}}\partial_{\theta^i}(\tilde{g}^{\theta^i\theta^j}\sqrt{|\tilde{g}^k|}\partial_{\theta^j}).$$

We have,

$$-\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \partial_r\log J(x,r,\theta)\partial_r + \frac{1}{r^2\sqrt{|\tilde{g}^k|}}\partial_{\theta^i}(\tilde{g}^{\theta^i\theta^j}\sqrt{|\tilde{g}^k|}\partial_{\theta^j}).$$

We write the laplacian (radial and angular decomposition),

$$-\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + \partial_r[\log J(x,r,\theta)]\partial_r - \Delta_{S_r(x)},$$

where $\Delta_{S_r(x)}$ is the laplacian on the sphere $S_r(x)$.

We set $L_\theta(x,r)(...) = r^2\Delta_{S_r(x)}(...)[\exp_x(r\theta)]$, clearly, this operator is a laplacian on $\mathbb{S}_{n-1}$ for particular metric. We write,

$$L_\theta(x,r) = \Delta_{g_{x,r,\mathbb{S}_{n-1}}},$$

and,

$$\Delta=\partial_{rr}+\frac{n-1}{r}\partial_r+\partial_r[J(x,r,\theta)]\partial_r-\frac{1}{r^2}L_\theta(x,r).$$

If, u is function on M, then, $\bar u(r,\theta)=uo\exp_x(r\theta)$ is the corresponding function in polar coordinates centred in x. We have,

$$-\Delta u=\partial_{rr}\bar u+\frac{n-1}{r}\partial_r\bar u+\partial_r[J(x,r,\theta)]\partial_r\bar u-\frac{1}{r^2}L_\theta(x,r)\bar u.$$

Part II: "Blow-up" and "Moving-plane" methods

The "blow-up" technic

Let, $(u_i)_i$ a sequence of functions on M such that,

$$\Delta u_i+R_g u_i=V_i{u_i}^5,\ u_i>0,\qquad (E')$$

We argue by contradiction and we suppose that $\sup^{1/3}\times\inf$ is not bounded.

We assume that:

$\forall\ c,R>0\ \exists\ u_{c,R}$ solution of (E') such that:

$$R[\sup_{B(x_0,R)}u_{c,R}]^{1/3}\times\inf_M u_{c,R}\geq c,\qquad (H)$$

Proposition 2:

There exist a sequence of points $(y_i)_i$, $y_i\to x_0$ and two sequences of positive real number $(l_i)_i,(L_i)_i$, $l_i\to 0$, $L_i\to+\infty$, such that if we consider $v_i(y)=\dfrac{u_io\exp_{y_i}(y)}{u_i(y_i)}$, we have:

$$0<v_i(y)\leq\beta_i\leq 2^{1/2},\ \beta_i\to 1.$$

$$v_i(y)\to\left(\frac{1}{1+|y|^2}\right)^{1/2},\ \text{uniformly on every compact set of } \mathbb{R}^3.$$

$$l_i^{1/2}[u_i(y_i)]^{1/3}\times\inf_M u_i\to+\infty$$

Proof:

We use the hypothesis (H), we can take two sequences $R_i>0, R_i\to 0$ and $c_i\to+\infty$, such that,

$$R_i[\sup_{B(x_0,R_i)}]^{1/3}u_i\times\inf_M u_i\geq c_i\to+\infty,$$

Let, $x_i\in B(x_0,R_i)$, such that $\sup_{B(x_0,R_i)}u_i=u_i(x_i)$ and $s_i(x)=[R_i-d(x,x_i)]^{1/2}u_i(x), x\in B(x_i,R_i)$. Then, $x_i\to x_0$.

