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Integrable geodesic flows and Multi-Centre versus Bianchi A metrics

Galliano VALENT [†] [*]

Hamed Ben YAHIA [†]

[†] *Laboratoire de Physique Théorique et des Hautes Energies*
CNRS, Unité associée URA 280
2 Place Jussieu, F-75251 Paris Cedex 05, France

[*] *Département de Mathématiques*
UFR Sciences-Luminy
Case 901 163 Avenue de Luminy
13258 Marseille Cedex 9, France

Abstract

It is shown that most, but not all, of the four dimensional metrics in the Multi-Centre family with integrable geodesic flow may be recognized as belonging to spatially homogeneous Bianchi type A metrics. We show that any diagonal bi-axial Bianchi II metric has an integrable geodesic flow, and that the simplest hyperkähler metric in this family displays a finite dimensional W-algebra for its observables. Our analysis puts also to light non-diagonal Bianchi VI_0 and VII_0 metrics which seem to be new. We conclude by showing that the elliptic coordinates advocated in the literature do not separate the Hamilton-Jacobi equation for the tri-axial Bianchi IX metric.

1 Introduction

The study of the integrable geodesic flows of the Multi-Centre metrics, initiated in [16], has been worked out completely in [27]. Let us recall that this family of metrics has the local form

$$g = \frac{1}{V}\,(dt + \Theta)^2 + V\,\gamma_0, \qquad \star d\Theta = \pm\, dV, \qquad (1)$$

where $\gamma_0 = dX^2 + dY^2 + dZ^2$ is the flat metric and $V(X, Y, Z)$ is any harmonic function in this flat space.

These metrics have self-dual Riemann tensor and are therefore Ricci-flat: they realize an exact linearization of euclidean empty space Einstein equations, each four dimensional euclidean metric being "parametrized" by the harmonic function V. The geodesic flow is Liouville integrable only for very special potentials V as proved in [16] and [27]. All these cases correspond to metrics with two commuting Killing vectors. It is therefore interesting to ascertain for what particular potentials the infinitesimal isometries algebra increases to three or more Killing vectors. For three Killing vectors the situation is quite interesting since the corresponding metrics could be related with the so-called Bianchi "spatially homogeneous" metrics (most popular in the cosmology field) which are co-homogeneity one metrics, with a 3-dimensional "space" acted on homogeneously by the Bianchi isometries. These were studied in [20], [21] and [2].

Even if for some particular (Riemann self-dual) Bianchi metrics, their Multi-Centre form is known, some items were still missing. It is the aim of this article to give a complete description of this correspondence and, as a consequence of the results in [16] and [27], to ascertain which Bianchi A self-dual metrics do have an integrable geodesic flow.

Among these Bianchi A metrics with integrable geodesic flow, the Bianchi II exhibits a quite remarkable algebraic structure: for any diagonal and bi-axial metric the geodesic flow is integrable! For the simplest metric, with anti-self-dual spin connection, the set of conserved quantities quadratic in the momenta (induced by Killing-Stäckel tensors) generate a finite dimensional W-algebra with respect to the Poisson bracket which seems to appear for the first time in problems related to General Relativity.

The structure of the article is the following: in Section 2 we have gathered some background material and then, in Section 3 we begin with Bianchi II and display in Section 4 its finite dimensional W-algebra for the conserved quantities. In Section 5 we consider other Bianchi II geometries which all share geodesic integrability. In Section 6 we discuss Bianchi VI_0 and Bianchi VII_0. Another integrable metric is shown to give rise, in Section 7, to a non-diagonal Bianchi VII_0 metric, for which we derive its Bianchi VI_0 partner. After a quick review, in section 8, of the Bianchi VIII and IX metrics we show that the elliptic coordinates are not separating ones for the Hamilton-Jacobi equation on Bianchi IX. After a short discussion of the quantum integrability aspects within minimal quantization in Section 9, we present some concluding remarks.

2 Background material

We follow the more modern classification of Bianchi Lie algebras given in [10]. The Bianchi A Lie algebras have 3 generators which we denote by $\mathcal{L}_i$, $i = 1, 2, 3$ with commutation relations

$$[\mathcal{L}_1, \mathcal{L}_2] = n_3\,\mathcal{L}_3, \qquad [\mathcal{L}_2, \mathcal{L}_3] = n_1\,\mathcal{L}_1, \qquad [\mathcal{L}_3, \mathcal{L}_1] = n_2\,\mathcal{L}_2, \qquad (2)$$

with the invariant 1-forms σ_i, $i = 1, 2, 3$ such that

$$d\sigma_1 = n_1 \sigma_2 \wedge \sigma_3, \qquad d\sigma_2 = n_2 \sigma_3 \wedge \sigma_1, \qquad d\sigma_3 = n_3 \sigma_1 \wedge \sigma_2, \qquad \mathcal{L}_i \sigma_j = 0. \quad (3)$$

For type A algebras the structure coefficients are given by the triplets (n_1, n_2, n_3):

$$\text{type } I \;\rightarrow\; (0,0,0), \qquad \text{type } II \;\rightarrow\; (1,0,0), \qquad \text{type } VI_0 \;\rightarrow\; (1,-1,0),$$

$$\text{type } VII_0 \;\rightarrow\; (1,1,0), \qquad \text{type } VIII \;\rightarrow\; (1,1,-1), \qquad \text{type } IX \;\rightarrow\; (1,1,1).$$

The type I, which is fully abelian, leads only to the flat metric and will be skipped. In this paper we will consider *diagonal* spatially homogeneous metrics of the form

$$g = \alpha^2 \, ds^2 + \beta^2 \, \sigma_1^2 + \gamma^2 \, \sigma_2^2 + \delta^2 \, \sigma_3^2, \tag{4}$$

where α, β, γ and δ depend solely on s, and our task will be to bring them to the Multi-Centre form.

Just to settle our notations we will use the natural vierbein

$$e_0 = \alpha \, ds, \qquad e_1 = \beta \, \sigma_1, \qquad e_2 = \gamma \, \sigma_2, \qquad e_3 = \delta \, \sigma_3, \tag{5}$$

and the SD two forms

$$F_i^{\pm} = e_0 \wedge e_i \pm \frac{1}{2} \, \epsilon_{ijk} \, e_j \wedge e_k. \tag{6}$$

Similarly the self-dual components of the spin-connection are defined by

$$\omega_i^{\pm} = \omega_{0i} \pm \frac{1}{2} \, \epsilon_{ijk} \, \omega_{jk}, \tag{7}$$

and similarly for the SD curvature components. The matrices describing the curvature in the self-dual basis are then A and C, which are symmetric, and B. They are defined by

$$R_i^+ \;=\; A_{ij} \, F_j^+ \;+\; B_{ij} \, F_j^-, \qquad R_i^- \;=\; (B^t)_{ij} \, F_j^+ \;+\; C_{ij} \, F_j^-. \tag{8}$$

The self-dual components of the Weyl tensor are obtained from

$$W^- = A - \frac{1}{3} \, (\mathrm{tr}\, A) \, \mathbb{I}, \qquad W^+ = C - \frac{1}{3} \, (\mathrm{tr}\, C) \, \mathbb{I}. \tag{9}$$

As observed in [20] there are two different ways of being Riemann self-dual:

1. The spin connection is itself antiself-dual (ASD), i.e.

$$\omega_i^+ = 0, \quad i = 1, 2, 3 \qquad \implies \qquad R^+ = W^+ = 0. \tag{10}$$

2. The curvature itself is ASD but not the spin connection. In this case, since the metric is diagonal, we can write the spin connection as

$$\omega_1^+ = \lambda_1(s) \, \sigma_1, \qquad \omega_2^+ = \lambda_2(s) \, \sigma_2, \qquad \omega_3^+ = \lambda_3(s) \, \sigma_3. \tag{11}$$

Then imposing $R_i^+ = 0$ shows that the functions λ_i are independent of s and are algebraically constrained by

$$n_1 \lambda_1 = \lambda_2 \, \lambda_3, \qquad n_2 \lambda_2 = \lambda_3 \, \lambda_1, \qquad n_3 \lambda_3 = \lambda_1 \, \lambda_2. \tag{12}$$

For each metric we will consider successively both cases.

We will use Killing-Yano (K-Y) and Killing-Stäckel (K-S) tensors, for which the reader could consult the references [16] and [27]. The first one contains also many useful information on the Multi-Centre metrics.

Let us conclude by mentioning an interesting result, proved by Hitchin [18]. It allows to compute the cartesian coordinates X, Y, Z, given the tri-holomorphic Killing vector $K = \partial_t$ and the complex structures 2-forms J_i, according to

$$dX = i(K) J_1, \qquad dY = i(K) J_2, \qquad dZ = i(K) J_3. \tag{13}$$

In fact these coordinates are the moment maps of the complex structures under the tri-holomorphic action of the Killing vector ∂_t.

