HAL
archives-ouvertes.fr

A second order SDE for the Langevin process reflected at a completely inelastic boundary

Jean Bertoin

▶ **To cite this version:**

Jean Bertoin. A second order SDE for the Langevin process reflected at a completely inelastic boundary. 2006. <hal-00106365>

HAL Id: hal-00106365
https://hal.archives-ouvertes.fr/hal-00106365

Submitted on 14 Oct 2006

A second order SDE
for the Langevin process reflected
at a completely inelastic boundary

Jean Bertoin

Laboratoire de Probabilités et Modèles Aléatoires
Université Pierre et Marie Curie
and DMA, Ecole Normale Supérieure
Paris, France

Summary. It was shown in [2] that a Langevin process can be reflected at an energy absorbing boundary. Here, we establish that the law of this reflecting process can be characterized as the unique weak solution to a certain second order stochastic differential equation with constraints, which is in sharp contrast with a deterministic analog.

Key words. Langevin process, reflection, stochastic differential equation.

A.M.S. Classification. Primary 60 H 10 , 60 J 55. Secondary 34 A 12

e-mail. jbe@ccr.jussieu.fr

1 Introduction

Consider the motion of a particle in the half-line $\mathbb{R}_+$ under an external force that governs its acceleration. Assume that the energy of the particle is instantaneously absorbed at the boundary point 0, meaning that the velocity of the particle is always 0 immediately after hitting 0. In other words, the trajectory $(X_t)_{t \geq 0}$ of the particle fulfills the constraints of completely inelastic impacts

$$\begin{cases} X_t \geq 0 \,, \\ X_t = 0 \ \Rightarrow \ \dot{X}_{t+} = 0 \,. \end{cases} \tag{1}$$

and solves the second order differential equation

$$\mathrm{d}X_t = \dot{X}_t \mathrm{d}t \,, \quad \mathrm{d}\dot{X}_t = F_t \mathrm{d}t + \mathrm{d}A_t \quad , \quad A_t = -\sum_{0 < s \leq t} \dot{X}_{s-} \mathbf{1}_{\{X_s = 0\}} \,, \tag{2}$$

where $(F_t)_{t \geq 0}$ denotes the external force. More precisely, $t \to A_t$ is right-continuous non-decreasing and accounts for the kick induced by the boundary. Specifically, if the particle

hits 0 at time t with incoming velocity $\dot{X}_{t-} < 0$, then $A_t - A_{t-} = -\dot{X}_{t-} > 0$ so that $\dot{X}_{t+} = \dot{X}_{t-} + (A_t - A_{t-}) = 0$.

Equation (2) can be viewed as a special case of differential measure inclusions which have been studied initially by Schatzman [13]; see also Ballard [1] and the references therein. It is quite remarkable that multiple solutions may exist even in situations when the external force is $\mathcal{C}^\infty$.

Following a question raised by Bertrand Maury, we are interested in the case when the external force is a generalized function given by a white noise, i.e. when $F_t = \dot{B}_t$ with $(B_t)_{t \geq 0}$ a standard Brownian motion. In this setting, it is natural to consider first the much simpler situation when there is no obstacle at 0, that is to introduce the process with values in $\mathbb{R}$

$$Y_t = y_0 + t\dot{y}_0 + \int_0^t B_s \mathrm{d}s, \qquad t \geq 0.$$

The latter will be called here a *free Langevin process*, started from location $y_0 \in \mathbb{R}$ and with initial velocity $\dot{y}_0 \in \mathbb{R}$; we refer to Lachal [5] for a rich source of results and references in this area. It is easily seen that for $y_0 = \dot{y}_0 = 0$, 0 is an accumulation point of the set of times at which the free Langevin process returns to 0. Informally, this may suggest that if the energy of the Langevin particle is absorbed at each visit to 0, then the particle might never be able to reach a strictly positive velocity, and thus might never take off the boundary. It turns out that this intuition is not correct as we shall see.

It is convenient to agree that throughout this work, all random processes are implicitly càdlàg, i.e. their sample paths are right-continuous and possess left-limits everywhere, a.s. In a preceding work [2], we established the following result of existence and uniqueness in distribution.

Theorem 1 *There exists a strong Markov process $(X_t, \dot{X}_t)_{t \geq 0}$ with values in $\mathbb{R}_+ \times \mathbb{R}$, started from $X_0 = \dot{X}_0 = 0$, such that*

$$\mathrm{d}X_t = \dot{X}_t \mathrm{d}t, \quad \int_0^\infty \mathbf{1}_{\{X_t=0\}} \mathrm{d}t = 0 \text{ and } X_t = 0 \implies \dot{X}_t = 0 \qquad a.s., \tag{3}$$

and which evolves as a free Langevin process as long as $X > 0$. Specifically, for every stopping time S in the natural filtration of X (after the usual completions), if we define $\zeta_S = \inf\{t \geq S : X_t = 0\}$, then conditionally on $X_S = x_0 > 0$ and $\dot{X}_S = \dot{x}_0$, the process $(X_{S+t})_{0 \leq t \leq \zeta_S}$ is independent of $\mathcal{F}_S$ and has the same distribution as $(Y_t)_{0 \leq t \leq \zeta}$, where

$$Y_t = x_0 + t\dot{x}_0 + \int_0^t B_s \mathrm{d}s, \quad \zeta = \inf\{t \geq 0 : Y_t = 0\}$$

and $(B_t)_{t \geq 0}$ is a standard Brownian motion. Further, the preceding requirements determine the distribution of $(X_t, \dot{X}_t)_{t \geq 0}$.

We stress that the strong Markov process $(X_t, \dot{X}_t)_{t \geq 0}$ has jumps at predictable stopping times (namely, the hitting times by X of the boundary point 0), and thus fails to be standard; in particular, the Feller property does not hold.

The main purpose of this work is to connect the process characterized in Theorem 1 to Equation (2) when the external force F is a white noise. In this direction, it is convenient to rewrite (2) in the form

$$\mathrm{d}X_t = \dot{X}_t \mathrm{d}t \quad , \quad \dot{X}_t = B_t + A_t \quad , \quad A_t = -\sum_{0<s\leq t} \dot{X}_{s-} \mathbf{1}_{\{X_s=0\}} . \tag{4}$$

We are now able to state the main result of this work.

