Asymptotic Estimates for Perturbed Scaiar Curvature Equation

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ASYMPTOTIC ESTIMATE FOR PERTURBED SCALAR CURVATURE EQUATION.

SAMY SKANDER BAHOURA

ABSTRACT. We consider the equation $\Delta u_\epsilon = V_\epsilon {u_\epsilon}^{(n+2)/(n-2)} + \epsilon W_\epsilon {u_\epsilon}^\alpha$ with $\alpha \in]\dfrac{n}{n-2}, \dfrac{n+2}{n-2}[$ and we give some minimal conditions on ∇V and ∇W to have an uniform estimate for their solutions when $\epsilon \to 0$.

1. INTRODUCTION AND RESULTS.

We denote $\Delta = -\sum_i \partial_{ii}$ the geometric Laplacian on $\mathbb{R}^n, n \geq 3$.

Let us consider on open set Ω of $\mathbb{R}^n, n \geq 3$, the following equation:

$$\Delta u_\epsilon = V_\epsilon {u_\epsilon}^{(n+2)/(n-2)} + \epsilon W_\epsilon {u_\epsilon}^\alpha \qquad (E_\epsilon)$$

where V_ϵ and W_ϵ are two regular functions and $\alpha \in]\dfrac{n}{n-2}, \dfrac{n+2}{n-2}[$.

We assume:

$$0 < a \leq V_\epsilon(x) \leq b, \ ||\nabla V_\epsilon||_{L^\infty} \leq A \qquad (C_1)$$

$$0 < c \leq W_\epsilon(x) \leq d, \ ||\nabla W_\epsilon||_{L^\infty} \leq B \qquad (C_2)$$

Problem: Can we have an $\sup \times \inf$ estimate with the minimal conditions (C_1) and (C_2) ?

Note that for $W \equiv 0$, the equation (E_ϵ) is the wellknowen scalar curvature equation on open set of $\mathbb{R}^n$, $n \geq 3$. In this case, there is many results about this equation, see for example [B] and [C-L 1].

When $\Omega = \mathbb{S}_n$ YY. Li, give a flatness condition to have the boundedness of the energy and the existence of the simple blow-up points, see [L1] and [L2].

In [C-L 2], Chen and Lin gave a conterexample of solutions of the scalar curvature equation with unbounded energy. The conditions of Li are minimal in heigh dimension.

Note that, in [C-L 1] and [C-L 3], there is some results concerning Harnack inequalities of type $\sup \times \inf$ with the "Li-flatness" conditions for the following equation:

$$\Delta u = V u^{(n+2)/(n-2)} + g(u)$$

where g is a regular function (at least C^1) such that $g(t)/[t^{(n+2)/(n-2)}]$ is deacrising and tends to 0 when $t \to +\infty$. They extend Li result ([L1]) to any open set of the euclidian space.

We can find in [A], some existence results for the presribed scalar curvature equation.

In our work we have no assumption on the energy. We use the blow-up analysis and the moving-plane method, developped by Gidas-Ni-Nirenberg, see [G-N-N]. This method was used by different authors to have a priori estimates, look for example, [B], [B-L-S] (in dimension 2), [C-L 1], [C-L 3], [L 1] and [L 2].

We set $\delta = [(n+2) - \alpha(n-2)]/2$, $\delta \in]0, 1[$. We have:

Theorem 1. *For all* $a, b, c, d, A, B > 0$, *for all* $\alpha \in]\frac{n}{n-2}, \frac{n+2}{n-2}[$ *and all compact set* K *of* Ω, *there is a positive constant* $c = c(a, b, c, d, A, B, \alpha, K, \Omega, n)$ *such that:*

$$\epsilon^{(n-2)/2(1-\delta)} (\sup_K u_\epsilon)^{1/3} \times \inf_\Omega u_\epsilon \leq c$$

for all u_ϵ *solution of* (E_ϵ) *with* V_ϵ *and* W_ϵ *satisfying the conditions* (C_1) *and* (C_2).

