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A QUANTITATIVE METHOD FOR DETERMINING OPTIMAL PLANT LAYOUT

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SUMMARY

The objective of this study is to develop a procedure which will yield an optimal arrangement of machines or departments by minimizing material handling costs. A solution is suggested through the use of a linear programming approach. Necessary input information includes: size, shape, and orientation of each machine or department; also required are the material handling costs between every possible pair of machines (or departments) as a linear function of the distance separating the two (measured as the sum of the differences of the two coordinates of the center of the machine area).

An objective function is written as a total of all of the material handling relationships between machines (or departments) and constraints of four types are written — non-overlap; physical location; "D" definitional; and "U-V" definitional.[1] For the general case this presents 3.5 m(m-1) constraints where m is the number of machines to be located.

The algorithm developed consists of solving the objective function and the first three types of constraints by using the simplex method. The solution is then completed, if necessary, by using mixed integer programming and the dual simplex method. The layout is then finished by the addition of facilities not included in the mathematical model.

[1]"D" variables are inserted in the model in order to eliminate "and/or" pairs of constraints; "U-V" variables are inserted in order to eliminate absolute values from constraints.

The model may be adapted to most physical situations. Facilities may be located against walls or in corners as desired, the length or width of the layout may be restricted, and facilities may be located a minimum or specific distance apart. Various materials may flow between the same machines in both directions, fixed or semi-fixed machine locations may be assigned, and other costs such as that of the flow of personnel between two machines or departments, may be incorporated into the model.

In addition to its adaptability, other advantages of the model include: it may be solved by use of a computer program, it is relatively unrestricted, it does not require an initial solution, it is based on cost, and it uses rectangular distances. However, it has the disadvantage that it may or may not yield an optimal solution within a reasonable amount of computer time.

CHAPTER I

INTRODUCTION

There have been many different methods of approach to the problem of plant layout design, but in general it still remains an empirical procedure. Layout design is actually part of the overall problem of facilities planning, which also includes plant location, process selection, capacity estimation, materials handling methods selection, material flow, work methods design, etc.

The most important of these in relation to plant layout is material flow planning, because the design of the materials flow is a requisite to designing the layout. It is necessary to study the movements of each item of material between work centers or departments in a plant because the flow not only influences the sequence and orientation of the equipment arrangement but also the materials handling cost.

Given the required flow of materials throughout the entire manufacturing process, the problem is to design the most effective and economical plant layout. Usually the best plant layout is one that has the least materials handling cost. Therefore, it would be desirable to have a method for the determination of a plant layout which yields the minimum materials handling cost for a given pattern of material flow between departments. The purpose of this study is to develop such a method.

CHAPTER II

LITERATURE SURVEY

Of the many methods that have been devised for plant layout, those most commonly practiced have been qualitative in nature. These qualitative methods have been notably lacking in exactness, and therefore there remains a need for a general analytical method of approach.

This need was first recognized about 1950, and since then a number of quantitative techniques have been proposed. In general, many of the analytical methods are based on the From - To Chart concept, which provides quantitative information on the movements of materials between each pair of facilities. The numerical elements of the From - To Chart are employed to analyze and plan the movements of materials in a plant.

The general concept of plant layout is based on the process type of equipment arrangement, the product type of arrangement, or a combination of the two. Smith (24) developed a procedure for solving the problem involving a process type of layout, where similar machines are grouped together into one department. He analyzed the processing of different parts through the same groups of machines, with the sequence of operations differing for the several parts. Smith's objective was to design a layout and investigate the efficiency of this layout with respect to materials handling. He suggested that the movements or the frequency of handling between two departments should be used to determine these relationships. Smith's method is based on the fact that the backtracking is bad and from this he calculated efficiencies for particular layouts. However, his efficiencies seem to bear no relationship to the effectiveness of the layouts. Also he did not prove that the least backtracking will yield the best layout. Smith's method also has very limited applications because it is only concerned with the arrangement of departments in one straight line.

Several textbooks such as those of Moore (18) and Reed (22), repeat Smith's method. Apple (1) does not assume that backtracking is necessarily twice as harmful as moving work forward as Smith does. Although Apple does not give a specific penalty to backtracking, he does indicate that improvement can be obtained by its elimination. This may be an erroneous assumption when backtracking is not a significant factor. The optimum movement may depend more heavily on other materials handling factors, such as distance between departments, characteristics and quantity of product, and method of handling.

Farr (6) represents the From - To Chart with another name - the Cross Chart. He applies the chart to the analysis of a layout by combining the Flow Diagram and the Operation Process Chart and then "translating" the quantitative information of the From - To Chart into qualitative relative importance. He then classifies the relative locations of departments into four groups according to the degree of closeness. Farr's method, however, has only limited application and still uses a qualitative judgment.

Muther (20) follows the same principle as Farr in employing the From - To Chart, which he combines with the Activity Relationship Chart, replacing the quantitative values with qualitative considerations.

In 1955 Buffa (3) presented another method for analyzing plant layouts. He developed the From - To Chart for analyzing functional layouts, in which the main idea was to determine the relationships of one work center with every other work center. He suggested that these relationships can be represented by the magnitude of the materials handling cost between a given work center and every other work center.

Buffa's objective was to determine the position of each work center by interpreting the data of the From - To Chart into a schematic relationship. The application of this method is limited, however, if the number of products and amount of backtracking increase. Such situations are complex problems, and Buffa presented only a simple situation in the application of his method. Buffa, however, has contributed a significant principle in determining the best flow of materials. The principle is the analysis and investigation of each work center as it is related to the magnitude of the materials handling cost with all other work centers. His method is applicable in analyzing the movements of materials if there is no backtracking but is not satisfactory in solving complex problems.

In 1958, Llewellyn (14) combined Smith's method and Buffa's method. He multiplied the number of movements by average aisle distance per move and then compared the total of these multiplications with the actual travel from each department in order to obtain the efficiency of the layout, that is, by comparing the actual condition with the optimal condition. He then attempted to reduce the distance by locating two departments which have high materials handling relationships as close together as possible. The significance of Llewellyn's method is that the layout can be improved by reducing the travel distance.

In 1960, Schneider (23 and 24) tried to extend the application of the From - To Chart to different types of projects. Schneider's concept is based on the number of units of volume transferred between departments and is similar to the C.P.M approach. If the movements of material are changed to be "movements of time" in Schneider's method, the graphical representation of a C.P.M. network will be obtained. His method however, is very inadequate for analyzing complex flow in a materials handling problem.