Theorem 2 (i) *One can construct on some filtered probability space* $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *an adapted process* $(X_t)_{t\geq 0}$ *distributed as in Theorem 1 and an* $(\mathcal{F}_t)$-*Brownian motion* $(B_t)_{t\geq 0}$, *such that Equations* (1) *and* (4) *hold.*

(ii) *Conversely, if on some filtered probability space* $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, *there is an* $(\mathcal{F}_t)$-*Brownian motion* $(B_t)_{t\geq 0}$ *and an adapted process* $(X_t)_{t\geq 0}$ *which satisfies Equations* (1) *and* (4) *and starts with initial conditions* $X_0 = \dot{X}_0 = 0$, *then* $(X_t)_{t\geq 0}$ *is distributed as in Theorem 1.*

We shall refer to the process X which appears in Theorems 1 and 2 as a *Langevin process reflected at a completely inelastic boundary.* Note that we implicitly restrict our attention to the case when the process starts from 0 with initial velocity 0, which is obviously the most interesting situation and induces no loss of generality. In some loose sense, Theorems 1 and 2 both state the existence and uniqueness in law of the Langevin process reflected at completely inelastic boundary, but viewed from two different perspectives. Theorem 1 belongs to the framework of the theory of Markov processes and their excursions, whereas Theorem 2 is expressed in terms of stochastic differential equations. It is well-known that these two theories are intimately connected, and one may expect that a soft argument should enable us to deduce Theorem 2 from Theorem 1.

In this direction, the existence of a weak solution to (4) and (1) is rather easy and will be established in the first part of Section 2 by investigating, in the framework of stochastic calculus, the explicit construction given in [2] of the process specified by Theorem 1. More precisely, the latter is obtained from the free Langevin process associated to some standard Brownian motion $(W_t)_{t\geq 0}$ first by a reflection *à la* Skorohod and then by a non-invertible random time-substitution.

However, establishing weak uniqueness in Theorem 2 is less straightforward. Indeed, if we aim at applying Theorem 1, then we have to check *a priori* that any weak solution $(X_t, \dot{X}_t)_{t\geq 0}$ to (4) and (1) enjoys the strong Markov property. But it is well-known that solutions of an SDE fulfill the Markov property only in the situation when weak uniqueness holds for the SDE, and thus Theorem 1 cannot help. We also stress that weak uniqueness is the most striking aspect of Theorem 2 as it is in sharp contrast with the deterministic situation for which (strong) uniqueness can fail even with a smooth forcing.

In the second part of Section 2, we shall observe a key point which lies at the heart of the proof of weak uniqueness. From the same Brownian motion $(W_t)_{t\geq 0}$ which is used to construct a weak solution $(X_t, B_t)_{t\geq 0}$, one can also built another standard Brownian motion $(B'_t)_{t\geq 0}$ which is independent of $(B_t)_{t\geq 0}$, and such that $(W_t)_{t\geq 0}$ can be recovered from $(X_t, B_t)_{t\geq 0}$ and $(B'_t)_{t\geq 0}$.

3

Weak uniqueness is established in Section 3. We consider any solution $(X_t, B_t)_{t \geq 0}$ to (4) and (1) where $(B_t)_{t \geq 0}$ is some Brownian motion. We then introduce an independent standard Brownian motion $(B'_t)_{t \geq 0}$, and using the analysis developed in Section 2, we construct from $(X_t, B_t)_{t \geq 0}$ and $(B'_t)_{t \geq 0}$ another Brownian motion $(W_t)_{t \geq 0}$, such that $(X_t)_{t \geq 0}$ can be recovered from $(W_t)_{t \geq 0}$ in the same way as in Section 2.

In the final Section, we first make some brief historical comments about the question of uniqueness in the -deterministic- setting of mechanical systems with perfect constraints. For the reader's convenience, we also provide a simple example showing that uniqueness of the solution to (1) and (2) may fail even when the external force is smooth. Finally, we discuss some open questions regarding strong solutions to (4) and (1).

Nota Bene. In this paper, I will use the same notation $X, B, W,...$ for processes which, *in fine*, will be shown to have the same distributions. However the initial definition and assumptions for these processes may be different in different sections. I hope that the reader will find this helpful and not confusing.

2   A weak solution

The first purpose of this section is to check that the construction of Section 2 in [2] also provides a solution to (4) and (1). Then we shall study this construction in further details to gain insight for the proof of weak uniqueness.

2.1   Construction of a weak solution

We start by recalling the construction of Section 2 in [2] and some of its properties.

Let $W = (W_t, t \geq 0)$ be a standard Wiener process started from $W_0 = 0$ and write $(\mathcal{W}_t)_{t \geq 0}$ for its natural filtration after the usual completions. Define the free Langevin process

$$Y_t := \int_0^t W_s \mathrm{d}s, \qquad t \geq 0,$$

its infimum process

$$I_t := \inf\{Y_s : 0 \leq s \leq t\},$$

and the random closed set of times when Y coincides with its infimum

$$\mathcal{I} := \{t \geq 0 : Y_t = I_t\}.$$

We write $\mathcal{I}^o$ for the interior of $\mathcal{I}$ and recall from Lemma 2 in [2] that with probability one, the boundary $\partial \mathcal{I} = \mathcal{I} \backslash \mathcal{I}^o$ has zero Lebesgue measure. Further the canonical decomposition of the open set $\mathcal{I}^o$ into disjoint open intervals is given by

$$\mathcal{I}^o = \bigcup_{u \in \mathcal{J}}]u, d_u[,$$

where $\mathcal{J}$ is the set of times at which Y reaches its infimum for the first time during some negative excursion of W, and d_u the first return time to 0 for W after the instant u. That is

$$\mathcal{J} := \{t \geq 0 : W_t < 0, Y_t = I_t \text{ and } Y_{t-\varepsilon} > I_{t-\varepsilon} \text{ for all } \varepsilon > 0 \text{ sufficiently small}\}$$

and

$$d_u := \inf\{s > u : W_s = 0\}.$$

It is readily seen that $\mathcal{J}$ can be expressed in the form of a countable family of stopping times in the filtration $(\mathcal{W}_t)_{t\geq 0}$. For instance $\mathcal{J} = \{S_{k,n} : k, n \in \mathbb{N}\}$, where $S_{k,n}$ is the k-th instant t such that $Y_t = I_t$, $Y_{t-\varepsilon} > I_{t-\varepsilon}$ for all $\varepsilon > 0$ sufficiently small and the velocity at time t fulfills $\dot{Y}_t \in]-1/(n-1), -1/n]$. Note that the $d_{S_{k,n}}$ are then also stopping times.

