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ABSTRACT. Stochastic evolution equations in Banach spaces with unbounded nonlinear drift and diffusion operators are considered. Under some regularity condition assumed for the solution, the rate of convergence of implicit Euler approximations is estimated under strong monotonicity and Lipschitz conditions. The results are applied to a class of quasilinear stochastic PDEs of parabolic type.

1. INTRODUCTION

Let $V \hookrightarrow H \hookrightarrow V^*$ be a *normal triple* of spaces with dense and continuous embeddings, where V is a separable and reflexive Banach space, H is a Hilbert space, identified with its dual by means of the inner product in H, and V^* is the dual of V. Thus $\langle v,h\rangle = (v,h)$ for all $v \in V$ and $h \in H^* = H$, where $\langle v, v^*\rangle = \langle v^*, v\rangle$ denotes the duality product of $v \in V$, $v^* \in V^*$, and (h_1,h_2) denotes the inner product of $h_1, h_2 \in H$. Let $W = \{W(t) : t \geq 0\}$ be a d_1-dimensional Brownian motion carried by a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Consider the stochastic evolution equation

$$u(t) = u_0 + \int_0^t A(s,u(s))\, ds + \sum_{k=1}^{d_1} \int_0^t B_k(s,u(s))\, dW^k(s)\,, \tag{1.1}$$

where u_0 is a V-valued $\mathcal{F}_0$-measurable random variable, A and B are (non-linear) adapted operators defined on $[0,\infty[\times V \times \Omega$ with values in V^* and $H^{d_1} := H \times ... \times H$, respectively.

It is well-known, see [7], [10] and [13], that this equation admits a unique solution if the following conditions are met: There exist constants $\lambda > 0$, $K \geq 0$ and an $\mathcal{F}_t$-adapted non-negative locally integrable stochastic process $f = \{f_t : t \geq 0\}$ such that

(i) (Monotonicity) There exists a constant K such that

$$2\langle u - v, A(t,u) - A(t,v)\rangle + \sum_{k=1}^{d_1} |B_k(t,u) - B_k(t,v)|_H^2 \leq K|u - v|_H^2, \tag{1.2}$$

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(ii) (Coercivity)

$$2\langle v, A(t,v)\rangle + \sum_{k=1}^{d_1} |B_k(t,v)|_H^2 \le -\lambda |v|_V^2 + K|v|_H^2 + f(t),$$

(iii) (Linear growth)

$$|A(t,v)|_{V^*}^2 \le K|v|_{V^*}^2 + f(t),$$

(iv) (Hemicontinuity)

$$\lim_{\lambda\to 0}\langle w, A(t, v+\lambda u)\rangle = \langle w, A(t,v)\rangle$$

hold for all for $u, v, w \in V$, $t \in [0,T]$ and $\omega \in \Omega$.

Under these conditions equation (1.1) has a unique solution u on $[0,T]$. (See Definition 2.3 below for the definition of the solution.) Moreover, if $E|u_0|_H^2 < \infty$ and $E\int_0^T f(t)\,dt < \infty$, then

$$E\sup_{t\le T} |u(t)|_H^2 + E\int_0^T |u(t)|_V^2\,dt < \infty.$$

In [5] it is shown that under these conditions, approximations defined by various implicit and explicit schemes converge to u.

Our aim is to prove rate of convergence estimates for these approximations. To achieve this aim we require stronger assumptions: a strong monotonicity condition on A, B and a Lipschitz condition on B in $v \in V$. In the present paper we consider implicit time discretizations. Note that without space discretizations, in general, explicit time discretizations do not converge. Consider, for example, the heat equation $du(t) = \Delta u(t)$, with initial condition $u(0) = u_0 \in L_2(\mathbb{R}^d)$. Then the explicit time discretization on the grid $\{k/n\}_{k=0}^n$ gives the approximation $u_n(k/n) := (I+\Delta/n)^k u_0$ at time $t = k/n$. Hence clearly, if $u_0 \notin \cap_{i=1}^\infty W_2^i(\mathbb{R}^d)$, then $u(k/n)$ does not belong to the Sobolev space $W_2^l(\mathbb{R}^d)$, with any fixed negative index l, when k is sufficiently large.

The study of various space-time discretization schemes will be done in the continuation of the present paper.

We require also the following time regularity from the solution u (see condition (T2)): $E|u_0|_V^2 < \infty$, almost surely $u_t \in V$ for every $t \in T$, and there exist some constants C and $\nu > 0$ such that

$$E|u(t) - u(s)|_V^2 \le C\,|t-s|^{2\nu},$$

for all $s, t \in [0,T]$. Note that unlike the solutions to stochastic differential equations, the solutions to stochastic PDEs can satisfy this condition with a variety of exponents ν, different from $1/2$, due to the interplay between space and time regularities of the solutions. (See [9] for space and time regularity of the solutions to stochastic parabolic PDEs of second order.) Note also that our general setting allows us to cover a large class of stochastic parabolic PDEs of order $2m$ for any $m \ge 1$ (see [7] for the class of stochastic parabolic SPDEs of order $2m$ and see [1] for the stochastic Cahn-Hilliard equation).