3 Bianchi II metrics

The Bianchi II Lie algebra is generated by the vector fields

$$\mathcal{L}_1 = \partial_t, \qquad \mathcal{L}_2 = \partial_y - z\partial_t, \qquad \mathcal{L}_3 = \partial_z, \tag{14}$$

and the invariant 1-forms (3) are

$$\sigma_1 = dt + ydz, \qquad \sigma_2 = dy, \qquad \sigma_3 = dz. \tag{15}$$

The metric with self-dual connection, given by [20], reads

$$g_{\mathrm{II}} = ms\, ds^2 + \frac{1}{ms}\sigma_1^2 + s(\sigma_2^2 + \sigma_3^2), \qquad m > 0, \quad s > 0. \tag{16}$$

The parameter m is not essential and will be scaled out to 1 from now on. The global properties are not good: there is a curvature singularity at $s = 0$ while infinity is flat as can be seen from the curvature

$$W^+ = A = B = 0, \qquad W^- = C = \frac{1}{s^3}\operatorname{diag}(-2, 1, 1). \tag{17}$$

It is therefore Petrov type D^-.

As a side remark, in [27][p.592] an apparently different metric was given

$$g = \frac{1}{V}(d\tau - \frac{\mathcal{E}}{2}ydx + \frac{\mathcal{E}}{2}xdy)^2 + V(dx^2 + dy^2 + dz^2), \qquad V = v_0 + \mathcal{E}z. \tag{18}$$

By a translation of z we can set $v_0 = 0$ and by a scaling we can take $\mathcal{E} = 1$. Then exchanging the variables x and z and defining $t = -\tau - \frac{1}{2}yz$ brings (18) to the form (16), showing the identity of these two metrics.

The triplet of covariantly constant complex structures is given by

$$J_1 = ds \wedge \sigma_1 + s\,\sigma_2 \wedge \sigma_3, \quad J_2 = s\,ds \wedge \sigma_2 + \sigma_3 \wedge \sigma_1, \quad J_3 = s\,ds \wedge \sigma_3 + \sigma_1 \wedge \sigma_2. \tag{19}$$

There is an extra Killing vector for this metric because the coefficients of σ_2^2 and σ_3^2 are equal. Its generator is

$$\mathcal{L}_4 = y\partial_z - z\partial_y - \frac{1}{2}(y^2 - z^2)\partial_t, \tag{20}$$

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and the full algebra closes under commutation according to

$$[\mathcal{L}_4, \mathcal{L}_1] = 0, \qquad [\mathcal{L}_4, \mathcal{L}_2] = -\mathcal{L}_3, \qquad [\mathcal{L}_4, \mathcal{L}_3] = \mathcal{L}_2. \tag{21}$$

The Killing vectors $\mathcal{L}_i$, $i = 1, 2, 3$ are tri-holomorphic, while $\mathcal{L}_4$ is just holomorphic since it rotates (J_2, J_3) as a doublet. This metric is therefore some Multi-Centre: taking for convenience $\mathcal{L}_1 = \partial_t$ as tri-holomorphic Killing vector, it is trivial to reduce this metric to the form (1) via the identifications:

$$\begin{cases} V = X, & \Theta = Y\, dZ, \\[2mm] X = s, & Y = y, \qquad Z = z. \end{cases} \tag{22}$$

This metric is nothing but the metric written in [27]

$$\frac{1}{V}(dt + my\,dz)^2 + V(dx^2 + dy^2 + dz^2), \qquad V = v_0 + mx.$$

Indeed by a translation of x we can set $v_0 = 0$ and scale out m to 1.

As pointed out in section 2, we may also have a non SD connection. Solving the equations (12) one gets [1]

$$\lambda_1 = \lambda_3 = 0 \quad \& \quad \lambda_2 = \lambda, \qquad \lambda \neq 0$$

where λ is some real constant. This gives rise to the tri-axial Bianchi II metric [20]

$$G_{II} = s e^{-2\lambda s}\left[ds^2 + \sigma_2^2\right] + \frac{1}{s}\,\sigma_1^2 + s\,\sigma_3^2, \qquad \lambda \neq 0. \tag{23}$$

For this metric too $s = 0$ is a curvature singularity. The curvature is Petrov type I:

$$W^+ = R^+ = 0, \qquad W^- = C = \frac{e^{2\lambda s}}{s^3}\,\mathrm{diag}\left(-2 + \lambda s, 1, 1 - \lambda s\right),$$

and only for $\lambda < 0$ is the geometry flat for $s \to +\infty$.

One can check that the complex structures are now

$$\widetilde{J}_1 + i\widetilde{J}_3 = e^{i\lambda y}\left(J_1 + iJ_3\right), \qquad \widetilde{J}_2 = J_2, \tag{24}$$

where the J_i are defined by (19). Due to the tri-axial nature of this metric, the vector field (20) is no longer an isometry of G_{II}. The vector fields $\mathcal{L}_1$ and $\mathcal{L}_3$ are tri-holomorphic, while $\mathcal{L}_2$ is just holomorphic. Since the Killing vector $\mathcal{L}_1$ is still tri-holomorphic, the metric (23) remains a Multi-Centre.

To determine the coordinates X, Y, Z the most convenient procedure is to use Hitchin's result [18] stating that these coordinates are the moment maps of the circle action of the tri-holomorphic vector ∂_t. Taking for it $\partial_t = \mathcal{L}_1$, and using the complex structures (24) the identification with the Multi-Centre form (1) is then easily obtained:

$$\begin{cases} V = -\frac{1}{2\lambda}\ln\left((1 + \lambda X)^2 + \lambda^2 Y^2\right), & \Theta = \frac{1}{\lambda}\arctan\left(\frac{\lambda Y}{1 + \lambda X}\right)\, dZ \\[3mm] X + iY = \frac{1}{\lambda}\left(e^{-\lambda s + i\lambda y} - 1\right), & Z = z. \end{cases} \tag{25}$$

---

[1]The other solution, corresponding to $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda \neq 0$, corresponds to the interchange $\sigma_2 \leftrightarrow \sigma_3$.

The Killing vector $\mathcal{L}_2$, which is translational when acting on the metric (16), acquires a rotational part when acting on the metric (23), according to

$$g_{II} \qquad\qquad\qquad\qquad G_{II}$$

$$\mathcal{L}_1 = \partial_t \qquad\qquad\qquad \mathcal{L}_1 = \partial_t$$

$$\mathcal{L}_2 = \partial_Y - Z\partial_t \quad \longrightarrow \quad \mathcal{L}_2 = \partial_Y - Z\partial_t + \lambda(X\partial_Y - Y\partial_X)$$

$$\mathcal{L}_3 = \partial_Z \qquad\qquad\qquad \mathcal{L}_3 = \partial_Z$$

Let us observe, to conclude this section, that both metrics (16) and (23) are in fact special cases of the most general Ricci-flat Bianchi II metric given by Taub [25]. Its euclidean version is

$$g_T = \frac{1}{X}\,\sigma_1^2 + X\left[e^{as}\sigma_2^2 + e^{bs}\sigma_3^2 + e^{(a+b)s}ds^2\right], \qquad X = \frac{\sinh(\sqrt{ab}\,s)}{\sqrt{ab}}.$$

Taking the $b \to 0$ limit we get the self-dual metric (23) with $\lambda = -a/2$ and $m = 1$.

4   The W-algebra for the observables

Let us now consider the metric g_{II}. Its geodesic flow has for Hamiltonian

$$H = \frac{1}{2s}\left((\Pi_z - y\Pi_t)^2 + s^2\Pi_t^2 + \Pi_s^2 + \Pi_y^2\right). \tag{26}$$

The Poisson bracket induced by the symplectic form $\Omega = d\Pi_i \wedge dx^i$ is

$$\{A, B\} = \frac{\partial A}{\partial \Pi_i}\frac{\partial B}{\partial x^i} - \frac{\partial A}{\partial x^i}\frac{\partial B}{\partial \Pi_i}.$$

The isometry algebra with generators $\{\mathcal{L}_i\}$, $i = 1, 2, 3, 4$ produces four conserved quantities linear in the momenta:

$$K_1 = \Pi_t, \quad K_2 = \Pi_y - z\Pi_t, \quad K_3 = \Pi_z, \quad K_4 = y\Pi_z - z\Pi_y - \frac{1}{2}(y^2 - z^2)\Pi_t. \tag{27}$$

Obviously, their algebra is isomorphic to the isometry algebra (2):

$$\{K_1, K_2\} = 0, \qquad \{K_2, K_3\} = K_1, \qquad \{K_3, K_1\} = 0, \tag{28}$$

and for the extra Killing

$$\{K_4, K_1\} = 0, \qquad \{K_4, K_2\} = -K_3, \qquad \{K_4, K_3\} = K_2. \tag{29}$$