Finally we introduce the right-continuous time-substitution

$$T_t := \inf\left\{s \geq 0 : \int_0^s \mathbf{1}_{\{Y_v > I_v\}} \mathrm{d}v > t\right\}, \qquad t \geq 0,$$

and then the free Langevin process reflected at its infimum (in the sense of Skorohod) and time-changed by T_t, that is we set for every $t \geq 0$

$$X_t := (Y - I) \circ T_t.$$

We mention that the process $t \to Y_t - I_t$ has been studied first by Lapeyre [6]. Clearly, the process $t \to X_t$ only takes nonnegative values, is continuous, and it can be shown that it possess a right-derivative at every $t \geq 0$ given by

$$\dot{X}_t = W \circ T_t;$$

see Equation (7) in [2]. We also set $\mathcal{F}_t = \mathcal{W}_{T_t}$.

The following proposition establishes the existence stated in Theorem 2(i).

Proposition 1 *Define*

$$A_t := \int_0^{T_t} \mathbf{1}_{\{Y_s = I_s\}} \mathrm{d}W_s \quad and \quad B_t := \int_0^{T_t} \mathbf{1}_{\{Y_s > I_s\}} \mathrm{d}W_s, \qquad t \geq 0,$$

so that

$$\dot{X}_t = A_t + B_t, \qquad t \geq 0.$$

Then there is the identity

$$A_t = -\sum_{0 < s \leq t} \dot{X}_{s-}\mathbf{1}_{\{X_s=0\}}, \qquad t \geq 0,$$

and $(B_t)_{t\geq 0}$ is an $(\mathcal{F}_t)$-Brownian motion. As a consequence, $(X_t, \dot{X}_t)_{t\geq 0}$ is a weak solution to (4) and (1) with initial condition $X_0 = \dot{X}_0 = 0$.

5

Remark : The fact that the series $\sum_{0<s\leq t} \dot{X}_{s-}\mathbf{1}_{\{X_s=0\}}$ converges for every $t \geq 0$ a.s. can be deduced from Corollary 2 in [2]. However this fact shall be established directly in the present analysis.

Proof: We express $\dot{X}_t = W \circ T_t = A_t + B_t$, where A_t and B_t are defined in the statement. The basic facts that have been recalled above imply the identities

$$\int_0^t \mathbf{1}_{\{Y_s=I_s\}}\mathrm{d}W_s = \int_0^t \mathbf{1}_{\{s\in\mathcal{I}\}}\mathrm{d}W_s = \int_0^t \mathbf{1}_{\{s\in\mathcal{I}^o\}}\mathrm{d}W_s = \sum_{u\in\mathcal{J}}(W_{d_u\wedge t} - W_{u\wedge t}). \tag{5}$$

On the one hand, the assertion that $(B_t)_{t\geq 0}$ is an $(\mathcal{F}_t)$-Brownian motion is seen from the very definition of the time-substitution T_t and the Dambis-Dubins-Schwarz theorem (see e.g. [12] on its page 181). On the other hand, again by definition, $T_t \notin \mathcal{I}^o$ and $W_{d_u} = 0$ for every $u \in \mathcal{J}$. We deduce from (5) the identity

$$A_t = - \sum_{u\in\mathcal{J}, u\leq T_t} W_u.$$

Further, it is easily checked that $\mathcal{J}$ coincides with the set of times of the form $u = T_{s-}$ with $s > 0$ an instant at which X hits the boundary point 0 with a negative incoming velocity (i.e. $X_s = 0$ and $\dot{X}_{s-} < 0$). Since $\dot{X}_{s-} = W \circ T_{s-}$, we conclude that

$$A_t = - \sum_{0<s\leq t} \dot{X}_{s-}\mathbf{1}_{\{X_s=0\}}, \qquad t \geq 0.$$

To complete the proof, either we observe that if t is an instant at which $X_t = 0$, then $\dot{X}_{t-} = B_t + A_{t-}$ and thus

$$\dot{X}_t = B_t + A_t = \dot{X}_{t-} + (A_t - A_{t-}) = \dot{X}_{t-} - \dot{X}_{t-} = 0,$$

or we just recall from Equation (3) in Theorem 1 that $X_t = 0 \Rightarrow \dot{X}_t = 0$. $\qquad\square$

2.2 Some further properties

We introduce the time-substitution

$$T'_t := \inf\left\{s \geq 0 : \int_0^s \mathbf{1}_{\{Y_v=I_v\}}\mathrm{d}v > t\right\}, \qquad t \geq 0,$$

which can be thought of as the dual to T_t. Next we set

$$B'_t := \int_0^{T'_t} \mathbf{1}_{\{Y_v=I_v\}}\mathrm{d}W_v, \qquad t \geq 0,$$

and then, for every $x \geq 0$,

$$\sigma'(x) := \inf\{t \geq 0 : B'_t > x\}$$

for the first passage time of B' above the level x.

Lemma 1 *With probability one, there is the identity*

$$T_t = t + \sigma'(A_t), \qquad t \geq 0.$$

Proof: It will be convenient to write $B'(t) := B'_t$ and observe from the definition of B' and A_t (in Proposition 1) the identities

$$
\begin{aligned}
\sigma'(A_t) &= \inf\left\{ \int_0^s \mathbf{1}_{\{Y_v=I_v\}}\mathrm{d}v : B'\left(\int_0^s \mathbf{1}_{\{Y_v=I_v\}}\mathrm{d}v\right) > A_t \right\} \\
&= \inf\left\{ \int_0^s \mathbf{1}_{\{Y_v=I_v\}}\mathrm{d}v : \int_0^s \mathbf{1}_{\{Y_v=I_v\}}\mathrm{d}W_v > \int_0^{T_t} \mathbf{1}_{\{Y_v=I_v\}}\mathrm{d}W_v \right\}.
\end{aligned}
$$

Then recall (5). Observe that for every $u \in \mathcal{J}$, the process $s \to W_{d_u \wedge s} - W_{u \wedge s}$ is 0 before time u, takes some strictly positive values immediately after time u, reaches its overall maximum for the first time at d_u, and remains constant after d_u. Note furthermore that the intervals $[u, d_u]$ for $u \in \mathcal{J}$ are pairewise disjoint. It follows that whenever $t \notin \mathcal{I}^o$, the stochastic integral $s \to \int_0^s \mathbf{1}_{\{Y_v=I_v\}}\mathrm{d}W_v$ attains its overall maximum on the time-interval $[0, t]$ at time t, and if we define $r(t) = \inf\{s > t : s \in \mathcal{I}^o\}$, then the first instant when this stochastic integral exceeds its value at time t is $r(t)$. Further this stochastic integral remains constant on $[t, r(t)]$.