In a recent research project, A. Z. Gani (7), under the supervision of J. M. Apple, used the From - To Chart as the basis for a more quantitative method of planning and/or analyzing material flow and activity location. The procedure involves the development of an Inflow Chart and an equivalent Outflow Chart, based on the From - To Chart. The entries in the From - To Chart combine (1) the number of moves per time period, (2) the distance moved and (3) the cost per foot of moving by a predetermined method into a single number representing handling costs. Then

$$Y_i = \sum_{j=1}^{n} X_{ji}$$

is calculated where "n" is the number of departments and "X_{ji}" represents the material handling costs moving from Department j to Department i, that is, the entries in the From - To Chart. The elements

(I_{ji}) in the Inflow Chart are calculated as follows:

$$I_{ji} = \frac{X_{ji}}{Y_i}$$

These elements are then used as the basis for planning relative locations and/or evaluating layouts or flow patterns. Although Gani seems to take many of the appropriate conditions into account, he gives little explanation or logical basis for his numerical calculations.

Although the From-To Chart is very helpful in analyzing the flow of materials between facilities, it does not include the distance of the move as one of the optimal location criteria nor does it actually prescribe the spatial locations for facilities. Wimmert (27,28) pointed out the necessity of including this location dependent distance as a criterion and then presented a method for locating n facilities in n available locations so that the product of the volume of the move multiplied by the distance between the facilities is minimized. Wimmert's method does not guarantee optimality as Conway and Maxwell's method (5), which is similar, also does not. In both cases the application to a relatively small practical problem would be impossible even on the latest computers.

Another class of methods uses process charting concepts to optimally relate a number of products and facilities into sequential line flow. The concept was introduced by Ireson (12) and expanded by Young (29) to include consideration of the capacity required for cyclic scheduling. Noy (21) and Michel (15), in apparently independent attempts, both begin with a process chart tableau but arrange the facilities in line sequence by using weighting factors.

Moore (16,17) has considered a somewhat different problem, that of locating n new facilities among m existing facilities. In his method the n new facilities are to be assigned to n locations such that the sum of the product of the volume of each move multiplied by the distance between each of the n new facilities and m existing facilities is minimized. The problem seems easily defined in this manner, and in this form an optimal solution is readily obtained by means of the assignment algorithm. However, flows between the new facilities are not considered, and only point location for facilities instead of areas are considered.

In recent times many researchers have formulated the plant layout problem as a quadratic assignment problem, and thus the literature contains several algorithms for treating the plant layout problem in this form. These methods rank among the most promising of those developed to date. In particular, Gilmore (8) and Steinberg (26) have derived suboptimal algorithms while Lawler (13) and also Gilmore (8) have determined optimal algorithms which can be modified to yield suboptimal solutions within a specified accuracy.

For dealing with special cases, Armour and Buffa (2) and Hiller (10) have also developed algorithms. These algorithms consider only pair-wise exchanges of facilities and locations instead of permutations of the facilities among the available locations. Although these methods are inherently suboptimal, they have the advantage that their application can attempt to improve any suboptimal solution, including those yielded by other algorithms.

The very similar Gilmore and Lawler optimal algorithms are unfortunately not computationally feasible for realistic problems. Gilmore states that his algorithm is not feasible for more than $n = 15$ facilities to be arranged, and an upper bound on the computational effort for the Lawler algorithm is that

$$\sum_{i=0}^{n-2} (n!/(n - 1)!)$$

linear assignment problems must be set up and solved — one with n assignments, n with $n - 1$ assignments, $(n - 1)n$ with $n - 2$ assignments, ..., $n!/2$ with 2 assignments. However, the efficiency and amount of computer storage required for either of these algorithms can be greatly improved if a good suboptimal solution is obtained initially.

The quadratic assignment problem may be solved by regarding it as n separate decisions in which one of the n facility assignments is made at each of the decisions. Each decision would then consist of assigning one unassigned facility to one unused location. Several methods for doing this have been presented at various levels of generality by Gilmore (8), Hillier (10), and Lawler (13). Also a new suboptimal technique has been present by Hillier and Connors (11). It utilizes a sequential decision technique which begins each individual decision by obtaining the contribution to the expected total cost associated with assigning work center i to location j, given the previous assignments. This is done for each possible pair of unassigned facility i and unused location j until all possibilites have been tried and then the minimum cost pair is selected.

CHAPTER III

OBJECTIVES

As can be seen from the literature cited in the previous chapter, there exists no formulation of the plant layout problem which permits complete solution in a practical amount of time. In general, all optimal techniques take too much time even on the latest computers, while techniques which can be carried out in a reasonable amount of time are only suboptimal. Both types of techniques also present certain other disadvantages. The objective of this research is to develop a technique which, if carried to conclusion, will yield an optimal solution or, if not, will yield a suboptimal solution, and which will overcome most of the other disadvantages of present techniques. The relation of the proposed technique to these disadvantages will be discussed in Chapter VIII.

The criterion of minimum materials handling cost has been chosen for this study since the effectiveness of any layout is described accurately only by its cost. When materials handling costs are minimized, the layout will be as small and compact as possible, and plant construction costs will also be minimized — at least as far as space requirements are concerned. Thus, the two major cost considerations in most layout problems are accounted for.

Although the proposed method will give a solution to the layout problem, additional constraints will have to be incorporated, as revisions to the resulting layout, using other analytical techniques or subjective methods. The proposed approach is not intended to give and will not give a finished layout in most cases. It will provide an exact formulation to the major portion of the problem. From this formulation a solution may be obtained into which it will generally be necessary to incorporate various changes and additions such as aisles, machine aligments, etc. Although this adjustment procedure must be done with any of the existing layout algorithms, it is hoped that the point at which it is done will be closer to the correct solution than that arrived at by using other approaches.

As in the development of any mathematical model, numerous attempts were made before a satisfactory method was discovered. Therefore, in the Appendix is a very brief description of the more important approaches unsuccessfully tried by the author in exploring a quantitative solution to the plant layout problem.

CHAPTER IV

DEVELOPMENT OF THE MODEL

One of the major concerns when a mathematical model is formulated for a real world situation is that of the type of approach taken. Primarily for the ease of handling and the availability of a general solution technique, a linear programming approach was chosen for this study of the plant layout problem. In addition to the advantages cited above, the great amount of flexibility afforded by a linear programming approach makes it an appealing algorithm for the problem at hand.

As has been stated previously, the objective taken is that of minimizing materials handling cost which assumes all materials handling equipment has been previously selected. Thus, it is necessary to write an objective function for the linear programming algorithm in terms of materials handling cost. This cost is defined as the sum of the initial, operating, maintenance, and all other costs connected with the materials handling equipment, prorated on a per unit time basis.

