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THE EMERGENCE OF THE CABIBBO ANGLE IN NON-DEGENERATE COUPLED SYSTEMS OF FERMIONS

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(Dated: October 3rd 2006)

Investigating, in direct continuation of our previous paper [1], the implications of the non-unitarity of mixing matrices for non-degenerate coupled systems that we demonstrated there, we examine more accurately the vicinity of Cabibbo-like mixing in quantum field theory. We show that it is possible to preserve one of its main features, namely that, in the space of mass eigenstates, the two requirements – of universality for weak diagonal currents and – of the absence of their non-diagonal counterparts, although not fulfilled separately any more, can however reduce to a single condition for a unique mixing angle θ_c. This leads to $\tan(2\theta_c) = \pm 1/2$, or $\cos\theta_c \approx 0.9732$, only $7/10000$ away from experimental results. No mass ratio appears in the argumentation.

PACS numbers: 12.15.Ff 12.15.Hh 13.20.-v

INTRODUCTION

Why mixing angles are what they are is, in addition to their more and more precise experimental determination, one of the most active present domain of research, both in the leptonic and hadronic sectors. The attempts that have been proposed up to now in order to address this fundamental question, such as the linking of the sine of the Cabibbo angle [2] to the d and s quark mass ratio [3], are often based on empirical evidence, and so remain unsatisfying [1] . Indeed, there are actually too many free parameters in the most general Yukawa couplings of the standard model to infer mixing angles from mass ratios.

In [1], we have demonstrated the important property that, in quantum field theory, unlike in the Wigner-Weisskopf approximation of quantum mechanics, the mixing matrix of non-degenerate coupled systems of particles should never be parametrized as unitary ; it had already been checked for neutral kaons [5], and holds in particular for fermions which are, in the standard model, coupled to each other through the Higgs boson. From this fondamental - although often ignored - feature, we have shown that two sets of solutions arise, which satisfy the combined requirements of universality for the diagonal neutral currents of *mass eigenstates* (which we will refer to hereafter as condition (C1)) and of

[1] The rigorous treatment [4] shows that only weaker "asymptotic" relations hold, which involve both m_d/m_s and m_u/m_c.

the absence of their off-diagonal counterparts, that we call hereafter "mass changing neutral currents" – MCNC's – (condition (C2)) :

$*$ two-parameter solutions, for which (C1) and (C2) are independent ; two different mixing angles θ_1 and θ_2 then occur ; they include in particular the so-called "maximal mixing" ;

$*$ one-parameter (also called Cabibbo-like) solutions, for which (C1) and (C2) coincide, and the two mixing angles are related by $\theta_2 = \pm\theta_1 + k\pi$.

We shall focus here our attention on the latter, investigating more thoroughly their close neighborhood through the study of neutral weak currents of mass eigenstates at

$$\theta_2 = \pm\theta_1 + \epsilon. \tag{1}$$

While in the Cabibbo case $\theta_2 = \pm\theta_1$, (C1) is exactly satisfied together with (C2) (which is expressed, as will be shown below, by the vanishing of two algebraic expressions $R = 0$, S or $T = 0$), this is no more the case when one moves at (1). Nevertheless, it is still possible to maintain the alignment of (C1) and (C2), that is their being approximately fulfilled with the same accuracy (which is expressed by the condition $|R| = |S|$ or $|R| = |T|$). It turns out that this requirement, which is trivial in the exact Cabbibo-case, provides in its $\mathcal{O}(\epsilon)$ neighborhood a very accurate estimate of the Cabbibo angle.

GENERALITIES ; STATES AND (WEAK) CURRENTS

In quantum field theory, the states of definite mass for a system of particles are defined as the eigenstates, at its poles, of the full (renormalized) propagator, which is given by a matrix of dimension $2n_f$ in flavour space. For non-degenerate coupled systems, like neutral kaons, leptons or quarks, these mass eigenstates belong to different bases [1]. Indeed, one finds only one such eigenstate at each of the poles of the propagator ; it is, among the $2n_f$ eigenstates of the latter, the one corresponding to the vanishing eigenvalue. Hence, the mass eigenstates do not make up an orthonormal basis, and the mixing matrix, which by definition connects the set of mass eigenstates to that of flavour eigenstates, is non-unitary.

Nevertheless, for systems of coupled fields which, like quarks, are never on-shell, a natural basis appears : the one that occurs at any given $z = q^2$. It is then orthonormal as soon as the inverse propagator $L^{(2)}(z)$ is hermitian. Thus, the mixing matrix relating it to the flavour basis is unitary (assuming that the flavour basis is orthonormal, too). This has, in the simple case of two generations, the following consequences. While, in general, (2×2) mixing matrices for coupled systems are to be parametrized with two mixing angles θ_1 and θ_2, quark-like (Cabibbo-like) systems finally shrink to one dimension.

