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Abstract. This paper analyzes stochastic networks consisting of finite capacity nodes with different classes of requests which move according to some routing policy. The Markov processes describing these networks do not have, in general, reversibility properties so that the explicit expression of their invariant distribution is not known. A heavy traffic limit regime is considered: the arrival rates of calls as well as the capacities of the nodes are proportional to a factor going to infinity. It is proved that, in the limit, the associated rescaled Markov process converges to a deterministic dynamical system with a unique equilibrium point characterized by a non-standard fixed point equation.

Contents

1. Introduction

In this paper a new class of stochastic networks is introduced and analyzed. Their dynamics combine the key characteristics of the two main classes of queueing networks: loss networks and Jackson type networks.

(1) Each node of the network has finite capacity so that a request entering a saturated node is rejected as in a loss network.
(2) Requests visit a subset of nodes along some (possibly) random route as in Jackson or Kelly's networks.

This class of networks is motivated by the mathematical representation of cellular wireless networks. Such a network is a group of base stations covering some geographical area. The area where *mobile users* communicate with *a base station* is referred to as *a cell*. A base station is responsible for the bandwidth management concerning mobiles in its cell. New calls are initiated in cells and calls are handed over (transfered) to the corresponding neighboring cell when mobiles move through the network. A new or a handoff call is accepted if there is available bandwidth in the cell, otherwise, it is rejected.

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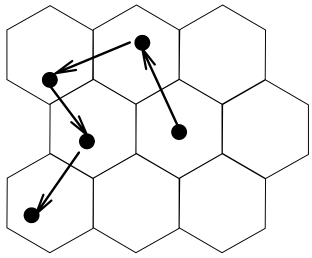

FIGURE 1. The motion of a mobile among the cells of the network

Up to now these networks have been modeled at a macroscopic level as loss networks characterized by call arrival rates, mean call lengths, handoff rates and capacity restrictions on the number of calls in the case of exponential times. One of the main quantity of interest is the blocking probability of the network. Approximations have been used to analyze these networks. See Antunes *et al.* [2], Boucherie and van Dijk [5] and Sidi and Starobiski [11] and the references therein.

Assuming Poisson arrivals and exponential dwell times at each node, the time evolution of such a network with N nodes can be represented as a Markov jump process $(X(t))$ with values in some finite (but large) set $\mathcal{S}$. It turns out that, contrary to loss networks, the Markov process $(X(t))$ is not in general *reversible* or quasi reversible. Consequently, contrary to Jackson networks and the like or pure loss networks, *these networks do not have a stationary distribution with a product form.*

In this paper the time evolution of these networks is analyzed by considering heavy traffic limits. The arrival rates and capacities at nodes are proportional to some factor N which gets large. This scaling has been introduced by Kelly [7] to study the invariant distribution of loss networks. A study of the time evolution of loss networks under this scaling has been achieved by Hunt and Kurtz [6]. See Kelly [8] for a survey of these questions. A different scaling is considered in Antunes *et al.* [1].

The equilibrium points. The time evolution of the network can be (roughly) described as follows. A stochastic process $(\overline{X}_N(t))$ associated with the state of the network for the parameter N is introduced: $\overline{X}_N(t)$ is the vector describing the numbers of requests of different classes at the nodes of the network. The equation of evolution of the network is

$$\frac{d}{dt}\overline{X}_N(t) = F_N\big(\overline{X}_N(t)\big) + \overline{M}_N(t), \qquad t \geq 0,$$

where $(\overline{M}_N(t))$ is a martingale which vanishes as N gets large, F_N is a quite complicated functional (associated with the generator of the corresponding Markov process) converging to some limit F. As N goes to infinity, it is proved that $(\overline{X}_N(t))$ converges to some function $(x(t))$, satisfying the deterministic equation

$$\frac{d}{dt}x(t) = F\big(x(t)\big), \qquad t \geq 0. \tag{1}$$

The equilibrium points of the limiting process are the solutions x of the equation $F(x) = 0$. It is shown in this paper, and this is a difficult point, that there is only one equilibrium point in the heavy traffic regime.

Related Work. For classical loss networks, the invariant probability has a product form representation. Nevertheless, the heavy traffic scaling of the evolution of these networks turns out to be quite intricate. Hunt and Kurtz [6] has shown that at any x the vector field $F(x)$ driving the limiting dynamical system is given in terms of some reflected random walk in $\mathbb{R}_+^d$ with jump rates depending in x. At points x at which this random walk is ergodic, $F(x)$ is expressed in terms of the invariant distribution and at x at which it is transient, the exit paths to infinity determine $F(x)$. It is not known, in general, whether there always exists a unique limiting dynamical system. Hunt and Kurtz [6], Bean *et al.* [3, 4] and Zachary [12] analyzed several examples with one or two nodes where uniqueness is shown to hold.

Results of the Paper. Using the terminology of cellular networks: users arriving in the network correspond to new requests for a connection in a cell. Different classes of customers access the network; Classes differ by their arrival rate, by their *dwell time* at the nodes (i.e. the amount of time that a mobile of on going call remains in a given cell) and by their call duration and also by their routing through the network. During a call, a user moves from one cell to another according to some Markovian mechanism depending on his class. When a user moves to another cell (node), this cell has to be non-saturated to accommodate the user, otherwise the user is rejected (the call is lost). If it is not rejected during the travel through the network, the user call terminates after the call duration time has elapsed.

For the networks analyzed in this paper, the uniqueness of the limiting dynamical system is not difficult to establish. The main difficulty lies in the complexity of the system of equations defining the equilibrium points of the dynamical system. Since there does not seem to exist some reasonably simple contracting scheme to solve these equations, the uniqueness of the equilibrium points is therefore a quite challenging problem. As an example, in Section 4.2 the case of a very simple network with two nodes and two deterministic routes is investigated, the explicit representation of the equilibrium point is obtained, it is expressed with quite complicated polynomial expressions of the parameters.

The paper is organized as follows: Section 2 introduces the Markovian description of these networks, Section 3 gives the convergence results together with the description of the limiting dynamical system. Section 4 is devoted to the main results of the paper, it is shown that, in the limit, there exists a unique stable point for the network. The ingredients used to obtain this uniqueness result are:

- a dual approach for the problem of uniqueness: find the set of parameters such that a given point is an equilibrium point of the dynamical system;
- a key inequality proved in Section 5;
- a convenient probabilistic representation of a set of linear equations.

The inequality proved in the appendix involves a quantity related to relative entropy, but, curiously, it does not seem to be a consequence of a standard convex inequality as it is usually the case in this kind of situation.