In order to write this objective function, it is first necessary to define what is meant by "the distance between two machines." If the coordinates of the centers of two machines are given by (X1, Y1) and (X2, Y2), then the "distance" between machines one and two is given by

$$|X2 - X1| - |Y2 - Y1|$$

that is to say, the distance is given by the difference of the X coordinates plus the difference of the Y coordinates. This is one of several possible forms of the distance function, all of which are merely approximations of the actual distance from one machine to the next. However, this would appear to be realistic since most goods in moving from one machine to another must travel down one aisle and then perpendicularly down another to reach their destination, as represented by the dashed line in Figure 1 a. The difference of the coordinates also approaches the mean of the two other possible distances actually travelled by goods going from one machine to the next as shown by the arrows in Figure 1 b and dots in Figure 1 c. The dotted line (Figure 1 c) depicts the actual path containing the additional distance necessary to move out of a department into an aisle and then back into a second department. The arrows in Figure 1 b depict the shorter direct distance possible when a material handling device such as an over-head conveyor or bridge crane is used. Thus, the rectangular distance chosen for this model is a median between the other two. Although this form is only an approximation, it is far more realistic than the minimum possible or direct distance ($= \sqrt{(X2 - X1)^2 + (Y2 - Y1)^2}$) so often used in plant layout calculations.

Now that the distance between machines has been defined, an objective function may be formulated in terms of this distance. The assumption is made here that it is realistic to write this as a linear function. This is necessary in order to use the linear programming approach. Although, again, this form does not conform perfectly to reality, it appears to be reasonably close in most cases. For example, per unit time a belt conveyor system costs (roughly) a fixed amount for motor and drive and then a fixed amount per unit distance for belting and supports.

As an example for the development of the model, let three machines have dimensions 2 by 4, 2 by 2, and 4 by 4 and be designated machines 1, 2, and 3, respectively, as shown in Figure 2. If the coordinates of the center of these machines are designated (X1, Y1), (X2, Y2), and (X3, Y3) respectively, the materials handling costs per time period could be as follows:

Between Machines	Cost[1] as a Function of Distance
1 and 2	$25 \|X2 - X1\|+\|Y2 - Y1\| + 500$
1 and 3	$5 \|X3 - X1\|+\|Y3 - Y1\| + 100$
2 and 3	$10 \|X3 - X2\|+\|Y3 - Y2\| + 250.$

Thus, the total objective function to be minimized for this example would be

$$Z = 25 \; |X2 - X1|+|Y2 - Y1| + 500 + 5 \; |X3 - X1|+|Y3 - Y1|$$

$$+ 100 + 10 \; |X3 - X2|+|Y3 - Y2| + 250$$

$$= 25|X2 - X1|+ 25|Y2 - Y1|+ 5|X3- X1|+ 5|Y3 - Y1|$$

$$+ 10|X3 - X2|+ 10|Y3 - Y2|+ 850.$$

The next step is to write constraints for this formulation. The first of these to be treated are "non-overlap" constraints. These

[1] 25, 5, and 10 represent arbitrary variable costs per foot per unit time of three methods of handling. 500, 100, 250 represent arbitrary fixed costs per unit time period.

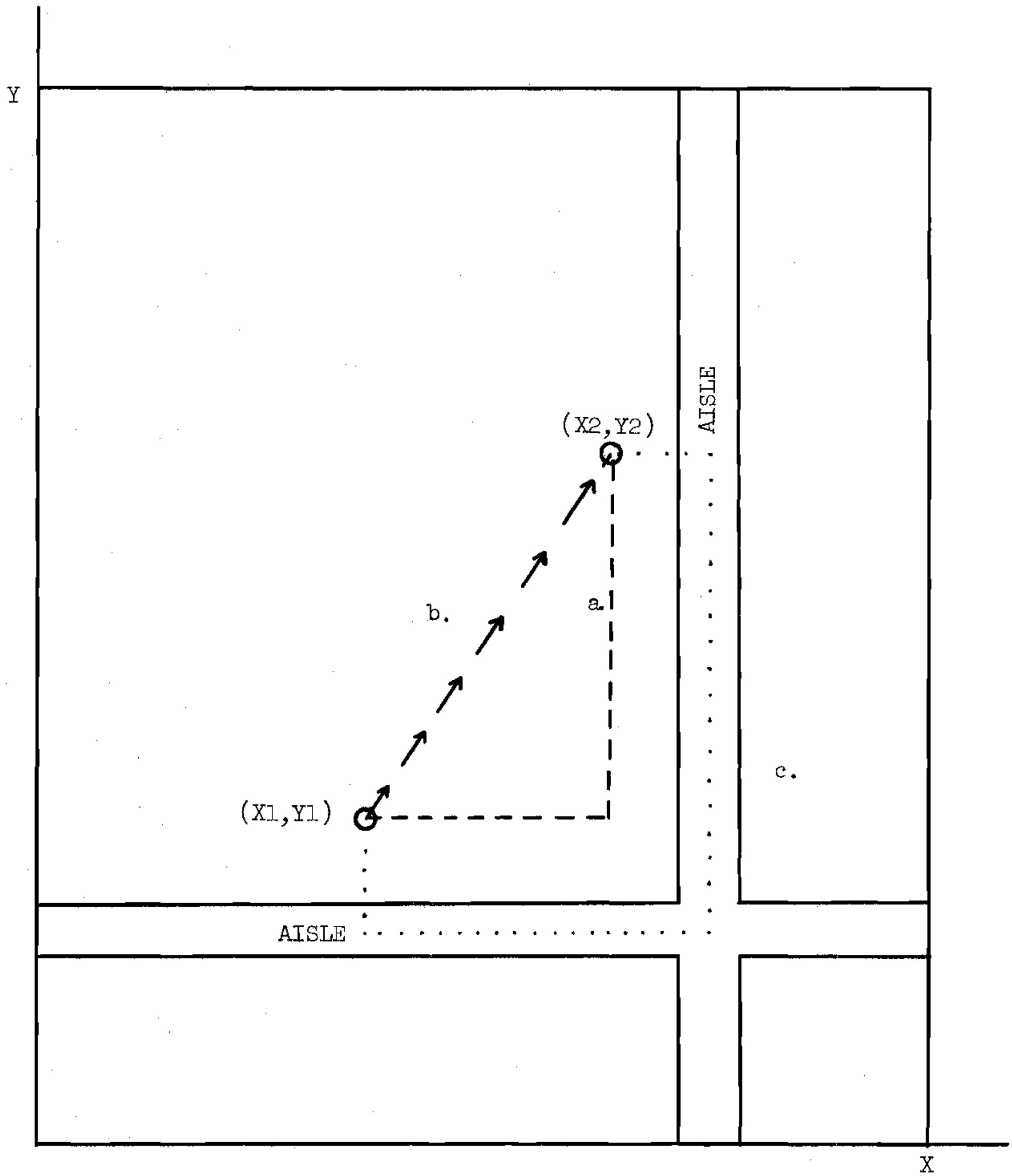

Figure 1. Possible Interpretations of "the Distance Between Two Machines."

non-overlap constraints insure that no machine is located in any part of an area occupied by another machine. However, before these constraints can be written it is necessary to assume an orientation for each machine. One possibility for the machines in the example above is given in Figure 2.