In the following we shall mainly deal with weak currents in the space of mass eigenstates which, as stressed in [1], underlie the physics of mixing angles. Neutral (left-handed) weak currents for mass eigenstates are determined by the combinations $K_1^\dagger K_1$ and $K_2^\dagger K_2$, where K_1 and K_2 are the mixing matrices for the two types of fermions concerned (for example neutral and charged leptons or quarks of the u and d-type), whereas charged currents involve the combination $K = K_1^\dagger K_2$. As soon as neither K_1 nor K_2 is unitary, the conditions for universality (equality of diagonal neutral currents) and for the absence of off-diagonal neutral currents, which are built-in properties in flavour space, appear no longer trivial in the space of mass eigenstates, but give rise to specific constraints which we will study in detail.

We will only deal hereafter with K_1, but the process would be exactly the same for K_2.
Let us parametrize K_1 like in [1], with two mixing angles θ_1 and θ_2 :

$$K_1 = \begin{pmatrix} e^{i\alpha} c_1 & e^{i\delta} s_1 \\ -e^{i\beta} s_2 & e^{i\gamma} c_2 \end{pmatrix}. \tag{2}$$

In the mass basis, the weak neutral current couplings of the weak Lagrangian write

$$\overline{\Psi}_m \gamma^\mu \Big(\frac{1-\gamma^5}{2}\Big) W_\mu^3 K_1^\dagger K_1 \Psi_m, \tag{3}$$

where Ψ may stand for $\begin{pmatrix} u_m \\ c_m \end{pmatrix}$. From the explicit expression of $K_1^\dagger K_1$ obtained with (2), it is straightforward to see that (C1) imposes the vanishing of

$$R = c_1^2 + s_2^2 - c_2^2 - s_1^2, \tag{4}$$

while (C2) requires that of

$$S = c_1 s_1 - c_2 s_2 \quad \text{or} \quad T = c_1 s_1 + c_2 s_2. \tag{5}$$

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Due to the smallness of the expected deviation of K_1 from unitarity, we shall only explore the vicinity (1) of the Cabibbo solutions [2] .

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[2] The "$+$" sign corresponds, there, to the condition S and the "$-$" sign to T.

Calling hereafter $\theta_1 = \theta_c$, $K_1^\dagger K_1$ expands then at $\mathcal{O}(\epsilon)$ as

$$K_1^\dagger K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} \sin(2\theta_c) & -a\cos(2\theta_c) \\ -a^*\cos(2\theta_c) & -\sin(2\theta_c) \end{pmatrix} + \mathcal{O}(\epsilon^2), \tag{6}$$

where $a = e^{i\theta_a}$ is a complex number of unit modulus related to the phases $\alpha, \beta, \gamma, \delta$ present in the original parametrization (2) of K_1 [3] . As for $|R|$, $|S|$ and $|T|$, they respectively become [4] [5]

$$|R| = 2\epsilon|\sin 2\theta_c|, \quad |S| = |T| = \epsilon|\cos 2\theta_c|. \tag{7}$$

One obviously cannot ensure at $\mathcal{O}(\epsilon)$ the simultaneous fulfillment of (C1) and (C2). However it is still possible to preserve one important feature of the Cabbibo-like solutions, i.e. that the two conditions (C1) and (C2) reduce to a single one, which translates into

$$\tan(2\theta_c) = \pm\frac{1}{2}. \tag{8}$$

By the change of variables [7] $y = \tan(\pi/4 - \theta_c) \Leftrightarrow \tan\theta_c = \dfrac{1-y}{1+y}$, (8) becomes equivalent to [6]

$$\frac{1}{y} - y = \pm 1, \tag{9}$$

one of the solutions of which is the golden number

$$\varphi = \frac{1+\sqrt{5}}{2}. \tag{10}$$

The four solutions of (8) can then be rewritten in terms of φ :

$$\tan\theta_c = \pm\frac{\varphi-1}{\varphi+1},\ \pm\frac{\varphi+1}{\varphi-1}. \tag{11}$$

The first two, which have opposite signs, correspond to $\cos\theta_c = 0.9732$; this lies only $7/10000$ away from the present [9] experimental range $[0.9739, 0.9751]$ for the Cabibbo angle generally refered to.