Now, we suppose that V_ϵ satisfies:

$$0 < a \leq V_\epsilon(x) \leq b \ \text{ and } \ ||\nabla V_\epsilon||_{L^\infty(\Omega)} \leq k\epsilon \qquad (C_3)$$

We have:

Theorem 2. *For all* $a, b, c, d, k, B > 0$, *for all* $\alpha \in]\frac{n}{n-2}, \frac{n+2}{n-2}[$ *and all compact set* K *of* Ω, *there is a positive constant* $c = c(a, b, c, d, k, B, \alpha, K, \Omega, n)$ *such that:*

$$\sup_K u_\epsilon \times \inf_\Omega u_\epsilon \leq c$$

for all u_ϵ *solution of* (E_ϵ) *with* V_ϵ *and* W_ϵ *satisfying the conditions* (C_3) *and* (C_2).

Note that in [B], we have some results as the previous but for prescribed scalar curvature equation with subcritical exponent tending to the critical. Here, we have a $\sup \times \inf$ inequality for the scalar curvature equation, with critical exponent, perturbed by a nonlinear term. We can see the influence of this non-linear term.

2. PROOFS OF THE THEOREMS.

Proof of the theorem 1.

Without loss of generality, we suppose $\Omega = B_1$ the unit ball of $\mathbb{R}^n$. We want to prove an a priori estimate around 0. We can also suppose $\epsilon \to 0$, the case $\epsilon \not\to 0$ is solved in [B].

Let (u_i) and (V_i) be a sequences of functions on Ω such that:

$$\Delta u_i = V_i {u_i}^{(n+2)/(n-2)} + \epsilon_i W_i u_i^\alpha, \ u_i > 0,$$

with $0 < a \leq V_i(x) \leq b$, $0 < a \leq W_i(x) \leq d$, $||V_i||_{L^\infty} \leq A$ and $||W_i||_{L^\infty} \leq B$.

We argue by contradiction and we suppose that the $\sup \times \inf$ is not bounded.

We have:

$\forall \ c, R > 0 \ \exists \ u_{c,R}$ solution of (E_1) such that:

$$\epsilon^{(n-2)/2(1-\delta)} R^{n-2} (\sup_{B(0,R)} u_{\epsilon,c,R})^{1/3} \times \inf_\Omega u_{\epsilon,c,R} \geq c, \qquad (H)$$

Proposition :*(blow-up analysis)*

There is a sequence of points $(y_i)_i$, $y_i \to 0$ and two sequences of positive real numbers $(l_i)_i, (L_i)_i$, $l_i \to 0$, $L_i \to +\infty$, such that if we set $v_i(y) = \frac{u_i(y + y_i)}{u_i(y_i)}$, we have:

$$0 < v_i(y) \leq \beta_i \leq 2^{(n-2)/2}, \ \beta_i \to 1.$$

$$v_i(y) \to \left(\frac{1}{1 + |y|^2} \right)^{(n-2)/2}, \text{ uniformly on all compact set of } \mathbb{R}^n.$$

$$l_i^{(n-2)/2} {\epsilon_i}^{(n-2)/2(1-\delta)} [u_i(y_i)]^{1/3} \times \inf_{B_1} u_i \to +\infty,$$

Proof of the proposition:

We use the hypothesis (H), we take two sequences $R_i > 0$, $R_i \to 0$ and $c_i \to +\infty$, such that,

$$\epsilon_i{}^{(n-2)/2(1-\delta)} R_i{}^{(n-2)} (\sup_{B(0,R_i)} u_i)^{1/3} \times \inf_{B_1} u_i \geq c_i \to +\infty,$$

Let $x_i \in B(x_0, R_i)$ be a point such that $\sup_{B(0,R_i)} u_i = u_i(x_i)$ and $s_i(x) = (R_i - |x - x_i|)^{(n-2)/2} u_i(x), x \in B(x_i, R_i)$. Then, $x_i \to 0$.