2. The Stochastic Model

The network consists of a finite set I of nodes, node $i \in I$ has capacity $\lfloor c_i N \rfloor$, where $c_i > 0$ and $N \in \mathbb{N}$. This network receives a finite number of classes of customers indexed by a finite set R; class $r \in R$ customers enter the network according to a Poisson process with rate $\lambda_r N$ with $\lambda_r > 0$.

- *Call duration.* A class r customer who is never rejected during his travel through the network spends an exponentially distributed time with rate μ_r in the network (call duration in the context of a cellular network). The case $\mu_r = 0$ is not excluded, it corresponds to the case of customers staying forever in the network as long as they are not rejected, i.e. as long as they do not arrive at a saturated node.
- *Dwell time.* The residence time of a customer of class r at any node $i \in I$ is exponentially distributed with parameter γ_r. Such a customer can leave the node before the end of his dwell time, due to the end of call, at rate μ_r.
- *Routing.* A class r customer entering the network arrives at some random node in I whose distribution is q_r, and then he moves from one node to another or to the outside (referred to as node 0) according to some transition matrix $p^{(r)}$ on $I \times I \cup \{0\}$. By changing the parameter of the residence time, it can be assumed without loss of generality that the matrix $p^{(r)}$ is 0 on the diagonal.
- *Capacity Requirements.* All customers require one unit of capacity at each node.

All random variables used for arrivals, residence times or call durations are assumed to be independent.

This class of networks includes the case of classes of customers with deterministic routing as in Kelly's networks and also classes of customers with Markovian routing as in Jackson networks. Figure 2 represents a network with two classes of customers, class 1 jobs follow a deterministic route while class 2 customers can either reach node 1 or node 3 from node 4, the capacities of the nodes are 5. Note that no assumption has been done on the transition matrices $p^{(r)}(\cdot,\cdot)$, so that some classes of jobs may achieve infinite loops in the network.

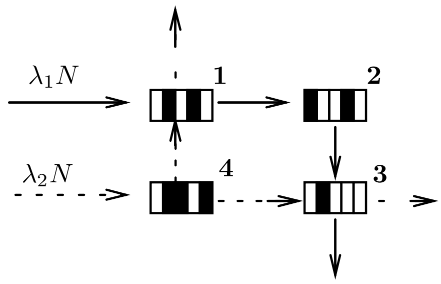

FIGURE 2. A Network with Two Classes of Customers

Notations. For $i \in I$, $r \in R$ and $t \geq 0$, $X^N_{i,r}(t)$ denotes the number of class r customers at node i at time t, $(X^N(t)) = \left(X^N_{i,r}(t), i \in I, r \in R\right)$ is the corresponding process. The renormalized process is defined as follows,

$$\overline{X}^N_{i,r}(t) = \frac{1}{N} X^N_{i,r}(t)$$

and $\overline{X}^N(t) = \left(\overline{X}^N_{i,r}(t), i \in I, r \in R\right)$.

Denote by $I_r \subset I$ the set of nodes which can be visited by a class r customer, i.e. $i \in I_r$ when i is visited with positive probability by the Markov chain with transition matrix $p^{(r)}$ and initial distribution q_r. It is assumed that $I = \cup_{r\in R} I_r$. The state space of the Markov process $(X^N(t))$ is

$$\mathcal{S}_N = \Big\{x = (x_{i,r}) \in \mathbb{N}^{I\times R} : \sum_r x_{i,r} \le c_i N \text{ and } x_{i,r} = 0 \text{ if } i \notin I_r\Big\}.$$

The Q-matrix $(A_N(x,y))$ of $(X^N(t))$

Arrival of a class r customer at node i:

$$A_N(x, x + e_{i,r}) = \lambda_r N q_r(i) \mathbb{1}_{\{x+e_{i,r}\in\mathcal{S}_N\}},$$

Service completion, rejection by a cell or a transition to the outside:

$$A_N(x, x - e_{i,r}) = x_{i,r}\Big(\mu_r + \gamma_r \sum_{j\in I} p^{(r)}(i,j)\mathbb{1}_{\{x+e_{j,r}\notin\mathcal{S}_N\}} + \gamma_r p^{(r)}(i,0)\Big)$$

Transfer from node i to node j:

$$A_N(x, x - e_{i,r} + e_{j,r}) = \gamma_r x_{i,r} p^{(r)}(i,j),$$

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where $e_{i,r}$ is the unit vector at coordinate (i,r). The state space of the renormalized process is given by

$$\mathcal{X}_c = \Big\{x = (x_{i,r}) \in \mathbb{R}_+^{I\times R} : \sum_r x_{i,r} \le c_i \text{ and } x_{i,r} = 0 \text{ if } i \notin I_r\Big\},$$

the subscript $c = (c_i)$ of $\mathcal{X}_c$ stands for the vector of capacities.

3. Convergence results

The following proposition establishes the deterministic behavior of $\overline{X}^N(t)$ as N goes to infinity. This result is the consequence of the fact that the stochastic perturbations of the original system are of the order $\sqrt{N}$ and therefore vanish because of the scaling in $1/N$.

To describe the time evolution of the network, one introduces the following Poisson processes: $\mathcal{N}_\xi$ denotes a Poisson process with parameter $\xi > 0$, an upper index $\mathcal{N}_\xi^p$, $p \in \mathbb{N}^d$, $d \in \mathbb{N}$, is added when several such Poisson processes are required. For example, for $i \in I, r \in R$, $\mathcal{N}_{\lambda_r q_r(i)}$ is the external arrival Poisson process of class r customers at node i. In a similar way, for $k \ge 1$, $\mathcal{N}^k_{\gamma_r p^{(r)}(i,j)}$ is the Poisson process associated with the transfer of the kth class r customers from node i to $j \in I \cup \{0\}$.

For $t \ge 0$ and $(i,r) \in I \times R$, denote by $Y_i^N(t) = \lfloor c_i N \rfloor - \sum_r X_{i,r}^N(t)$, $Y_i^N(t)$ is the size of the free space at node i. The process $(X^N(t))$ can then be represented as the solution of the following stochastic integral equation,

$$(2)\quad X_{i,r}^N(t) = X_{i,r}^N(0) + \int_0^t \mathbb{1}_{\{Y_i^N(s-)>0\}}\,\mathcal{N}_{\lambda_r N q_r(i)}(ds)$$
$$+ \sum_{j\in I-\{i\}}\sum_{k\geq 1}\int_0^t \mathbb{1}_{\{k\leq X_{j,r}^N(s-),Y_i^N(s-)>0\}}\,\mathcal{N}^k_{\gamma_r p^{(r)}(j,i)}(ds)$$
$$-\sum_{\substack{j\in I\cup\{0\}\\ j\neq i}}\sum_{k\geq 1}\int_0^t \mathbb{1}_{\{k\leq X_{i,r}^N(s-)\}}\,\mathcal{N}^k_{\gamma_r p^{(r)}(i,j)}(ds)$$
$$-\sum_{k\geq 1}\int_0^t \mathbb{1}_{\{k\leq X_{i,r}^N(s-)\}}\,\mathcal{N}^{i,k}_{\mu_r}(ds),$$