Applying these observations to the random time $T_t \notin \mathcal{I}^o$, we conclude that

$$\sigma'(A_t) = \int_0^{r(T_t)} \mathbf{1}_{\{Y_v=I_v\}}\mathrm{d}v = \int_0^{T_t} \mathbf{1}_{\{Y_v=I_v\}}\mathrm{d}v,$$

and thus

$$T_t = \int_0^{T_t} \mathbf{1}_{\{Y_v>I_v\}}\mathrm{d}v + \int_0^{T_t} \mathbf{1}_{\{Y_v=I_v\}}\mathrm{d}v = t + \sigma'(A_t),$$

as it has been stated. $\qquad \square$

We are now able to establish the following statement, which will provides us with the hint for establishing weak uniqueness in the next section.

Proposition 2 *The process $(B'_t)_{t\geq 0}$ is a standard Brownian which is independent of $(B_t)_{t\geq 0}$. Further, W can be recovered from (X, B, B') as*

$$W_t = B_{\tau(t)} + B'_{\tau'(t)},$$

where

$$\tau(t) := \inf\{s \geq 0 : s + \sigma'(A_s) > t\} \quad and \quad \tau'(t) := 1 - \tau(t).$$

Proof: That $(B'_t)_{t\geq 0}$ is a Brownian motion which is independent of $(B_t)_{t\geq 0}$ follows immediately from the definition of B and B' and Knight's extension of the Dambis-Dubins-Schwarz theorem (see e.g. [12] on its page 183).

Then we simply write

$$W_t = \int_0^t \mathbf{1}_{\{Y_s > I_s\}} \mathrm{d}W_s + \int_0^t \mathbf{1}_{\{Y_s = I_s\}} \mathrm{d}W_s = B_{\tau(t)} + B'_{\tau'(t)}\,,$$

where

$$\tau(t) := \int_0^t \mathbf{1}_{\{Y_s > I_s\}} \mathrm{d}s \quad \text{and} \quad \tau'(t) := \int_0^t \mathbf{1}_{\{Y_s = I_s\}} \mathrm{d}s\,.$$

The identity $\tau(t) + \tau'(t) = t$ is obvious. By definition, $T_t = \inf\{s \geq 0 : \tau(s) > t\}$ and thus $\tau(\cdot)$ coincides with the continuous left-inverse of the strictly increasing time-change $T_\cdot$, i.e. $\tau(T_t) \equiv t$. We get from Lemma 1 that $\tau(t) = \inf\{s \geq 0 : s + \sigma'(A_s) > t\}$. $\qquad\square$

Remark : By a more careful analysis, one could also establish a stronger result of independence, namely that X and B' are independent processes. Nonetheless, as this will not be needed in this work and also follows from the analysis in the next section, we leave the direct proof to the interested reader. In this direction, we also stress that the Brownian motion B is adapted to the natural filtration of X, as one sees from Proposition 1. But we do not know whether, conversely, X is adapted to the natural filtration of the Brownian motion B, that is whether the solution to (4) is strong.

3 Uniqueness in distribution

In this Section, we consider some filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We assume there is an $(\mathcal{F}_t)$-Brownian motion $(B_t)_{t\geq 0}$ and an adapted process $(X_t)_{t\geq 0}$ which satisfies Equations (1) and (4) and starts with initial conditions $X_0 = \dot{X}_0 = 0$. Our goal is to establish that $(X_t)_{t\geq 0}$ has the distribution of the process in the preceding Section. In this direction, Proposition 2 points at the role of an independent Brownian motion, so we assume that the same probability space Ω can be endowed with another filtration $(\mathcal{F}'_t)_{t\geq 0}$ such that the terminal sigma-fields $\mathcal{F}_\infty$ and $\mathcal{F}'_\infty$ are independent, and that there exists an $(\mathcal{F}'_t)$-Brownian motion $(B'_t)_{t\geq 0}$. Clearly, these assumptions induce no loss of generality (as it suffices to enlarge the initial probability space).

Just as in the preceding Section, we then write

$$\sigma'(x) := \inf\{t \geq 0 : B'_t > x\}$$

for the first passage time of B' above level $x \geq 0$, and define

$$T_t := t + \sigma'(A_t)\,, \qquad t \geq 0\,.$$

The process $t \to T_t$ is strictly increasing and thus possesses a continuous left-inverse

$$\tau(t) := \inf\{s \geq 0 : s + \sigma'(A_s) > t\}\,, \qquad t \geq 0\,,$$

i.e. $\tau(T_t) \equiv t$. Clearly $0 \leq \tau(t) \leq t$, and we also set

$$\tau'(t) := t - \tau(t)\,, \qquad t \geq 0\,.$$

Finally we define
$$W_t := B_{\tau(t)} + B'_{\tau'(t)}, \qquad t \geq 0.$$

The weak uniqueness stated in Theorem 2(ii) is now a consequence of the following.