We have:

$$\max_{B(x_i,R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i{}^{(n-2)/2} u_i(x_i) \geq \sqrt{c_i} \to +\infty.$$

We set:

$$l_i = R_i - |y_i - x_i|, \ \bar{u}_i(y) = u_i(y_i + y), \ v_i(z) = \frac{u_i[y_i + \left(z/[u_i(y_i)]^{2/(n-2)}\right)]}{u_i(y_i)}.$$

Clearly we have, $y_i \to x_0$. We also obtain:

$$L_i = \frac{l_i}{(c_i)^{1/2(n-2)}} [u_i(y_i)]^{2/(n-2)} = \frac{[s_i(y_i)]^{2/(n-2)}}{c_i^{1/2(n-2)}} \geq \frac{c_i^{1/(n-2)}}{c_i^{1/2(n-2)}} = c_i^{1/2(n-2)} \to +\infty.$$

If $|z| \leq L_i$, then $y = [y_i + z/[u_i(y_i)]^{2/(n-2)}] \in B(y_i, \delta_i l_i)$ with $\delta_i = \dfrac{1}{(c_i)^{1/2(n-2)}}$ and $|y - y_i| < R_i - |y_i - x_i|$, thus, $|y - x_i| < R_i$ and, $s_i(y) \leq s_i(y_i)$. We can write:

$$u_i(y)(R_i - |y - y_i|)^{(n-2)/2} \leq u_i(y_i)(l_i)^{(n-2)/2}.$$

But, $|y - y_i| \leq \delta_i l_i$, $R_i > l_i$ and $R_i - |y - y_i| \geq R_i - \delta_i l_i > l_i - \delta_i l_i = l_i(1 - \delta_i)$. We obtain,

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \left[\frac{l_i}{l_i(1-\delta_i)}\right]^{(n-2)/2} \leq 2^{(n-2)/2}.$$

We set, $\beta_i = \left(\dfrac{1}{1-\delta_i}\right)^{(n-2)/2}$, clearly, we have, $\beta_i \to 1$.

The function v_i satisfies:

$$\Delta v_i = \tilde{V}_i v_i{}^{(n+2)/(n-2)} + \epsilon_i \tilde{W}_i \frac{v_i^{n/(n-2)}}{[u_i(y_i)]^{[(n+2)/(n-2)]-\alpha}}$$

where, $\tilde{V}_i(y) = V_i \left[y + y/[u_i(y_i)]^{2/(n-2)}\right]$ and $\tilde{W}_i(y) = W_i \left[y + y/[u_i(y_i)]^{2/(n-2)}\right]$.

Without loss of generality, we can suppose that $\tilde{V}_i \to V(0) = n(n-2)$.

We use the elliptic estimates, Ascoli and Ladyzenskaya theorems to have the uniform convergence of (v_i) to v on compact set of $\mathbb{R}^n$. The function v satisfies:

$$\Delta v = n(n-2) v^{N-1}, \ v(0) = 1, \ 0 \leq v \leq 1 \leq 2^{(n-2)/2},$$

By the maximum principle, we have $v > 0$ on $\mathbb{R}^n$. If we use Caffarelli-Gidas-Spruck result, (see [C-G-S]), we obtain, $v(y) = \left(\dfrac{1}{1+|y|^2}\right)^{(n-2)/2}$. We have the same properties that in [B].

<u>Polar Coordinates</u> *(Moving-Plane method)*

Now, we must use the same method than in the Theorem 1 of [B]. We will use the moving-plane method.

We must prove the lemma 2 of [B].

We set $t \in]-\infty, -\log 2]$ and $\theta \in \mathbb{S}_{n-1}$:

$w_i(t,\theta)=e^{(n-2)t/2}u_i(y_i+e^t\theta),\;\; \bar V_i(t,\theta)=V_i(y_i+e^t\theta)$ and $\bar W_i(t,\theta)=W_i(y_i+e^t\theta)$.

We consider the following operator $L=\partial_{tt}-\Delta_\sigma-\dfrac{(n-2)^2}{4}$, with Δ_σ the Laplace-Baltrami operator on $\mathbb{S}_{n-1}$.