We have,

$$\max_{B(x_i,R_i)}s_i(x)=s_i(y_i)\geq s_i(x_i)={R_i}^{1/2}u_i(x_i)\geq\sqrt{c_i}\to+\infty.$$

Set :

$$l_i=R_i-d(y_i,x_i),\ \bar u_i(y)=u_io\exp_{y_i}(y),\ v_i(z)=\frac{u_io\exp_{y_i}\left(z/[u_i(y_i)]^2\right)}{u_i(y_i)}.$$

Clearly, $y_i\to x_0$. We obtain:

$$L_i=\frac{l_i}{(c_i)^{1/2}}[u_i(y_i)]^2=\frac{[s_i(y_i)]^2}{c_i^{1/2}}\geq\frac{c_i^1}{c_i^{1/2}}=c_i^{1/2}\to+\infty.$$

If $|z| \leq L_i$, then $y = \exp_{y_i}[z/[u_i(y_i)]^2] \in B(y_i, \delta_i l_i)$ with $\delta_i = \dfrac{1}{(c_i)^{1/2}}$ and $d(y, y_i) < R_i - d(y_i, x_i)$, thus, $d(y, x_i) < R_i$ and, $s_i(y) \leq s_i(y_i)$, we can write,

$$u_i(y)[R_i - d(y, y_i)]^{1/2} \leq u_i(y_i)(l_i)^{1/2}.$$

But, $d(y, y_i) \leq \delta_i l_i$, $R_i > l_i$ and $R_i - d(y, y_i) \geq R_i - \delta_i l_i > l_i - \delta_i l_i = l_i(1-\delta_i)$, we obtain,

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \left[\frac{l_i}{l_i(1-\delta_i)}\right]^{1/2} \leq 2^{1/2}.$$

We set, $\beta_i = \left(\dfrac{1}{1-\delta_i}\right)^{1/2}$, clearly $\beta_i \to 1$.

The function v_i is solution of:

$$-g^{jk}[\exp_{y_i}(y)]\partial_{jk} v_i - \partial_k \left[g^{jk}\sqrt{|g|}\right][\exp_{y_i}(y)]\partial_j v_i + \frac{R_g[\exp_{y_i}(y)]}{[u_i(y_i)]^4} v_i = V_i {v_i}^5,$$

By elleptic estimates and Ascoli, Ladyzenskaya theorems, $(v_i)_i$ converge uniformely on each compact to the function v solution on $\mathbb{R}^3$ of,

$$\Delta v = V(x_0) v^5, \;\; v(0) = 1, \;\; 0 \leq v \leq 1 \leq 2^{1/2},$$

Without loss of generality, we can suppose $V(x_0) = 3$.

By using maximum principle, we have $v > 0$ on $\mathbb{R}^3$, the result of Caffarelli-Gidas-Spruck (see [C-G-S]) give, $v(y) = \left(\dfrac{1}{1+|y|^2}\right)^{1/2}$. We have the same propreties for v_i in the previous paper [B2].

<u>Polar coordinates and ”moving-plane” method</u>

Let,

$$w_i(t,\theta) = e^{1/2}\bar{u}_i(e^t,\theta) = e^{t/2} u_i o \exp_{y_i}(e^t\theta), \;\; \text{et} \;\; a(y_i,t,\theta) = \log J(y_i, e^t, \theta).$$

<u>Lemma 1:</u>

The function w_i is solution of:

$$-\partial_{tt} w_i - \partial_t a \partial_t w_i - L_\theta(y_i, e^t) + c w_i = V_i w_i^5,$$

with,

$$c = c(y_i, t, \theta) = \left(\frac{1}{2}\right)^2 + \frac{1}{2}\partial_t a - \lambda e^{2t},$$

<u>Proof:</u>

We write:

$$\partial_t w_i = e^{3t/2}\partial_r \bar{u}_i + \frac{1}{2} w_i, \;\; \partial_{tt} w_i = e^{5t/2}\left[\partial_{rr}\bar{u}_i + \frac{2}{e^t}\partial_r \bar{u}_i\right] + \left(\frac{1}{2}\right)^2 w_i.$$

$$\partial_t a = e^t \partial_r \log J(y_i, e^t, \theta), \partial_t a \partial_t w_i = e^{5t/2}\left[\partial_r \log J \partial_r \bar{u}_i\right] + \frac{1}{2}\partial_t a w_i.$$

the lemma is proved.