Proposition 3 (i) *The process* $(W_t)_{t\geq 0}$ *is a standard Brownian motion.*

(ii) The process $(X_t)_{t\geq 0}$ *can be recovered as*
$$X_t = (Y - I) \circ T_t, \qquad t \geq 0,$$

where
$$Y_t := \int_0^t W_s \mathrm{d}s \ \text{ and } \ I_t := \inf\{Y_s : 0 \leq s \leq t\}.$$

(iii) Finally, there is the identity
$$T_t = \inf\left\{ s \geq 0 : \int_0^s \mathbf{1}_{\{Y_v > I_v\}}\mathrm{d}v > t \right\}, \qquad t \geq 0.$$

Remark : Proposition 3 shows that X is distributed as the process which appears in Theorem 1, and as a consequence, we must have $\int_0^\infty \mathbf{1}_{\{X_t=0\}}\mathrm{d}t = 0$ a.s. It may be interesting to point out that this property can be checked directly from (4) and (1). More precisely, the set of times t when $X_t = 0$ is contained into the zero set of the Brownian semi-martingale $\dot{X} = B + A$. That the latter has zero Lebesgue measure a.s. can be seen from the occupation density formula for Brownian semi-martingales, see e.g. Corollary 1 in [11] on its page 216.

The rest of this Section is devoted to the proof of Proposition 3; we start with the first part.

Proof of (i): Although one may perhaps establish the result more directly by stochastic calculus, we shall use an approximation, as this makes the proof more intuitive. Specifically, pick $\varepsilon > 0$ and introduce
$$A_t^{(\varepsilon)} := -\sum_{0<s\leq t} \dot{X}_{s-}\mathbf{1}_{\{X_s=0, \dot{X}_{s-}<-\varepsilon\}}, \qquad t \geq 0,$$

so that $t \to A_t^{(\varepsilon)}$ is a non-decreasing process and
$$\lim_{\varepsilon\downarrow 0} \uparrow A_t^{(\varepsilon)} = A_t, \qquad t \geq 0.$$

Set also
$$\tau^{(\varepsilon)}(t) := \inf\{s \geq 0 : s + \sigma'(A_s^{(\varepsilon)}) > t\},$$

so $t \to \tau^{(\varepsilon)}(t)$ is a continuous non-decreasing process with $0 \leq \tau^{(\varepsilon)}(t) \leq t$ and
$$\lim_{\varepsilon\downarrow 0} \downarrow \tau^{(\varepsilon)}(t) = \tau(t), \qquad t \geq 0.$$

Thus, if we define
$$W_t^{(\varepsilon)} := B_{\tau^{(\varepsilon)}(t)} + B'_{t-\tau^{(\varepsilon)}(t)}, \qquad t \geq 0,$$

9

then we have

$$\lim_{\varepsilon \to 0+} W_t^{(\varepsilon)} = W_t \,. \tag{6}$$

Hence, it now suffices to check that for every $\varepsilon > 0$, the process $(W_t^{(\varepsilon)})_{t \geq 0}$ is a standard Brownian motion. Let us first provide an intuitive explanation. The time-change $t \to \tau^{(\varepsilon)}(t)$ is an absolutely continuous process, and its derivative $\mathrm{d}\tau^{(\varepsilon)}(t)/\mathrm{d}t := \dot{\tau}^{(\varepsilon)}(t)$ is a step process which takes alternately the values 1 and 0. The dual time-change $t \to t - \tau^{(\varepsilon)}(t)$ is also absolutely continuous with derivative $1 - \dot{\tau}^{(\varepsilon)}(t) \in \{0, 1\}$. The process $W^{(\varepsilon)}$ is thus obtained by following alternately the paths of two independent Brownian motions, B and B', in a way which may remind us of the classical two-arm bandit (see, e.g. [4]). More precisely, $W^{(\varepsilon)}$ follows B when $\dot{\tau}^{(\varepsilon)}(t) = 1$ and follows B' otherwise. The instants when $W^{(\varepsilon)}$ switches from B to B' correspond to the jump times of $A^{(\varepsilon)}$, whereas the instants when $W^{(\varepsilon)}$ switches from B' to B correspond to certain first passage times of B'. These switching times form an increasing sequence of predictable random times for $W^{(\varepsilon)}$, and we can then deduce from the strong Markov property that $W^{(\varepsilon)}$ is a standard Brownian motion.

More precisely, the assumption that X solves (4) and elementary properties of the free Langevin process easily imply that with probability 1, the set of times t at which X_t hits the boundary 0 with incoming velocity $\dot{X}_{t-} < -\varepsilon$ is both discrete and unbounded. Thus the set of jump times of $A^{(\varepsilon)}$ can be expressed as an increasing sequence of stopping times $J_1^{(\varepsilon)} < J_2^{(\varepsilon)} < \cdots$ where $J_0^{(\varepsilon)} = 0$ and

$$J_n^{(\varepsilon)} := \inf\{t > J_{n-1}^{(\varepsilon)} : X_t = 0 \text{ and } \dot{X}_{t-} < -\varepsilon\}, \qquad n \in \mathbb{N}\,,$$

and $\lim_{n \to \infty} J_n^{(\varepsilon)} = \infty$.

Write for simplicity $a_n^{(\varepsilon)} = A_{J_n^{(\varepsilon)}}^{(\varepsilon)}$, and consider the increasing sequence $(\sigma'(a_n^{(\varepsilon)}))_{n \in \mathbb{N}}$. As each $a_n^{(\varepsilon)}$ is a random variable which is measurable with respect to $\mathcal{F}_\infty$ and thus independent of B', the $\sigma'(a_n^{(\varepsilon)})$ form an increasing sequence of randomized $(\mathcal{F}_t')$-stopping times. The strong Markov property entails that conditionally on $(a_n^{(\varepsilon)})_{n \in \mathbb{N}}$, the pieces of Brownian paths

$$(B'_{t+\sigma'(a_{n-1}^{(\varepsilon)})} - a_{n-1}^{(\varepsilon)} : 0 \leq t < \sigma'(a_n^{(\varepsilon)}) - \sigma'(a_{n-1}^{(\varepsilon)}))$$

are independent, and for each fixed $n \in \mathbb{N}$, the conditional law of this n-th piece is that of a standard Brownian motion killed when it exceeds $a_n^{(\varepsilon)} - a_{n-1}^{(\varepsilon)}$.