In the case of time independent operators A and B we obtain the rate of convergence for the implicit approximation u^τ corresponding to the mesh size $\tau = T/m$ of the partition of $[0, T]$

$$E \max_{i \le m} |u(i\tau) - u^\tau(i\tau)|_H^2 + E \sum_{i \le m} |u(i\tau) - u^\tau(i\tau)|_V^2 \tau \le C\tau^\nu,$$

where C is a constant independent of τ. If in addition to the above assumptions A is also Lipschitz continuous in $v \in V$ then the order of convergence is doubled,

$$E \max_{i \le m} |u(i\tau) - u^\tau(i\tau)|_H^2 + E \sum_{i \le m} |u(i\tau) - u^\tau(i\tau)|_V^2 \tau \le C\tau^{2\nu}.$$

In the case of time dependent A and B it is natural to assume that they are Hölder continuous in t in order to control the error due to their discretization in time. However, it is possible to control this discretization error when the operator A is not even continuous in t, if we discretize it by taking the average of $A(s)$ over the intervals $[t_i, t_{i+1}]$. This explains the discretization of $A(t)$ and condition (T1) below. If both operators A and B are Hölder continuous in time then we use also the obvious discretization: $A^\tau_{t_i} = A(t_{i+1}, .)$ and $B^\tau_{k,t_i} = B(t_i, .)$.

As examples we present a class of quasi-linear stochastic partial differential equations (SPDEs) of parabolic type, and show that it satisfies our assumptions. Thus we obtain rate of convergence results also for implicit approximations of linear parabolic SPDEs, in particular, for the Zakai equation of nonlinear filtering. We refer to [8], [12], [11] and [13] for basic results for the stochastic PDEs of nonlinear filtering.

We will extend these results to degenerate parabolic SPDEs, and to space-time explicit and implicit schemes for stochastic evolution equations in the continuation of this paper.

In Section 2 we give a precise description of the schemes and state the assumptions on the coefficients which ensure the convergence of these schemes to the solution u of (1.1). In Section 3 estimates for the speed of convergence of time implicit schemes are stated and proved. Finally, in the last section, we give a class of examples of quasi-linear stochastic PDEs for which all the assumptions of the main theorem, Theorem 3.4, are fulfilled.

As usual, we denote by C a constant which can change from line to line.

2. Preliminaries and the approximation scheme

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, P)$ be a stochastic basis, satisfying the usual conditions, i.e., $(\mathcal{F}_t)_{t \ge 0}$ is an increasing right-continuous family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains every P-null set. Let $W = \{W(t) : t \ge 0\}$ be a d_1-dimensional Wiener martingale with respect to $(\mathcal{F}_t)_{t \ge 0}$, i.e., W is an $\mathcal{F}_t$-adapted Wiener process with values in $\mathbb{R}^{d_1}$ such that $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \le s \le t$.

Let T be a given positive number. Consider the stochastic evolution equation (1.1) for $t \in [0, T]$ in a triplet of spaces

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

satisfying the following conditions: V is a separable and reflexive Banach space over the real numbers, embedded continuously and densely into a Hilbert space H, which is identified with its dual H^* by means of the inner product $(\cdot, \cdot)$ in H, such that $(v, h) = \langle v, h \rangle$ for all $v \in V$ and $h \in H$, where $\langle \cdot, \cdot \rangle$ denotes the duality product between V and V^*, the dual of V. Such triplet of spaces is called a normal triplet.

Let us state now our assumptions on the initial value u_0 and the operators A, B in the equation. Let

$$A : [0,T] \times V \times \Omega \to V^*, \quad B : [0,T] \times V \times \Omega \to H^{d_1}$$

be such that for every $v, w \in V$ and $1 \le k \le d_1$, $\langle A(s,v), w\rangle$ and $(B_k(s,v), w)$ are adapted processes and the following conditions hold:

(C1) The pair (A, B) satisfies the *strong monotonicity condition*, i.e., there exist constants $\lambda > 0$ and $L > 0$ such almost surely

$$2\,\langle u - v, A(t,u) - A(t,v)\rangle + \sum_{k=1}^{d_1} |B_k(t,u) - B_k(t,v)|_H^2$$
$$+\lambda\,|u - v|_V^2 \le L\,|u - v|_H^2 \tag{2.1}$$

for all $t \in]0,T]$, u and v in V.

(C2) (*Lipschitz condition on* B) There exists a constant L_1 such that almost surely

$$\sum_{k=1}^{d_1} |B_k(t,u) - B_k(t,v)|_H^2 \le L_1\,|u - v|_V^2 \tag{2.2}$$

for all $t \in [0,T]$, u and v in V.

(C3) (*Lipschitz condition on* A) There exists a constant L_2 such that almost surely

$$|A(t,u) - A(t,v)|_{V^*}^2 \le L_2\,|u - v|_V^2 \tag{2.3}$$

for all $t \in [0,T]$, u and v in V.