Now we have, $\partial_t a = \dfrac{\partial_t b_1}{b_1}$, $b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0$,

We can write,

$$-\frac{1}{\sqrt{b_1}}\partial_{tt}(\sqrt{b_1} w_i) - L_\theta(y_i, e^t) w_i + [c(t) + b_1^{-1/2} b_2(t,\theta)] w_i = V_i {w_i}^{N-1},$$

where, $b_2(t,\theta)=\partial_{tt}(\sqrt{b_1})=\dfrac{1}{2\sqrt{b_1}}\partial_{tt}b_1-\dfrac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2$.

Let,

$$\tilde w_i=\sqrt{b_1}w_i.$$

Lemma 2:

The function $\tilde w_i$ is solution of:

$$-\partial_{tt}\tilde w_i+\Delta_{g_{y_i,e^t,\mathbb{S}_2}}(\tilde w_i)+2\nabla_\theta(\tilde w_i).\nabla_\theta\log(\sqrt{b_1})+(c+b_1^{-1/2}b_2-c_2)\tilde w_i=$$

$$=V_i\left(\frac{1}{b_1}\right)^2\tilde w_i^5,$$

where, $c_2=[\dfrac{1}{\sqrt{b_1}}\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}(\sqrt{b_1})+|\nabla_\theta\log(\sqrt{b_1})|^2]$.

Proof:

We have:

$$-\partial_{tt}\tilde w_i-\sqrt{b_1}\Delta_{g_{y_i,e^t,\mathbb{S}_2}}w_i+(c+b_2)\tilde w_i=V_i\left(\frac{1}{b_1}\right)^2\tilde w_i^5,$$

But,

$$\Delta_{g_{y_i,e^t,\mathbb{S}_2}}(\sqrt{b_1}w_i)=\sqrt{b_1}\Delta_{g_{y_i,e^t,\mathbb{S}_2}}w_i-2\nabla_\theta w_i.\nabla_\theta\sqrt{b_1}+w_i\Delta_{g_{y_i,e^t,\mathbb{S}_2}}(\sqrt{b_1}),$$

and,

$$\nabla_\theta(\sqrt{b_1}w_i)=w_i\nabla_\theta\sqrt{b_1}+\sqrt{b_1}\nabla_\theta w_i,$$

we deduce,

$$\sqrt{b_1}\Delta_{g_{y_i,e^t,\mathbb{S}_2}}w_i=\Delta_{g_{y_i,e^t,\mathbb{S}_2}}(\tilde w_i)+2\nabla_\theta(\tilde w_i).\nabla_\theta\log(\sqrt{b_1})-c_2\tilde w_i,$$

with $c_2=[\dfrac{1}{\sqrt{b_1}}\Delta_{g_{y_i,e^t,\mathbb{S}_2}}(\sqrt{b_1})+|\nabla_\theta\log(\sqrt{b_1})|^2]$. The lemma is proved.

The "moving-plane" method:

Let ξ_i a real number, and suppose $\xi_i\leq t$, we set $t^{\xi_i}=2\xi_i-t$ and $\tilde w_i^{\xi_i}(t,\theta)=\tilde w_i(t^{\xi_i},\theta)$.

We have,

$$-\partial_{tt}\tilde w_i^{\xi_i}+\Delta_{g_{y_i,e^{t^{\xi_i}}\mathbb{S}_2}}(\tilde w_i)+2\nabla_\theta(\tilde w_i^{\xi_i}).\nabla_\theta\log(\sqrt{b_1})\tilde w_i^{\xi_i}+[c(t^{\xi_i})+b_1^{-1/2}(t^{\xi_i},.)b_2(t^{\xi_i})-c_2^{\xi_i}]\tilde w_i^{\xi_i}=$$

$$=V_i^{\xi_i}\left(\frac{1}{b_1^{\xi_i}}\right)^2(\tilde w_i^{\xi_i})^5.$$

By using the same arguments than in [B], we have:

Proposition 3:

We have:

$$1)\ \tilde w_i(\lambda_i,\theta)-\tilde w_i(\lambda_i+4,\theta)\geq\tilde k>0,\ \forall\ \theta\in\mathbb{S}_2.$$

For all $\beta>0$, there exist $c_\beta>0$ such than:

$$2)\ \frac{1}{c_\beta}e^{t/2}\leq\tilde w_i(\lambda_i+t,\theta)\leq c_\beta e^{t/2},\ \forall\ t\leq\beta,\ \forall\ \theta\in\mathbb{S}_2.$$

