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Character decomposition of Potts model partition functions.

II. Toroidal geometry

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Abstract

We extend our combinatorial approach of decomposing the partition function of the Potts model on finite two-dimensional lattices of size $L \times N$ to the case of toroidal boundary conditions. The elementary quantities in this decomposition are characters $K_{l,D}$ labelled by a number of bridges $l = 0, 1, \ldots, L$ and an irreducible representation D of the symmetric group S_l. We develop an operational method of determining the amplitudes of the eigenvalues as well as some of their degeneracies.

1 Introduction

The Q-state Potts model on a graph $G = (V, E)$ with vertices V and edges E can be defined geometrically through the cluster expansion of the partition function [1]

$$Z = \sum_{E' \subseteq E} Q^{n(E')} (\mathrm{e}^J - 1)^{b(E')} \,, \tag{1}$$

where $n(E')$ and $b(E') = |E'|$ are respectively the number of connected components (clusters) and the cardinality (number of links) of the edge subsets E'. We are interested in the case where G is a finite regular two-dimensional lattice of width L and length N, so that Z can be constructed by a transfer matrix T_L propagating in the N-direction.

In a companion paper [2], we studied the case of cyclic boundary conditions (periodic in the N-direction and non-periodic in the L-direction). We decomposed Z into linear combinations of certain restricted partition functions (characters) K_l (with $l = 0, 1, \ldots, L$) in which l *bridges* (that is, marked non-contractible clusters) wound around the periodic lattice direction. We shall often refer to l as the *level*. Unlike Z itself, the K_l could be written as (restricted) traces of the transfer matrix, and hence be directly related to its eigenvalues. It was thus straightforward to deduce from this decomposition the amplitudes in Z of the eigenvalues of T_L.

The goal of this second part of our work is to repeat this procedure in the case of toroidal boundary conditions. This case has been a lot less studied than the cyclic case (a noticeable exception is [3]). Indeed, when the boundary conditions are toroidal, the transfer matrix (of the related six-vertex model, to be precise) does no longer commute with the generators of the quantum group $U_q(sl(2))$. Therefore, there is no simple algebraic way of obtaining the amplitudes of eigenvalues, although some progress has been made by considering representations of the periodic Temperley-Lieb algebra (see for instance [4]). But the representations of this algebra are not all known, and therefore we choose to pursue here another approach than the algebraic one.

We use instead the combinatorial approach we developed in [2], as it is for now the only approach which can be easily extended to the toroidal case. There are however several complications due to the boundary conditions, the first of which is that the bridges can now be permuted (by exploiting the periodic L-direction). In the following

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this leads us to consider decomposition of Z into more elementary quantities than K_l, namely characters $K_{l,C}$ labeled by l *and* a class C of permuations of the symmetric group S_l. However, $K_{l,C}$ is not simply linked to the eigenvalues of T, and thus we will further consider its expansion over related quantities $K_{l,D}$, where D labels an irreducible representation (irrep) of S_l. It is $K_{l,D}$ which are the elementary quantities in the case of toroidal boundary conditions.

The second complication comes from the fact that, due to the planarity of the lattice, not all the permutations between bridges can be realised. It follows that the $K_{l,D}$ are not all independent, and so there are eigenvalue degeneracies inside and between levels. Finally, there can be additional degeneracies because of the particular symmetry of the lattice, and even accidental degeneracies[1]. We have therefore not been able to go as far as in the cyclic case, where the amplitude of any eigenvalue in K_l was given by a simple expression, depending only on l. We do however establish an operational method of determining, for any fixed (but in practice small) L, the amplitudes and degeneracies of eigenvalues in the case of a generic lattice[2].

The structure of the article is as follows. In section 2, we define appropriate generalisations of the quantities we used in the cyclic case [2]. Then, in section 3, we decompose restricted partition functions—and as a byproduct the total partition function—into characters K_l and $K_{l,C}$. Finally, in section 4, we expose a method of determining the amplitudes of eigenvalues.

2 Preliminaries

2.1 Definition of the $Z_{j,n1,P}$

As in the cyclic case, the existence of a periodic boundary condition allows for non-trivial clusters (henceforth abbreviated NTC), i.e., clusters which are not homotopic to a point.

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[1]An example occurs for the square lattice of width $L = 4$, where an eigenvalue at level 1 coincides with an eigenvalue at level 2 [3], without any apparent reason.

[2]By a generic lattice we understand one without mirror symmetry with respect to the transfer axis, i.e., without any accidental degeneracies. An example of a generic lattice is the triangular lattice, drawn as a square lattice with diagonals added.
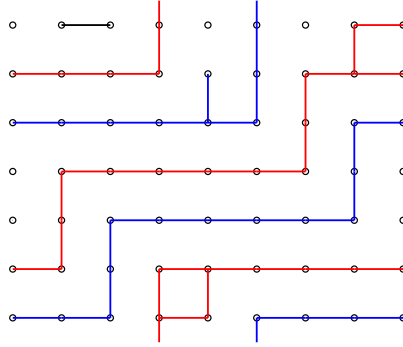
Figure 1: Cluster configuration with $j = 2$ non-trivial clusters (NTC), here represented in red and blue colours. Each NTC is characterised by its number of branches, $n_1 = 2$, and by the permutation it realises, $P = (12)$. Within a given configuration, all NTC have the same topology.

However, the fact that the torus has *two* periodic directions means that the topology of the NTC is more complicated that in the cyclic case. Indeed, each NTC belongs to a given homotopy class, which can be characterised by two coprime numbers (n_1, n_2), where n_1 (resp. n_2) denotes the number of times the cluster percolates horizontally (resp. vertically) [5]. The fact that all clusters (non-trivial or not) are still constrained by planarity to be non-intersecting induces a convenient simplification: all NTC in a given configuration belong to the same homotopy class. For comparison, we recall that in the cyclic case the only possible homotopy class for a NTC was $(n_1, n_2) = (1, 0)$.

It is a well-known fact [6, 7] that the difficulty in decomposing the Potts model partition function—or relating it to partition functions of locally equivalent models (of the six-vertex or RSOS type)—is due solely to the weighing of the NTC. Although a typical cluster configuration will of course contain trivial clusters (i.e., clusters that are homotopic to a point) with seemingly complicated topologies (e.g., trivial clusters can surround other trivial clusters, or be surrounded by trivial clusters or by NTC), we shall therefore tacitly disregard such clusters in most of the arguments that follow. Note also that the so-called degenerate clusters of Ref. [7] in the present context correspond to $n_1 = 1$.

Consider therefore first the case of a configuration having a single NTC. For the purpose of studying its topology, we can imagine that is has been shrunk to a line that winds the two periodic directions (n_1, n_2) times. In our approach we focus on the the proper-

ties of the NTC along the direction of propagation of the transfer matrix T_L, henceforth taken as the horizontal direction. If we imagine cutting the lattice along a vertical line, the NTC will be cut into n_1 horizontally percolating parts, which we shall call the n_1 *branches* of the NTC. Seen horizontally, a given NTC realises a permutation P between the vertical coordinates of its n_1 branches, as shown in Fig. 1. Up to a trivial relabelling of the vertical coordinate, the permutation P is independent of the horizontal coordinate of the (imaginary) vertical cut, and so, forms part of the topological description of the NTC. We thus describe totally the topology along the horizontal direction of a NTC by n_1 and the permutation $P \in S_{n_1}$.

Note that there are restrictions on the admissible permutations P. Firstly, P cannot have any proper invariant subspace, or else the corresponding NTC would in fact correspond to several distinct NTC, each having a smaller value of n_1. For example, the case $n_1 = 4$ and $P = (13)(24)$ is not admissible, as P corresponds in fact to two distinct NTC with $n_1 = 2$. In general, therefore, the admissible permutations P for a given n_1 are simply cyclic permutations of n_1 coordinates. Secondly, planarity implies that the different branches of a NTC cannot intersect, and so not all cyclic permutations are admissible P. For example, the case $n_1 = 4$ and $P = (1324)$ is not admissible. In general the admissible cyclic permutations are characterised by having a constant coordinate difference between two consecutive branches, i.e., they are of the form $(k, 2k, 3k, \ldots)$ for some constant k, with all coordinates considered modulo n_1. For example, for $n_1 = 4$, the only admissible permutations are then finally (1234) and (1432).

Consider now the case of a configuration with several NTC. Recalling that all NTC belong to the same homotopy class, they must all be characterised by the same n_1 and P. Alternatively one can say that the branches of the different NTC are entangled. Henceforth we denote by j the number of NTC with $n_1 \geq 1$ in a given configuration. Note in particular that, seen along the horizontal direction, configurations with no NTC and configurations with one or more NTC percolating only vertically are topologically equivalent. This is an important limitation of our approach.