It was proved in [27] that there is a K-Y tensor

$$Y = s\,ds \wedge (-z\,\sigma_2 + y\,\sigma_3) + \sigma_1 \wedge (y\,\sigma_2 + z\,\sigma_3) - 2s^2\,\sigma_2 \wedge \sigma_3. \tag{30}$$

It follows that Y^2 and the symmetrized products of Y with the triplet of complex structures give rise to four K-S tensors. This means that we have a set of four conserved quantities quadratic in the momenta:

$$
\begin{cases}
L_1 = \Pi_y^2 + (\Pi_z - y\Pi_t)^2 \\[2mm]
L_2 = \Pi_s\Pi_y - s\Pi_t(\Pi_z - y\Pi_t) - yH \\[2mm]
L_3 = \Pi_s(\Pi_z - y\Pi_t) + s\Pi_t\Pi_y - zH \\[2mm]
L_4 = sL_1 - yL_2 - zL_3 - \frac{1}{2}(y^2 + z^2)H
\end{cases}
\Rightarrow \; \{H, L_i\} = 0, \; i = 1, 2, 3, 4. \quad (31)
$$

The isometries action on these K-S tensors is [2]

$$
\begin{aligned}
\{K_2, L_2\} &= -H, & \{K_2, L_4\} &= -L_2, \\
\{K_3, L_3\} &= -H, & \{K_3, L_4\} &= -L_3, \\
\{K_4, L_2\} &= -L_3, & \{K_4, L_3\} &= L_2.
\end{aligned}
\quad (32)
$$

The Liouville integrability of the geodesic flow is ensured by the set of observables [3]

$$
K_2 \qquad K_3 \qquad H \qquad L_1,
$$

in involution for the Poisson bracket.

The remaining brackets, bilinear with respect to the $\{L_i\}$, exhibit the nice structure

$$
\begin{aligned}
\{L_1, L_2\} &= -2K_2H + 2K_1L_3 & \{L_2, L_3\} &= 2K_1L_1 \\
\{L_1, L_3\} &= -2K_3H - 2K_1L_2 & \{L_2, L_4\} &= 2K_2L_1 \\
\{L_1, L_4\} &= -2K_2L_2 - 2K_3L_3 & \{L_3, L_4\} &= 2K_3L_1
\end{aligned}
\quad (33)
$$

So we have obtained a new finite W-algebra out of 9 conserved quantities: H, K_i, L_i. If we compare with the superintegrable geodesic flows in the two-dimensional Darboux spaces discussed in [19] we observe that its observable algebra, made out of 3 conserved quantities, closes up with observables which are *quartic* with respect to the momenta, while here the closing occurs with *cubic* quantities.

Finite W-algebras can also be constructed using Poisson reduction [5]. It seems quite unclear whether this method could lead to the W-algebra obtained here.

5   Other Bianchi II metrics

This section is intended to describe some general properties of the metrics of this class, and to give examples with different geometries: Kähler scalar-flat, Einstein with self-dual Weyl tensor and Kähler-Einstein.

---

[2]The omitted brackets are vanishing.

[3]Using (27) one can check that L_1 is indeed irreducible.

5.1 Separation of Hamilton-Jacobi equation

Let us begin with the proof of

Proposition 1 *The geodesic flow of* any *diagonal and bi-axial Bianchi II metric with isometries $\mathcal{L}_i$, $i = 1, \ldots, 4$ is integrable in Liouville sense.*

Proof:
The metric considered in this proposition must have the following form

$$g = A^2(s)\, ds^2 + B^2(s)\, \sigma_1^2 + C^2(s)(\sigma_2^2 + \sigma_3^2). \tag{34}$$

In the sequel we will use the vierbein

$$e_0 = A\, ds, \qquad e_1 = B\, \sigma_1, \qquad e_2 = C\, \sigma_2, \qquad e_3 = C\, \sigma_3. \tag{35}$$

The hamiltonian governing the geodesic flow is

$$2H = \frac{\Pi_s^2}{A^2} + \frac{\Pi_y^2}{C^2} + \frac{(\Pi_z^2 - y\Pi_t)^2}{C^2} + \frac{\Pi_t^2}{B^2}. \tag{36}$$

The Hamilton-Jacobi equation is seen to be

$$\frac{1}{A^2}(\partial_s S)^2 + \frac{1}{C^2}(\partial_y S)^2 + \frac{1}{B^2}(\partial_t S)^2 + \frac{1}{C^2}(\partial_z S - y\partial_t S)^2 = 2E. \tag{37}$$

Defining

$$S = t\Pi_t + z\Pi_z + \lambda(s) + \mu(y), \qquad \Pi_t = q, \quad \Pi_z = J, \tag{38}$$

leads to the separation of variables in the form

$$\left(\frac{d\mu}{dy}\right)^2 + (J - qy)^2 = C^2 \left(2E - \frac{q^2}{B^2} - \frac{1}{A^2}\left(\frac{d\lambda}{ds}\right)^2 \right). \tag{39}$$

The separation constant gives a quadratic conserved quantity

$$L = \Pi_y^2 + (\Pi_z - y\Pi_t)^2, \tag{40}$$

which we already encountered (as L_1) in section 4 for the metric g_{II}. It is easy to ascertain that this conserved quantity cannot be obtained from a quadratic form of the Killing vectors, so we conclude to the integrability of the geodesic flow, with H, Π_t, Π_z, L_1 in involution with respect to the Poisson bracket, and this ends the proof. $\square$

5.2 Killing-Yano versus Killing-Stäckel tensors

The integration of the K-Y and of the K-S equations are quite easy if the corresponding tensors are form invariant under the isometries, and leads to the following:

Proposition 2 *The metric (34) exhibits the K-Y tensor*

$$Y = e_0 \wedge e_1 + \mu(s)\, e_2 \wedge e_3, \qquad \mu = \frac{(C^2)'}{AB} \tag{41}$$

provided that the following relation holds:

$$AB - \mu\, (C^2)' + 2C^2\, \mu' = 0. \tag{42}$$

It exhibits also the high-symmetry K-S tensor

$$S = e_0^2 + (1 + \beta B^2)e_1^2 + (1 + \gamma C^2)(e_2^2 + e_3^2), \tag{43}$$

with two real constants β and γ.

7

5.3 Kähler scalar-flat metric

As explained in [7], [26] it is possible to construct Einstein generalizations with self-dual Weyl tensor. The procedure is the following: one first looks for a Kähler metric

$$g = \frac{ds^2}{f} + f\sigma_1^2 + s(\sigma_2^2 + \sigma_3^2), \qquad \Omega = ds \wedge \sigma_1 + s\sigma_2 \wedge \sigma_3, \tag{44}$$

where $f(s)$ is some free function. Imposing the vanishing of the scalar curvature leads to a self-dual Weyl tensor. This gives for f the very simple equation $sf'' + 2f' = 0$, and so $f(s) = a + b/s$.

In particular, if we take $a = -b = 1$ the resulting metric

$$g_K = \frac{s}{s-1} ds^2 + \frac{s-1}{s} \sigma_1^2 + s(\sigma_2^2 + \sigma_3^2), \quad s > 1, \tag{45}$$

is seen to be complete if t has period 4π, since for $s \sim 1$ its local approximate form is

$$\frac{g}{4} \approx d\rho^2 + \rho^2 \left(\frac{dt}{2}\right)^2 + \left(\frac{dy}{2}\right)^2 + \left(\frac{dz}{2}\right)^2, \qquad \rho = \sqrt{s-1}.$$

The curvature, using the vierbein (35), is Petrov D

$$A = 0, \qquad B = \frac{1}{2s^2} \operatorname{diag}(1,0,0), \qquad C = \frac{(s-2)}{2s^3} \operatorname{diag}(-2,1,1). \tag{46}$$

For this metric the K-Y tensor (41) reduces to the complex structure so its square is trivial, but (43) gives two extra conserved quantities:

$$S_{KSF} = e_0^2 + \left(1 + \beta \frac{s-1}{s}\right) e_1^2 + (1 + \gamma s)(e_2^2 + e_3^2). \tag{47}$$

5.4 Einstein metric with self-dual Weyl tensor

Starting from the metric (45), it is easy to find a conformal factor $\rho(s)$ which transforms the scalar flat Kähler metric into an Einstein one, with self-dual Weyl tensor:

$$g_E = \rho \left(\frac{s}{as+b} ds^2 + \frac{as+b}{s} \sigma_1^2 + s(\sigma_2^2 + \sigma_3^2)\right), \qquad \rho = \frac{3a^3}{2\lambda(as+2b)^2}. \tag{48}$$

For $a = -b = 1$ this metric is seen to be complete. Indeed, taking for variable $r = s-1$ it becomes

$$g_E = \frac{3}{2\lambda(1-r)^2} \left(\frac{r+1}{r} dr^2 + \frac{r}{r+1} \sigma_1^2 + (r+1)(\sigma_2^2 + \sigma_3^2)\right), \qquad 0 < r < 1. \tag{49}$$