We set,

$$\bar Z_i = -\partial_{tt}(...) + \Delta_{g_{y_i,e^t,\mathbb{S}_2}}(...) + 2\nabla_\theta(...).\nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2} b_2 - c_2)(...)$$

Remark: In the opertor $\bar Z_i$, by using the proposition 3, the coeficient $c + b_1^{-1/2} b_2 - c_2$ verify:

$$c + b_1^{-1/2} b_2 - c_2 \geq k' > 0, \text{ pour } t << 0,$$

it is fundamental if we want to apply Hopf maximum principle.

Goal:

Like in [B], we have elliptic second order operator, here it's $\bar Z_i$, the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$\bar Z_i(\tilde w_i^{\xi_i} - \tilde w_i) \leq 0, \text{ if } \tilde w_i^{\xi_i} - \tilde w_i \leq 0.$$

We write:

$$\bar Z_i(\tilde w_i^{\xi_i} - \tilde w_i) = (\Delta_{g_{y_i,e^{t\xi_i},\mathbb{S}_2}} - \Delta_{g_{y_i,e^t,\mathbb{S}_2}})(\tilde w_i^{\xi_i}) +$$

$$+2(\nabla_{\theta,e^{t\xi_i}} - \nabla_{\theta,e^t})(w_i^{\xi_i}).\nabla_{\theta,e^{t\xi_i}} \log(\sqrt{b_1^{\xi_i}}) + 2\nabla_{\theta,e^t}(\tilde w_i^{\xi_i}).\nabla_{\theta,e^{t\xi_i}}[\log(\sqrt{b_1^{\xi_i}}) - \log\sqrt{b_1}] +$$

$$+2\nabla_{\theta,e^t} w_i^{\xi_i}.(\nabla_{\theta,e^{t\xi_i}} - \nabla_{\theta,e^t})\log\sqrt{b_1} - [(c + b_1^{-1/2} b_2 - c_2)^{\xi_i} - (c + b_1^{-1/2} b_2 - c_2)]\tilde w_i^{\xi_i} +$$

$$+V_i^{\xi_i}\left(\frac{1}{b_1^{\xi_i}}\right)^2 (\tilde w_i^{\xi_i})^5 - V_i\left(\frac{1}{b_1}\right)^2 \tilde w_i^5. \qquad (***1)$$

Clearly, we have:

Lemma 3 :

$$b_1(y_i,t,\theta) = 1 - \frac{1}{3} Ricci_{y_i}(\theta,\theta)e^{2t} + \dots,$$

$$R_g(e^t\theta) = R_g(y_i) + < \nabla R_g(y_i)|\theta > e^t + \dots.$$

According to proposition 1 and lemma 3,

Propostion 4 :

$$\bar Z_i(\tilde w_i^{\xi_i} - \tilde w_i) \leq V_i b_1{}^{(-2}[(\tilde w_i^{\xi_i})^5 - \tilde w_i^5] + 2(\tilde w_i^{\xi_i})^5 |V_i^{\xi_i} - V_i| +$$

$$+C|e^{2t} - e^{2t^{\xi_i}}|\left[|\nabla_\theta \tilde w_i^{\xi_i}| + |\nabla_\theta^2(\tilde w_i^{\xi_i})| + |Ricci_{y_i}|[\tilde w_i^{\xi_i} + (\tilde w_i^{\xi_i})^5] + |R_g(y_i)|\tilde w_i^{\xi_i}\right] + C'|e^{3t^{\xi_i}} - e^{3t}|.$$

Proof:

We use proposition 1, we have:

$$a(y_i,t,\theta) = \log J(y_i,e^t,\theta) = \log b_1, |\partial_t b_1(t)| + |\partial_{tt} b_1(t)| + |\partial_{tt} a(t)| \leq Ce^{2t},$$

and,

$$|\partial_{\theta_j} b_1| + |\partial_{\theta_j,\theta_k} b_1| + \partial_{t,\theta_j} b_1| + |\partial_{t,\theta_j,\theta_k} b_1| \leq Ce^{2t},$$

then,

$$|\partial_t b_1(t^{\xi_i}) - \partial_t b_1(t)| \leq C'|e^{2t} - e^{2t^{\xi_i}}|, \text{ sur }]-\infty, \log\epsilon_1] \times \mathbb{S}_2, \forall\ x \in B(x_0,\epsilon_1)$$