Let us denote by $Z_{j,n_1,P}$ the partition function of the Potts model on an $L \times N$ torus, restricted to configurations with exactly j NTC characterised by the index $n_1 \geq 1$ and the permutation $P \in S_{n_1}$; if P is not admissible, or if $n_1 j > L$, we set $Z_{j,n_1,P} =$

0. Further, let Z_{j,n_1} be the partition function restricted to configurations with j NTC of index n_1, let Z_j be the partition function restricted to configurations with j NTC *percolating horizontally*, and let Z be the total partition function. Obviously, we have $Z_{j,n_1} = \sum_{P \in S_{n_1}} Z_{j,n_1,P}$, and $Z_j = \sum_{n_1=1}^{L} Z_{j,n_1}$, and $Z = \sum_{j=0}^{L} Z_j$. In particular, Z_0 corresponds to the partition function restricted to configurations with no NTC, or with NTC percolating only vertically.

In the case of a generic lattice all the $Z_{j,n_1,P}$ are non-zero, provided that P is an admissible cyclic permutation of length n_1, and that $n_1 j \leq L$. The triangular lattice is a simple example of a generic lattice. Note however that other regular lattices may be unable to realise certain admissible P. For example, in the case of a square lattice or a honeycomb lattice, all $Z_{j,n_1,P}$ with $n_1 j = L$ and $n_1 > 1$ are zero, since there is not enough "space" on the lattice to permit all NTC branches to percolate horizontally while realising a non-trivial permutation. Such non-generic lattices introduce additional difficulties in the analysis which have to be considered on a case-to-case basis. In the following, except when explicitly stated, we consider therefore the case of a generic lattice.

2.2 Structure of the transfer matrix

The construction and structure of the transfer matrix T can be taken over from the cyclic case [2]. In particular, we recall that T acts towards the right on states of connectivities between two time slices (left and right) and has a block-trigonal structure with respect to the number of *bridges* (connectivity components linking left and right) and a block-diagonal structure with respect to the residual connectivity among the non-bridged points on the left time slice. As before, we denote by T_l the diagonal block with a fixed number of bridges l and a trivial residual connectivity. Each eigenvalue of T is also an eigenvalue of one or more T_l. In analogy with [3] we shall sometimes call T_l the transfer matrix at level l. It acts on connectivity states which can be represented graphically as a partition of the L points in the right time slice with a special marking (represented as a *black point*) of precisely l distinct components of the partition (i.e., the components that are linked to the left time slice via a bridge).

A crucial difference with the cyclic case is that for a given partition of the right time

slice, there are more possibilities for attributing the black points. Namely, a connectivity component which is not *apparently* accessible from the left (and thus markable) may in fact be so due to the periodic boundary conditions identifying the top and the bottom rows. This will obviously increase the dimension of the level l subspace of connectivities (for $0 < l < L$). Considering for the moment the black points to be indistinguishable, we denote the corresponding dimension as $n_{\text{tor}}(L, l)$. It can be shown [3] that

$$
n_{\text{tor}}(L, l) = \begin{cases} \frac{1}{L+1}\binom{2L}{L} & \text{for } l = 0 \\ \binom{2L-1}{L-1} & \text{for } l = 1 \\ \binom{2L}{L-l} & \text{for } 2 \leq l \leq L \end{cases} \tag{2}
$$

and clearly $n_{\text{tor}}(L, l) = 0$ for $l > L$.

Suppose now that a connectivity state at level l is time evolved by a cluster configuration of index n_1 and corresponding to a permutation P. This can be represented graphically by adjoining the initial connectivity state to the left rim of the cluster configuration, as represented in Fig. 1, and reading off the final connectivity state as seen from the right rim of the cluster configuration. Evidently, the positions of the black points in the final state will be permuted with respect to their positions in the intial state, according to the permutation P. As we have seen, not all P are admissible, but it turns out to be advantageous to consider formally also the action of non-admissible permutations. This is permissible since in any case T_l will have only zero matrix elements between states which are related by a non-admissible permutation. Since $n_{\text{tor}}(L, l)$ was just defined as the number of possible connectivity states without taking into account the possible permutations between black points, the dimension of T_l is $l!\, n_{\text{tor}}(L, l)$.

Let us denote by $|v_{l,i}\rangle$ (where $1 \leq i \leq n_{\text{tor}}(L, l)$) the $n_{\text{tor}}(L, l)$ standard connectivity states at level l. The full space of connectivities at level l, i.e., with l distinguishable black points, can then be obtained by subjecting the $|v_{l,i}\rangle$ to permutations of the black points. It is obvious that T_l commutes with the permutations between black points (the physical reason being that T_l cannot "see" to which positions on the left time slice each bridge is attached). Therefore T_l itself has a block structure in a appropriate basis. Indeed, T_l can be decomposed into $\text{T}_{l,D}$ where $\text{T}_{l,D}$ is the restriction of T_l to the states transforming according to the irreducible representation (irrep) of S_l corresponding to the Young diagram D. One can obtain the corresponding basis by applying the projectors

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p_D on all the connectivity states at level l, where p_D is given by

$$p_D = \frac{\dim(D)}{l!} \sum_P \chi_D(P) \, P \, . \tag{3}$$

Here $\dim(D)$ is the dimension of the irrep D and $\chi_D(P)$ the character of P in this irrep. We have used the fact that all characters of S_l are real. The application of all possible permutations on any given standard vector $|v_{l,i}\rangle$ generates a regular representation of S_l, which contains therefore $\dim(D)$ representations D (each of dimension $\dim(D)$). As there are $n_{\text{tor}}(L, l)$ standard vectors, the dimension of $\mathrm{T}_{l,D}$ is thus $[\dim(D)]^2 \, n_{\text{tor}}(L, l)$. Furthermore, using Schur's lemma, we deduce that each of its eigenvalues is (at least) $\dim(D)$ times degenerate. Therefore $\mathrm{T}_{l,D}$ has (at most) $\dim(D) \, n_{\text{tor}}(L, l)$ different eigenvalues, which we shall denote $\lambda_{l,D,k}$.[3]

2.3 Definition of the $K_{l,D}$

We now define, as in the cyclic case [2], K_l as the trace of $(\mathrm{T}_l)^N$. Since T_l commutes with S_l, we can write

$$K_l = l! \sum_{i=1}^{n_{\text{tor}}(L,l)} \langle v_{l,i}| \, (\mathrm{T}_l)^N \, |v_{l,i}\rangle \, . \tag{4}$$

In distinction with the cyclic case, we cannot decompose the partition function Z over K_l because of the possible permutations of black points (see below). We shall therefore resort to more elementary quantities, the $K_{l,D}$, which we define as the trace of $(\mathrm{T}_{l,D})^N$. Since both T_l and the projectors p_D commute with S_l, we have

$$K_{l,D} = l! \sum_{i=1}^{n_{\text{tor}}(L,l)} \langle v_{l,i}|p_D \, (\mathrm{T}_l)^N \, |v_{l,i}\rangle \, . \tag{5}$$

Obviously one has

$$K_l = \sum_D K_{l,D} \, , \tag{6}$$

the sum being over all the irreps D of S_l. Recall that in the cyclic case the amplitudes of the eigenvalues at level l are all identical. This is no longer the case, since the amplitudes depend on D as well. Indeed

$$K_{l,D} = \sum_{k=1}^{\dim(D)n_{\text{tor}}(L,l)} \dim(D) \, (\lambda_{l,D,k})^N \, . \tag{7}$$

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[3]It turns out that there are more degeneracies than warranted by this argument. The reason is that the cluster configurations cannot realise all the permutations of S_l (recall our preceeding discussion), and thus some $\lambda_{l,D,k}$ with different l and/or D are in fact equal. We shall come back to this point later.