The process $(W_t^{(\varepsilon)})_{t \geq 0}$ is obtained by splicing the sequence of pieces of paths

$$(B_t : 0 \leq t < J_1^{(\varepsilon)}),\, (B_t' : 0 \leq t < \sigma'(a_1^{(\varepsilon)})),\, (B_{t+J_1^{(\varepsilon)}} - B_{J_1^{(\varepsilon)}} : 0 \leq t < J_2^{(\varepsilon)} - J_1^{(\varepsilon)}),\, \ldots$$

In particular

$$W_t^{(\varepsilon)} = \begin{cases} B_t & \text{when } 0 \leq t < J_1^{(\varepsilon)}\,, \\ B'_{t-J_1^{(\varepsilon)}} + B_{J_1^{(\varepsilon)}} & \text{when } J_1^{(\varepsilon)} \leq t < J_1^{(\varepsilon)} + \sigma'(a_1^{(\varepsilon)})\,, \end{cases}$$

and the strong Markov property of Brownian motion shows that the process $(W_t^{(\varepsilon)} : 0 \leq t < J_1^{(\varepsilon)} + \sigma'(a_1^{(\varepsilon)}))$ has the same law as $(B_t : 0 \leq t < \rho_1)$ where ρ_1 is the $(\mathcal{F}_t)$-stopping time defined

10

as $\rho_1 := \inf\{t > J_1^{(\varepsilon)} : B_t - B_{J_1^{(\varepsilon)}} > a_1^{(\varepsilon)}\}$. Splicing more and more pieces, we now see by that an iteration of this argument that $(W_t^{(\varepsilon)} : 0 \leq t < J_n^{(\varepsilon)} + \sigma'(a_n^{(\varepsilon)}))$ has the same distribution as $(B_t : 0 \leq t < \rho_n)$ where $\rho_n := \inf\{t > J_n^{(\varepsilon)} : B_t - B_{J_n^{(\varepsilon)}} > a_n^{(\varepsilon)}\}$. Letting $n \to \infty$, we conclude that $(W_t^{(\varepsilon)})_{t \geq 0}$ is a standard Brownian motion, and an appeal to (6) completes the proof. $\quad\square$

Next, let us write

$$\mathcal{D}_A := \{t > 0 : A_t > A_{t-}\}$$

for the set of times where A is discontinuous. Observe from (4) that we have also the identification

$$\mathcal{D}_A = \{t > 0 : X_t = 0 \text{ and } \dot{X}_{t-} < 0\}$$

as the set of instants when X hits the boundary 0 with a strictly negative incoming velocity. An important step in the proof of Proposition 3 is provided by the following representation of the processes $\tau(\cdot)$ and $\tau'(\cdot)$.

Lemma 2 *Introduce the random open set*

$$\mathcal{O} := \bigcup_{t \in \mathcal{D}_A}]t + \sigma'(A_{t-}), t + \sigma'(A_t)[\,,$$

and write $\mathcal{O}^c := [0, \infty[\backslash\mathcal{O}$ for its complementary set.

(i) The processes $t \to \tau(t)$ and $t \to \tau'(t)$ are both absolutely continuous non-decreasing processes with Stieltjes measures given by

$$\mathrm{d}\tau(t) = \mathbf{1}_{\mathcal{O}^c}\mathrm{d}t \quad , \quad \mathrm{d}\tau'(t) = \mathbf{1}_{\mathcal{O}}\mathrm{d}t\,.$$

(ii) We have $W_t \leq 0$ and $X_{\tau(t)} = 0$ for every $t \in \mathcal{O}$, a.s.

Proof: (i) Write λ for the Lebesgue measure on $\mathbb{R}_+$. We have to show that

$$\lambda\left(\mathcal{O} \cap [0, t]\right) = \tau'(t)\,, \qquad t \geq 0\,. \tag{7}$$

In this direction, let $\mathcal{A}$ denote the closed range of the process $A_\cdot$, viz. the set of points of the type A_t or A_{t-} for some $t \geq 0$. The complementary set $\mathcal{A}^c := \mathbb{R}_+\backslash\mathcal{A}$ has a canonical decomposition as union of disjoint open intervals given by

$$\mathcal{A}^c = \bigcup_{t \in \mathcal{D}_A}]A_{t-}, A_t[\,.$$

Observe that, since $A_\cdot$ is a pure jump process, then for every $t \geq 0$

$$\lambda([0, A_t]\backslash\mathcal{A}) = \sum_{s \in \mathcal{D}_A \cap [0,t]} (A_s - A_{s-}) = A_t\,,$$

and hence $\lambda(\mathcal{A}) = 0$.

11

On the other hand, it is well-known that the first passage process σ' is a stable subordinator with index $1/2$. In particular, it is purely discontinuous, and, by the Lévy-Itô decomposition, the process of its jumps is a Poisson point process. Since $\mathcal{A}$ has zero Lebesgue measure and is independent of σ', $\mathcal{A}$ does not contain any jump time of σ', a.s. It follows that for every $v \geq 0$

$$\lambda\left(\mathcal{O} \cap [0, v + \sigma'(A_v)]\right) = \sum_{s \in \mathcal{D}_A \cap [0,v]} \left(\sigma'(A_s) - \sigma'(A_{s-})\right) = \sigma'(A_v).$$

Recall now that $T_v = v + \sigma'(A_v)$ and observe that

$$\tau'(T_v) = T_v - \tau(T_v) = T_v - v = v + \sigma'(A_v) - v = \sigma'(A_v).$$

We have thus checked that (7) holds for every t of the form $t = T_v$ for some $v \geq 0$, and hence, by approximation, also for every t of the form $t = T_{v-}$ for some $v > 0$.

Finally, suppose that $t \in]T_{v-}, T_v[$, where $v = \tau(t)$. We get from above

$$\lambda\left(\mathcal{O} \cap [0, t]\right) = \lambda\left(\mathcal{O} \cap [0, T_{v-}]\right) + t - T_{v-} = \tau'(T_{v-}) + t - T_{v-}.$$

But

$$\tau'(T_{v-}) + t - T_{v-} = T_{v-} - \tau(T_{v-}) + t - T_{v-} = t - v = t - \tau(t) = \tau'(t),$$

and we conclude that (7) holds for all $t \geq 0$.

(ii) If $t \in \mathcal{O}$, then $t \in]T_{v-}, T_v[$ where $v = \tau(t)$. By definition, we have

$$W_t = B_v + B'_{t-v}.$$

On the other hand, $v \in \mathcal{D}_A$ and thus $X_v = 0$. Further, by (4) and (1), we have $\dot{X}_v = B_v + A_v = 0$. We deduce that

$$W_t = B'_{t-v} - A_v \leq 0,$$

as $t - v < \sigma'(A_v)$ (because $t < T_v = v + \sigma'(A_v)$). $\qquad\square$

We are now able to establish the second part of Proposition 3.