With size, shape and orientation given, each pair of machines requires two non-overlap constraints. These two restraints require that the minimum possible distance between machines be maintained in the X - coordinate direction and in the Y - coordinate direction. See Figure 3. Thus, if there are m machines to be located, m(m-1) non-overlap constraints are required.

However as can be seen in Figure 3, these are "and/or" pairs- only one or the other of the two constraints <u>must</u> hold but both <u>may</u> hold. Thus, if the X - constraint holds in a given situation, the Y - constraint may or may not hold. It is not necessary for the Y - coordinates to differ at all if the X - coordinates differ enough to prevent the two machines from occupying the same area.

Referring to the three machine example the non-overlap constraints are given as:

$$\text{and/or} \begin{cases} |X2 - X1| \geq 3 \\ |Y2 - Y1| \geq 2 \end{cases}$$

$$\text{and/or} \begin{cases} |X3 - X1| \geq 4 \\ |Y3 - Y1| \geq 3 \end{cases}$$

$$\text{and/or} \begin{cases} |X3 - X2| \geq 3 \\ |Y3 - Y2| \geq 3 \end{cases}$$

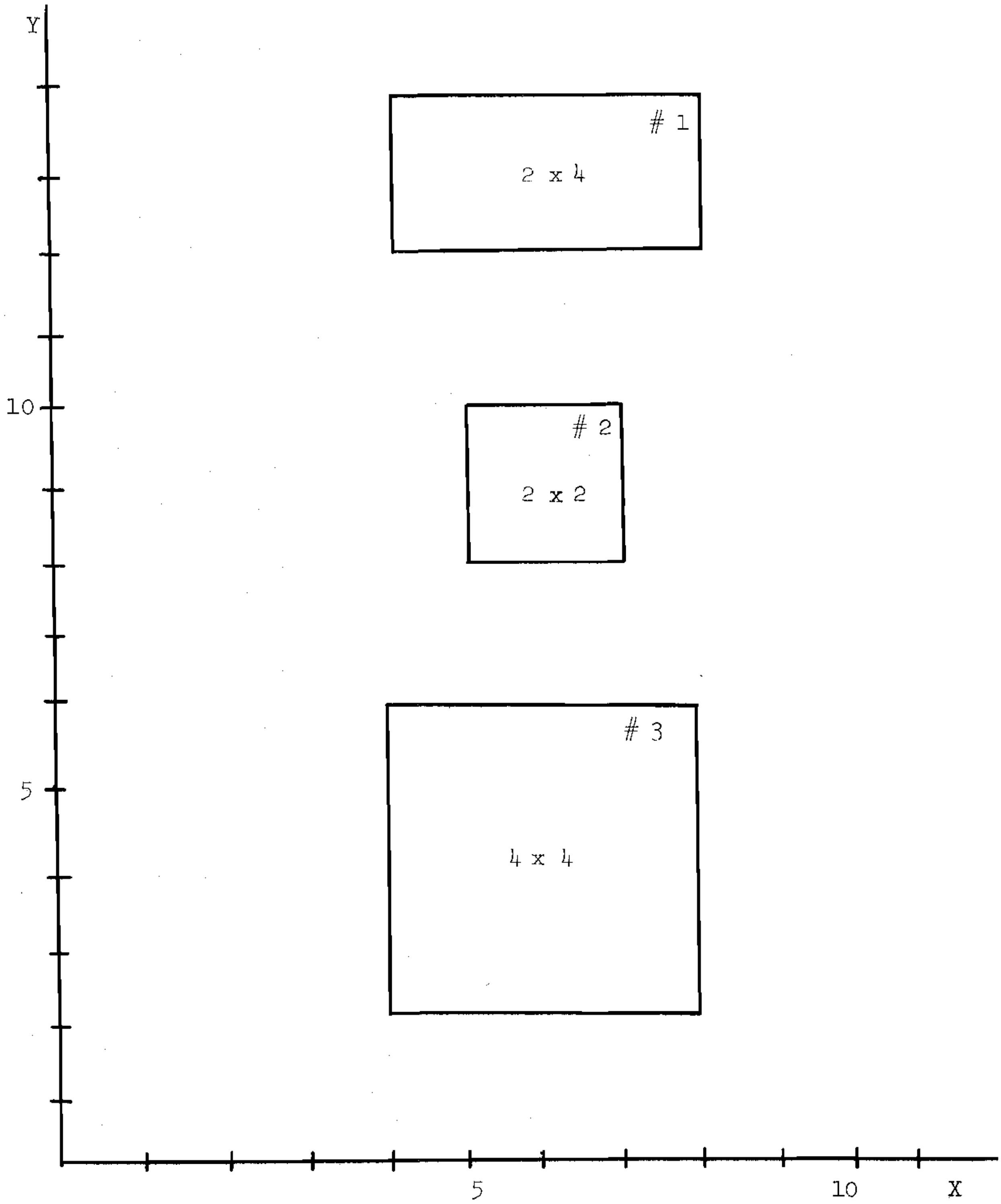

Figure 2. Possible Size, Shape, and Orientation for the Three Machines in the Example.