where $f(t-)$ denotes the limit on the left of the function f at t. By compensating the Poisson processes, i.e. by replacing the differential term $\mathcal{N}_\xi(ds)$ by the martingale increment $\mathcal{N}_\xi(ds) - \xi\, ds$, one gets the identity

$$(3)\quad X_{i,r}^N(t) = X_{i,r}^N(0) + M_{i,r}^N(t) + \lambda_r N q_r(i)\int_0^t \mathbb{1}_{\{Y_i^N(s^-)>0\}}\,ds$$
$$+\gamma_r\sum_{j\in I} p^{(r)}(j,i)\int_0^t X_{j,r}^N(s-)\mathbb{1}_{\{Y_i^N(s-)>0\}}\,ds - (\gamma_r+\mu_r)\int_0^t X_{i,r}^N(s-)\,ds$$

where $\left(M_{i,r}^N(t)\right)$ is the martingale obtained from the compensated integrals of the previous expression.

Denote the renormalized martingale $\overline{M}_{i,r}^N(t) = M_{i,r}^N(t)/N$, one finally gets

$$(4)\quad \overline{X}_{i,r}^N(t) = \overline{X}_{i,r}^N(0) + \overline{M}_{i,r}^N(t) + \lambda_r q_r(i)\int_0^t \mathbb{1}_{\{Y_i^N(s-)>0\}}\,ds$$
$$+\gamma_r\sum_{j\in I} p^{(r)}(j,i)\int_0^t \overline{X}_{j,r}^N(s-)\mathbb{1}_{\{Y_i^N(s-)>0\}}\,ds - (\gamma_r+\mu_r)\int_0^t \overline{X}_{i,r}^N(s-)\,ds.$$

The evolution equations for the renormalized process being written, one can establish the main convergence result.

Theorem 1. *If the initial state $\overline{X}^N(0)$ converges to $x\in\mathcal{X}_c$ as N goes to infinity, then $(\overline{X}^N(t))$ converges in the Skorohod topology to the solution $(x(t))$ of the following differential equation: for $(i,r)\in I\times R$,*

$$(5)\quad \frac{d}{dt}x_{i,r}(t) = \Big(\lambda_r q_r(i) + \gamma_r\sum_j x_{j,r}(t)p^{(r)}(j,i)\Big)\tau_i(x(t)) - (\gamma_r+\mu_r)x_{i,r}(t)$$

with $x(0)=x$ and

$$\tau_i(x) = \begin{cases} 1 & \text{if } \sum_r x_{i,r} < c_i,\\ \rho_x^i\wedge 1 & \text{otherwise,}\end{cases}$$

where $a\wedge b = \min(a,b)$ for a, $b\in\mathbb{R}$ and

$$\rho_x^i \overset{def.}{=} \frac{\sum_r(\gamma_r+\mu_r)x_{i,r}}{\sum_r[\lambda_r q_r(i) + \gamma_r\sum_j x_{j,r}p^{(r)}(j,i)]}.$$

By convergence in the Skorohod topology, one means the convergence in distribution for Skorohod topology on the space of trajectories.

Proof. Recall that if $\mathcal{N}_{\xi_1}$ and $\mathcal{N}_{\xi_2}$ are two independent Poisson processes and if $M_p(t) = \mathcal{N}_{\xi_p}((0,t]) - \xi_p t,\ p = 1,\ 2$ are their associated martingales, they are orthogonal in the sense that $(M_1(t)M_2(t))$ is a martingale, i.e. the bracket process $\langle M_1, M_2\rangle(t)$ is 0 for all $t \geq 0$. See Rogers and Williams [10]. The same property holds for stochastic integrals of previsible processes $(H_1(t))$ and $(H_2(t))$, for $t \geq 0$,

$$\left\langle \int_0^\cdot H_1(s)\, dM_1(s), \int_0^\cdot H_2(s)\, dM_2(s) \right\rangle (t) = 0.$$

The increasing process of the renormalized martingale defined above is

$$\left\langle \overline{M}^N_{i,r}, \overline{M}^N_{i,r}\right\rangle (t) = \frac{1}{N^2} \left\langle M^N_{i,r}, M^N_{i,r}\right\rangle (t),$$

the increasing process in the right hand side of the last equation can be evaluated by using the orthogonality of independent Poisson processes mentioned above. By using the fact that, for $(i,r) \in I \times R$ and $t \geq 0$, $X^N_{i,r}(t) \leq \lfloor c_i N\rfloor$, one gets that there exists some constant K such that

$$\mathbb{E}\left(\left[M^N_{i,r}(t)\right]^2\right) = E\left(\left\langle M^N_{i,r}, M^N_{i,r}\right\rangle (t)\right) \leq KNt,$$

Doob's Inequality implies that the martingale $(\overline{M}^N_{i,r}(t))$ converges a.s. to 0 uniformly on compact sets. Hence the stochastic fluctuations represented by the martingales vanish in the limit.

Now, by using the results of Kurtz [9], similarly as in Hunt and Kurtz [6] for loss networks, one can prove that any weak limit $X = (X_{i,r})$ of the process $\overline{X}^N$ satisfies the following equations: for $(i,r) \in I \times R$,

$$X_{i,r}(t) = X_{i,r}(0) + \int_0^t \Big(\lambda_r q_r(i) + \gamma_r \sum_{j\in I} p^{(r)}(j,i) X_{j,r}(s)\Big) \pi_{X(s)}\left(\overline{\mathbb{N}}^I_i\right)\, ds - (\gamma_r + \mu_r)\int_0^t X_{i,r}(s)\, ds \tag{6}$$

where $\overline{\mathbb{N}}^I_i = \{m = (m_j) \in (\mathbb{N}\cup\{+\infty\})^I : m_i \geq 1\}$ and for $x = (x_{ir}) \in \mathcal{X}_c$, π_x is *some* stationary probability measure on $\overline{\mathbb{N}}^I = (\mathbb{N}\cup\{+\infty\})^I$ of the Markov jump process whose Q-matrix $(B_x(\cdot,\cdot))$ is defined as

$$\begin{aligned}
B_x(m, m - e_i) &= \sum_r \lambda_r q_r(i), \qquad \text{if } m_i \geq 1,\\
B_x(m, m + e_i) &= \sum_r x_{i,r}\Big(\mu_r + \gamma_r\big(p^{(r)}(i,0) + \sum_{j\in I} p^{(r)}(i,j)\mathbb{1}_{\{m_j=0\}}\big)\Big),\\
B_x(m, m - e_i + e_j) &= \sum_r \gamma_r x_{j,r} p^{(r)}(j,i), \qquad \text{if } m_i \geq 1,
\end{aligned}$$

where e_i denotes the ith unit vector of $\mathbb{R}^I$. Moreover, the probability distribution π_x has to satisfy the following condition

$$\pi_x\left(m \in \overline{\mathbb{N}}^I : m_i = +\infty\right) = 1 \qquad \text{if } \sum_r x_{i,r} < c_i. \tag{7}$$