Its curvature is Petrov D

$$A = -\frac{\lambda}{3} I, \qquad W^+ = 0, \qquad W^- = -\frac{\lambda}{3}\left(\frac{1-r}{1+r}\right)^3 \operatorname{diag}(-2,1,1). \tag{50}$$

Using (41) we get now the K-Y tensor

$$Y_E = e_0 \wedge e_1 + \mu(s) e_2 \wedge e_3, \qquad \mu(s) = \frac{2b - as}{2b + as}, \tag{51}$$

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with the vierbein

$$e_0 = \sqrt{\frac{s\rho}{as+b}}\, ds, \qquad e_1 = \sqrt{\frac{(as+b)\rho}{s}}\, \sigma_1, \qquad e_2 = \sqrt{s\rho}\, \sigma_2, \qquad e_3 = \sqrt{s\rho}\, \sigma_3.$$

It is now interesting to compare the K-S tensor obtained by squaring the Yano tensor. We get

$$Y_E^2 - g_E = (\mu(s)^2 - 1)(e_2^2 + e_3^2) = -\frac{16\lambda ab}{3a^3}\, s\rho(s)(e_2^2 + e_3^2), \tag{52}$$

which is just a piece of the more general K-S tensor given by (43):

$$S - g_E = \beta\, \frac{as+b}{s}\, \rho(s)\, e_1^2 + \gamma\, s\, \rho(s)(e_2^2 + e_3^2). \tag{53}$$

5.5 Kähler-Einstein metric

There is a last Bianchi II metric, due to Dancer and Strachan [7], which is Kähler-Einstein and can be written [4]:

$$g_{KE} = \frac{ds^2}{\Delta} + \Delta\, \sigma_1^2 + s(\sigma_1^2 + \sigma_2^2), \qquad \Delta = \frac{\delta}{s} - \frac{2\lambda}{3} s^2, \tag{54}$$

with the complex structure

$$J = e_0 \wedge e_1 + e_2 \wedge e_3 = ds \wedge \sigma_1 + s\, \sigma_2 \wedge \sigma_3.$$

The situation is similar to the Kähler scalar-flat metric : the K-Y tensor reduces to the complex structure and the K-S tensor is

$$S_{KE} - g_{KE} == \beta\, \Delta\, e_1^2 + \gamma\, s\, (e_2^2 + e_3^2). \tag{55}$$

6 The Bianchi VI$_0$ and VII$_0$ self-dual metrics

We will consider successively both cases.

6.1 The Bianchi VI$_0$ metrics

One has for Killing vectors

$$\mathcal{L}_1 = \partial_\theta + z\, \partial_y + y\, \partial_z, \qquad \mathcal{L}_2 = \partial_y, \qquad \mathcal{L}_3 = \partial_z. \tag{56}$$

The invariant 1-forms are

$$\sigma_1 = \cosh\theta\, dy - \sinh\theta\, dz, \qquad \sigma_2 = -\sinh\theta\, dy + \cosh\theta\, dz, \qquad \sigma_3 = d\theta. \tag{57}$$

The metric with ASD connection was first given in [20] and writes [5]

$$g_{VI} = c^2 \sin\chi \cos\chi \left[d\chi^2 + \sigma_3^2 \right] + \cot\chi\, \sigma_1^2 + \tan\chi\, \sigma_2^2. \tag{58}$$

---

[4]We have set $s = r^2/4$.

[5]The partner metric obtained by the interchange of the coefficients of σ_1 and σ_2 is not different since it corresponds to the change of coordinate $\chi \to \pi/2 - \chi$.

The positivity of the metric requires $\chi \in]0, \pi/2[$, and both end-points are curvature singularities.

The complex structures are

$$
\begin{cases}
J_1 = c(\cos \chi \, d\chi \wedge \sigma_1 + \sin \chi \, \sigma_2 \wedge \sigma_3), \\[2mm]
J_2 = c(\sin \chi \, d\chi \wedge \sigma_2 + \cos \chi \, \sigma_3 \wedge \sigma_1), \\[2mm]
J_3 = c^2 \sin \chi \cos \chi \, d\chi \wedge \sigma_3 + \sigma_1 \wedge \sigma_2.
\end{cases}
\tag{59}
$$

It follows that the three Killing vectors $\mathcal{L}_i$ are tri-H and therefore this metric is again a Multi-Centre.

For convenience we take $\partial_t = \mathcal{L}_2$. The canonical form (1) is obtained with

$$
V = \frac{\sin \chi \cos \chi}{\cosh^2 \theta - \sin^2 \chi}, \qquad \Theta = -\frac{\sinh \theta \cosh \theta}{\cosh^2 \theta - \sin^2 \chi} dz, \qquad \star d\Theta = -dV, \tag{60}
$$

and the 3 dimensional metric

$$
\gamma_0 = c^2(\cosh^2 \theta - \sin^2 \chi)(d\chi^2 + d\theta^2) + dz^2,
$$

Using Hitchin's result [18] it is easy to get the coordinates

$$
X = c \cosh \theta \sin \chi, \quad Y = c \sinh \theta \cos \chi, \quad X + iY = c \sin(\theta + i\chi), \quad Z = z.
$$

The potential V becomes:

$$
V = \frac{1}{4}\left(\frac{1}{R_-} - \frac{1}{R_+} \right) \sqrt{4c^2 - (R_+ - R_-)^2}, \qquad R_\pm = \sqrt{(X \pm c)^2 + Y^2}. \tag{61}
$$

This relation shows clearly that the coordinates X, Y are quite unnatural to look for Bianchi metrics. Also the check that V is a solution of Laplace equation is hairy!

It is convenient to examine the potential using elliptic coordinates

$$
\xi = c \cosh \theta, \qquad \eta = c \sin \chi \qquad \Rightarrow \qquad V = \frac{\eta \sqrt{c^2 - \eta^2}}{\xi^2 - \eta^2}. \tag{62}
$$

Now we can compare with the more general potential [27][p. 590] leading to an integrable geodesic flow:

$$
V = v_0 + \frac{a\xi \sqrt{\xi^2 - c^2} + b\eta \sqrt{c^2 - \eta^2}}{\xi^2 - \eta^2}. \tag{63}
$$

So, in the special case $v_0 = a = 0$, we recover the Bianchi VI_0 metric.

It is also interesting to have a look at the more general case where the SD connection does not vanish

$$
\omega_1^+ = 0, \qquad \omega_2^+ = 0, \qquad \omega_3^+ = \lambda \, \sigma_3, \qquad \lambda \in \mathbb{R} \backslash \{0\}, \tag{64}
$$

still leading to an ASD curvature. The metric, given in [20], is

$$
G_{VI} = c^2 \sin \chi \cos \chi \, e^{-2\lambda \chi} \left[d\chi^2 + \sigma_3^2 \right] + \cot \chi \, \sigma_1^2 + \tan \chi \, \sigma_2^2. \tag{65}
$$

In view of relation (64) the complex structures $\widetilde{J}_i$ are now

$$\widetilde{J}_1 + i\widetilde{J}_2 = e^{-i\lambda\theta}(J_1 + iJ_2), \qquad \widetilde{J}_3 = J_3, \tag{66}$$

where the J_i were defined in (59). From this we conclude that $\mathcal{L}_2$, $\mathcal{L}_3$ are tri-H while $\mathcal{L}_1$ is not. Hence this metric is still a Multi-Centre, with the potential and connection still given by (60), but with cartesian coordinates

$$X + iY = c\,\frac{e^{-\lambda(\chi+i\theta)}}{1+\lambda^2}\left[\sin(\chi+i\theta) - \lambda\cos(\chi+i\theta)\right], \qquad Z = z. \tag{67}$$

The potential V and the 1-form Θ are still given by (60) but it is no longer possible to get an explicit form in terms of these new coordinates.

The curvature is such that $A = B = 0$ so that $W^+ = R^+ = 0$, and

$$W^- = C = \frac{1}{f(\chi)}\,\mathrm{diag}\,(\,1 - 3\cos^2\chi + \lambda\sin\chi\cos\chi,\; -2 + 3\cos^2\chi - \lambda\sin\chi\cos\chi,\; 1\,),$$

with $f(\chi) = c^2 e^{-2\lambda\chi}\sin^3\chi\cos^3\chi$. So it is Petrov I.