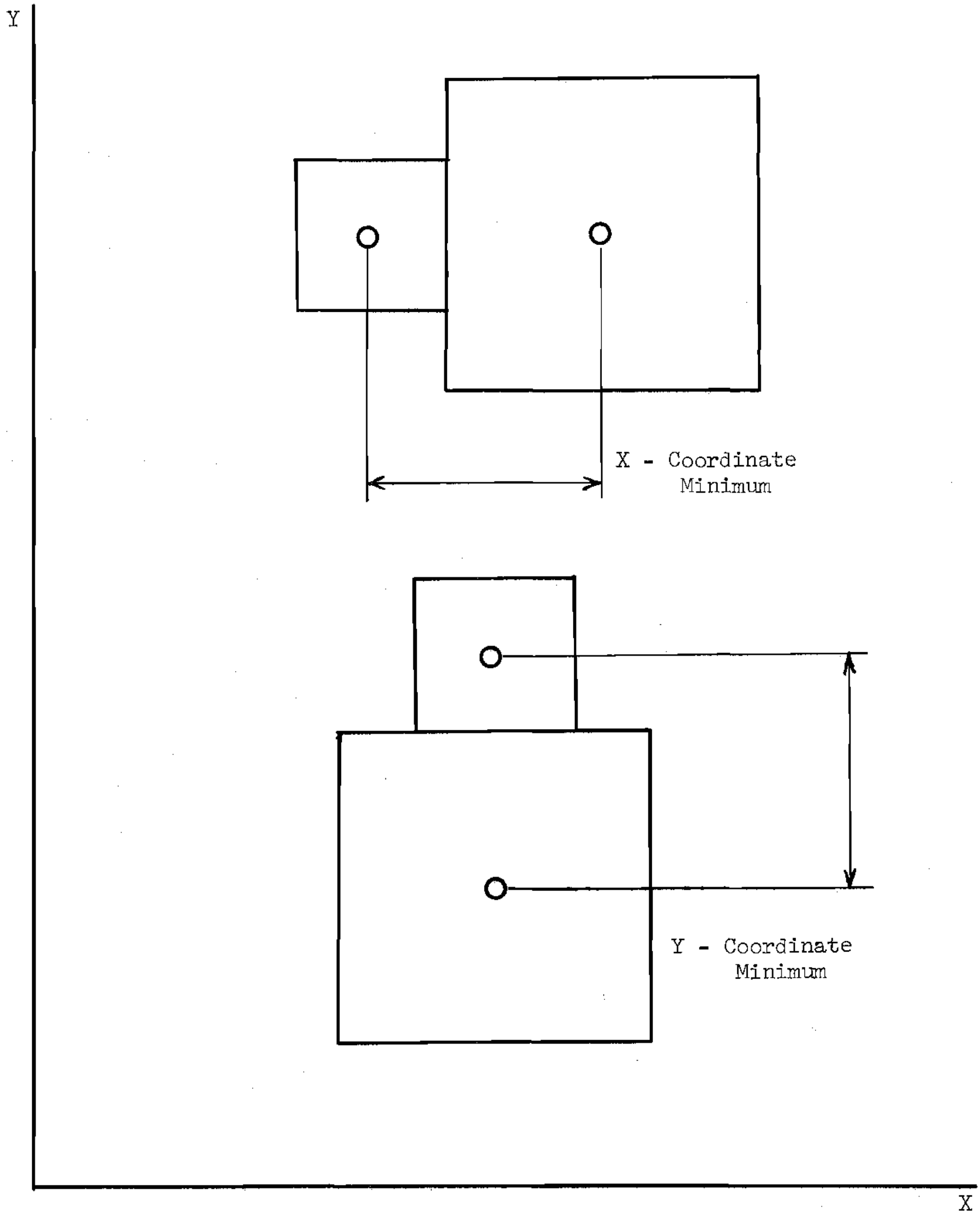

Figure 3. X and Y Minimum Coordinate Distances.

The reasoning behind these constraints can be seen in Figure 4 for machines one and two. The other relationships are similar.

In order for absolute values to be treated by linear programming means, certain substitutions are necessary. First, define two variables U12X and V12X for machines one and two as follows:

$$U12X = \begin{cases} X2 - X1 \text{ if } X2 \geq X1 \\ 0, \text{ otherwise} \end{cases}$$

and

$$V12X = \begin{cases} X1 - X2 \text{ if } X1 \geq X2 \\ 0, \text{ otherwise.} \end{cases}$$

Thus $|X2 - X1| = U12X + V12X$.

Also let

$$U12Y = \begin{cases} Y2 - Y1 \text{ if } Y2 \geq Y1 \\ 0, \text{ otherwise} \end{cases}$$

and

$$V12Y = \begin{cases} Y1 - Y2 \text{ if } Y1 \geq Y2 \\ 0, \text{ otherwise} \end{cases}$$

Therefore $|Y2 - Y1| = U12Y + V12Y$. Similar definitions may be constructed for other machine parts.

Returning to the example, the model may now be stated as follows (with U's and V's in place of the absolute values):

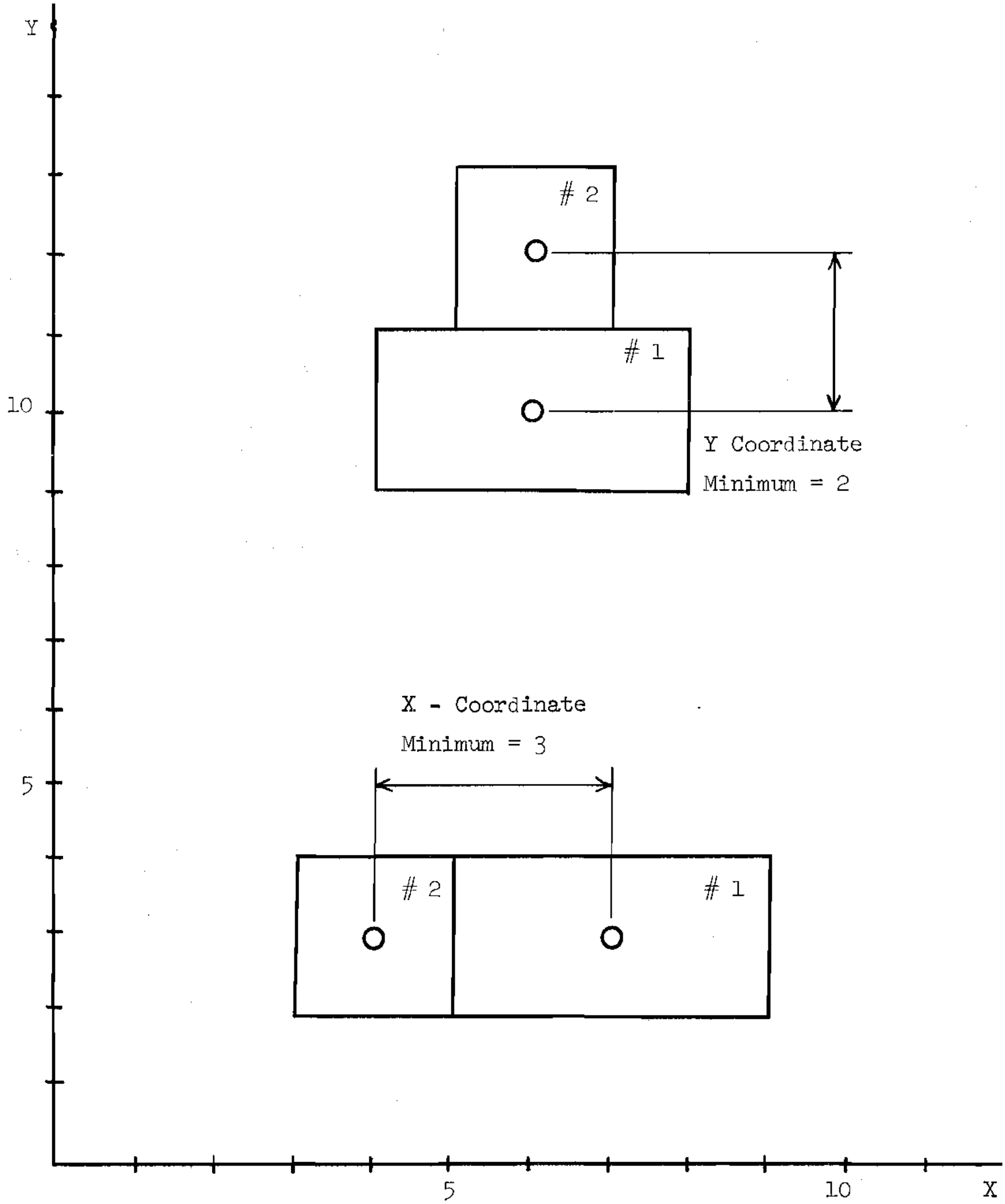

Figure 4. Minimum X and Y Coordinate Distances for Machines 1 and 2 in the Example.