In order to decompose Z over $K_{l,D}$ we first introduce the auxiliary quantities

$$K_{l,C_l} = \sum_{P_l \in C_l} K_{l,P_l} \,, \tag{8}$$

the sum being over permutations $P_l \in S_l$ belonging to the class C_l. We then have

$$K_{l,P_l} = \sum_{i=1}^{n_{\mathrm{tor}}(L,l)} \langle v_{l,i} | \, (P_l)^{-1} \, (T_l)^N \, | v_{l,i} \rangle \,. \tag{9}$$

So K_{l,P_l} (resp. K_{l,C_l}) can be thought of as modified traces in which the final state differs from the initial state by the application of the permutation P_l (resp. the class C_l). Note that $K_{l,\mathrm{Id}}$ is simply equal to $\frac{K_l}{l!}$. Since the character is the same for all permutations belonging to the same class, Eqs. (5) and (3) yield a relation between $K_{l,D}$ and K_{l,P_l}:

$$K_{l,D} = \dim(D) \sum_{C_l} \chi_D(C_l) K_{l,C_l} \,. \tag{10}$$

These relations can be inverted so as to obtain K_{l,C_l} in terms of $K_{l,D}$, since the number of classes equals the number of irreps D:

$$K_{l,C_l} = \sum_D \frac{c(D,C_l)}{l!} K_{l,D} \tag{11}$$

With the chosen normalisation, the coefficients $c(D,C_l)$ are integer. Multiplying Eq. (10) by $\chi_D(C_l')$ and summing over D, and using the orthogonality relation $\sum_D \chi_D(C_l)\chi_D(C_l') = \frac{l!}{|C_l|}\delta_{C_l,C_l'}$ one easily deduces that

$$c(D,C_l) = \frac{|C_l| \, \chi_D(C_l)}{\dim(D)} \,. \tag{12}$$

We also note that

$$\sum_D [\dim(D)]^2 \, c(D,C_l) = l! \, \delta_{C_l,\mathrm{Id}} \tag{13}$$

3 Decomposition of the partition function

3.1 The characters K_l

By generalising the working for the cyclic case, we can now obtain a decomposition of the K_l in terms of the Z_{j,n_1}. To that end, we first determine the number of states $|v_{l,i}\rangle$ which are *compatible* with a given configuration of Z_{j,n_1}, i.e., the number of initial states
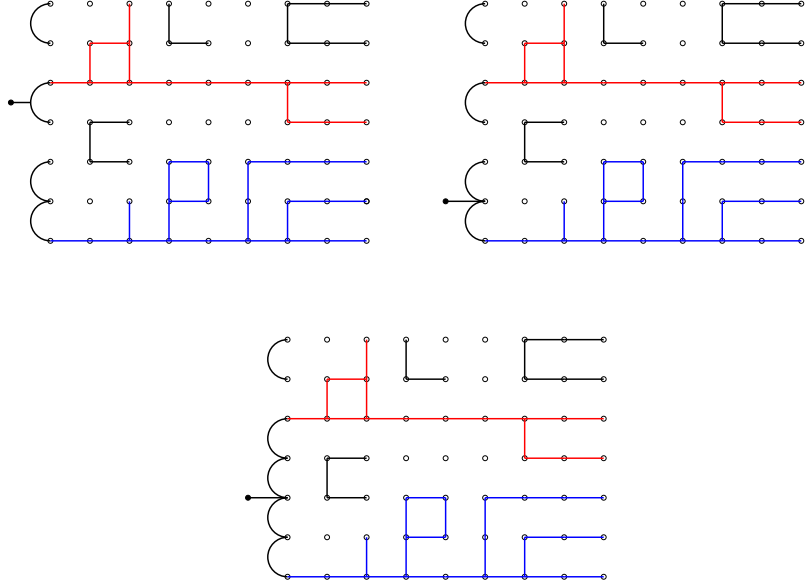
Figure 2: Standard connectivity states at level $l = 1$ which are compatible with a given cluster configuration contributing to $Z_{2,1}$.

$|v_{l,i}\rangle$ which are thus that the action by the given configuration produces an identical final state. The notion of compatability is illustrated in Fig. 2.

We consider first the case $n_1 = 1$ and suppose that the k'th NTC connects onto the points $\{y_k\}$. The rules for constructing the compatible $|v_{l,i}\rangle$ are identical to those of the cyclic case:

1. The points $y \notin \cup_{k=1}^{j}\{y_k\}$ must be connected in the same way in $|v_{l,i}\rangle$ as in the cluster configuration.

2. The points $\{y_k\}$ within the same bridge must be connected in $|v_{l,i}\rangle$.

3. One can independently choose to associate or not a black point to each of the sets $\{y_k\}$. One is free to connect or not two distinct sets $\{y_k\}$ and $\{y_{k'}\}$.

The choices mentioned in rule 3 leave $n_{\text{tor}}(j,l)$ possibilities for constructing a compatible $|v_{l,i}\rangle$. The coefficient of $Z_{j,1}$ in the decomposition of K_l is therefore $\frac{l!\, n_{\text{tor}}(j,l)}{Q^j}$, since the permutation of black points in a standard vector $|v_{l,i}\rangle$ allows for the construction of $l!$ distinct states, and since the weight of the j NTC in K_l is 1 instead of Q^j. It follows that

$$K_l = \sum_{j=l}^{L} l!\, n_{\text{tor}}(j,l)\frac{Z_{j,1}}{Q^j} \qquad \text{for } n_1 = 1. \tag{14}$$
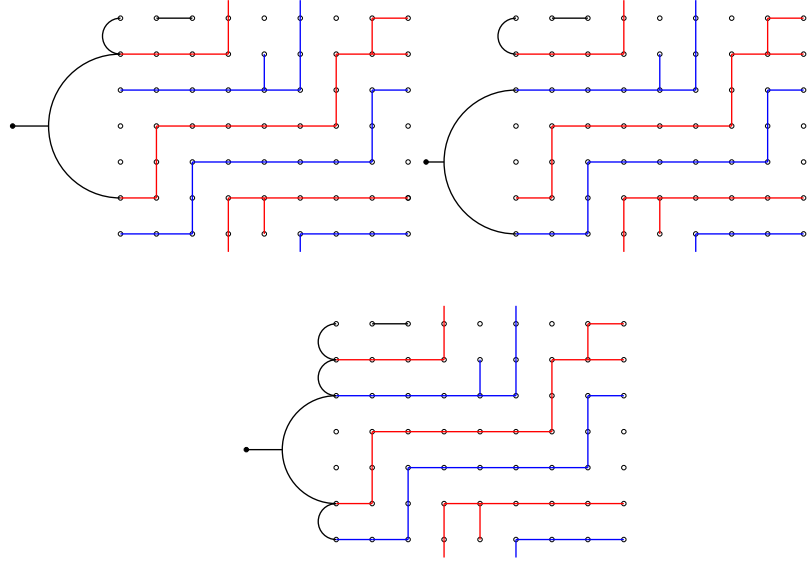
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Figure 3: Standard connectivity states at level $l = 1$ which are compatible with a given cluster configuration contributing to $Z_{2,2}$.

We next consider the case $n_1 > 1$. Let us denote by $\{y_{k,m}\}$ the points that connect onto the m'th branch of the k'th NTC (with $1 \leq m \leq n_1$ and $1 \leq k \leq j$), and by $\{y_k\} = \cup_{m=1}^{n_1}\{y_{k,m}\}$ all the points that connect onto the k'th NTC. As shown in Fig. 3, the $|v_{l,i}\rangle$ which are compatible with this configuration are such that

1. The connectivities of the points $y \notin \cup_{k=1}^{j}\{y_k\}$ are identical to those appearing in the cluster configuration.

2. All points $\{y_{k,m}\}$ corresponding to the branch of a NTC must be connected.

3. For each of the k NTC there are two possibilities. A) Either one connects all $\{y_{k,m}\}$ (with $1 \leq m \leq n_1$) corresponding to all n_1 branches of the NTC, obtaining what we shall henceforth call a *big block*. B) Or alternatively one connects none of the n_1 branches.

4. Because of the constraint of planarity and the fact that the NTC are entangled, all the different big blocks are automatically connected among themselves. One can therefore attribute at most one black point to the collection of big blocks.

To obtain rule 3 we have used the fact that the permutations P characterising the NTC do not have any proper invariant subspace. Note that rule 4 implies that the decomposition

of K_l with $l \geq 2$ does not contain any of the Z_{j,n_1} with $n_1 > 1$. We therefore have simply

$$K_l = \sum_{j=l}^{L} l!\, n_{\text{tor}}(j,l) \frac{Z_{j,1}}{Q^j} \qquad \text{for } l \geq 2 \,. \tag{15}$$

It remains to obtain the decomposition of K_1 and K_0. The number of standard connectivities $|v_{l,i}\rangle$ compatible with r big blocks is 0 for $l \geq 2$ (because of rule 4); $\binom{j}{r}$ for $l = 1$ and $r \geq 1$ (by rule 3 we independently choose to link up r of the j NTC, and by rule 4 the resulting big block must carry the black point); 0 for $l = 1$ and $r = 0$ (since one needs a big block to attribute the black point); and $\binom{j}{r}$ for $l = 0$. Summing over r, we finally obtain the number of compatible $|v_{l,i}\rangle$: 0 for $l \geq 2$; $\sum_{r=1}^{j} \binom{j}{r} = 2^j - 1$ for $l = 1$; and $\sum_{r=0}^{j} \binom{j}{r} = 2^j$ for $l = 0$. The decomposition of K_1 reads therefore

$$K_1 = \sum_{j=1}^{L} n_{\text{tor}}(j,1) \frac{Z_{j,1}}{Q^j} + \sum_{j=1}^{\lfloor \frac{L}{2} \rfloor} (2^j - 1) \frac{Z_{j,n_1>1}}{Q^j} \tag{16}$$

and that of K_0 is

$$K_0 = \sum_{j=0}^{L} n_{\text{tor}}(j,1) \frac{Z_{j,1}}{Q^j} + \sum_{j=1}^{\lfloor \frac{L}{2} \rfloor} 2^j \frac{Z_{j,n_1>1}}{Q^j} \,. \tag{17}$$

Note that the coefficients in front of Z_{j,n_1} do not depend on the precise value of n_1 when $n_1 > 1$. To simplify the notation we have defined $Z_{0,1} = Z_0$.