The Markov process $(m^x(t))$ associated with the matrix $B_x(\cdot,\cdot)$ describes the evolution of $Y^N(t/N) = (Y^N_i(t/N))$, i.e. the time-rescaled process of the numbers of free units of capacity at different nodes, during a time interval $[t, t+Ndt[$ when the renormalized process $\overline{X}^N$ is around x on the normal time scale. Compared to $(X^N(t))$, the process $(Y^N(t))$ indeed evolves on a rapid time scale, so that quantities

$$\int_t^{t+dt} \mathbb{1}_{\{Y_i^N(s^-)>0\}}\, ds \sim \pi_x\left(\overline{\mathbb{N}}_i^I\right) dt$$

i.e. can be replaced, in the limit, by the average values of indicator functions under some limiting regime π_x of Y^N when $\overline{X}^N(t) \sim x$. Hunt and Kurtz [6] gives a detailed treatment of these interesting questions. See also Bean *et al.* [3, 4] and Zachary [12] for the analysis of several examples.

In our case the marginals of $(m^x(t))$ are also Markov, due to the fact that each customer occupies only one node at a time so that the acceptance at node i only depends on the number of free units at node i. For $i \in I$, the process $(m_i^x(t))$ of the number of free units at node i when the renormalized process is around x, is a classical birth and death process on $\overline{\mathbb{N}}$ whose rates are given by

$$q(m, m+1) = N \sum_r (\gamma_r + \mu_r) x_{i,r},$$
$$q(m, m-1) = N \sum_r \Big(\lambda_r q_r(i) + \gamma_r \sum_j x_{j,r} p^{(r)}(j,i)\Big) \qquad \text{if } m \geq 1.$$

The point $+\infty$ is an absorbing point. Under the condition

$$\sum_r (\gamma_r + \mu_r) x_{i,r} < \sum_r \Big(\lambda_r q_r(i) + \gamma_r \sum_j x_{j,r} p^{(r)}(j,i)\Big) \tag{8}$$

the geometric distribution with parameter

$$\sum_r (\gamma_r + \mu_r) x_{i,r} \Big/ \sum_r \Big(\lambda_r q_r(i) + \gamma_r \sum_j x_{j,r} p^{(r)}(j,i)\Big) = \rho_x^i$$

and the probability $\delta_{+\infty}$ are the two extreme invariant measures of this process. If $\sum_r x_{i,r} = c_i$ and if Condition (8) holds, then the quantity $\pi_x(\overline{\mathbb{N}}_i^I)$ is necessarily some convex combination of 1 and ρ_x^i. For such an $i \in I$, by summing up Equations (6) over r it is easy to check that the quantity $\pi_x(\overline{\mathbb{N}}_i^I)$ cannot be more than ρ_x^i, otherwise the finite capacity condition $\sum_r x_{i,r} \leq c_i$ would be violated. One gets that $\pi_x(\overline{\mathbb{N}}_i^I) = \rho_x^i$.

The other cases follow from Condition (7) or the transience of the process $(m_i^x(t))$. Since the differential (5) clearly has a unique solution, the theorem is proved. □

4. Equilibrium Points

Theorem 1 shows that equilibrium points $x \in \mathcal{X}_c$ of the limiting dynamical system, that is those x that satisfy $x_{i,r}'(t) = 0$ for any $(i,r) \in I \times R$ and $t \geq 0$ when $(x_{i,r}(0)) = x$, are the solutions of the following set of equations

$$(\gamma_r + \mu_r) x_{i,r} = \Big(\lambda_r q_r(i) + \gamma_r \sum_j x_{j,r} p^{(r)}(j,i)\Big) \tau_i(x), \qquad (i,r) \in I \times R, \tag{9}$$

where $\tau_i(x)$ is defined in Theorem 1. Note that $\tau_i(x) \in (0,1]$ and that the following dichotomy holds: either $\tau_i(x) = 1$ or $\sum_r x_{i,r} = c_i$.

4.1. **Characterizations and existence of Equilibrium Points.** Conversely, assume that some point $x \in \mathcal{X}_c$ satisfies the relation

$$(\gamma_r + \mu_r) x_{i,r} = \Big(\lambda_r q_r(i) + \gamma_r \sum_j x_{j,r} p^{(r)}(j,i)\Big)\, t_i, \qquad \forall (i,r) \in I \times R, \tag{10}$$

for some $t = (t_i) \in (0,1]^I$ and that for any $i \in I$ either $t_i = 1$ or $\sum_r x_{i,r} = c_i$.

For a fixed $i \in I$, if $\lambda_r q_r(i) + \gamma_r \sum_j x_{j,r} p_r(j,i) = 0$ for all $r \in R$, then $x_{i,r} = 0$ for all r and Relations (10) hold with $\tau_i(x)$ replaced by t_i.
Otherwise, by adding up these relations over $r \in R$, one gets the identity

$$t_i = \frac{\sum_r (\gamma_r + \mu_r) x_{i,r}}{\sum_r \big(\lambda_r q_r(i) + \gamma_r \sum_j x_{j,r} p^{(r)}(j,i)\big)}.$$

If $t_i = 1$ then $\rho_x^i = 1$, and so, by definition of $\tau_i(x)$, $\tau_i(x) = 1 = t_i$. If $t_i < 1$, due to the above assumption, necessarily $\sum_r x_{i,r} = c_i$ so

$$\tau_i(x) = \frac{\sum_r (\gamma_r + \mu_r) x_{i,r}}{\sum_r \big(\lambda_r q_r(i) + \gamma_r \sum_j x_{j,r} p^{(r)}(j,i)\big)} \wedge 1 = t_i.$$

Equations (9) are thus satisfied for x. The following characterization of equilibrium points of the system has thus been obtained.