Minimize: $Z = 25\ (U12X + V12X) + 25\ (U12Y + V12Y)$

$+ 5\ (U13X + V13X) + 5\ (U13Y + V13Y)$

$+ 10\ (U23X + V23X) + 10\ (U23Y + V23Y + 850$

Subject to:

$$\text{and/or} \left\{ \begin{array}{l} U12X + V12X \geq 3 \\ U12Y + V12Y \geq 2 \end{array} \right.$$

$$\text{and/or} \left\{ \begin{array}{l} U13X + V13X \geq 4 \\ U13Y + V13Y \geq 3 \end{array} \right.$$

$$\text{and/or} \left\{ \begin{array}{l} U23X + V23X \geq 3 \\ U23Y + V23Y \geq 3 \end{array} \right.$$

This presents m(m-1) constraints where "m" is the number of facilities to be located.

However the addition of U's to V's causes the necessity of adding two new types of contraints: "physical location" constraints and "U-V definitional" constraints. Since the U's and V's give only distances and directions between machines, physical location constraints insure that the solution yields actual physical (or coordinate) locations. These constraints may be arrived at as follows:

$V12X = X1 - X2$ or 0 by definition

$-V12X = X2 - X1$ or 0 multiply above by -1

$$U12X = X2 - X1 \text{ or } 0 \quad \text{by definition}$$

Since the right side of the second and third equations must be X2 - X1 for one equation, and zero for the other, the two may be added, giving:

$$U12X - V12X = X2 - X1.$$

Similarly

$$U12Y - V12Y = Y2 - Y1$$

$$U13X - V13X = X3 - X1$$

$$U13Y - V13Y = Y3 - Y1$$

$$U23X - V23X = X3 - X2$$

$$U23Y - V23Y = Y3 - Y2$$

This adds two more constraints per machine pair giving $2\ m(m - 1)$ total constraints. The above constraints insure that the solution yields actual coordinate locations for each machine, precluding the possibility of all machines being located as close as possible to each other with no regard to machines in between. This prevents a solution such as:

$$U12X = 0,$$

$$V12X = 3,$$

$$U13X = 0,$$

$$V13X = 4,$$

$$U23X = 0,$$

$$V23X = 3$$

and all U-V variables ending in "Y" equal to zero which satisfies the U-V modified non-overlap constraints but is physically impossible.

In addition to the physical location constraints, it is necessary to completely describe the definitions of the U and V variables in terms of constraints. Since this has already been partially done by the physical location constraints, it is only necessary to insure that either U or V will be zero for each machine pair in each coordinate direction. If this condition holds the physical location constraints define the other half of the U-V pair.

Thus if $U12X = 0$ and $U12X - V12X = X2 - X1$ then

$$V12X = X1 - X2 \text{ as intended.}$$

Unfortunately the constraint that U or V must equal zero can not be written in linear form — it is necessary to write the constraints as $U \cdot V = 0$. For the example

$$U12X \cdot V12X = 0,$$

$$U12Y \cdot V12Y = 0,$$

$$U13X \cdot V13X = 0,$$

$$U13Y \cdot V13Y = 0,$$

$$U23X \cdot V23X = 0,$$

$$U23Y \cdot V23Y = 0.$$

This adds m (m-1) more constraints to the general case, giving a total of 3 m (m-1) constraints. The treatment of the U-V definitional constraints will be given in Chapter V.

Finally to complete the model, the "and/or" restrictions on the non-overlap constraints must be removed. This may be done by defining a variable "D" which may take on only values of one or zero (the Kronecker delta). This variable may be incorporated in the non-overlap constraints for the example in the following manner:

$$U12X + V12X \geq 3 - 3\ D12, \tag{1}$$

$$U12Y + V12Y \geq 2 - 2(1 - D12). \tag{2}$$

If D12 = 1, the first equation (1) becomes $U12X + V12X \geq 0$ which gives no additional information since the left side represents an absolute value which is obviously greater than zero. Thus, this equation is in effect nullified. At the same time for D12 = 1, the second equation (2) is unchanged from its original form, that is $U12Y + V12Y \geq 2$. However, the fact that the first equation (1) is nullified does not keep it from holding. It nullification means only that it does not necessarily hold. Also the reverse is true when D12 = 0; equation (1) holds, and equation (2) is nullified.

The only remaining problem is to force the D's to take on the integer values of one or zero. This may be done by inserting another set of constraints of the form $D \leq 1$ (such as $D12 \leq 1$) and by using mixed integer programming[2] to force the D's to take on integer values. This combined with the non-negativity restraint imposed on all variables by the simplex algorithm will cause the D's to take on only values of zero or one. The remainder of the non-overlap constraints for the example are formulated as follows:

$$U13X + V13X \geq 4 - 4\ D13$$

$$U13Y + V13Y \geq 3 - 3\ (1 - D13)$$

$$D13 \leq 1$$

$$U23X + V23X \geq 3 - 3\ D23$$

$$U23Y + V23Y \geq 3 - 3\ (1 - D23)$$

$$D23 \leq 1$$

---

[2] For an explanation of mixed integer programming, see Ralph E. Gomory, "Outline of an Algorithm for Interger Solutions to Linear Programs."

In general the formulation of the complete model is as follows:

$$\text{Minimize } Z = \sum_{I=1}^{m} \sum_{J=I+1}^{m} [AIJ\ (UIJX + VIJX + UIJY + VIJY) + BIJ]$$

Subject to the following restraints for all unique I-J pairs, I = 1 through m, J = I+1 through m:

$$UIJX + VIJX \geq (LI/2 + LJ/2) - (LI/2 - LJ/2)\ DIJ$$

$$UIJY + VIJY \geq (WI/2 + WJ/2) - (WI/2 + WJ/2)(1 - DIJ)$$

$$DIJ \leq 1 \text{ (Integer Values Only)}$$

$$UIJX - VIJX = XJ - XI$$

$$UIJY - VIJY = YJ - YI$$

$$UIJX \cdot VIJX = 0$$

$$UIJY \cdot VIJY = 0$$

where:

m = the total number of facilities to be located.