3.2   The coefficients $b^{(l)}$

Since the coefficients in front of $Z_{j,1}$ and $Z_{j,n_1>1}$ in Eqs. (16)–(17) are different, we cannot invert the system of relations (15)–(17) so as to obtain $Z_j \equiv Z_{j,1} + Z_{j,n_1>1}$ in terms of the K_l. It is thus precisely because of NTC with several branches contributing to $Z_{j,n_1>1}$ that the problem is more complicated than in the cyclic case.

In order to appreciate this effect, and compare with the precise results that we shall find later, let us for a moment assume that Eq. (15) were valid also for $l = 0, 1$. We would then obtain

$$Z_{j,1} = \sum_{l=j}^{L} b_j^{(l)} \frac{K_l}{l!} \tag{18}$$

where

$$b^{(l)} \equiv \sum_{j=0}^{l} b_j^{(l)} = \begin{cases} \sum_{j=0}^{l} (-1)^{l-j} \frac{2l}{l+j} \binom{l+j}{l-j} Q^j + (-1)^l (Q-1) & \text{for } l \geq 2 \\ \sum_{j=0}^{l} (-1)^{l-j} \binom{l+j}{l-j} Q^j & \text{for } l \leq 2 \end{cases} \tag{19}$$

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The coefficients $b^{(l)}$ play a role analogous to those denoted $c^{(l)}$ in the cyclic case [2]; note also that $b^{(l)} = c^{(l)}$ for $l \leq 2$. Chang and Schrock have developed a diagrammatic technique for obtaining the $b^{(l)}$ [3].

Supposing still the unconditional validity of Eq. (15), one would obtain for the full partition function

$$Z = \sum_{l=0}^{L} b^{(l)} \frac{K_l}{l!} \ . \tag{20}$$

This relation will be modified due to the terms $Z_{j,n_1>1}$ realising permutations of the black points, which we have here disregarded. To get things right we shall introduce Young diagram dependent coefficients $b^{(l,D)}$ and write $Z = \sum_{l=0}^{L} \sum_D b^{(l,D)} K_{l,D}$. Neglecting $Z_{j,n_1>1}$ terms would lead, according to Eq. (20), to $b^{(l,D)} = \frac{b^{(l)}}{l!}$ independently of D. We shall see that the $Z_{j,n_1>1}$ will lift this degeneracy of amplitudes in a particular way, since there exists certain relations between the $b^{(l,D)}$ and the $b^{(l)}$.

3.3   Decomposition of the K_{l,C_l}

The relations (15)–(17) were not invertible due to an insufficient number of elementary quantities K_l. Let us now show how to produce a development in terms of K_{l,C_l}, i.e., taking into account the possible permutations of black points. This development turns out to be invertible.

A standard connectivity state with l black points is said to be C_l-*compatible* with a given cluster configuration if the action of that cluster configuration on the connectivity state produces a final state that differs from the initial one just by a permutation C_l of the black points. This generalises the notion of compatibility used in Sec. 3.1 to take into account the permutations of black points.

Let us first count the number of standard connectivities $|v_{l,i}\rangle$ which are C_l-compatible with a cluster configuration contributing to $Z_{j,n_1,P}$. For $n_1 = 1$, S_{n_1} contains only the identity element Id, and so the results of Sec. 3.1 apply: the $Z_{j,1}$ contribute only to $K_{l,\mathrm{Id}}$. We consider next a configuration contributing to $Z_{j,n_1,P}$ with $n_1 > 1$. The $|v_{l,i}\rangle$ which are C_l-compatible with this configuration satisfy the same four rules as given in Sec. 3.1 for the case $n_1 > 1$, with the slight modification of rule 4 that the black points must be attributed to the big blocks in such a way that *the final state differs from the initial one*

by a permutation C_l.

This modification makes the attribution of black points considerably more involved than was the case in Sec. 3.1. First note that not all C_l are allowed. To be precise, the cycle decomposition of the allowed permutations can only contain id (the identity acting on a single black point) or P (recall that P is the permutation of coordinates realised by the branches of a single NTC). Indeed, if one attributes a black point to a big block its position remains unchanged by action of the cluster configuration, whereas if one attributes n_1 black points to the n_1 branches of one same NTC these points will be permuted by P. Furthermore, since the big blocks are automatically connected among themselves, one can at most attribute to them a single black point, and so id is contained in the cycle decomposition 0 or 1 times. Note also that the entanglement of the NTC will imply the entanglement of the structure of the allowed permutations, but this fact is of no importance here since we are only interested in C_l, i.e., the *classes* of allowed permutations.

Denoting by n_P the number of times the permutations of class C_l contains P, the two types of allowed C_l are: 1) those associated with permutations that only contain P, i.e., such that $l = n_P n_1$, and 2) those associated with permutations that contain id once, i.e., such that $l = n_P n_1 + 1$. In the following we denote these two types as (n_P, n_1) and $(n_P, n_1)'$, respectively, and the corresponding K_{l,C_l} will be denoted $K_{(n_P,n_1)}$ and $K_{(n_P,n_1)'}$ respectively.

Let us consider the first case, which is depicted in Fig. 4. If the $|v_{l,i}\rangle$ have r big blocks, there are $\binom{j}{r}$ ways of choosing them among the j NTC, and $\binom{j-r}{n_P}$ ways to attribute the black points. Indeed one needs to distribute $l = n_P n_1$ black points among n_P groups of n_1 non-connected blocks corresponding to the same NTC, out of a total of $j - r$. Since the $|v_{l,i}\rangle$ can contain at most $j - n_P$ big blocks, the number of C_l-compatible standard connectivities is

$$\sum_{r=0}^{j-n_P} \binom{j}{r}\binom{j-r}{n_P} = \sum_{r=0}^{j-n_P} \binom{j}{n_P}\binom{j-n_P}{r} = \binom{j}{n_P}2^{j-n_P} . \tag{21}$$

From this we infer the decomposition of $K_{(n_P,n_1)}$:

$$K_{(n_P,n_1)} = \sum_{j=n_P}^{\left\lfloor \frac{L}{n_1} \right\rfloor} \binom{j}{n_P}2^{j-n_P}\frac{Z_{j,n_1}}{Q^j} . \tag{22}$$
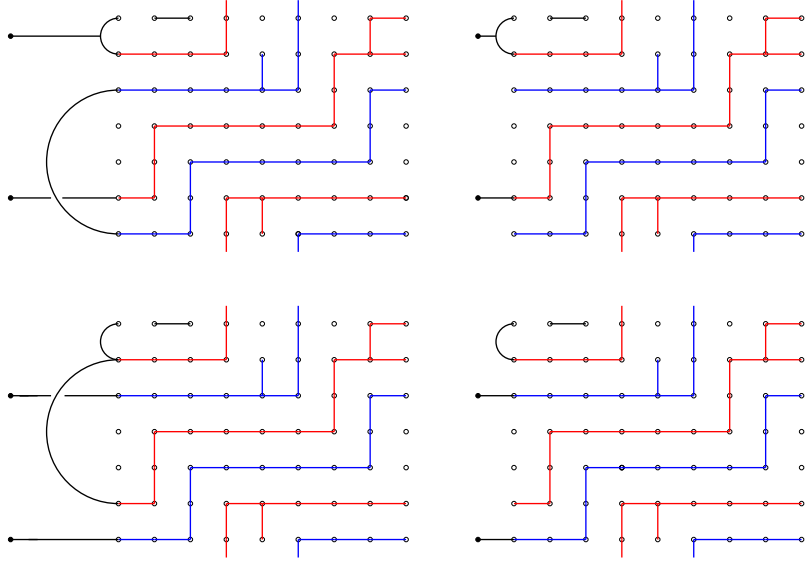
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Figure 4: Standard connectivity states at level $l = 2$ which are (12)-compatible with a given cluster configuration contributing to $Z_{2,2}$. The action of the cluster configuration on the connectivity states permutes the positions of the two black points.