Proposition 1 (Characterization of Equilibrium Points). *The equilibrium points of the limiting dynamical system are the elements $x \in \mathcal{X}_c$ such that there exists some $t \in (0,1]^I$ satisfying*

(1) *For any $(i,r) \in I \times R$*

$$x_{i,r} = \Big(\alpha_r q_r(i) + \beta_r \sum_j x_{j,r} p^{(r)}(j,i)\Big)\, t_i \tag{11}$$

(2) *For any $i \in I$, either $t_i = 1$ or $\sum_r x_{i,r} = c_i$,*

where $\alpha_r = \lambda_r/(\gamma_r + \mu_r)$ and $\beta_r = \gamma_r/(\gamma_r + \mu_r)$ for $r \in R$.

To prove existence of a fixed point, a second characterization of equilibrium points will be useful.

Proposition 2 (Existence of Equilibrium Points). *The equilibrium points of the dynamical system* (5) *of Theorem 1 are the fixed points in $\mathcal{X}_c$ of the function Φ_c defined by, for $x \in \mathcal{X}_c$,*

$$\Phi_c(x) = \Big(\Theta_{c_i}\Big(\big(\alpha_r q_r(i) + \beta_r \sum_j x_{j,r} p^{(r)}(j,i), r \in R\big)\Big), i \in I\Big), \tag{12}$$

where, for $z > 0$ and $u \in [0,+\infty)^{|R|}$,

$$\Theta_z(u) = \left(\frac{z}{\sum_r u_r} \wedge 1\right) u.$$

The function Φ_c has at least one fixed point.

Proof. Note that the function Θ_c maps $[0,+\infty)^R$ into the subset $\{u \in [0,+\infty)^R : \sum_r u_r \leq c\}$ and that $\Phi_c(x)$ indeed belongs to $\mathcal{X}_c$: its (i,r)th coordinate is 0 whenever $i \notin I_r$.

The characterization of equilibrium points follows from Proposition 1 and by noting that, for $u \in [0,+\infty)^R$, $z > 0$ and $v \in [0,+\infty)^R$ such that $\sum_r v_r \leq z$, there is an equivalence between the identity $\Theta_z(u) = v$ and the fact that there exists some $t \in (0,1]$ such that $v = tu$ and that either $t = 1$ or $\sum_r v_r = z$.

The existence of a fixed point is then a consequence of Brouwer's fixed point theorem since $\mathcal{X}_c$ is a convex compact subset of $\mathbb{R}^{I\times R}$ and Φ_c is a continuous function from $\mathcal{X}_c$ into itself. □

4.2. **The Example of Deterministic Routes.** Requests of class r use a deterministic route of length $L \in \mathbb{N}\cup\{+\infty\}$ consisting of a sequence $I_r = (i_p, 0 \le p < L)$ with values in I such that

$$q_r(i_0) = 1, p^{(r)}(i_p, i_{p+1}) = 1, \text{ for } 0 \le p < L-1$$

and $p^{(r)}(i_{L-1}, 0) = 1$ if $L < +\infty$. Note that, since I is finite, the case $L = +\infty$ necessarily corresponds to a route r which is periodic after some point. Equilibrium points as described in Proposition 1 can be written more explicitly in terms of t solving Equations (11)

$$x_{i,r} = \Big(\alpha_r q_r(i) + \beta_r \sum_j x_{j,r} p^{(r)}(j,i)\Big)\, t_i.$$

as follows:

(1) For a non periodic deterministic route, $L < +\infty$, these equations reduce to a recursion, for $0 \le p < L$,

$$x_{i_p,r} = \alpha_r \beta_r^p \prod_{k=0}^{p} t_{i_k}.$$

(2) For a periodic route r consisting in nodes $i_0, i_1, \dots i_{k-1}$ and then the infinite loop $i_k, i_{k+1}, \dots, i_{k+l-1}, i_k, i_{k+1}, \dots$ these equations have a solution if and only if $(\beta_r)^l t_k \dots t_{k+l-1} < 1$ and in this case

$$x_{i_h,r} = \alpha_r \beta_r^h \prod_{0\le m\le h} t_{i_m}, \qquad 0 \le h \le k-1,$$

$$x_{i_h,r} = \frac{\alpha_r \beta_r^h t_{i_0} t_{i_1} \dots t_{i_h}}{1 - \beta_r^l t_{i_k} \dots t_{i_{k+l-1}}}, \qquad h \ge k. \tag{13}$$

The above calculations show that an equilibrium point $(x_{i,r})$ has a polynomial expression in $t = (t_j)$ whose degree is related to the rank of i along the route in the case of a non-periodic route; and $x_{i,r}$ is given by a power series in t when the route r is periodic. Moreover, these quantities have to satisfy the following constraints: for $i \in I$, then either $t_i = 1$ or $\sum_r x_{i,r} = c_i$. The exact expression of fixed points in the case of deterministic routes is therefore very likely to be non tractable. As it will be seen, even the uniqueness is not a simple problem.

The complexity of exact expressions is illustrated by a simple example of a network with two nodes: $I = \{1,2\}$ and two deterministic non periodic routes: first class enters at node 1, goes to node 2 then exits, second class does the opposite. Take $\mu_1 = \mu_2 = 0$ so that $\beta_1 = \beta_2 = 1$, then it is easy to show that:

(1) An equilibrium point associated to (t_1, t_2) with $t_1 = t_2 = 1$ exists if and only if $\alpha_1 + \alpha_2 \le c_1$ and $\alpha_1 + \alpha_2 \le c_2$, in this case $x_{1,1} = x_{2,1} = \alpha_1$ and $x_{1,2} = x_{2,2} = \alpha_2$.

(2) An equilibrium point exists with $t_1 = 1$, $t_2 < 1$ if and only if

$$\alpha_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} c_2 \le c_1 \quad \text{ and } \quad \alpha_1 + \alpha_2 > c_2,$$

under these conditions it is then unique

$$x_{1,1} = \alpha_1, \quad x_{2,1} = \alpha_1 \frac{c_2}{\alpha_1 + \alpha_2,}$$
$$x_{1,2} = x_{2,2} = \alpha_2 \frac{c_2}{\alpha_1 + \alpha_2},$$

(3) By symmetry analogous results hold with $t_1 = 1$ and $t_2 < 1$.
(4) An equilibrium point exists with $t_1 < 1$ and $t_2 < 1$ if and only if

$$\alpha_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} c_2 > c_1 \quad \text{and } \alpha_2 + \frac{\alpha_1}{\alpha_1 + \alpha_2} c_1 > c_2,$$

in this case the solution is unique:

$$x_{1,1} = \alpha_1 t_1, \quad x_{2,1} = \alpha_1 t_1 t_2$$
$$x_{1,2} = \alpha_2 t_1 t_2, \quad x_{2,2} = \alpha_2 t_2,$$

with

$$t_1 = \frac{(\alpha_1 c_1 - \alpha_2 c_2 - \alpha_1 \alpha_2) + \sqrt{(\alpha_1 c_1 - \alpha_2 c_2 - \alpha_1 \alpha_2)^2 + 4 c_1 \alpha_2 \alpha_1^2}}{2\alpha_1^2},$$

and t_2 has a similar expression with the subscripts 1 and 2 exchanged.