AIJ = the total variable material handling cost per unit time per unit distance between departments I & J.

BIJ = the total fixed material handling cost per unit time between departments I & J.

(XI,YI) = the coordinate of the center of department I.

$$\left.\begin{array}{l} UIJX = XJ - XI \text{ if } XJ \geq XI, = 0 \text{ otherwise} \\ VIJX = XI - XJ \text{ if } XI \geq XJ, = 0 \text{ otherwise} \end{array}\right\} \text{likewise for Y.}$$

LI = the length of department I (in the X direction).

WI = the width of department I (in the Y direction).

DIJ = 1 or 0 - the Kronecker delta.

This presents $7\ [\frac{m(m-1)}{2}] = 3.5\ m(m-1)$ constraints.

Thus the total formulation of the example is as follows:

Minimize:

$$Z = 25\ (U12X + V12X) + 25\ (U12Y + V12Y)$$
$$+ 5\ (U13X + V13X) + 5\ (U13Y + V13Y)$$
$$+ 10\ (U23X + V23X) + 10\ (U23Y + V23Y)$$
$$+ 850$$

Subject to:

$$U12X + V12X \geq 3 - 3\ D12$$
$$U12Y + V12Y \geq 2 - 2\ (1 - D12)$$
$$D12 \leq 1$$
$$U13X + V13X \geq 4 - 4\ D13$$
$$U13Y + V13Y \geq 3 - 3\ (1 - D13)$$
$$D13 \leq 1$$
$$U23X + V23X \geq 3 - 3\ D23$$
$$U23Y + V23Y \geq 3 - 3\ (1 - D23)$$
$$D23 \leq 1$$
$$U12X - V12X = X2 - X1$$
$$U12Y - V12Y = Y2 - Y1$$
$$U13X - V13X = X3 - X1$$
$$U13Y - V13Y = Y3 - Y1$$
$$U23X - V23X = X3 - X2$$
$$U23Y - V23Y = Y3 - Y2$$
$$U12X \cdot V12X = 0$$
$$U12Y \cdot V12Y = 0$$
$$U13X \cdot V13X = 0$$
$$U13Y \cdot V13Y = 0$$
$$U23X \cdot V23X = 0$$
$$U23Y \cdot V23Y = 0$$

CHAPTER V

SOLUTION TECHNIQUE

In order to solve the formulation of the model developed, four steps are necessary. They are as follows:

1. Pick an arbitrary machine and assign values to its center coordinates (X1, Y1). This causes the layout to be located on the coordinate axis and all machine coordinates to take on actual values. However, the values assigned must be large enough so that no overlapping of any machines with a coordinate axis is possible, for this would cause a restriction of the layout. This can be done in the example by letting (X1, Y1) = (10, 10). As can be seen from Figure 5, this is a legitimate assignment since no overlapping with either coordinate axis is possible.

2. Solve the model with one departmental coordinate assigned and without the non-linear U-V definitional constraints by use of the simplex method. At the termination of the simplex algorithm one of the following illegal conditions may exist: there exists in the solution a U-V pair neither member of which is zero, which violates the U-V definitional constraints, or the existence in the solution of one or more D's that are not equal to zero or one.

3. If the solution obtained in step 2 yields a U-V pair neither member of which is zero, one member of the pair may be removed from the basis by using the dual simplex algorithm.[3] This may be repeated until

[3] An explanation of this technique may be found in An-min Chung, <u>Linear Programming.</u>

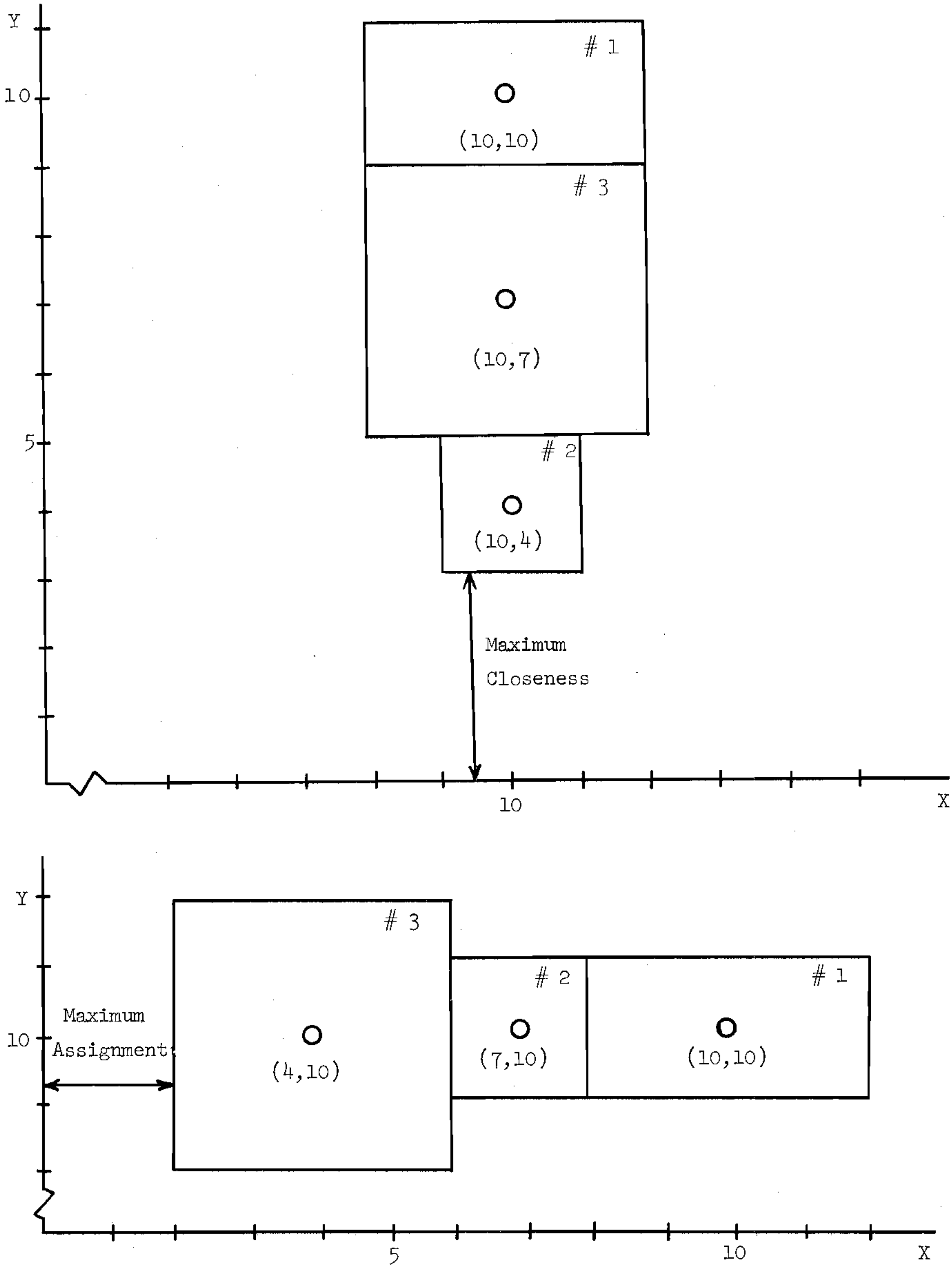

Figure 5. Illustration of the Legitimacy of (X1, Y1) = (10,10) Assignment in the Example.

all U-V pairs have one member equal to zero, which causes the U-V definitional constraints to be satisfied.