Consider next the second case. The $|v_{l,i}\rangle$ can still contain at most $j - n_P$ big blocks, but they are now required to contain at least one, as one black point needs to be attributed. Therefore, the sums in Eq. (21) start from $r = 1$, leading to the following result for the decomposition of $K_{(n_P, n_1)'}$:

$$K_{(n_P, n_1)'} = \sum_{j=n_P+1}^{\left\lfloor \frac{L}{n_1} \right\rfloor} \binom{j}{n_P} (2^{j-n_P} - 1) \frac{Z_{j,n_1}}{Q^j} \ . \tag{23}$$

It remains to study the special case of $n_P = 0$, i.e., the case of $C_l = \text{Id}$. This is in fact trivial. Indeed, in that case, the value of n_1 in Z_{j,n_1} is no longer fixed, and one must sum over all possible values of n_1, taking into account that the case of $n_1 = 1$ is particular (absense of big blocks). Since $K_{l,\text{Id}} = \frac{K_l}{l!}$, one obtains simply Eqs. (15)–(17) of Sec. 3.1 up to a global factor.

3.4 Decomposition of Z_j over the K_{l,C_l}

To obtain the decomposition of Z_j in terms of the K_{l,C_l}, one would need to invert Eqs. (22)–(23) obtained above. But we now encounter the opposite problem of that announced in the beginning of Sec. 3.3: there are too many K_{l,C_l}. Indeed, the elemen-

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tary quantities K_{l,C_l} are not independent, since a given cluster configuration can realise different permutations depending on the way in which the black points are attributed. We must therefore select an independent set of K_{l,C_l}, and we make the choice of selecting the $K_{(n_P,n_1)}$, i.e., the C_l of the first type. Inverting Eq. (22) for varying n_P and fixed $n_1 > 1$ one obtains:

$$Z_{j,n_1} = Q^j \sum_{n_P=j}^{\left\lfloor \frac{L}{n_1} \right\rfloor} \binom{n_P}{j} (-2)^{n_P-j} K_{(n_P,n_1)} \qquad \text{for } n_1 > 1 \, . \tag{24}$$

Since the coefficients in this sum do not depend on n_1 (provided that $n_1 > 1$), we can sum this relation over n_1 and write it as

$$Z_{j,n_1>1} = Q^j \sum_{n_P=j}^{\left\lfloor \frac{L}{2} \right\rfloor} \binom{n_P}{j} (-2)^{n_P-j} K_{(n_P,n_1>1)} \tag{25}$$

where we recall the notations $Z_{j,n_1>1} = \sum_{n_1=2}^{L} Z_{j,n_1}$ and $K_{(n_P,n_1>1)} = \sum_{n_1=2}^{L} K_{(n_P,n_1)}$, corresponding to permutations composed of n_P cycles of the same length > 1.

Consider next the case $n_1 = 1$. For $j \geq 2$ one has simply

$$Z_{j,1} = \sum_{l=j}^{L} \frac{b_j^{(l)}}{l!} K_l \, , \tag{26}$$

recalling Eq. (18) and the fact that for $l \geq 2$ the $Z_{j,n_1>1}$ do not appear in the decomposition of K_l. However, according to Eqs. (16)–(17), the $Z_{j,n_1>1}$ do appear for $l = 0$ and $l = 1$, and one obtains

$$Z_{1,1} = \left(QK_1 - Q \sum_{j=1}^{L} (2^j - 1) \frac{Z_{j,n_1>1}}{Q^j} \right) + \sum_{l=2}^{L} \frac{b_j^{(l)}}{l!} K_l \, . \tag{27}$$

Inserting the decomposition (25) of $Z_{j,n_1>1}$ into Eq. (27) one obtains the decomposition of $Z_{1,1}$ over K_l and $K_{(n_P,n_1)}$:

$$Z_{1,1} = \sum_{l=1}^{L} \frac{b_j^{(l)}}{l!} K_l + \sum_{n_P=1}^{L} Q(-1)^{n_P} K_{(n_P,n_1>1)} \, . \tag{28}$$

We proceed in the same fashion for the decomposition of $Z_0 \equiv Z_{0,1}$, finding

$$Z_0 = \sum_{l=0}^{L} \frac{b_j^{(l)}}{l!} K_l - \sum_{j=1}^{L} \frac{Z_{j,n_1>1}}{Q^j} \, . \tag{29}$$

Upon insertion of the decomposition (25) of $Z_{j,n_1>1}$, one arrives at

$$Z_0 = \sum_{l=0}^{L} \frac{b_j^{(l)}}{l!} K_l + \sum_{n_P=1}^{L} \left[(-1)^{n_P+1} + (-2)^{n_P} \right] K_{(n_P,n_1>1)} \, . \tag{30}$$

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Since $Z_j = Z_{j,1} + Z_{j,n_1>1}$, we conclude from Eqs. (26)–(25) that, for any j,

$$Z_j = \sum_{l=j}^{L} \frac{b_j^{(l)}}{l!} K_l + \sum_{n_P=j}^{L} b_j^{(n_P,n_1>1)} K_{(n_P,n_1>1)} , \qquad (31)$$

with the coefficients

$$b_j^{(n_P,n_1>1)} = \begin{cases} Q^j \binom{n_P}{j}(-2)^{n_P-j} & \text{for } j \geq 2 \\ Q\left[n_P(-2)^{n_P-1} + (-1)^{n_P}\right] & \text{for } j = 1 \\ (-1)^{n_P+1} + (-2)^{n_P} & \text{for } j = 0 \end{cases} \qquad (32)$$

The decomposition of $Z \equiv \sum_{0 \leq j \leq L} Z_j$ is therefore

$$Z = \sum_{l=0}^{L} \frac{b^{(l)}}{l!} K_l + \sum_{n_P=1}^{L} b^{(n_P,n_1>1)} K_{(n_P,n_1>1)} \qquad (33)$$

with

$$
\begin{aligned}
b^{(l)} &= \sum_{0 \leq j \leq l} b_j^{(l)} , \\
b^{(n_P,n_1>1)} &= \sum_{0 \leq j \leq n_P} b_j^{(n_P,n_1>1)} .
\end{aligned}
\qquad (34)
$$

Note that $b_j^{(l)}$ (resp. $b_j^{(n_P,n_1>1)}$) is just the term multiplying Q^j in $b^{(l)}$ (resp. $b^{(n_P,n_1>1)}$). Computing the sum over j, we obtain the simple result

$$b^{(n_P,n_1>1)} = (Q-2)^{n_P} + (-1)^{n_P}(Q-1) . \qquad (35)$$

4 Amplitudes of the eigenvalues

4.1 Decomposition of Z over the $K_{l,D}$

The culmination of the preceeding section was the decomposition (31) of Z_j in terms of K_{l,C_l}. However, it is the $K_{l,D}$ which are directly related to the eigenvalues of the transfer matrix T. For that reason, we now use the relation (11) between K_{l,C_l} and $K_{l,D}$ to obtain the decomposition of Z_j in terms of $K_{l,D}$. The result is:

$$Z_j = \sum_{l,D} b_j^{(l,D)} K_{l,D} \qquad (36)$$

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where the coefficients $b_j^{(l,D)}$ are given by

$$b_j^{(l,D)} = \frac{b_j^l}{l!} + \sum_{(n_1>1)|l} \frac{b_j^{\left(\frac{l}{n_1}, n_1>1\right)}}{l!} \, c\left(D, \left(\frac{l}{n_1}, n_1\right)\right) . \tag{37}$$

Indeed, $K_l = \sum_D K_{l,D}$, and since $K_{(n_P,n_1)}$ corresponds to the level $l = n_P n_1$, we have $K_{(n_P,n_1)} = \sum_{D \in S_{n_P n_1}} \frac{c(D,(n_P,n_1))}{l!} K_{n_P n_1, D}$. (Recall that (n_P, n_1) is the class of permutations composed of n_P cycles of the same length n_1.) As explained in Sec. 3.2, the $b_j^{(l,D)}$ are not simply equal to $\frac{b_j^l}{l!}$ because of the $n_1 > 1$ terms. Using Eq. (13) we find that they nevertheless obey the following relation

$$\sum_{D \in S_l} [\dim(D)]^2 \, b_j^{(l,D)} = b_j^{(l)} . \tag{38}$$

But from Eq. (37) the $b_j^{(l,D)}$ with $l < 2j$ are trivial, i.e., equal to $\frac{b_j^l}{l!}$ independently of D. This could have been shown directly by considering the decomposition (15) of K_l. Finally, since $b_1^{(1,n_1>1)} = 0$ from Eq. (32), only $b_0^{(l,D)}$ is non-trivial for $l = 2$ or $l = 3$.