Proof of (ii): We decompose

$$Y_t = \int_0^t W_s \mathrm{d}s = \int_0^t W_s \mathrm{d}\tau(s) + \int_0^t W_s \mathrm{d}\tau'(s).$$

The change of variables $s = T_v$ enables us to rewrite the first integral in the sum as

$$\int_0^t W_s \mathrm{d}\tau(s) = \int_0^{\tau(t)} W_{T_v} \mathrm{d}v = \int_0^{\tau(t)} (B_{\tau(T_v)} + B'_{T_v - \tau(T_v)}) \mathrm{d}v.$$

Since $\tau(T_v) = v$ and $T_v - \tau(T_v) = T_v - v = \sigma'(A_v)$, the right-hand side equals

$$\int_0^{\tau(t)} (B_v + A_v) \mathrm{d}v = \int_0^{\tau(t)} \dot{X}_v \mathrm{d}v = X_{\tau(t)},$$

where the first equality follows from (4).

Next, we write $I'_t := \int_0^t W_s \mathrm{d}\tau'(s)$, so that the process $t \to Y_t - I'_t = X_{\tau(t)}$ is nonnegative. Further, we know from Lemma 2(ii) that $t \to I'_t$ is a non-increasing process and that the Stieltjes measure $\mathrm{d}(-I'_t)$ assigns no mass to $\mathcal{O}^c$, and *a fortiori* is supported on the set of times t such that $Y_t - I'_t = X_{\tau(t)} = 0$. An application of Skorohod's reflection principle (see for instance [12] on its page 239) enables to make the identification $I'_t = \inf\{Y_s : 0 \le s \le t\} = I_t$. We conclude that $X_t = X_{\tau(T_t)} = Y_{T_t} - I_{T_t}$. $\qquad\square$

Finally, we turn our attention to the third part of Proposition 3.

Proof of (iii): On the one hand, we have seen in the proof of part (ii) above that the infimum I of the free Langevin process Y can be expressed as

$$I_t = \int_0^t W_s \mathrm{d}\tau'(s) = \int_0^t \mathbf{1}_{\mathcal{O}}(s) W_s \mathrm{d}s \,,$$

where the second identity follows from Lemma 2.

On the other hand, we must have $W_t \le 0$ for every t such that $Y_t = I_t$. Indeed, if we had $W_t > 0$ for such a time t, then Y would be strictly increasing on some neighborhood of t, which is absurd. Now for every $t \ge 0$ such that $Y_t = I_t$ and $W_t < 0$, Y is strictly decreasing on some interval $[t, t']$ with $t' > t$, and thus $Y = I$ on $[t, t']$. Since the total time that W spends at 0 is zero, we also obtain

$$I_t = \int_0^t \mathbf{1}_{\{Y_s = I_s\}} W_s \mathrm{d}s \,.$$

Using again the fact that the total time that W spends at 0 is zero, we deduce by comparison of these two expressions that with probability one, the random sets $\mathcal{O}$ and $\{s \ge 0 : Y_s = I_s\}$ coincide λ-almost everywhere. More precisely, recall the notation $\mathcal{I} := \{s \ge 0 : Y_s = I_s\}$ and that the boundary $\partial \mathcal{I} = \mathcal{I} \backslash \mathcal{I}^o$ has zero Lebesgue measure. We now see that the open sets $\mathcal{O}$ and $\mathcal{I}^o$ coincide a.s. As a consequence of Lemma 2,

$$\tau(t) = \int_0^t \mathbf{1}_{\mathcal{O}^c}(s) \mathrm{d}s = \int_0^t \mathbf{1}_{\{Y_s > I_s\}} \mathrm{d}s \,, \qquad t \ge 0 \,,$$

and since $T_.$ is the right-continuous inverse of $\tau(\cdot)$, this completes the proof. $\qquad\square$

4 Some comments and questions

We mentioned in the Introduction that second order differential equations with constraints of the type (1) and (2) may have multiple solutions even in the situation when the external force F is smooth. This was first pointed out by Bressan [3], who also made the conjecture that uniqueness holds when the force is a polynomial function of time. Schatzman [13] formulated the general setting of second order differential inclusions, and established a general theorem of existence of solutions. She also recovered independently Bressan's example of a force of class $\mathcal{C}^\infty$ for which such a system possesses multiple solutions. Percivale [10] was the first to show

that uniqueness holds for systems with only one degree of freedom, when the force is given by an analytic function of the time and depends neither on the position nor on the velocity of the particle, and then Schatzman [14] extended this to the much harder case when the force is an analytic function of time, position and velocity. Finally Ballard [1] considered more general discrete systems with several degrees of freedom and established that uniqueness always holds in the case when the force is analytic. We also refer to Maury [7, 8], Moreau [9] and Stewart [15] for numerical schemes for the computation of the motion of bodies systems with inelastic impacts.

For the convenience of the reader, we shall propose here simple example of an external force of class $\mathcal{C}^k$ (for any fixed $k \in \mathbb{N}$) for which multiple solutions to (1) and (2) exist. Consider the increasing sequences $0 < \cdots < s_n < t_n < s_{n+1} < \cdots$ given by

$$s_n := 2^{2n} \quad \text{and} \quad t_n := 2^{2n+1}, \qquad n \in \mathbb{Z}.$$

Then introduce the convex increasing function $\alpha : [0, \infty[\to [0, \infty[$ which is linear on the intervals $[s_n, s_{n+1}]$ and such that $\alpha(s_n) = s_n^{k+3}$. Similarly, let $\beta : [0, \infty[\to [0, \infty[$ denote the convex increasing function which is linear on the intervals $[t_n, t_{n+1}]$ and such that $\beta(t_n) = t_n^{k+3}$. Observe that α and β enjoy a property of self-similarity, namely

$$\alpha(4u) = 4^{k+3}\alpha(u) \quad \text{and} \quad \beta(4u) = 4^{k+3}\beta(u), \qquad u > 0.$$