4. If one or more D's have values not equal to zero or one, mixed integer programming may be used to force the D's to take on integer values. However, it appears that this is rarely if ever necessary.

Steps 1, 2, and 4 (if necessary) always lead to an optimal solution. Step 3 (if necessary) leads to an optimal solution only when all possible combinations of variables to be removed have been tried and the lowest minimum among the trails has been selected. However, since each trial represents a suboptimal solution the process may be stopped at any time and the best layout obtained to that point may be employed. In general, the efficiency of the overall method cannot be stated since it depends on the outcome of the simplex solution obtained in step 2.

CHAPTER VI

COMPLETING THE LAYOUT

As mentioned previously, various additions and changes must be made to the layout resulting from the mathematical model. All areas not included in the mathematical calculations must be inserted; for example offices, service areas, rest rooms, tool room, water fountains, etc. Any of these may be included in the calculations of the model by quantifying the cost of each in terms of distance from each machine or department. Since this cost is usually in terms of the total average cost per unit time of personnel walking back and forth to the various facilities, it is hard to quantify and therefore may or may not be included in the model. Aisles must also be put in, for which the layout may be expanded or space may be included as a fixed percentage of the area of each machine or department before writing the restraints for the model. In addition, the overall shape of the layout must be formed into a rectangle or some other suitable shape.

CHAPTER VII

ADJUSTING THE MODEL

One of the reasons for choosing a linear programming approach to the plant layout problem was the many adaptations possible. Should a truck dock or other facility necessarily be located on a wall or in a corner of the building, this may be incorporated into the model. All that is necessary is to choose this facility to which to assign coordinate values as described in Chapter V, Step 1, and to place it either on an axis for a wall or at the origin for a corner location. This will restrict the layout as desired since no machine or department may be located out of the first quadrant. Also, should it be desired to restrict the length (L) or width (W) of the layout or both, it may be accomplished by including restraint equations of the form $XI \leq L$ for all I and/or $YJ \leq W$ for all J.

If it is undesirable to locate two areas near each other, this may be incorporated into the model in two ways. If there is a minimum possible separation (S) a restraint of the form $|XI - XJ| + |YI - YJ| \geq S$ may be incorporated. And if the greater the separation the more desirable the situation, a decreasing cost with increasing separation of the form $A-B(|XI - XJ| + |YI - YJ|)$ may be incorporated into the objective function.

Should more than one item travel from one facility to the next and possibly a different item or items travel in return, the materials handling cost may be figured for each and then added to give a total materials handling cost function per time unit for insertion in the objective function. It is also possible to create fixed or semi-fixed machine locations by assigning one or both coordinates of a machine's center, and it is possible to incorporate other distance related or fixed costs in addition to materials handling costs. An example might be the cost of the flow of personnel if the model were used for a non-manufacturing layout such as the arrangement of an office.

CHAPTER VIII

CONCLUSIONS AND RECOMMENDATIONS

As was stated in Chapter III, the hope is that the proposed method overcomes some of the disadvantages of other methods and possibly presents a better overall method of plant layout than is currently in existence. To enumerate some of its advantages:

1. The proposed method is quantitative and may be computerized.
2. It is not restricted in any way except for the orientation of machines, which must be decided at the outset in most cases.
3. It does not require an initial solution as an input as many quantitative methods do.
4. It is based on cost, the primary criteria of a layout.
5. It uses the more realistic rectangular distances.
6. It is generally adaptable to almost any condition imposed by a physical situation.

There are also some disadvantages; specifically, it may or may not yield an optimal solution within a reasonable amount of computer time. This will depend on the individual problem, and the best suboptimal solution arrived at in a reasonable amount of time may or may not be realistic. In addition it is untested.

In a recent article, Hillier and Conner (11) compared the best existing quantitative layout techniques in terms of their efficiency and solutions on sixteen test layouts. It would be beneficial to test the model and solution procedure presented here on the same test layouts to gain some insight into its efficiency and effectiveness. If this test yields realistic results in a reasonable amount of time, then a computer program for the entire method should be written, giving a practical solution to the plant layout problem.

APPENDIX

APPENDIX

The following is a very brief description of the more important theories unsuccessfully tried by the author in exploring a quantitative solution to the plant layout problem. These are presented in the hope that others will not make the same mistakes. The last three (5, 6 and 7) pertain to the linear programming approach contained herein.

1. Extend the work of A.Z. Gani (7). This was given up because no adequate basis could be found for Gani's numerical calculations.
2. Develop a graphical "nearest neighbor" technique. The "nearest neighbor" concept, which is too complex to describe here, was suggested by Gani (7). A graphical takeoff on this method was attempted but the "nearest neighbor" concept proved to be unrealistic.
3. Extend James M. Moore's method as given on pages 130-141 of his Ph.D. dissertation (19). The model proved to be impossible to solve since the necessary non-linear solution techniques were not available.
4. Take a simplex approach to Moore's method as described in No. 3 above. Direct distances were replaced by rectangular distances and other changes were made but the model could not be made linear so that it could be treated by the simplex method.

5. Change equations of the type $|XJ - XI| \geq a$ into "or" pairs. This was attempted as follows:

$$\text{or} \begin{cases} XJ - XI \geq a \\ XI - XJ \geq a \end{cases}$$

However, no solution technique could be found which could be used within the framework of the simplex approach.

6. Insert Linear equations to force either "U" or "V" of a "U - V" pair to be zero. Pairs of many types were tried, such as

$$U12X \geq XI - XJ$$

$$V12X \geq XJ - XI$$

but all proved to be unsuccessful. Thus the non-linear $U \cdot V = 0$ type constraints were added to the model described herein.

7. Insert orientation into the final layout solution. This was attempted by inclusion of equations of the form $XJ \geq XI$, $YJ \geq YI$ but these proved to be unnecessary.

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