The decomposition of Z over $K_{l,D}$ is obviously given by

$$Z = \sum_{l,D} b^{(l,D)} K_{l,D} \tag{39}$$

where

$$b^{(l,D)} = \sum_{j=1}^{l} b_j^{(l,D)} . \tag{40}$$

The $b^{(l,D)}$ then satisfy

$$\sum_{D \in S_l} [\dim(D)]^2 \, b^{(l,D)} = b^{(l)} . \tag{41}$$

4.2   Relations among the $K_{l,D}$

Just like the K_{l,C_l}, the $K_{l,D}$ are not independent, and for the same reasons. Indeed, the number of $K_{l,D}$ which are independent among themselves, and independent of $K_{l',D'}$ at higher levels $l' > l$, equals the number of independent K_{l,C_l}. This number in turn equals the number of integers dividing l, since the independent K_{l,C_l} are $K_{l,\mathrm{Id}}$ and the $K_{(n_P,n_1)}$ with $l = n_P n_1$.

Therefore one can write relations between the $K_{l,D}$, by selecting an independent number of $K_{l,D}$ and expressing the others in terms of those selected. This produces

relations of the form

$$K_{l,D_l} = \sum_{D_{l'}} e(D_l, D_{l'}) K_{l',D_{l'}} \tag{42}$$

where the K_{l,D_l} are now those not selected, and the sum of $K_{l',D_{l'}}$ is over the $D_{l'}$ selected with $l' \geq l$. The expressions of the coefficients $e(D_l, D_{l'})$ depend of the choice of $K_{l,D}$ made. Note in particular that to obtain the $e(D_l, D_{l'})$, the $K_{(n_P,n_1)'}$ must be expressed in terms of the $K_{(n_P,n_1)}$. By combining Eqs. (23) and (24) we obtain

$$K_{(n_P,n_1)'} = \sum_{n'_P=n_P+1}^{\left\lfloor \frac{L}{n_1} \right\rfloor} \sum_{j=n_P+1}^{n'_P} \binom{j}{n_P} \binom{n'_P}{j} (2^{j-n_P} - 1)(-2)^{n'_P-j} K_{(n'_P,n_1)} , \tag{43}$$

and performing the sum over j this becomes

$$K_{(n_P,n_1)'} = \sum_{n'_P=n_P+1}^{\left\lfloor \frac{L}{n_1} \right\rfloor} \binom{n'_P}{n_P} (-1)^{n'_P-n_P+1} K_{(n'_P,n_1)} . \tag{44}$$

Let us give an example of this relation: for $L = 4$ we find that $K_{3,(1,2)'} = 2K_{4,(2,2)}$.

The coefficients $e(D_l, D_{l'})$ have the following properties:

$$\sum_{D_{l'} \in S_{l'=l}} [\dim(D_{l'})]^2 e(D_l, D_{l'}) = [\dim(D_l)]^2 \tag{45}$$

$$\sum_{D_{l'} \in S_{l'>l}} [\dim(D_{l'})]^2 e(D_l, D_{l'}) = 0 \tag{46}$$

which can be proved using the fact that the $e(D_l, D_{l'})$ are independent of L and that the number of eigenvalues, including degeneracies, corresponding to K_{l,D_l} is $[\dim(D_l)]^2 n_{\text{tor}}(L,l)$. These relations between K_{l,D_l} have strong physical implications: additional degeneracies inside a level and between different levels. We shall give in the next subsection a method to determine these degeneracies, but note for now that they depend of L although the $e(D_l, D_{l'})$ are independent of L.

We can now repeat the decompositions of the preceeding subsection, but expanding only over the selected independent $K_{l,D}$. To that end, we define the coefficients $\tilde{c}(D, C_l)$ by

$$K_{l,C_l} = \sum_{\text{indep. } D \in S_{l' \geq l}} \frac{\tilde{c}(D, C_l)}{(l')!} K_{l',D} . \tag{47}$$

Note that contrary to Eq. (11), the sum carries over all independent (selected) D at levels $l' \geq l$. Because of Eq. (46), the $\tilde{c}(D, C_l)$ have the following properties: if $C_l \neq \text{Id}$

$$\sum_{\text{indep. } D \in S_{l'}} [\dim(D)]^2 \tilde{c}(D, C_l) = 0 , \tag{48}$$

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whereas if $C_l = \mathrm{Id}$

$$
\begin{aligned}
\sum_{\text{indep. } D \in S_{l'=l}} [\dim(D)]^2 \tilde{c}(D, Id) &= |C_l| \,, \\
\sum_{\text{indep. } D \in S_{l'>l}} [\dim(D)]^2 \tilde{c}(D, Id) &= 0 \,.
\end{aligned}
\tag{49}
$$

Inserting the decomposition (47) of K_{l,C_l} into Eq. (33), we obtain the decomposition of Z over independent $K_{l,D}$:

$$
Z = \sum_{l,D} \frac{\tilde{b}^{(l,D)}}{l!} K_{l,D}
\tag{50}
$$

where the $\tilde{b}$ can be obtained using the $\tilde{c}$.

We do not have any general closed-form expression[4] for $\tilde{b}$, but in the next subsection we show how they can be determined in practice by a straightforward, though somewhat lengthy, procedure. More precisely, we determine all the $\tilde{b}^{(l,D)}$ up to $l = 4$, with a given convention for the choice of independent $K_{l,D}$. As the $b^{(l,D)}$, the $\tilde{b}^{(l,D)}$ verify

$$
\sum_{\text{indep. } D \in S_l} [\dim(D)]^2 \tilde{b}^{(l,D)} = b^{(l)}
\tag{51}
$$

except that now the sum is over independent D. This is a consequence of the properties of the $\tilde{c}$.

4.3 Method to obtain the amplitudes of the eigenvalues

Because of the additional degeneracies between the $K_{l,D}$, we have not been able to find a general formula giving the total degeneracies of the eigenvalues. But, using Eq. (33) and the fact that the $c(D, C_l)$ defined by Eq. (11) are integers, we deduce that the amplitudes of the eigenvalues are integer combinations of the $\frac{b^{(l)}}{l!}$ and the $\frac{b^{(n_P, n_1>1)}}{(n_P n_1)!}$. Determining precisely with which integers is not an easy task, and we give here a method which is operational for all values of L, though in practice it will probably become quite cumbersome for large L.

One must begin at the highest possible level, $l = L$. Since not all permutations are admissible, one can write relations between the $K_{L,D}$ and deduce which eigenvalues

---

[4]The best one could hope for would be an explicit formula relating $\tilde{b}$ to the characters of the symmetric group.

are shared by several different $K_{L,D}$. One then proceeds to the next lower-lying level, $l = L - 1$. Since not all permutations are admissible, and as some permutations are not independent of those at level $l + 1$, one can write relations between the $K_{l,D}$ and the $K_{l+1,D}$. These relations permit to deduce which eigenvalues appearing at level l are new and what are their degeneracies. This method is then iterated until one attains level $l = 3$. Considering $l \le 2$ is not necessary: all the eigenvalues at these levels are new as there are no relations between the corresponding $K_{l,D}$. Finally, using Eq. (33) where all $K_{l,D}$ have been expressed in terms of an independent number of $K_{l,D}$, we deduce the amplitudes of the eigenvalues.

Let us consider in detail the case $L = 4$. At level 4, the possible $K_{4,D}$ are $K_{(4)}$, $K_{(1^4)}$, $K_{(3,1)}$, $K_{(2,1,1)}$ and $K_{(2,2)}$, while the admissible K_{l,C_l} are $K_{4,\mathrm{Id}}$, $K_{4,(2,2)}$ and $K_{4,(1,4)}$. Using the table of characters of S_4, we can write:

$$K_{(4)} \;=\; K_{4,\mathrm{Id}} + K_{4,(2,2)} + K_{4,(1,4)} \tag{52}$$

$$K_{(1^4)} \;=\; K_{4,\mathrm{Id}} + K_{4,(2,2)} - K_{4,(1,4)} \tag{53}$$

$$K_{(3,1)} \;=\; 9K_{4,\mathrm{Id}} - 3K_{4,(2,2)} - 3K_{4,(1,4)} \tag{54}$$

$$K_{(2,1,1)} \;=\; 9K_{4,\mathrm{Id}} - 3K_{4,(2,2)} + 3K_{4,(1,4)} \tag{55}$$

$$K_{(2,2)} \;=\; 4K_{4,\mathrm{Id}} + 4K_{4,(2,2)} \tag{56}$$

We choose $K_{(4)}$, $K_{(3,1)}$ and $K_{(2,2)}$ as independent $K_{4,D}$, and we express the K_{4,C_l} in terms of those $K_{4,D}$:

$$K_{4,\mathrm{Id}} \;=\; \frac{K_{(4)}}{4} + \frac{K_{(3,1)}}{12} \tag{57}$$

$$K_{4,(2,2)} \;=\; -\frac{K_{(4)}}{4} - \frac{K_{(3,1)}}{12} + \frac{K_{(2,2)}}{4} \tag{58}$$

$$K_{4,(1,4)} \;=\; K_{(4)} - \frac{K_{(2,2)}}{4} \tag{59}$$

Next, using these expressions, we obtain $K_{(1^4)}$ and $K_{(2,1,1)}$ in terms of the independent $K_{4,D}$ chosen:

$$K_{(1^4)} \;=\; -K_{(4)} + \frac{K_{(2,2)}}{2} \tag{60}$$

$$K_{(2,1,1)} \;=\; 6K_{(4)} + K_{(3,1)} - \frac{3}{2}K_{(2,2)} \tag{61}$$

With these two relations we can determine the eigenvalue degeneracies between the chosen $K_{4,D}$. Recall first that according to Eq. (7) the number of eigenvalues contributing to $K_{l,D}$ is $n_{\text{tor}}(L,l)\dim(D)$ and that each eigenvalue has multiplicity $\dim(D)$. Further, $\dim(\square\square\square) = 1$, $\dim(\boxplus\boxminus) = 3$ and $\dim(\boxplus) = 2$. Consider now Eq. (60), recalling that the $K_{l,D}$ have been defined in Eq. (7) as a trace. We deduce that the corresponding eigenvalues must satisfy

$$\left(\lambda_{\yng}\right)^N = -(\lambda_{\square\square\square})^N + \sum_{i=1}^{2}\left(\lambda_{\boxplus,i}\right)^N \tag{62}$$

for any positive integer N. This implies that $\lambda_{\boxplus,1} = \lambda_{\square\square\square}$ and that $\lambda_{\boxplus,2} = \lambda_{\yng}$. Using this, Eq. (61) then yields

$$\sum_{i=1}^{3}\left(\lambda_{\boxplusplus,i}\right)^N = (\lambda_{\square\square\square})^N - \left(\lambda_{\yng}\right)^N + \sum_{i=1}^{3}\left(\lambda_{\boxplusplus,i}\right)^N \tag{63}$$

for any N. This is possible provided that either $\lambda_{\yng} = \lambda_{\square\square\square}$ or $\lambda_{\yng} = \lambda_{\boxplusplus,1}$. But the first possibility can be excluded since, by Eqs. (52)–(53), it would imply $K_{4,(1,4)} = 0$ which is inconsistent with our hypothesis that we work on a generic lattice where all admissible K are non-zero. We conclude that $\lambda_{\yng} = \lambda_{\boxplusplus,1}$ and hence that $\lambda_{\square\square\square} = \lambda_{\boxplus,1}$ and $\lambda_{\boxplusplus,i} = \lambda_{\boxplusplus,i}$ for $i = 2, 3$. There are therefore only 4 different eigenvalues at level 4 instead of 10.

Consider now the level $l = 3$. From the character table of S_3 we obtain:

$$K_{\square\square\square} = K_{3,\text{Id}} + K_{3,(1,2)'} + K_{3,(1,3)} \tag{64}$$

$$K_{\yng} = K_{3,\text{Id}} - K_{3,(1,2)'} + K_{3,(1,3)} \tag{65}$$

$$K_{\boxplusplus} = 4K_{3,\text{Id}} - 2K_{3,(1,3)} \tag{66}$$

$K_{3,(1,2)'}$ must be expressed terms of the independent K_{4,D_l} chosen:

$$K_{3,(1,2)'} = 2K_{4,(2,2)} = -\frac{K_{\square\square\square}}{2} - \frac{K_{\boxplusplus}}{6} + \frac{K_{\boxplus}}{2} \tag{67}$$

We choose $K_{\square\square\square}$ and $K_{\boxplusplus}$ as independent $K_{3,D}$, and we express the K_{3,C_l} in terms of the independent $K_{3,D}$ and $K_{4,D}$ chosen:

$$K_{3,\text{Id}} = \frac{K_{\square\square\square}}{3} + \frac{K_{\boxplusplus}}{6} + \frac{K_{\square\square\square}}{6} + \frac{K_{\boxplusplus}}{18} - \frac{K_{\boxplus}}{6} \tag{68}$$

$$K_{3,(1,3)} = \frac{2}{3}K_{\square\square\square} - \frac{K_{\boxplusplus}}{6} + \frac{K_{\square\square\square}}{3} + \frac{K_{\boxplusplus}}{9} - \frac{K_{\boxplus}}{3} \tag{69}$$

We obtain then the expression of $K_{\tiny\yng(1,1,1)}$:

$$K_{\tiny\yng(1,1,1)} = K_{\tiny\yng(3)} + K_{\tiny\yng(4)} + \frac{K_{\tiny\yng(2,1,1)}}{3} - K_{\tiny\yng(2,2)} \tag{70}$$

Using again Eq. (7) and the eigenvalue identities obtained at level $l = 4$, this becomes

$$\sum_{i=1}^{8} \left(\lambda_{\tiny\yng(1,1,1),i}\right)^N = \sum_{i=1}^{8} (\lambda_{\tiny\yng(3),i})^N - (\lambda_{\tiny\yng(4)})^N - \left(\lambda_{\tiny\yng(2,2),1}\right)^N + \left(\lambda_{\tiny\yng(2,2),2}\right)^N + \left(\lambda_{\tiny\yng(2,2),3}\right)^N \tag{71}$$

from which we deduce that $\lambda_{\tiny\yng(3),1} = \lambda_{\tiny\yng(4)}$, and that $\lambda_{\tiny\yng(3),2} = \lambda_{\tiny\yng(2,2),1}$. (Note that we cannot have, for example, $\lambda_{\tiny\yng(4)} = \lambda_{\tiny\yng(2,2),2}$ since these eigenvalues were shown to be independent in the preceeding analysis at level $l = 4$.) We can then further deduce that $\lambda_{\tiny\yng(1,1,1),1} = \lambda_{\tiny\yng(2,2),2}$, that $\lambda_{\tiny\yng(1,1,1),2} = \lambda_{\tiny\yng(2,2),3}$, and that $\lambda_{\tiny\yng(1,1,1),i} = \lambda_{\tiny\yng(3),i}$ for $i = 3, 4, \ldots, 8$. So among the 8 eigenvalues participating in $K_{\tiny\yng(3)}$ and the 8 participating in $K_{\tiny\yng(1,1,1)}$ only a total of 6 are new. On the other hand, all 16 eigenvalues participating in $K_{\tiny\yng(2,2)}$ are new, since $K_{\tiny\yng(2,2)}$ did not appear in an identity such as Eq. (70).

The eigenvalues for $l \leq 2$ are all new, as there are no relations between the $K_{l,D}$. Therefore, there are 28 new eigenvalues associated to $K_{\tiny\yng(3)}$, 28 to $K_{\tiny\yng(1,1,1)}$, 35 to K_1 and 14 to K_0.

To obtain the amplitudes associated to the eigenvalues, we use Eq. (33):

$$
\begin{aligned}
Z = \; & K_0 + b^{(1)} K_1 + b^{(2)} K_{2,\text{Id}} + b^{(1,n_1>1)} K_{2,(1,2)} + b^{(3)} K_{3,\text{Id}} + b^{(1,n_1>1)} K_{3,(1,3)} \\
& + \; b^{(4)} K_{4,\text{Id}} + b^{(1,n_1>1)} K_{4,(1,4)} + b^{(2,n_1>1)} K_{4,(2,2)}
\end{aligned}
\tag{72}
$$

and insert the expressions of the K_{l,C_l} in terms of the independent $K_{l,D}$ chosen above:

$$
\begin{aligned}
Z = \; & K_0 + b^{(1)} K_1 + \tilde{b}^{(\tiny\yng(2))} K_{\tiny\yng(2)} + \tilde{b}^{(\tiny\yng(1,1))} K_{\tiny\yng(1,1)} + \tilde{b}^{(\tiny\yng(3))} K_{\tiny\yng(3)} + \tilde{b}^{(\tiny\yng(2,1))} K_{\tiny\yng(2,1)} \\
& + \; \tilde{b}^{(\tiny\yng(4))} K_{\tiny\yng(4)} + \tilde{b}^{(\tiny\yng(3,1))} K_{\tiny\yng(3,1)} + \tilde{b}^{(\tiny\yng(2,2))} K_{\tiny\yng(2,2)}
\end{aligned}
\tag{73}
$$

where the amplitudes associated to the independent $K_{l,D}$ are:

$$\tilde{b}^{(\tiny\yng(2))} = \frac{b^{(2)}}{2} + \frac{b^{(1,n_1>1)}}{2} \tag{74}$$

$$\tilde{b}^{(\tiny\yng(1,1))} = \frac{b^{(2)}}{2} - \frac{b^{(1,n_1>1)}}{2} \tag{75}$$

$$\tilde{b}^{(\tiny\yng(3))} = \frac{b^{(3)}}{3} + \frac{2b^{(1,n_1>1)}}{3} \tag{76}$$