The function w_i is solution of:

$$-Lw_i=\bar V_i {w_i}^{N-1}+\epsilon_i e^{[(n+2)-(n-2)\alpha]t/2}\bar W_i {w_i}^{\alpha}.$$

For $\lambda\leq 0$ we set :

$t^\lambda=2\lambda-t$ $w_i^\lambda(t,\theta)=w_i(t^\lambda,\theta)$, $\bar V_i^\lambda(t,\theta)=\bar V_i(t^\lambda,\theta)$ et $\bar W_i^\lambda(t,\theta)=\bar W_i(t^\lambda,\theta)$.

Remark: Here we work on $[\lambda,t_i]\times\mathbb{S}_{n-1}$, with $\lambda\leq -\dfrac{2}{n-2}\log u_i(y_i)+2$ and $t_i\leq \log\sqrt{l_i}$, where l_i is chooses as in the proposition.

First, like in [B], we have the following lemma:

Lemma 1:

Let A_λ be the following property:

$A_\lambda=\{\lambda\leq 0,\ \exists\ (t_\lambda,\theta_\lambda)\in]\lambda,t_i]\times\mathbb{S}_{n-1},\ \bar w_i^\lambda(t_\lambda,\theta_\lambda)-\bar w_i(t_\lambda,\theta_\lambda)\geq 0\}$.

Then, there is $\nu\leq 0$, such that for $\lambda\leq\nu$, A_λ is not true.

Like in the proof of the Theorem 1 of [B], we want to prove the following lemma:

Lemma 2:

For $\lambda\leq 0$ we have :

$$ {w_i}^\lambda-w_i\leq 0\Rightarrow -L({w_i}^\lambda-w_i)\leq 0,$$

on $]\lambda,t_i]\times\mathbb{S}_{n-1}$.

Like in [B], we have:

A useful point:

$\xi_i=\sup\{\lambda\leq\bar\lambda_i=2+\log\eta_i, {w_i}^\lambda-w_i<0$, on $]\lambda,t_i]\times\mathbb{S}_{n-1}\}$. The real ξ_i exists.

First, we have:

$$w_i(2\xi_i-t,\theta)=w_i[(\xi_i-t+\xi_i-\log\eta_i-2)+(\log\eta_i+2)],$$

the definition of w_i and the fact that, $\xi_i\leq t$, we obtain:

$$w_i(2\xi_i-t,\theta)=e^{[(n-2)(\xi_i-t+\xi_i-\log\eta_i-2)]/2}e^{n-2}v_i[\theta e^2e^{(\xi_i-t)+(\xi_i-\log\eta_i-2)}]\leq 2^{(n-2)/2}e^{n-2}=\bar c.$$

Proof of the Lemma 2:

We know that:

$$-L(w_i^{\xi_i}-w_i)=[\bar V_i^{\xi_i}(w_i^{\xi_i})^{N-1}-\bar V_i {w_i}^{N-1}]+\epsilon_i[e^{\delta t^{\xi_i}}\bar W_i^{\xi_i}(w_i^{\xi_i})^\alpha-e^{\delta t}\bar W_i {w_i}^\alpha],$$

with $\delta=[(n+2)-(n-2)\alpha]/2$.

We denote by Z_1 and Z_2 the following terms:

$$Z_1=(\bar V_i^{\xi_i}-\bar V_i)(w_i^{\xi_i})^{N-1}+\bar V_i[(w_i^{\xi_i})^{N-1}-{w_i}^{N-1}],$$

and

$$Z_2 = \epsilon_i(\bar W_i^{\xi_i} - \bar W_i)(w_i^{\xi_i})^\alpha e^{\delta t^{\xi_i}} + \epsilon_i e^{\delta t^{\xi_i}} \bar W_i[(w_i^{\xi_i})^\alpha - {w_i}^\alpha] + \epsilon_i \bar W_i {w_i}^\alpha (e^{\delta t^{\xi_i}} - e^{\delta t}).$$