It is not difficult to check that these four cases are disjoint and cover all the situations. Therefore the uniqueness of the equilibrium point holds in this case.

It does not seem possible to carry out a similar approach for a more complicated system of deterministic routes. Even proving uniqueness in such a context is challenging.

4.3. **Uniqueness of Equilibrium Points.** In view of Proposition 2, to prove the uniqueness of equilibrium points, a contraction property of Φ_c would be enough. But it can be shown that Φ_c is generally not a contraction for classical norms.

For example, in the simple network considered above with $\beta_1 = \beta_2 = 1$, the equation $\Phi_c(x) = y$ is

$$(y_{1,1}, y_{1,2}) = \Theta_{c_1}(\alpha_1, x_{2,2}) \quad \text{and} \quad (y_{2,1}, y_{2,2}) = \Theta_{c_2}(x_{1,1}, \alpha_2).$$

When $c_1 > \alpha_1$ and $c_2 > \alpha_2$, one can choose x and x' in $\mathcal{X}_c$ such that

$$\begin{cases} \alpha_1 + x_{2,2} \leq c_1, \quad \alpha_1 + x'_{2,2} \leq c_1, \quad x_{1,1} + \alpha_2 \leq c_2, \\ x'_{1,1} + \alpha_2 \leq c_2, \quad x_{1,2} = x'_{1,2}, \qquad x_{2,1} = x'_{2,1} \end{cases}$$

then, in this case $\|\Phi_c(x) - \Phi_c(x')\|_p = \|x - x'\|_p$ for $p \in [1, +\infty]$, where $\|x\|_p$ is the L_p-norm $(\|x\|_p)^p = \sum_{i,r} |x_{i,r}|^p$ for $p < +\infty$ and $\|x\|_\infty = \max\{|x_{i,r}| : (i,r) \in I \times R\}$.

Under the condition $\max\{\beta_r : r \in R\} < 1$, in the case of deterministic non-periodic routes, the function

$$x \to \left(\alpha_r q_r(i) + \beta_r \sum_j x_{j,r} p^{(r)}(j,i),\ (i,r) \in I \times R\right)$$

is a contraction for any L_p-norm but the same property does not necessarily hold for Φ_c since it can be shown that the function Θ_c, $c > 0$, is not a contraction for any L_p-norm on $[0, +\infty)^R$, except when $|R| = 1$ or when $|R| = 2$ and $p = +\infty$.

A Dual Approach. To prove uniqueness in the general case, the point of view is changed: instead of looking for $x \in \mathcal{X}_c$ which are equilibrium points of the limiting dynamics associated to a given vector $c = (c_i, i \in I) \in (0, +\infty)^I$ of capacities, an element x is given and one looks for the set of vectors c such that x is a equilibrium point of the limiting dynamics. The uniqueness of the equilibrium point for a given c is then equivalent to the property that those sets associated to two different values of x do not intersect.

Define

$$\mathcal{X}_\infty \stackrel{\text{def.}}{=} \left\{x \in [0, +\infty)^{I\times R} : x_{i,r} = 0 \text{ if } i \notin I_r\right\},$$

it is of course enough to consider the solutions x in $\mathcal{X}_\infty$ that satisfy Equations (11) for some $t \in (0, 1]^I$. The first step of this analysis is to show that for any $t \in (0, 1]^I$, a solution x to the system of equations (11) is at most unique.

Proposition 3 (Probabilistic Representation)**.** *If $t \in (0, 1]^I$ is such that Equations* (11) *have a solution in $\mathcal{X}_\infty$, it is unique and can be expressed as*

$$x^t_{i,r} = \alpha_r \,\mathbb{E}\left(\sum_{k=0}^{+\infty} \beta_r^k \prod_{p=0}^{k} t_{Z_p^{(r)}} \mathbb{1}_{\{Z_k^{(r)}=i\}}\right), \quad \forall (i, r) \in I \times R, \tag{14}$$

where $\left(Z_n^{(r)}\right)$ is a (possibly killed) Markov chain with transition matrix $p^{(r)}(\cdot,\cdot)$ and initial distribution q_r.

Note that the above expression for $(x_{i,r})$ generalizes the formula obtained for periodic deterministic Markovian routes since, using the same notations as in the example of periodic deterministic routes, Equation (14) gives, for $h \geq k$,

$$x_{i_h} = \alpha_r\beta_r^h t_{i_0}t_{i_1}\ldots t_{i_h} \sum_{j=0}^{+\infty} \left(\beta_r^p t_{i_k}\ldots t_{i_{k+l-1}}\right)^j = \frac{\alpha_r\beta_r^h t_{i_0}t_{i_1}\ldots t_{i_h}}{1-\beta_r^p t_{i_k}\ldots t_{i_{k+l-1}}},$$

which is Formula (13), and $x_{i_h} = \alpha_r\beta_r^h t_{i_0}\cdots t_{i_h}$ for $h < k$.

Proof. The system of equations (11) splits into $|R|$ systems, one for each $r \in R$, with unknown variables $(x_{i,r},\ i \in I_r)$. So, just consider one of these $|R|$ systems and remove the index r for simplicity: J is defined as the range in I of the Markov chain (Z_k) with initial distribution q and transition matrix $p(\cdot,\cdot)$. The system then writes:

$$x_i = \Big(\alpha q(i) + \beta \sum_j x_j p(j, i)\Big)\, t_i, \qquad i \in J.$$

These equations have a solution since the system of equations (11) is assumed to have one. Set, for $i \in J$, $y_i = x_i/(\alpha t_i)$ (remember that both α and t_i are positive), then the vector $y = (y_i)$ solves the equations

$$y_i = q(i) + \sum_j y_j \widetilde{P}(j, i), \qquad i \in J. \tag{15}$$

with $\widetilde{P}(j, i) = \beta t_j p(j, i)$. The matrix $\widetilde{P} = (\widetilde{P}(i, j))$ is sub-Markovian, $(\widetilde{Z}_n)$ denotes the Markov chain with initial distribution $(q(i))$ and transition matrix $\widetilde{P}$. Set, for $i \in J$, clearly $y_i \geq q(i) = \mathbb{P}(\widetilde{Z}_0 = i)$, the above equation gives by induction that, for $n \geq 1$,

$$y_i \geq \mathbb{E}\left(\mathbb{1}_{\{\widetilde{Z}_0=i\}} + \mathbb{1}_{\{\widetilde{Z}_1=i\}} + \cdots + \mathbb{1}_{\{\widetilde{Z}_n=i\}}\right),$$

by letting n goes to infinity, one gets that

$$y_i \geq u_i \stackrel{\text{def.}}{=} \mathbb{E}\left(\sum_{k=0}^{+\infty} \mathbb{1}_{\{\widetilde{Z}_k=i\}}\right), \quad \forall i \in J.$$

For any $i \in J$, the above inequality implies that

$$\sum_{k=0}^{+\infty} \widetilde{P}^k(i,i) < +\infty,$$

one concludes that the state i is transient for the Markov chain $(\widetilde{Z}_n)$.