6.2   The Bianchi VII$_0$ self-dual metrics

The Killing vectors are now

$$\mathcal{L}_1 = \partial_\theta + z\,\partial_y - y\,\partial_z, \qquad \mathcal{L}_2 = \partial_y, \qquad \mathcal{L}_3 = \partial_z, \tag{68}$$

and correspond to the choice $n_1 = 1$, $n_2 = 1$ and $n_3 = 0$ in relation (2). The invariant 1-forms are

$$\sigma_1 = \cos\theta\,dy - \sin\theta\,dz, \qquad \sigma_2 = \sin\theta\,dy + \cos\theta\,dz, \qquad \sigma_3 = d\theta. \tag{69}$$

The metric with ASD connection was first given in [20]. Another interesting derivation was given also in [2], which makes use of the relation between minimal surfaces in $\mathbb{R}^3$ and four dimensional self-dual metrics. If one takes for minimal surface the helicoid [6], then the corresponding self-dual metric is nothing but the Bianchi VII$_0$ one, which can be written

$$g_{VII} = c^2\sinh\chi\cosh\chi\left[d\chi^2 + \sigma_3^2\right] + \tanh\chi\,\sigma_1^2 + \coth\chi\,\sigma_2^2. \tag{70}$$

Positivity restricts $\chi \in [0, +\infty[$ and there is a partner metric obtained by the interchange of the coefficients of σ_1 and σ_2. The complex structures are

$$\begin{cases} J_1 = c(\sinh\chi\,d\chi \wedge \sigma_1 + \cosh\chi\,\sigma_2 \wedge \sigma_3), \\[2mm] J_2 = c(\cosh\chi\,d\chi \wedge \sigma_2 + \sinh\chi\,\sigma_3 \wedge \sigma_1), \\[2mm] J_3 = c^2\sinh\chi\cosh\chi\,d\chi \wedge \sigma_3 + \sigma_1 \wedge \sigma_2. \end{cases} \tag{71}$$

It follows that both three Killing vectors $\mathcal{L}_i$ are tri-H and therefore this metric is again a Multi-Centre.

---

[6]Taking the catenoid, one gets the Bianchi VI$_0$ metric.

For convenience we take $\partial_t = \mathcal{L}_2$. The canonical form (1) is obtained with

$$V = \frac{\sinh \chi \cosh \chi}{\cosh^2 \chi - \cos^2 \theta}, \qquad \Theta = -\frac{\sin \theta \cos \theta}{\cosh^2 \chi - \cos^2 \theta} \, dz, \qquad \star d\Theta = dV, \qquad (72)$$

while the 3 dimensional metric is

$$\gamma_0 = c^2 (\cosh^2 \chi - \cos^2 \theta)(d\chi^2 + d\theta^2) + dz^2.$$

We get for cartesian coordinates

$$X = c \cosh \chi \cos \theta, \qquad Y = c \sinh \chi \sin \theta, \qquad Z = z,$$

and the potential is quite complicated

$$V = \frac{1}{4} \left(\frac{1}{R_+} + \frac{1}{R_-} \right) \sqrt{(R_+ + R_-)^2 - 4c^2}. \qquad (73)$$

Switching to elliptic coordinates ξ, η defined by

$$\xi = c \cosh \chi, \qquad \eta = c \cos \theta,$$

the potential becomes

$$V = \frac{\xi \sqrt{\xi^2 - c^2}}{\xi^2 - \eta^2} \qquad (74)$$

which can be compared again with the potential (63) so we recover the Bianchi VII$_0$ metric for the parameters $v_0 = b = 0$. In this case the Killing tensor was first given in [2] as well as the proof of separability of the Hamilton-Jacobi and Schrödinger equations.

Let us examine the more general case with

$$\omega_1^+ = 0, \qquad \omega_2^+ = 0, \qquad \omega_3^+ = \lambda \, \sigma_3, \qquad \lambda \in \mathbb{R} \backslash \{0\}.$$

The corresponding metric was given in [20]:

$$G_{VII} = c^2 \sinh \chi \cosh \chi \, e^{-2\lambda\chi} [d\chi^2 + (\sigma_3)^2] + \tanh \chi \, (\sigma_1)^2 + \coth \chi \, (\sigma_2)^2. \qquad (75)$$

Now the complex structures are

$$\widetilde{J}_1 + i\widetilde{J}_2 = e^{-i\lambda\theta}(J_1 + iJ_2), \qquad \widetilde{J}_3 = J_3, \qquad (76)$$

where the J_i were defined in (71). The metric is still a Multi-Centre because $\mathcal{L}_1$ and $\mathcal{L}_2$ remain tri-holomorphic while $\mathcal{L}_3$ is just holomorphic. The potential and connection are still given by (72) while the new cartesian coordinates are

$$X + iY = \frac{c}{2} \left[\frac{e^{(1-\lambda)(\chi+i\theta)}}{1 - \lambda} + \frac{e^{-(1+\lambda)(\chi+i\theta)}}{1 + \lambda} \right].$$

The potential V and the 1-form Θ are still given by (72) but it is no longer possible to have an explicit form for them in terms of these new coordinates, quite similarly to the Bianchi VI$_0$ case.

The curvature is such that $A = B = 0$ so that $W^+ = R^+ = 0$, and

$$W^- = C = \frac{1}{f(\chi)} \, \mathrm{diag} \, (\, 1 - 3\cosh^2 \chi + \lambda \sinh \chi \cosh \chi, \; -2 + 3\cosh^2 \chi - \lambda \sinh \chi \cosh \chi, \; 1 \,),$$

with $f(\chi) = c^2 e^{-2\lambda\chi} \sinh^3 \chi \cosh^3 \chi$. So it is also Petrov I.

7 Non-diagonal Bianchi VI$_0$ and VII$_0$ metrics

In [27][p. 586] it was proved that the Multi-Centre metric (1) with the potential

$$V = v_0 + \frac{a\sqrt{\sqrt{X^2 + Y^2} + X} + b\sqrt{\sqrt{X^2 + Y^2} - X}}{2\sqrt{X^2 + Y^2}} \qquad (77)$$

has an integrable geodesic flow. The separation coordinates for the Hamilton-Jacobi equation were given, in the same reference, page 591, to be squared parabolic:

$$X = \frac{1}{2}(\xi^2 - \eta^2), \qquad Y = \xi\eta.$$

They simplify the metric to

$$g = \frac{1}{V}(dt + G\,dz)^2 + V dz^2 + V(\xi^2 + \eta^2)(d\xi^2 + d\eta^2), \qquad (78)$$

with

$$V = v_0 + \frac{a\xi + b\eta}{\xi^2 + \eta^2}, \qquad G = \frac{b\xi - a\eta}{\xi^2 + \eta^2}. \qquad (79)$$

The separation of the Hamilton-Jacobi gives in turn an extra quadratic conserved quantity $S = S^{ij}\Pi_i\Pi_j$ where S^{ij} are the components of a K-S tensor. So we have four independent conserved quantities [7]

$$H = \frac{1}{2}g^{ij}\Pi_i\Pi_j, \qquad \Pi_z, \qquad \Pi_t,$$
$$S = \Pi_\xi^2 + (\xi\Pi_z - b\Pi_t)^2 + v_0(v_0\xi^2 + 2a\xi)\Pi_t^2 - 2(v_0\xi^2 + a\xi)H, \qquad (80)$$

which are in involution with respect to the Poisson bracket.

It is the aim of this section to show that this metric is a non-diagonal Bianchi VII$_0$ metric.

Let us first observe that for $v_0 = 0$ it reduces to the Bianchi II metric given by (22). To achieve this identification the following change of coordinates:

$$T = (a^2 + b^2)z, \qquad Z = t, \qquad X = a\xi + b\eta, \qquad Y = b\xi - a\eta, \qquad (81)$$

allows to obtain

$$(a^2 + b^2)\,g(v_0 = 0) = \frac{1}{X}(dT + Y\,dZ)^2 + X(dX^2 + dY^2 + dZ^2), \qquad (82)$$

which does indeed coincide with the metric g_{II} of section 3.

For the more general three parameters metric, we will now show that it is a *non-diagonal* Bianchi VII$_0$ metric. The possibility of such metrics is known, but it seems that we are getting the first example of this kind.