But, using the same method as in the proof of the theorem 1 of [B], we have:

$${w_i}^{\xi_i} \leq w_i \;\text{ et }\; w_i^{\xi_i}(t,\theta) \leq \bar c \;\text{ pour tout }\; (t,\theta) \in [\xi_i, \log 2] \times \mathbb{S}_{n-1},$$

where $\bar c$ is a positive constantnot depending on i for $\xi_i \leq \log \eta_i + 2$;

$$|\bar V_i^{\xi_i} - \bar V_i| \leq A(e^t - e^{t^{\xi_i}}) \;\text{ et }\; |\bar W_i^{\xi_i} - \bar W_i| \leq B(e^t - e^{t^{\xi_i}}),$$

Then,

$$Z_1 \leq A\,(w_i^{\xi_i})^{N-1}\,(e^t - e^{t^{\xi_i}}) \;\text{ et }\; Z_2 \leq \epsilon_i B\,((w_i^{\xi_i})^\alpha\,(e^t - e^{t^{\xi_i}}) + \epsilon_i c\,(w_i^{\xi_i})^\alpha \times (e^{\delta t^{\xi_i}} - e^{\delta t}).$$

and,

$$-L(w_i^{\xi_i} - w_i) \leq (w_i^{\xi_i})^\alpha[(A\,{w_i^{\xi_i}}^{N-1-\alpha} + \epsilon_i B)\,(e^t - e^{t^{\xi_i}}) + \epsilon_i c\;(e^{\delta t^{\xi_i}} - e^{\delta t})].$$

But, $w_i^{\xi_i} \leq \bar c$, we obtain:

$$-L(w_i^{\xi_i} - w_i) \leq (w_i^{\xi_i})^\alpha[(A\bar c^{N-1-\alpha} + \epsilon_i B)\,(e^t - e^{t^{\xi_i}}) + \epsilon_i c\;(e^{\delta t^{\xi_i}} - e^{\delta t})].\;\; (1)$$

We must see the sign of:

$$\bar Z = [(A\bar c^{N-1-\alpha} + \epsilon_i B)\,(e^t - e^{t^{\xi_i}}) + \epsilon_i c\;(e^{\delta t^{\xi_i}} - e^{\delta t})].$$

But $\alpha \in]\dfrac{n}{n-2}, \dfrac{n+2}{n-2}[$, $\delta = \dfrac{n+2-(n-2)\alpha}{2} \in]0,1[$.

For $t \leq t_i < 0$, we have:

$$e^t \leq e^{(1-\delta)t_i} e^{\delta t}, \;\text{ for all }\; t \leq t_i.$$

and the fact that $t^{\xi_i} \leq t$ ($\xi_i \leq t$), by integration of the previous two members, we obtain:

$$e^t - e^{t^{\xi_i}} \leq \frac{e^{(1-\delta)t_i}}{\delta}(e^{\delta t} - e^{\delta t^{\xi_i}}), \;\text{ for all }\; t \leq t_i,$$

We can write:

$$(e^{\delta t^{\xi_i}} - e^{\delta t}) \leq \frac{\delta}{e^{(1-\delta)t_i}}\,(e^{t^{\xi_i}} - e^t).$$

Then,

$$-L(w_i^{\xi_i} - w_i) \leq (w_i^{\xi_i})^\alpha[-\frac{\epsilon_i \delta\, c}{e^{(1-\delta)t_i}} + A\,\bar c^{N-1-\alpha} + \epsilon_i B](e^t - e^{t^{\xi_i}}).$$

The term $\dfrac{\epsilon_i \delta\, c}{e^{(1-\delta)t_i}} - A\,\bar c^{N-1-\alpha} - \epsilon_i B$ is positive if:

$$\epsilon_i e^{-(1-\delta)t_i} \to +\infty,$$

then,

$$\epsilon_i^{(n-2)/2(1-\delta)} e^{-(n-2)/2t_i} \to +\infty.$$

If we take, $t_i = -\dfrac{2}{3(n-2)} \log u_i(y_i)$, we have:

$$\epsilon_i^{(n-2)/2(1-\delta)} [u_i(y_i)]^{1/3} \to +\infty.$$

It is given by our Hypothesis in the proposition.