It is easy to check that (u_i) is also a solution of Equations (15), consequently the non-negative vector $(v_i) = (y_i - u_i)$ satisfies the equation

$$v_i = \sum_j v_j \widetilde{P}(j,i), \qquad i \in J,$$

which is the invariant measure equation for this Markov chain. Since all the states are transient, necessarily $v_i = 0$ for all $i \in J$. The uniqueness is proved. It is easy to check that the representation of (x_i) in terms of the Markov chain (Z_n) is indeed given by the representation of (u_i) in terms of the Markov chain $(\widetilde{Z}_n)$. The proposition is proved. □

Definition. *The set $\mathcal{T}$ is the subset of $t \in (0,1]^I$ such that the system of Equations (11) has a solution, denoted by $x^t = (x^t_{i,r})$ since it is unique by the above proposition. For $t \in \mathcal{T}$ and $i \in I$, define*

$$\sigma_i(t) = \sum_r x^t_{i,r} = \sum_r \alpha_r \, \mathbb{E}\left(\sum_{k=0}^{+\infty} \beta_r^k \prod_{p=0}^{k} t_{Z_p^{(r)}} \mathbb{1}_{\{Z_k^{(r)}=i\}}\right),$$

where $\left(Z_n^{(r)}\right)$ is, as before, a Markov chain with transition matrix $p^{(r)}(\cdot,\cdot)$ and initial distribution q_r.

Lemma 1 (Strong monotonicity)**.** *If $t = (t_i)$ and $t' = (t'_i)$ are elements of $\mathcal{T}$ with the following property, for any $i \in I$,*

$$t_i < t'_i \Rightarrow \sigma_i(t) \geq \sigma_i(t') \quad \text{and} \quad t'_i < t_i \Rightarrow \sigma_i(t') \geq \sigma_i(t),$$

then $t = t'$.

Proof. The assumption on t and t' gives that the relation

$$\sum_{i \in I} \log(t'_i/t_i)\left(\sigma_i(t') - \sigma_i(t)\right) \leq 0 \tag{16}$$

holds. The definition of σ_i gives the following representation for the difference $\sigma_i(t') - \sigma_i(t)$,

$$\sigma_i(t') - \sigma_i(t) = \sum_r \alpha_r \, \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_r^k \left(\prod_{h=0}^{k} t'_{Z_h^{(r)}} - \prod_{h=0}^{k} t_{Z_h^{(r)}}\right) \mathbb{1}_{\{Z_k^{(r)}=i\}}\right].$$

Note that, as in the proof of Proposition 3, the infinite sums within the expectation are integrable, thereby allowing these algebraic operations. By plugging this expression into Relation (16) and first exchanging summations on $i \in I$ and $r \in R$ and then on $i \in I$ and $k \in \mathbb{N}$ (remember that I and R are finite), one gets,

$$\sum_r \alpha_r \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_r^k \log\left(t'_{Z_k^{(r)}} \big/ t_{Z_k^{(r)}}\right)\left(\prod_{h=0}^{k} t'_{Z_h^{(r)}} - \prod_{h=0}^{k} t_{Z_h^{(r)}}\right) \mathbb{1}_{\{Z_k^{(r)} \neq 0\}}\right] \leq 0,$$

and, by extending the definitions of t and t' to the coordinate 0 so that $t_0 = t'_0 = 1$,

$$\sum_r \frac{\alpha_r}{\beta_r} \mathbb{E}\left[\sum_{k=0}^{\infty} \log\left(\beta_r t'_{Z_k^{(r)}} \big/ \beta_r t_{Z_k^{(r)}}\right)\left(\prod_{h=0}^{k} \beta_r t'_{Z_h^{(r)}} - \prod_{h=0}^{k} \beta_r t_{Z_h^{(r)}}\right)\right] \leq 0.$$

Proposition 4 of the Appendix applied to the expression inside the expectation, implies that, with probability 1, this integrand should be 0. Consequently, the same proposition gives that, for any $r \in R$, the identity $t_{Z_k^{(r)}} = t'_{Z_k^{(r)}}$ holds almost surely for any $k \in \mathbb{N}$. Hence, $t_i = t'_i$ for any $i \in I_r$ and any $r \in R$ by definition of I_r. One concludes that $t = t'$ since $I = \cup_r I_r$. The lemma is proved. □

The main result concerning the equilibrium points of the limiting dynamical system (5) can now be established.

Theorem 2 (Uniqueness of Equilibrium)**.** *There is a unique equilibrium point of the dynamical system* $(x_{i,r}(t), (i,r) \in I \times R)$ *defined by Equations* (5)*.*

Proof. For $t \in \mathcal{T}$, define C_t as the set of vectors $c = (c_i) \in (0, +\infty[^I$ such that x^t is a fixed point for the dynamical system associated to capacities (c_i). For $t \in \mathcal{T}$ and $c \in (0, +\infty)^I$, Proposition 1 shows that if $c \in C_t$ then, for any $i \in I$, $\sigma_i(t) \leq c_i$ and when $t_i < 1$ then $\sigma_i(t) = c_i$.

For t, $t' \in \mathcal{T}$, assume that there exists some $c \in C_t \cap C_{t'}$. If $i \in I$, the relation $t_i < t'_i$ implies that $t_i < 1$ and therefore that $\sigma_i(t') \leq \sigma_i(t) = c_i$. From Lemma 1 one concludes that necessarily $t = t'$. The uniqueness of equilibrium points readily follows from this result: if z and z' are equilibrium points of the dynamical system (5) associated to some vector of capacities $c \in (0, +\infty)^I$, then there exist t and $t' \in \mathcal{T}$ such that $z = x^t$ and $z' = x^{t'}$. Since $c \in C_t \cap C_{t'}$, one gets that $t = t'$ and therefore $z = z'$. The theorem is proved. □

5. Appendix

This section is devoted to the proof of a key technical result for the proof of the uniqueness of equilibrium points. It involves an expression which bears some similarity with a relative entropy.