The proof of this fact relies on the existence of three isometries for (78), given by

$$\mathcal{L}_1 = -(b + 2v_0\eta)\partial_\xi + (a + 2v_0\xi)\partial_\eta + t\,\partial_z - v_0^2 z\,\partial_t, \qquad \mathcal{L}_2 = \partial_z, \qquad \mathcal{L}_3 = \partial_t, \qquad (83)$$

---
[7]Notice that for $v_0 = 0$ we recover the conserved quantity of formula (106) in [27].

with the Lie algebra

$$[\mathcal{L}_1, \mathcal{L}_2] = v_0^2 \, \mathcal{L}_3, \qquad [\mathcal{L}_2, \mathcal{L}_3] = 0, \qquad [\mathcal{L}_3, \mathcal{L}_1] = \mathcal{L}_2. \tag{84}$$

For $v_0 = 0$ it reduces to Bianchi II and this case has already been disposed of. For non-vanishing v_0 the algebra is Bianchi VII$_0$. The delicacy is now to relate the actual coordinates used for the metric (78) and the coordinates adapted to the Bianchi VII$_0$ isometries as defined by the vector fields (68). A comparison of the vector fields suggests the following coordinates change:

$$t \to Z, \qquad z \to \frac{Y}{v_0}, \qquad \xi \to -\frac{a}{2v_0} + r\cos(2\theta), \qquad \eta \to -\frac{b}{2v_0} + r\sin(2\theta). \tag{85}$$

After this change it is possible to express dY, dZ and $d\theta$ in terms of the 1-forms σ_1, σ_2 and σ_3 given by (69). For aesthetical reasons it is convenient to transform $a \to 2av_0$ and $b \to 2bv_0$ to get the final form

$$
\begin{aligned}
g_{ND} = v_0^2(r^2 - a^2 - b^2)\left(dr^2 + 4r^2\sigma_3^2\right) \\
+ \frac{[(r+a)^2 + b^2]\sigma_1^2 + 4br\sigma_1\sigma_2 + [(r-a)^2 + b^2]\sigma_2^2}{r^2 - a^2 - b^2}
\end{aligned}
\tag{86}
$$

Taking for vierbein

$$
\begin{cases}
e_0 = v_0\sqrt{f}\,dr, & e_3 = 2v_0 r\sqrt{f}\,\sigma_3, & f = r^2 - a^2 - b^2, \\[2mm]
e_1 = \sqrt{\dfrac{f}{g}}\,\sigma_1, & e_2 = \dfrac{2br}{\sqrt{fg}}\,\sigma_1 + \sqrt{\dfrac{g}{f}}\,\sigma_2, & g = (r-a)^2 + b^2
\end{cases}
$$

the spin connection has the structure

$$\omega_1^+ = \omega_2^+ = 0, \qquad \omega_3^+ = -\frac{b}{g}\,dr - 3\,\sigma_3,$$

which implies that the Riemann curvature is indeed anti-selfdual: $R_i^+ = 0$ for $i = 1, 2, 3$.

We expect that such a non-diagonal metric should exist also for Bianchi VI$_0$, so let us write the equations giving both Bianchi VI$_0$, and Bianchi VII$_0$, metrics. We take for vierbein

$$e_0 = \alpha(r)\,dr, \qquad e_3 = \beta(r)\,\sigma_3, \qquad e_1 = \lambda(r)\,\sigma_1, \qquad e_2 = \mu(r)\,\sigma_1 + \nu(r)\,\sigma_2, \tag{87}$$

and for connection

$$\omega_1^+ = \omega_2^+ = 0, \qquad \omega_3^+ = A(r)\,dr + C\,\sigma_3. \tag{88}$$

Imposing the hyperkähler structure is most conveniently done using the 2-forms F_i^+ defined in (6) for which we have

$$dF_1^+ = -\omega_3^+ \wedge F_2^+, \qquad dF_2^+ = \omega_3^+ \wedge F_1^+, \qquad dF_3^+ = 0.$$

This gives the differential system

$$a) \qquad \frac{1}{\alpha}(\beta\lambda)' = \epsilon\nu + \lambda\mu^2 - C\lambda,$$

$$b) \qquad \frac{1}{\alpha}(\beta\mu)' = -\mu\lambda^2 - C\mu, \qquad\qquad \lambda\nu = 1, \qquad \epsilon^2 = 1. \qquad (89)$$

$$c) \qquad \frac{1}{\alpha}(\beta\nu)' = \lambda - C\nu,$$

For $\epsilon = +1$ (resp. $\epsilon = -1$) we get the Bianchi VII$_0$ (resp. Bianchi VI$_0$) non diagonal metric. Let us take for coordinate fixing the relation $\beta = 2r\alpha$. Then relations (89)b and (89)c become

$$2r\frac{(\alpha\mu)'}{\alpha\mu} = -C - 2 - \lambda^2, \qquad 2r\frac{(\alpha\nu)'}{\alpha\nu} = -C - 2 + \lambda^2, \qquad (90)$$

implying $\alpha^2\mu\nu = K\,r^{-C-2}$. It is then convenient to parametrize α and ν according to

$$\alpha = v_0\sqrt{r^{-C-1}F}, \qquad \nu = \sqrt{\frac{G}{rF}} \quad \Rightarrow \quad \mu = \frac{K}{v_0^2}\sqrt{\frac{1}{rFG}}. \qquad (91)$$

Substituting these forms in relations (89)a and (89)b leaves us with

$$F = G', \qquad 2(r^2GG'' + rGG') - r^2G'^2 = \epsilon G^2 + (K/v_0^2)^2. \qquad (92)$$

It is convenient to define $H = \sqrt{G}$ and use the variable $t = \ln r$. The last differential equation becomes then

$$\ddot{H} = \frac{\epsilon}{4}H + \frac{b^2}{H^3}, \qquad \dot{H} = \frac{dH}{dt}.$$

Multiplying by $2\dot{H}$ and integrating leads to

$$\dot{H}^2 = \frac{L}{4} + \frac{\epsilon}{4}H^2 - \frac{b^2}{H^2},$$

where L is some constant. Then, multiplying by $4H^2$, one is left with

$$\dot{G}^2 = \epsilon\,G^2 + L\,G - 4b^2 \quad \Rightarrow \quad r^2\,G'^2 = \epsilon\,G^2 + LG - (K/v_0^2)^2, \qquad (93)$$

showing that only elementary functions will appear in the metric.

The metric itself can be written

$$g = v_0^2 r^{-C-2}\,rG'(dr^2 + 4r^2\,\sigma_3^2) + \frac{1}{rG'}\left((L + \epsilon G)\sigma_1^2 + 4b\,\sigma_1\,\sigma_2 + G\,\sigma_2^2\right).$$

It is then easy to integrate (93); up to simple algebra, the Bianchi VII$_0$ metric is recovered

$$g_{VII} = v_0^2\,r^{2c}(r^2 - a^2 - b^2)\left(dr^2 + 4r^2\sigma_3^2\right)$$
$$+\frac{[(r+a)^2 + b^2]\sigma_1^2 + 4br\,\sigma_1\sigma_2 + [(r-a)^2 + b^2]\sigma_2^2}{r^2 - a^2 - b^2}, \qquad (94)$$

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with the non-vanishing component of the self-dual spin connection

$$\omega_3^+ = A\,dr + C\,\sigma_3, \qquad A = -\frac{b}{(r-a)^2+b^2}, \qquad C = -3 - 2c.$$

For the Bianchi VI_0 metric we get

$$g_{VI} = v_0^2\,r^{2c}\,\cos\rho\left(dr^2 + 4r^2\sigma_3^2\right)$$
$$+ \frac{(a-\sin\rho)\sigma_1^2 + 2\sqrt{a^2-1}\,\sigma_1\sigma_2 + (a+\sin\rho)\sigma_2^2}{\cos\rho}, \tag{95}$$

where $\rho = \ln(r/r_0)$ and this time we have

$$\omega_3^+ = A\,dr + C\,\sigma_3, \qquad A = -\frac{\sqrt{a^2-1}}{2r(a+\sin\rho)}, \qquad C = -2 - 2c.$$

Remarks:

1. Notice that the integration process introduces an apparent fourth free parameter c in the solution. Its irrelevance is obvious since the potential V and the connection do not depend on it: its only effect is to change the form of the cartesian coordinates X and Y in terms of r and θ, while the coordinate $Z = z$ remains unchanged. So in what follows we will set $c = 0$.

2. The parameter v_0 allows for these metrics the euclidean as well as the lorentzian signature.

For the Bianchi VII_0 metric (94) the cartesian coordinates and the potential (77) are explicitly known, so it is a natural question to try to get the same information for the new Bianchi VI_0 metric (95). To this aim let us first obtain the triplet of complex structures: we define a new function ϕ by $\omega_3^+ = 2d\phi$ and then rotate the 2-forms F_i^+ into the J_i according to

$$J^+ \equiv J_1 + iJ_2 = e^{-2i\phi}(F_1^+ + iF_2^+), \qquad J_3 = F_3^+, \tag{96}$$

and since the J_i are closed, they are the complex structures we were looking for, as can be easily checked.

From these expressions we see that the Killing vectors $\mathcal{L}_2$ and $\mathcal{L}_3$ remain tri-hilomorphic while $\mathcal{L}_1$ is holomorphic. So we take $\partial_t = \mathcal{L}_2$ and transform the metric (95) into the Multi-Centre form (1). The potential and connection are now

$$V = \frac{\cos\rho}{D(\rho,\theta)}, \qquad G = \frac{-a\sinh 2\theta + \sqrt{a^2-1}\cosh 2\theta}{D(\rho,\theta)}\,dz, \tag{97}$$

with

$$D(\rho,\theta) = a\cosh 2\theta - \sqrt{a^2-1}\sinh 2\theta - \sin\rho,$$

and the flat 3-dimensional metric

$$\gamma_0 = dz^2 + v_0^2\,D(\rho,\theta)\,(dr^2 + 4r^2 d\theta^2). \tag{98}$$

We have checked the relation $dV = -\star dG$. The cartesian coordinates are on the one hand $Z = z$ and on the other hand

$$d(X + iY) = v_0 \, e^{2i\theta} \left\{ \frac{1+i}{2} \, A_+ \, e^{\frac{i}{2}(\rho + 2i\theta)} + \frac{1-i}{2} \, A_- \, e^{-\frac{i}{2}(\rho + 2i\theta)} \right\} (dr + 2ird\theta), \qquad (99)$$

with $A_\pm = \sqrt{a \pm \sqrt{a^2 - 1}}$. We were not able to express the potential in terms of the coordinates X and Y as was possible for the Bianchi VII_0 case.