Locally,

$$\Delta_{g_{y_i,e^t,\mathbb{S}_2}} = L_\theta(y_i,e^t) = -\frac{1}{\sqrt{|\tilde g^k(e^t,\theta)|}}\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^t,\theta)\sqrt{|\tilde g^k(e^t,\theta)|}\partial_{\theta^j}].$$

Thus, in $[0,\epsilon_1]\times U^k$, we have,

$$A_i=\left[\left[\frac{1}{\sqrt{|\tilde g^k|}}\partial_{\theta^l}(\tilde g^{\theta^l\theta^j}\sqrt{|\tilde g^k|}\partial_{\theta^j})\right]^{\xi_i}-\frac{1}{\sqrt{|\tilde g^k|}}\partial_{\theta^l}(\tilde g^{\theta^l\theta^j}\sqrt{|\tilde g^k|}\partial_{\theta^j})\right](\tilde w_i^{\xi_i})$$

then, $A_i=B_i+D_i$ with,

$$B_i=\left[\tilde g^{\theta^l\theta^j}(e^{t\xi_i},\theta)-\tilde g^{\theta^l\theta^j}(e^t,\theta)\right]\partial_{\theta^l\theta^j}\tilde w_i^{\xi_i}(t,\theta),$$

and,

$$D_i=\left[\frac{1}{\sqrt{|\tilde g^k|(e^{t\xi_i},\theta)}}\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^{t\xi_i},\theta)\sqrt{|\tilde g^k|}(e^{t\xi_i},\theta)]-\frac{1}{\sqrt{|\tilde g^k|(e^t,\theta)}}\partial_{\theta^l}[\tilde g^{\theta^l\theta^j}(e^t,\theta)\sqrt{|\tilde g^k|}(e^t,\theta)]\right]\partial_{\theta^j}\tilde w_i^{\xi_i}(t,\theta),$$

we deduce,

$$A_i\le C_k|e^{2t}-e^{2t\xi_i}|\left[|\nabla_\theta\tilde w_i^{\xi_i}|+|\nabla_\theta^2(\tilde w_i^{\xi_i})|\right],$$

If we take $C=\max\{C_i,1\le i\le q\}$ and if w use $(***1)$, we obtain proposition 4.
We have,

$$c(y_i,t,\theta)=\left(\frac{n-2}{2}\right)^2+\frac{n-2}{2}\partial_t a+R_g e^{2t},\qquad(\alpha_1)$$

$$b_2(t,\theta)=\partial_{tt}(\sqrt{b_1})=\frac{1}{2\sqrt{b_1}}\partial_{tt}b_1-\frac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2,\qquad(\alpha_2)$$

$$c_2=[\frac{1}{\sqrt{b_1}}\Delta_{g_{y_i,e^t,\mathbb{S}_{n-1}}}(\sqrt{b_1})+|\nabla_\theta\log(\sqrt{b_1})|^2],\qquad(\alpha_3)$$

Then,

$$\partial_t c(y_i,t,\theta)=\frac{1}{2}\partial_{tt}a+2e^{2t}R_g(e^t\theta)+e^{3t}<\nabla R_g(e^t\theta)|\theta>,$$

by proposition 1,

$$|\partial_t c_2|+|\partial_t b_1|+|\partial_t b_2|+|\partial_t c|\le K_1e^{2t},$$

Now, we consider the function, $\bar w_i(t,\theta)=\tilde w_i(t,\theta)-\dfrac{[u_i(y_i)]^{1/3}\times\min_M u_i}{2}e^t$, and $\lambda>2>0$.