Proposition 4. *Let* $u = (u_i)_{i \in \mathbb{N}}$ *and* $u' = (u'_i)_{i \in \mathbb{N}}$ *be two sequences of elements of* $(0, 1]$*. If the series*

$$\sum_{i=0}^{+\infty} \log\left(u'_i / u_i\right)\left(\prod_{j \leq i} u'_j - \prod_{j \leq i} u_j\right)$$

converges, its sum is non-negative and is 0 *if and only if* $u = u'$*.*

Proof. It is first proved by induction on $n \in \mathbb{N}$ that for any u, $u' \in (0, 1]^n$,

$$f_n(u, u') \stackrel{\text{def.}}{=} \sum_{i=0}^{n} \log\left(u'_i / u_i\right)\left(\prod_{j \leq i} u'_j - \prod_{j \leq i} u_j\right) \geq 0. \tag{17}$$

This is obviously true for $n = 0$. Now assume this inequality holds with for any integer $k < n$. Let u and u' be some fixed elements of $(0,1]^n$.

— If there exists some k such that $1 \le k \le n$ and

$$\Big(\prod_{j\le k-1} u'_j - \prod_{j\le k-1} u_j\Big)\Big(\prod_{j\le k} u'_j - \prod_{j\le k} u_j\Big) \le 0,$$

then $f_n(u,u')$ can be decomposed as follows,

$$\begin{aligned}(18)\quad f_n(u,u') = f_{k-1}&\Big((u_0,\dots,u_{k-1}),(u'_0,\dots,u'_{k-1})\Big)\\ &+ f_{n-k}\Big(\Big(\prod_{j\le k} u_j, u_{k+1},\dots,u_n\Big),\Big(\prod_{j\le k} u'_j, u'_{k+1},\dots,u'_n\Big)\Big)\\ &\qquad - \log\Big(\prod_{j\le k-1} u'_j \Big/ \prod_{j\le k-1} u_j\Big)\Big(\prod_{j\le k} u'_j - \prod_{j\le k} u_j\Big).\end{aligned}$$

From the induction hypothesis and the assumption on k, all terms of the right hand side of this identity are nonnegative, so $f_n(u,u') \ge 0$.

— Otherwise for any $0 \le k \le n$ the quantity $\prod_{j\le k} u'_j - \prod_{j\le k} u_j$ has a constant sign and is not 0 (positive say). There are two cases:

(1) if $u_k \le u'_k$ for all k such that $0 \le k \le n$, all terms in the sum defining $f_n(u,u')$ are non-negative, hence $f_n(u,u') \ge 0$.

(2) if not, let $k \le n$ be the first index such that $u_k > u'_k$. Since $u_0 < u'_0$ then $k \ge 1$ and we can write

$$\begin{aligned}f_n(u,u') = f_{n-1}\Big[&(u_0,\dots,u_{k-2},u_{k-1}u_k,u_{k+1},\dots,u_n),\\ &(u'_0,\dots,u'_{k-2},u'_{k-1}u'_k,u'_{k+1},\dots,u'_n)\Big]\\ &+ \log\Big(u'_{k-1}/u_{k-1}\Big)\Big((1-u'_k)\prod_{j\le k-1} u'_j - (1-u_k)\prod_{j\le k-1} u_j\Big)\end{aligned}$$

The first term is non-negative from the induction hypothesis, the second one also since $u_{k-1} \le u'_{k-1}$, $u'_k \le u_k$ and $\prod_{j\le k-1} u_j \le \prod_{j\le k-1} u'_j$, therefore $f_n(u,u') \ge 0$ also in this case. The proof by induction is completed.

Inequality (17) is thus true for any $n \in \mathbb{N}$, it implies that for any u, $u' \in (0,1]^{\mathbb{N}}$

$$f_\infty(u,u') \stackrel{\text{def.}}{=} \sum_{i=0}^{+\infty} \log\left(u'_i/u_i\right)\Big(\prod_{j\le i} u'_j - \prod_{j\le i} u_j\Big) \ge 0,$$

whenever the series converges. The first part of the proposition is proved.

Assume now that $f_\infty(u,u') = 0$ for some u, $u' \in (0,1]^{\mathbb{N}}$ such that the series converges. Using the same kind of decomposition as in Equation (18), $f_\infty(u,u')$

can be expressed as, for some fixed $k \geq 1$,

$$f_\infty(u,u') = f_{k-1}\Big((u_0,\ldots,u_{k-1}),(u'_0,\ldots,u'_{k-1})\Big) + f_\infty\Big(\Big(\prod_{j\leq k} u_j, u_{k+1},\ldots\Big),\Big(\prod_{j\leq k} u'_j, u'_{k+1},\ldots\Big)\Big) - \log\Big(\prod_{j\leq k-1} u'_j \Big/ \prod_{j\leq k-1} u_j\Big)\Big(\prod_{j\leq k} u'_j - \prod_{j\leq k} u_j\Big) = 0.$$

The second term of the right hand side is clearly well defined since $f_\infty(u,u')$ is. The first and second terms being non-negative, one gets that

$$\log\Big(\prod_{j\leq k-1} u'_j \Big/ \prod_{j\leq k-1} u_j\Big)\Big(\prod_{j\leq k} u'_j - \prod_{j\leq k} u_j\Big) \geq 0.$$

Consequently, the difference $u'_0u'_1\cdots u'_k - u_0u_1\cdots u_k$ has a constant sign for any $k \in \mathbb{N}$. It can be assumed that these expressions are non-negative.

(1) If $u_i \leq u'_i$ holds for any $i \geq 0$, then each term of the infinite sum defining $f_\infty(u,u')$ is non-negative and therefore null since $f_\infty(u,u') = 0$. It clearly implies that $u_i = u'_i$ for all $i \in \mathbb{N}$.

(2) Otherwise, since $u_0 \leq u'_0$, define $n \geq 1$ as the smallest integer such that $u_n > u'_n$. Since $u_0u_1\cdots u_n \leq u'_0u'_1\cdots u'_n$, there exists some index $i < n$ satisfying $u_i < u'_i$, define k as the largest one. In particular, for $k < i < n$, one has $u_i = u'_i$. Therefore,

$$f_\infty(u,u') = f_\infty\Big(\Big(u_0,\ldots,u_{k-1},\prod_{j=k}^{n} u_j, u_{n+1},\ldots\Big),\Big(u'_0,\ldots,u'_{k-1},\prod_{j=k}^{n} u'_j, u'_{n+1},\ldots\Big)\Big) + \log(u'_k/u_k)\Big(\Big(1-\prod_{k<j\leq n} u'_j\Big)\prod_{j\leq k} u'_j - \Big(1-\prod_{k<j\leq n} u_j\Big)\prod_{j\leq k} u_j\Big) = 0.$$

The first term is non-negative and it is easily checked by using the definitions of k and n that the second one is positive. This equality is therefore absurd. This second case is not possible.

The proposition is proved. □

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