Let us observe that for $\epsilon = 0$ we recover an apparently non-diagonal Bianchi II metric. However, up to some easy coordinates changes, it is possible to show that this metric is nothing but the tri-axial metric G_{II}, given by (23) in section 3.

8 The Bianchi VIII and IX self-dual case

We will begin with Bianchi IX metrics, which are the most popular and display the richest integrability properties, and present rather quickly the Bianchi VIII case, which is quite similar.

8.1 Bianchi IX case

Here we have the Maurer-Cartan 1-forms

$$\sigma_1 = -\sin\phi \, d\theta - \cos\phi \, \sin\theta \, d\psi, \quad \sigma_2 = \cos\phi \, d\theta - \sin\phi \, \sin\theta \, d\psi, \quad \sigma_3 = d\phi - \cos\theta \, d\psi,$$

with the relations

$$d\sigma_1 = \sigma_2 \wedge \sigma_3, \qquad d\sigma_2 = \sigma_3 \wedge \sigma_1, \qquad d\sigma_3 = \sigma_1 \wedge \sigma_2.$$

These forms are invariant under the vector fields

$$R_1 = \sin\psi \, \partial_\theta + \frac{\cos\psi}{\sin\theta}(\cos\theta \, \partial_\psi + \partial_\phi), \ R_2 = -\cos\psi \, \partial_\theta + \frac{\sin\psi}{\sin\theta}(\cos\theta \partial_\psi + \partial_\phi), \ R_3 = -\partial_\psi,$$

which generate the $su(2)$ Lie algebra.

The tri-axial metric

$$g = \frac{d\lambda^2}{4ABC} + \frac{BC}{A} \sigma_1^2 + \frac{CA}{B} \sigma_2^2 + \frac{AB}{C} \sigma_3^2, \qquad (100)$$

with

$$A = \sqrt{\lambda - \lambda_1}, \qquad B = \sqrt{\lambda - \lambda_2}, \qquad C = \sqrt{\lambda - \lambda_3},$$

was given in [4], [17]. Its hyperkähler nature follows from its triplet of complex structures

$$\Omega_1 = d(A \, \sigma_1), \qquad \Omega_2 = d(B \, \sigma_2), \qquad \Omega_3 = d(C \, \sigma_3). \qquad (101)$$

It follows that the vector fields R_i, $i = 1, 2, 3$ are tri-holomorphic. To write it in the Multi-Centre form (1) it is convenient to take $\partial_t = \partial_\psi$. One gets

$$\frac{1}{V} = \frac{AB}{C} \cos^2\theta + \frac{C}{AB}(A^2 \sin^2\phi + B^2 \cos^2\phi) \sin^2\theta,$$

$$-\frac{\Theta}{V} = \frac{AB}{C} \cos\theta \, d\phi + \frac{C}{AB}(A^2 - B^2) \sin\theta \sin\phi \cos\phi \, d\theta,$$

$$(102)$$

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and the cartesian coordinates

$$X = A \sin\theta \cos\phi, \quad Y = B \sin\theta \sin\phi, \quad Z = C \cos\theta, \tag{103}$$

with $\max(\lambda_1, \lambda_2) < \lambda_3 < \lambda$. This result was first given in [15], and using Hitchin's result in [17].

Its bi-axial limits $\lambda_1 = \lambda_2$ were discovered earlier by Eguchi and Hanson [9], and are best displayed using the coordinate $s = \sqrt{\lambda - \lambda_3}$ which gives

$$g = \frac{s}{s^2 + c^2}\, ds^2 + \frac{s^2 + c^2}{s}\, \sigma_3^2 + s(\sigma_1^2 + \sigma_2^2), \qquad c^2 = \lambda_3 - \lambda_1 > 0. \tag{104}$$

Notice that here positivity requires $s > 0$, and the metric is not complete due to the singularity at $s = 0$.

If $c^2 < 0$ we obtain, in the same bi-axial limit:

$$g_{EH} = \frac{s}{s^2 - c^2}\, ds^2 + \frac{s^2 - c^2}{s}\, \sigma_3^2 + s(\sigma_1^2 + \sigma_2^2). \tag{105}$$

Now positivity requires $s > c$ and $s = c$ is an apparent bolt singularity, leading to a complete metric. These two metrics enjoy the extra isometry ∂_ϕ with respect to the tri-axial metric, but it is only holomorphic.

The potential of its Multi-Centre form was discovered a long time ago; using as cartesian coordinates

$$X = \sqrt{s^2 - c^2} \sin\theta \cos\phi, \quad Y = \sqrt{s^2 - c^2} \sin\theta \sin\phi, \quad Z = s \cos\theta,$$

as well and the notation $r_\pm = \sqrt{X^2 + Y^2 + (Z \pm c)^2}$ one has

$$V = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right), \qquad \Theta = \frac{1}{2}\left(\frac{Z+c}{r_+} + \frac{Z-c}{r_-}\right) d\phi. \tag{106}$$

This is a particular 2-centre metric which displays the classical as well as the quantum integrability property [22].

As mentioned in section 2, there is also the possibility of having for the spin connection the form

$$\omega_1^+ = \sigma_1, \qquad \omega_2^+ = \sigma_2, \qquad \omega_3^+ = \sigma_3.$$

In the bi-axial case this leads to the Taub-NUT celebrated metric (still a Multi-Centre!) and its rich structure with respect to integrability, see [14],[11],[16]. The corresponding tri-axial metric was given by Atiyah and Hitchin [1] but is no longer in the Multi-Centre family and the integrability of its geodesic flow is an open problem.

8.2   Elliptic coordinates for tri-axial Bianchi IX

In [17] elliptic coordinates were used for the tri-axial Bianchi IX metric in the quest for separability of Hamilton-Jacobi equation. These coordinates (λ, μ, ν) are defined by

$$
\begin{aligned}
X^2 &= \frac{(\lambda - \lambda_1)(\mu - \lambda_1)(\nu - \lambda_1)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \\
Y^2 &= \frac{(\lambda - \lambda_2)(\mu - \lambda_2)(\nu - \lambda_2)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, \qquad 0 < \lambda_1 < \mu < \lambda_2 < \nu < \lambda_3 < \lambda. \\
Z^2 &= \frac{(\lambda - \lambda_3)(\mu - \lambda_3)(\nu - \lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)},
\end{aligned}
\tag{107}
$$

The flat metric γ_0 takes the diagonal form

$$\gamma_0 = dX^2 + dY^2 + dZ^2 = g_1\, d\lambda^2 + g_2\, d\mu^2 + g_3\, d\nu^2,$$

with

$$g_1 = \frac{(\lambda - \mu)(\lambda - \nu)}{4R(\lambda)}, \qquad R(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3),$$

$$g_2 = \frac{(\mu - \lambda)(\mu - \nu)}{4S(\mu)}, \qquad S(\mu) = (\mu - \lambda_1)(\mu - \lambda_2)(\mu - \lambda_3), \qquad (108)$$

$$g_3 = -\frac{(\nu - \lambda)(\nu - \mu)}{4T(\nu)}, \qquad T(\nu) = -(\nu - \lambda_1)(\nu - \lambda_2)(\nu - \lambda_3).$$

The potential and the 1-form Θ become:

$$V = \frac{\sqrt{R(\lambda)}}{(\lambda - \mu)(\lambda - \nu)}, \qquad \Theta = \frac{1}{2N(\mu,\nu)} \left(\sqrt{\frac{T}{S}}\, \frac{N(\lambda,\mu)}{\lambda - \nu}\, d\mu - \sqrt{\frac{S}{T}}\, \frac{N(\lambda,\nu)}{\lambda - \mu}\, d\lambda \right), \quad (109)$$

with $N(x,y) = (x - \lambda_3)(y - \lambda_3) - (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$. From these formulas we have checked the relation $d\Theta = - * dV$.