It should be obvious from a picture that there exists a function $\varphi :]0, \infty[\to \mathbb{R}$ of class $\mathcal{C}^{k+3}$, which is bounded from above by both α and β, enjoys the same property of self-similarity, viz. $\varphi(4u) = 4^{k+3}\varphi(u)$, and fulfills the following requirements :

$$\begin{cases} \varphi(u) = \alpha(u) \Leftrightarrow u = s_n \text{ for some } n \in \mathbb{Z}, \\ \varphi(u) = \beta(u) \Leftrightarrow u = t_n \text{ for some } n \in \mathbb{Z}, \\ \dot{\varphi}(s_n) = \dot{\alpha}(s_n+) \text{ for every } n \in \mathbb{Z}, \\ \dot{\varphi}(t_n) = \dot{\beta}(t_n+) \text{ for every } n \in \mathbb{Z}. \end{cases}$$

More precisely, one constructs first a function φ which satisfies the preceding requirements on the interval $[1, 4]$, in such a way that for every $\ell = 0, \ldots, k+3$, the ℓ-th derivative $\varphi^{(\ell)}$ of φ has $\varphi^{(\ell)}(4-) = 4^{k+3-\ell}\varphi^{(\ell)}(1+)$. Then φ is extended to $]0, \infty[$ by self-similarity, and we set $\varphi(0) = 0$. Again by self-similarity, we get that φ is now of class $\mathcal{C}^{k+2}$ on $[0, \infty[$ with $\varphi^{(\ell)}(0+) = 0$ for every $\ell = 0, \ldots, k+2$. The requirements implies that $X_u := \alpha(u) - \varphi(u)$ solves (1) and (2) with $F_u := -\ddot{\varphi}(u)$ and $A_u := \dot{\alpha}(u)$. Similarly, $X_u := \beta(u) - \varphi(u)$ solves (1) and (2) with the same external force and $A_u := \dot{\beta}(u)$. Hence Equations (1) and (2) have at least two distinct solutions for $F_u := -\ddot{\varphi}(u)$.

Self-similarity is merely used above as a convenient tool for checking the regularity of the external force at 0, and a perusal of the argument reveals that a large class of counter-examples to uniqueness can be built by mimicking the preceding construction, using now an arbitrary strictly convex increasing function $c : [0, \infty[\to [0, \infty[$ ($c(u) = u^{k+3}$ in the example above), and arbitrary increasing sequences $(s_n)_{n \in \mathbb{Z}}$ and $(t_n)_{n \in \mathbb{Z}}$ with no common point and such that $\lim_{n \to -\infty} s_n = \lim_{n \to -\infty} t_n = 0$. The external force $F = -\ddot{\varphi}$ may then no longer be smooth; note that in any case, F has strong oscillations near zero, in the sense that F takes negative

and positive values at times arbitrarily close to 0. This may suggest that, informally, existence of multiple solutions to (1) and (2) could hold for quite general external forces F with strong oscillations. In this direction, recall that uniqueness of the solution has only been established for analytic external forces, see Ballard [1].

Theorem 2 is thus in sharp contrast with the preceding observations, even though the uniqueness is only stated there in a weak sense. Hence an important open question is to ask whether *pathwise uniqueness* holds for equations (1) and (4).

Another interesting problem in this vein is to decide whether or not the solution which has been constructed in Section 2 is adapted to the natural filtration of the Brownian motion $(B_t)_{t \geq 0}$. One says that the solution is *strong* in the case when the answer is positive. We refer to Tsirel'son [16] for a classical example of an SDE which has a unique weak solution, but no strong solution.

References

[1] P. Ballard : The dynamics of discrete mechanical systems with perfect unilateral constraints *Arch. Rational Mech. Anal.* **154** (2000), 199-274.

[2] J. Bertoin : Reflecting a Langevin process at an absorbing boundary. *Ann. Probab.* (to appear). Available via `http://www.imstat.org/aop/future_papers.htm`

[3] A. Bressan : Incompatibilità dei teoremi di esistenza e di unicità del moto per un tipo molto comune e regolare di sistemi meccanici, *Ann. Scuola Norm. Sup. Pisa* Serie III **14** (1960), 333-348.

[4] J. C. Gittins : *Multi-armed bandit allocation indices.* Wiley-Interscience Series in Systems and Optimization. John Wiley & Sons, Chichester, 1989.

[5] A. Lachal : Applications de la théorie des excursions à l'intégrale du mouvement brownien. *Séminaire de Probabilités* **XXXVIII**, Lecture Notes in Math. 1801 (2003), pp. 109-195.

[6] B. Lapeyre : Une application de la théorie des excursions à une diffusion réfléchie dégénérée. *Probab. Theory Relat. Fields* **87** (1990), 189-207.

[7] B. Maury : Direct simulation of aggregation phenomena. *Comm Math. Sci.* supplemental issue No 1 (2004), 1-11. Available via `http://intlpress.com/CMS/issueS-1/S-1.pdf`

[8] B. Maury : A time-stepping scheme for inelastic collisions. *Numer. Math.* **102** (2006), 649-679.

[9] J.J. Moreau : Some numerical methods in multibody dynamics: Application to granular materials. *Eur. J. Mech., A* **13** (1994), 93-114.

[10] D. Percivale : Uniqueness in the elastic bounce problem, I,*J. Differential Equations* **56** (1985), 206-215.

[11] Ph. Protter : *Stochastic integration and differential equations.* Second edition. Applications of Mathematics **21**. Springer-Verlag, Berlin, 2004.

[12] D. Revuz and M. Yor : *Continuous martingales and Brownian motion.* Third edition. Grundlehren der Mathematischen Wissenschaften **293**. Springer-Verlag, Berlin, 1999.

[13] M. Schatzman : A class of nonlinear differential equations of second order in time. *Nonlinear Anal., Theory, Methods Appl.* **2** (1978), 355-373.

[14] M. Schatzman : Uniqueness and continuous dependence on data for one dimensional impact problems. *Math. Comput. Modelling* **28** (1998), 1-18.

[15] D. E. Stewart : Convergence of a time-stepping scheme for rigid-body dynamics and resolution of Painlevé's problem. *Arch. Ration. Mech. Anal.* **145** (1998), 215-260.

[16] B. S. Tsirel'son : An example of a stochastic differential equation having no strong solution. *Theory Probab. Appl.* **20** (1975), 416-418; translation from *Teor. Veroyatn. Primen.* **20** (1975), 427-430.