But the Hopf Maximum principle, gives:

$$\min_{\theta \in \mathbb{S}_{n-1}} w_i(t_i,\theta) \leq \max_{\theta \in \mathbb{S}_{n-1}} w_i(2\xi_i - t_i, \theta),$$

then,

$$e^{(n-2)t_i}u_i(y_i)\min_{B_2(0)} u_i \leq c,$$

and,

$$[u_i(y_i)]^{1/3}\min_{B_2(0)} u_i \leq c,$$

Contradiction.

Proof of the Theorem 2.

The proof is similar than the proof of the theorem 1. Only the end of the proof is different.

Step 1: The blow-up analysis give:

There is a sequence of points $(y_i)_i$, $y_i \to 0$ and two sequences of positive real numbers $(l_i)_i, (L_i)_i$, $l_i \to 0$, $L_i \to +\infty$, such that if we set $v_i(y) = \dfrac{u_i(y+y_i)}{u_i(y_i)}$, we have:

$$0 < v_i(y) \leq \beta_i \leq 2^{(n-2)/2},\ \beta_i \to 1.$$

$$v_i(y) \to \left(\frac{1}{1+|y|^2}\right)^{(n-2)/2},\ \text{uniformly on all compact set of } \mathbb{R}^n.$$

$$l_i^{(n-2)/2}u_i(y_i)\times \inf_{B_1} u_i \to +\infty,$$

Step 2: Application of the Hopf maximum principle.

We have the same notation that in the proof of the theorem 1. First, we take $t_i = \sqrt{l_i}$ as in the Step 1 and we look to the end of the proof of the theorem 1. We replace A by $k\epsilon_i$. We want to proof that:

$$w_i{}^{\lambda} - w_i \leq 0 \Rightarrow -L(w_i{}^{\lambda} - w_i) \leq 0,$$

on $]\xi_i, t_i] \times \mathbb{S}_{n-1}$. We have the same defintion for ξ_i (as in the proof of the theorem 1).

For $t \leq t_i < 0$, we have:

$$e^t \leq e^{(1-\delta)t_i}e^{\delta t},\ \text{ for all } t \leq t_i.$$

and the fact that $t^{\xi_i} \leq t$ ($\xi_i \leq t$), by integration of the previous two members, we obtain:

$$e^t - e^{t^{\xi_i}} \leq \frac{e^{(1-\delta)t_i}}{\delta}(e^{\delta t} - e^{\delta t^{\xi_i}}),\ \text{ for all } t \leq t_i,$$

We can write:

$$(e^{\delta t^{\xi_i}} - e^{\delta t}) \leq \frac{\delta}{e^{(1-\delta)t_i}}\,(e^{t^{\xi_i}} - e^t).$$

Then,

$$-L(w_i^{\xi_i} - w_i) \leq (w_i^{\xi_i})^{\alpha}[-\frac{\epsilon_i \delta\, c}{e^{(1-\delta)t_i}} + k\epsilon_i\,\bar{c}^{N-1-\alpha} + \epsilon_i B](e^t - e^{t^{\xi_i}}).$$

The term $\dfrac{\delta\, c}{e^{(1-\delta)t_i}} - k\,\bar{c}^{N-1-\alpha} - B$ is positive because $t_i \to -\infty$ and $\delta \in]0,1[$.

But the Hopf Maximum principle, gives:

$$\min_{\theta \in \mathbb{S}_{n-1}} w_i(t_i, \theta) \leq \max_{\theta \in \mathbb{S}_{n-1}} w_i(2\xi_i - t_i, \theta),$$

then,

$$e^{(n-2)t_i}u_i(y_i)\min_{B_2(0)} u_i \leq c,$$

and,

$$l_i^{(n-2)/2} u_i(y_i) \min_{B_2(0)} u_i \leq c,$$

Contradiction with the step 1.

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