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$$\tilde{b}^{(\boxplus)} = \frac{b^{(3)}}{6} - \frac{b^{(1,n_1>1)}}{6} \tag{77}$$

$$\tilde{b}^{(\boxminus\boxminus\boxminus\boxminus)} = \frac{b^{(3)}}{6} + \frac{4b^{(1,n_1>1)}}{3} + \frac{b^{(4)}}{4} - \frac{b^{(2,n_1>1)}}{4} \tag{78}$$

$$\tilde{b}^{(\boxplus\boxminus\boxminus)} = \frac{b^{(3)}}{18} + \frac{b^{(1,n_1>1)}}{9} + \frac{b^{(4)}}{12} - \frac{b^{(2,n_1>1)}}{12} \tag{79}$$

$$\tilde{b}^{(\boxplus\boxplus)} = -\frac{b^{(3)}}{6} - \frac{7b^{(1,n_1>1)}}{12} + \frac{b^{(2,n_1>1)}}{4} \tag{80}$$

These can now be calculated explicitly from Eqs. (19) and (35):

$$\tilde{b}^{(\boxminus\boxminus)} = \frac{Q^2}{2} - \frac{3Q}{2} \tag{81}$$

$$\tilde{b}^{(\boxminus\boxminus)} = \frac{Q^2}{2} - \frac{3Q}{2} + 1 \tag{82}$$

$$\tilde{b}^{(\boxminus\boxminus\boxminus)} = \frac{Q^3}{3} - 2Q^2 + \frac{8Q}{3} - 1 \tag{83}$$

$$\tilde{b}^{(\boxplus)} = \frac{Q^3}{6} - Q^2 + \frac{4Q}{3} \tag{84}$$

$$\tilde{b}^{(\boxminus\boxminus\boxminus\boxminus)} = \frac{Q^4}{4} - \frac{11Q^3}{6} + \frac{15Q^2}{4} - \frac{5Q}{3} - 2 \tag{85}$$

$$\tilde{b}^{(\boxplus\boxminus\boxminus)} = \frac{Q^4}{12} - \frac{11Q^3}{18} + \frac{15Q^2}{12} - \frac{5Q}{9} - \frac{1}{3} \tag{86}$$

$$\tilde{b}^{(\boxplus\boxplus)} = -\frac{Q^3}{6} + \frac{5Q^2}{4} - \frac{25Q}{12} + \frac{3}{2} \tag{87}$$

Note that the four first amplitudes in this list have been obtained by Chang and Schrock [3] using a different method.

We can finally give the amplitudes of the eigenvalues themselves. The 14 eigenvalues at level 0 have amplitude 1. The 35 eigenvalues at level 1 have amplitude $\tilde{b}^{(1)}$. At level 2, the 24 eigenvalues contributing to $K_{\boxminus\boxminus}$ have amplitude $\tilde{b}^{\boxminus\boxminus}$, and the 24 eigenvalues contributing to $K_{\boxminus}$ have amplitude $\tilde{b}^{\boxminus}$. At level 3, the 6 new eigenvalues contributing to $K_{\boxminus\boxminus\boxminus}$ have amplitude $\tilde{b}^{\boxminus\boxminus\boxminus}$, and the 16 eigenvalues contributing to $K_{\boxplus}$ have amplitude $2\tilde{b}^{\boxplus}$. At level 4, $\lambda_{\boxminus\boxminus\boxminus}$ has amplitude $\tilde{b}^{\boxminus\boxminus\boxminus} + 2\tilde{b}^{\boxplus\boxplus} + \tilde{b}^{\boxminus\boxminus}$, $\lambda_{\boxminus}$ has amplitude $3\tilde{b}^{\boxplus\boxminus\boxminus} + 2\tilde{b}^{\boxplus\boxplus} + \tilde{b}^{\boxminus\boxminus}$, and $\lambda_{\boxplus,1}$ and $\lambda_{\boxplus,2}$ both have the same amplitude $3\tilde{b}^{\boxplus\boxminus\boxminus}$.

Note that when we know the amplitudes of the $K_{l,D}$ in the expansion of Z and the relations between $K_{l,D}$ for a given width L, we know it for all widths smaller than L, as they do not change. The only difference is that for smaller widths some of the $K_{l,D}$

vanish, so the equations must be truncated. For example, the result (70) obtained here for $L = 4$, implies by truncation of the $K_{4,D}$ terms that for $L = 3$:

$$K_{\boxbar} = K_{\boxminus\boxminus\boxminus} \tag{88}$$

Likewise, the expansion (73) of Z obtained here for $L = 4$, implies by truncation that for $L = 3$:

$$Z = K_0 + b^{(1)} K_1 + \tilde{b}^{(\boxminus\boxminus)} K_{\boxminus\boxminus} + \tilde{b}^{(\boxbar)} K_{\boxbar} + \tilde{b}^{(\boxminus\boxminus\boxminus)} K_{\boxminus\boxminus\boxminus} + \tilde{b}^{(\boxplus)} K_{\boxplus} \tag{89}$$

From these equations, it is then simple to obtain the amplitudes of eigenvalues for $L = 3$.

4.4   Particular non-generic lattices

The degeneracies we have obtained apply to the case of a generic lattice. In the case of a specific lattice, i.e., one having extra non-generic symmetries, there might be additional degeneracies. An example is the case of a square or a honeycomb lattice, because of the invariance of the lattice under reflection by its symmetry axis.

Specifically, the transfer matrix T_L of the square lattice commutes with the dihedral group D_L, since the lattice enjoys both translational and reflectional symmetries in the space perpendicular to the transfer direction. Likewise, for the honeycomb lattice, the symmetry group is $D_{\frac{L}{2}}$. Those groups act on the right time slice, not on the left one (i.e., the black points), and thus commute with the symmetry group S_l at level l of the bridges. There are therefore additional degeneracies inside a given $K_{l,D}$.

In the case of the dihedral group, since all its irreps are of dimension 1 or 2, there are additional degeneracies between pairs of eigenvalues, as observed in Ref. [3]. The method to determine precisely these degeneracies is to decompose the space at level l of symmetry D into irreps of the dihedral group, and to count the number of irreps of dimension 2.

Furthermore, in the case of a square or a hexagonal lattice, there are yet additional degeneracies (which do not exist for a generic lattice with a dihedral symmetry). Indeed, for these two lattices, the Z_{L,n_1} with $n_1 > 1$ are found to be zero: at level $l = L$ no permutation between bridges is allowed. At this level there is thus only one eigenvalue of total degeneracy $\sum_{D \in S_l} [\dim(D)]^2 \tilde{b}^{(L,D)} = b^{(L)}$. Finally, in the case of a square lattice,

there appears to be additional "accidental" degeneracies: for example, an eigenvalue at level 1 coincides with an eigenvalue at level 2, as observed in Ref. [3].

5   Conclusion

To summarise, we have generalised the combinatorial approach developed in Ref. [2] for cyclic boundary conditions to the case of toroidal boundary conditions. In particular, we have obtained the decomposition of the partition function for the Potts model on finite tori in terms of the generalised characters $K_{l,D}$. This decomposition is considerably more difficult to interpret than in the cyclic case, as some eigenvalues coincide between different levels l for all values of Q. We have nevertheless succeeded in giving an operational method of determining the amplitudes of the eigenvalues as well as their *generic* degeneracies.

The eigenvalue amplitudes are instrumental in determining the physics of the Potts model, in particular in the antiferromagnetic regime [8]. Generically, this regime belongs to a so-called Berker-Kadanoff (BK) phase in which the temperature variable is irrelevant in the renormalisation group sense, and whose properties can be obtained by analytic continuation of the well-known ferromagnetic phase transition [9]. Due to the Beraha-Kahane-Weiss (BKW) theorem [10], partition function zeros accumulate at the values of Q where either the amplitude of the dominant eigenvalue vanishes, or where the two dominant eigenvalues become equimodular. When this happens, the BK phase disappears, and the system undergoes a phase transition with control parameter Q. Determining analytically the eigenvalue amplitudes is thus directly relevant for the first of the hypotheses in the BKW theorem.

For the cyclic geometry, the amplitudes are very simple, and the values of Q satisfying the hypothesis of the BKW theorem are simply the so-called Beraha numbers, $Q = B_n = (2\cos(\pi/n))^2$ with $n = 2, 3, \ldots$, independently of the width L. For the toroidal case, we have no general formula for the amplitudes, valid for any L. It is however clear from the amplitudes given for $L \leq 4$ in Sec. 4.3 that many of them vanish at $Q = 2$, and yet other differ just by a sign by virtue of Eq. (35). Indeed, it is consistent with simple physical arguments, that a phase transition in the antiferromagnetic regime must take

place at $Q = 2$. However, it remains to elucidate whether the BK phase exists for all other $Q \in (0, 4)$, and whether the Beraha numbers play any special role in the toroidal case.

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