Let us now use the necessary conditions for separability of the Hamilton-Jacobi equation due to Levi-Civita (see [24][p. 105]). They read

$$\frac{\partial H}{\partial \Pi_i} \frac{\partial H}{\partial \Pi_j} \frac{\partial^2 H}{\partial x^i \partial x^j} - \frac{\partial H}{\partial \Pi_i} \frac{\partial H}{\partial x^j} \frac{\partial^2 H}{\partial x^i \partial \Pi_j} - \frac{\partial H}{\partial x^i} \frac{\partial H}{\partial \Pi_j} \frac{\partial^2 H}{\partial \Pi_i x^j} - \frac{\partial H}{\partial x^i} \frac{\partial H}{\partial x^j} \frac{\partial^2 H}{\partial \Pi_i \partial \Pi_j} = 0, \; i \neq j. \; (110)$$

The Hamiltonian for the Bianchi IX metric is

$$2H = \frac{\Pi_\lambda^2}{V g_1} + \frac{\Pi_\mu^2}{V g_2} + \frac{\Pi_\nu^2}{V g_3} - 2q \left(\frac{\Theta_\mu}{V g_2} \Pi_\mu + \frac{\Theta_\nu}{V g_3} \Pi_\nu \right) + q^2 U, \qquad U = V + \frac{||\Theta||^2}{V}. \quad (111)$$

The conserved charge $q = \Pi_0$ may be used as an expansion parameter in (110): this gives five relations, according to the powers of q involved. For $q = 0$ we have checked that the Levi-Civita conditions hold as was to be expected. However, at the first order in q, taking $x^i = \lambda$ and $x^j = \mu$, the Levi-Civita conditions imply the constraint

$$\Pi_\mu \left(\alpha \Pi_\mu^2 + \beta \Pi_\nu^2 + \gamma \Pi_\mu \Pi_\nu \right) = 0.$$

The coefficients α, β, γ are complicated functions of the coordinates, but β can be seen to be non-vanishing. Hence we conclude that the elliptic coordinates are not separation coordinates for the Hamilton-Jacobi equation.

8.3 Bianchi VIII case

One has the Maurer-Cartan 1-forms

$$\sigma_1 = -\sin\phi\, d\tau - \cos\phi\, \sinh\tau\, d\psi, \quad \sigma_2 = \cos\phi\, d\tau - \sin\phi\, \sinh\tau\, d\psi, \quad \sigma_3 = d\phi - \cosh\tau\, d\psi,$$

which are invariant under the vector fields

$$R_1 = \sin\psi\, \partial_\tau + \frac{\cos\psi}{\sinh\tau}(\cosh\tau\, \partial_\psi + \partial_\phi), \; R_2 = -\cos\psi\, \partial_\tau + \frac{\sin\psi}{\sinh\tau}(\cosh\tau \partial_\psi + \partial_\phi), \; R_3 = -\partial_\psi,$$

generating the $su(1,1)$ Lie algebra.

The tri-axial metric was given in [23]. The only change in the metric (100) is that now $C = \sqrt{\mu_3 - \mu}$, from which we can take $\max(\mu_1, \mu_2) < \mu < \mu_3$. Its complex structures are

$$\Omega_1 = d(A\,\sigma_1), \qquad \Omega_2 = d(B\,\sigma_2), \qquad \Omega_3 = -d(C\,\sigma_3), \tag{112}$$

so the vector fields R_i, $i = 1, 2, 3$ are tri-holomorphic. Its Multi-Centre potential (1), taking again $\partial_t = \partial_\psi$, is still given by (102), with the cartesian coordinates

$$X = \sqrt{\mu - \mu_1}\,\sinh\tau\,\cos\phi, \quad Y = \sqrt{\mu - \mu_2}\,\sinh\tau\,\sin\phi, \quad Z = \sqrt{\mu_3 - \mu}\,\cosh\tau. \tag{113}$$

Here too, no definite conclusion is known about its integrability.

Its bi-axial limit, which enjoys the extra Killing vector ∂_ϕ, was derived earlier by Gegenberg and Das [12]:

$$g = \frac{s}{c^2 - s^2}\,ds^2 + \frac{c^2 - s^2}{s}\,\sigma_3^2 + s(\sigma_1^2 + \sigma_2^2), \qquad 0 < s < c. \tag{114}$$

It is not complete due to the $s = 0$ singularity. Taking for tri-holomorphic Killing vector ∂_ψ, this metric corresponds to a Multi-Centre with

$$V = \frac{s}{c^2\cosh^2\tau - s^2}, \qquad \Theta = -\frac{c^2 - s^2}{c^2\cosh^2\tau - s^2}\cosh\tau\,d\phi,$$

$$X = \sqrt{c^2 - s^2}\,\sinh\tau\,\cos\phi, \quad Y = \sqrt{c^2 - s^2}\,\sinh\tau\,\sin\phi, \quad Z = s\cosh\tau. \tag{115}$$

This time we have

$$V = \frac{1}{2}\left(\frac{1}{r_-} - \frac{1}{r_+}\right), \qquad \Theta = \frac{1}{2}\left(\frac{Z - c}{r_-} - \frac{Z + c}{r_+}\right)d\phi, \tag{116}$$

so we are back to a two-centre metric, with a positive and a negative mass, for which integrability is for sure. The work by Mignemi [22] could be adapted to this case to prove the classical (Hamilton-Jacobi) and quantum (Schrödinger) separability hence integrability.

As opposed to the Bianchi IX case, there is no possibility of having for the spin connection the form (11) because the relations

$$\lambda_1 = \lambda_2\lambda_3, \qquad \lambda_2 = \lambda_3\lambda_1, \qquad \lambda_3 = -\lambda_1\lambda_2$$

have no real solution: so there is neither a Taub-NUT like metric nor an Atiyah-Hitchin like metric for Bianchi VIII.

9 Quantum integrability aspects

Once the question of the *classical* integrability of some geodesic flow is obtained a natural question arises: what about its *quantum* integrability? This is quite a difficult question because there are many available quantization schemes. One of the most attractive is the so-called "minimal quantization" defined by Carter [6]. Simplifying somewhat, it uses the following quantization device up to quadratic classical observables

$$
\begin{aligned}
K(x) &\longrightarrow K(x)\,\mathbb{I} \\[2ex]
K^i(x)\,\Pi_i &\longrightarrow -\frac{i}{2}(K^i \circ \nabla_i + \nabla_i \circ K^i) \\[2ex]
K^{ij}(x)\,\Pi_i\,\Pi_j &\longrightarrow -\nabla_i \circ K^{ij} \circ \nabla_j,
\end{aligned}
\tag{117}
$$

where the formally symmetric operators act on the Hilbert space of wave functions, which are to be square summable for the invariant measure on the manifold. The quantized operator corresponding to the Hamiltonina is therefore the laplacian $\widehat{H} = -\frac{1}{2} \nabla^i \circ \nabla_i$.

These rules were completed in [8] to cover cubic observables, according to

$$K^{ijk} \, \Pi_i \, \Pi_j \, \Pi_k \quad \longrightarrow \quad \frac{i}{2} (\nabla_i \circ K^{ijk} \circ \nabla_j \circ \nabla_k + \nabla_i \circ \nabla_j \circ K^{ijk} \circ \nabla_k). \qquad (118)$$

We will denote by K_n some classical observable of degree $n \le 3$ in the momenta and by $\widehat{K}_n$ its quantum operator. If K_1 is generated by a Killing vector and K_2 by a K-S tensor, the following relations were proved in [6], [8]:

$$[\widehat{K_1}, \widehat{H}] = -i\widehat{\{K_1, H\}}, \qquad [\widehat{K_2}, \widehat{H}] = -i\widehat{\{K_2, H\}} + i\widehat{A_{K_2, H}}, \qquad (119)$$

with

$$A_{K_2, H} = \frac{2}{3} \left(\nabla_i B_{K_2, H}^{ij} \right) \Pi_l, \qquad B_{K_2, H}^{ij} = -K_2^{l[i} \operatorname{Ric}_l{}^{j]}. \qquad (120)$$

Now in all cases the classical integrability is ensured by two Killing vectors and one K-S tensor, so the relations (119) and the Ricci-flat character of these metrics show that within "minimal quantization" the classical integrability survives to quantization. In particular this means that the Schrödinger equation will be separable as well as the Hamilton-Jacobi one.

10   Conclusion

This article has mostly dealt with the integrability of hyperkähler Bianchi A metrics within the Multi-Centre class. Quite surprisingly this family exhibits most integrable models (albeit not all: recall, for instance, the tri-axial Bianchi VIII and Bianchi IX cases) among the Multi-Centre family. A striking fact is the emergence of a genuinely new W-algebra structure for the observables for the simplest Bianchi II metric. The appearance of such structures in problems related to General Relativity is somewhat surprising but could lead to further developements in the future. Nevertheless the problem of paramount importance remains the study of the classical (and quantum) integrability of the Atiyah-Hitchin geodesic flow, governing the dynamics of two-monopole states. Some qualitative results on the existence of closed geodesics [3], [28] are known and some perturbative arguments around the negative mass Taub-NUT [13]. If quadratic Killing-Stäckel tensors would exist for this metric one could separate the Hamilton-Jacobi equation (and possibly Schrödinger equation) leading to far-reaching consequences in our understanding of the classical (and quantum) monopole dynamics.

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