For $t\le t_i=-(2/3)\log u_i(y_i)$, we have:

$$\bar w_i(t,\theta)=e^t\left[b_1(t,\theta)e^{-t/2}u_i o\exp_{y_i}(e^t\theta)-\frac{[u_i(y_i)]^{1/3}\times\min_M u_i}{2}\right]\ge$$

$$\ge e^t\frac{[u_i(y_i)]^{1/3}\times\min_M u_i}{2}>0,$$

We set, $\mu_i=\dfrac{[u_i(y_i)]^{1/3}\times\min_M u_i}{2}$.

We use proposition 3 and the same arguments than in [B], we obtain:

Lemma 4:
There exists $\nu<0$ such that for $\mu\le\nu$:

$$\bar w_i^\mu(t,\theta)-\bar w_i(t,\theta)\le 0,\ \forall\ (t,\theta)\in[\mu,t_i]\times\mathbb{S}_2,$$

We set, $\lambda_i=-2\log u_i(y_i)$, then,

Lemma 5:

$$\bar w_i(\lambda_i,\theta)-\bar w_i(\lambda_i+4,\theta)>0.$$

Proof of lemma 5:

Clearly:

$$\bar{w}_i(\lambda_i,\theta)-\bar{w}_i(\lambda_i+4,\theta)=\tilde{w}_i(\lambda_i,\theta)-\tilde{w}_i(\lambda_i+4,\theta)+\mu_i(e^4-1),$$

we deduce lemma 6 from proposition 3.

Let, $\xi_i=\sup\{\mu\leq\lambda_i+2, \bar{w}_i^{\xi_i}(t,\theta)-\bar{w}_i(t,\theta)\leq 0,\ \forall\ (t,\theta)\in[\xi_i,t_i]\times\mathbb{S}_2\}$.

The real ξ_i exists (see [B]), if we use $(***2)$, we have:

$$\tilde{w}_i^{\xi_i}(t,\theta)+|\nabla_\theta\tilde{w}_i^{\xi_i}(t,\theta)|+|\nabla_\theta^2\tilde{w}_i^{\xi_i}(t,\theta)|\leq C(R),\ \ \forall\ (t,\theta)\in]-\infty,\log R]\times\mathbb{S}_2,$$

We can write:

$$\bar{Z}_i(\bar{w}_i^{\xi_i}-\bar{w}_i)=\bar{Z}_i(\tilde{w}_i^{\xi_i}-\tilde{w}_i)-\mu_i\bar{Z}_i(e^{2t^{\xi_i}}-e^{2t}),$$

$$-\bar{Z}_i(e^{2t^{\xi_i}}-e^{2t})=[1-\frac{1}{4}-\frac{3}{2}\partial_t a-R_g e^{2t}+b_1^{-1/2}b_2-c_2](e^{2t^{\xi_i}}-e^{2t})\leq c_1(e^{t^{\xi_i}}-e^t),$$

with $c_1>0$, car $|\partial_t a|+|\partial_t b_1|+|\partial_{tt}b_1|+|\partial_{t,\theta_j}b_1|+|\partial_{t,\theta_j,\theta_k}b_1|\leq C'e^{2t}<1$, for t very small.

We use proposition 4, to obtain on, $[\xi_i,t_i]\times\mathbb{S}_2$,

$$\bar{Z}_i(\bar{w}_i^{\xi_i}-\bar{w}_i)\leq c_2[V_i(\tilde{w}_i^{\xi_i})^5-\tilde{w}_i^5]+|V_i^{\xi_i}-V_i|(w_i^{\xi_i})^5+[\mu_i c_1-C'(R)](e^{t^{\xi_i}}-e^t)\leq 0,$$

Like in [B], after using Hopf maximum principle, we have,

$$\sup_{\theta\in\mathbb{S}_2}[\bar{w}_i^{\xi_i}(t_i,\theta)-\bar{w}_i(t_i,\theta)]=0.$$

We have:

$$\bar{w}_i^{\xi_i}(t_i,\theta_i)-\bar{w}_i(t_i,\theta_i)=0,\ \forall\ i.$$

We deduce,

$$[u_(y_i)]^{1/3}\times\inf_M u_i\leq c,\ \forall\ i.$$

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DEPARTMENT OF MATHEMATICS, PATRAS UNIVERSITY, 26500 PATRAS , GREECE
E-mail address: samybahoura@yahoo.fr, bahoura@ccr.jussieu.fr