Eq. (9) is symmetric by $y \to 1/y$ and $y \to -y$ (and likewise by their combination $y \to -1/y$). When $y \to 1/y$, $\theta_c \to -\theta_c$; when $y \to -y$, $\theta_c \to (\pi/2 - \theta_c)$ or $\theta_c \to -(\theta_c + \pi/2) \equiv -(\pi/2 - (-\theta_c))$;

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[3] See also section 3.2.1 of [1] which explains how these phases are linked to each other by the requirement of the absence of MCNC's.

[4] The expressions for S and T can be straightforwardly obtained from the relations called f_1 and g_1 in [1] for θ_2 in the vicinity of θ_1; the one for R is likewise obtained in the same conditions from the relation in the line just above.

[5] The identical normalization of the two diagonal terms proportional to ϵ in (6) is fortuitous, as shows the case of three generations [6]; universality is thus expressed on general grounds by the vanishing of $(2\sin\theta_c)$, which represents the *difference* between the two diagonal terms, and not by $\zeta\sin\theta_c$, where ζ would be an arbitrary number.

[6] Equations of this type were empirically advocated for in [8].

when $y \to -1/y$, $\theta_c \to \theta_c \pm \pi/2$. Owing to the invariance of the $\tan$ function by a translation by π of its argument, the set of solutions of (9) inside the $[0, 2\pi]$ interval appears as described on Fig. 1.

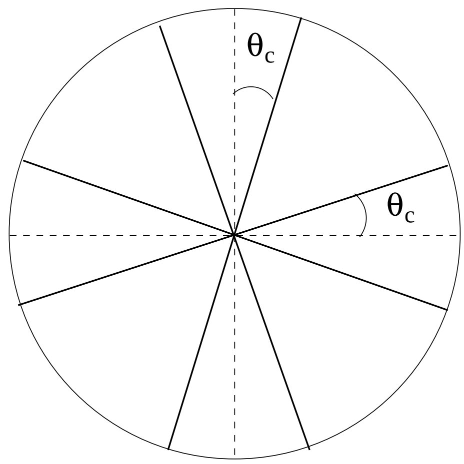

Fig. 1: solutions for the Cabibbo angle

The physical meaning of these symmetries appears on (3). It is invariant by the change $\theta_c \to -\theta_c$ if the following (unitary) transformation of the fields is simultaneously performed :

$$u_m \to e^{i(\theta_a \mp \frac{\pi}{2})} c_m \qquad \text{and} \qquad c_m \to e^{i(-\theta_a \mp \frac{\pi}{2})} u_m. \tag{12}$$

Likewise, (3) is invariant under the change $\theta_c \to \frac{\pi}{2} - \theta_c$ with

$$u_m \to e^{\pm i\frac{\pi}{2}} u_m \qquad \text{and} \qquad c_m \to e^{\mp i\frac{\pi}{2}} c_m. \tag{13}$$

Thus, the symmetry $y \to 1/y$ of (9) reflects the invariance of neutral currents of mass eigenstates by the exchange of families $u_m \leftrightarrow c_m, d_m \leftrightarrow s_m$, and, hence, the condition of universality for such currents, while the symmetry $y \to -y$ corresponds to a simple rephasing of the fields [7].

This approach yields a *constant* value for the Cabibbo angle, a feature which should be confronted with experiment. The most natural, *a priori* q^2-dependent, orthonormal basis related to Cabibbo-like systems mentioned in section 2, should then also exhibit special properties with respect to its q^2-dependence. This will be investigated in a forthcoming work.

[7] The interpretation above is deduced from considering weak *currents*; if one instead considers the mixing matrix of fermionic fields, the role of the symmetries is swapped: the transformation $\theta_c \to \pi/2 - \theta_c$, or $y \to -y$ gets associated with the exchange of families, while the transformation $y \to 1/y \Leftrightarrow \theta_c \to -\theta_c$ corresponds to a simple rephasing of the fields. Physically, it has however always been known that the currents are the relevant quantities.

CONCLUSION AND PROSPECTS

Realizing, as was first done in [5], that mixing matrices of non-degenerate coupled systems should not be parametrized as unitary, led in [1] to uncover maximal mixing of leptons as a class of solutions of very simple physical conditions for their mass eigenstates.

We have now shown that the measured value of the Cabibbo angle θ_c is the one ensuring, in its first order vicinity, the property, already stressed in [1], that universality of diagonal neutral currents for mass eigenstates and the absence of MCNC's reduce to a unique condition.

In this elementary algebraic calculation, no mass ratio appears. It may thus help to provide independent information on the latter. On another side, this feature is welcome for quark-like systems which cannot be defined on-shell and for which, accordingly, the notion of physical mass is ill-defined.

This work, together with [1], strongly suggests that the observed values of mixing angles for quarks and leptons follow from simple physical requirements. The generalization to three generations is currently under investigation [6].

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