(C4) $u_0 : \Omega \to V$ is $\mathcal{F}_0$-measurable and $E|u_0|_V^2 < \infty$. There exist non-negative random variables K_1 and K_2 such that $EK_i < \infty$, and

$$\sum_{k=1}^{d_1} |B_k(t,0)|_H^2 \le K_1 \tag{2.4}$$

$$|A(t,0)|_{V^*}^2 \le K_2 \tag{2.5}$$

for all $t \in [0,T]$ and $\omega \in \Omega$.

Remark 2.1. *If* $\lambda = 0$ *in* (2.1) *then one says that* (A, B) *satisfies the monotonicity condition. Notice that this condition together with the Lipschitz condition* (2.3) *on* A *implies the Lipschitz condition* (2.2) *on* B.

Remark 2.2. *(1) Clearly,* (2.3)–(2.5) *and* (2.2)–(2.4) *imply that* A *and* B *satisfy the growth condition*

$$\sum_{j=1}^{d_1} |B_k(t,v)|_H^2 \le 2L_1 |v|_V^2 + 2K_1, \tag{2.6}$$

and

$$|A(t,v)|_{V^*}^2 \le 2L_2\,|v|_V^2 + 2K_2 \tag{2.7}$$

respectively, for all $t \in [0,T]$, $\omega \in \Omega$ *and* $v \in V$.

(2) Condition (2.3) *obviously implies that the operator* A *is hemicontinuous:*

$$\lim_{\varepsilon \to 0} \langle A(t, u + \varepsilon v), w\rangle = \langle A(t,u), w\rangle \tag{2.8}$$

for all $t \in [0,T]$ and $u, v, w \in V$.

(3) The strong monotonicity condition (C1), (C2) and (2.4), (2.5) yield that the pair (A, B) satisfies the following coercivity condition: there exists a non-negative random variable K_3 such $EK_3 < \infty$ and almost surely

$$2\langle v, A(t,v)\rangle + \sum_{k=1}^{d_1} |B_k(t,v)|_H^2 + \tfrac{\lambda}{2}|v|_V^2 \le L|v|_H^2 + K_3 \tag{2.9}$$

for all $t \in]0,T]$, $\omega \in \Omega$ and $v \in V$.

Proof. We show only (3). By the strong monotonicity condition

$$2\langle v, A(t,v)\rangle + \sum_{k=1}^{d_1} |B_k(t,v)|_H^2 + \tfrac{\lambda}{2}|v|_V^2 \le L|v|_H^2 + R_1(t) + R_2(t) \tag{2.10}$$

with

$$\begin{aligned} R_1(t) &= 2\langle v, A(t,0)\rangle, \\ R_2(t) &= \sum_{k=1}^{d_1} |B_k(t,0)|_H^2 + 2\sum_{k=1}^{d_1}\Big(B_k(t,v) - B_k(t,0)\,,\, B_k(t,0)\Big). \end{aligned}$$

Using (C2) and (2.5), we have

$$\begin{aligned} |R_1| &\le \tfrac{\lambda}{4}|v|_V^2 + \tfrac{4K_2}{\lambda}, \\ |R_2| &\le 2\left(\sum_{j=1}^{d_1} |B_k(t,v) - B_k(t,0)|_H^2\right)^{\frac12}\left(\sum_{k=1}^{d_1} |B_k(t,0)|_H^2\right)^{\frac12} + K_1 \\ &\le \tfrac{\lambda}{4}|v|_V^2 + CK_1. \end{aligned}$$

Thus, (C1) concludes the proof of (2.9).

□

Definition 2.3. *An H-valued adapted continuous process $u = \{u(t) : t \in [0,T]\}$ is a solution to equation (1.1) on $[0,T]$ if almost surely $u(t) \in V$ for almost every $t \in [0,T]$,*

$$\int_0^T |u(t)|_V^2\, dt < \infty, \tag{2.11}$$

and

$$(u(t), v) = (u_0, v) + \int_0^t \langle A(s,u(s)), v\rangle\, ds + \sum_{k=1}^{d_1}\int_0^t (B_k(s,u(s)), v)\, dW^k(s) \tag{2.12}$$

holds for all $t \in [0,T]$ and $v \in V$. We say that the solution to (1.1) on $[0,T]$ is unique if for any solutions u and v to (1.1) on $[0,T]$ we have

$$P(\sup_{t\in[0,T]} |u(t) - v(t)|_H > 0) = 0.$$

The following theorem is well-known (see [7], [10] and [13]).

Theorem 2.4. *Let A and B satisfy the monotonicity, coercivity, linear growth and hemicontinuity conditions (i)-(iv) formulated in the Introduction. Then for every H-valued $\mathcal{F}_0$-measurable random variable u_0, equation (1.1) has a unique solution u on $[0,T]$. Moreover, if $E|u_0|_H^2 < \infty$ and $E\int_0^T f(t)\,dt < \infty$, then*

$$E\big(\sup_{t\in[0,T]} |u(t)|_H^2\big) + E\int_0^T |u(t)|_V^2\,dt < \infty \tag{2.13}$$

holds.

Hence by the previous remarks we have the following corollary.

Corollary 2.5. *Assume that conditions (C1), (C2) hold. Then for every H-valued random variable u_0 equation* (1.1) *has a unique solution u, and if $E|u_0|_H^2 < \infty$, then* (2.13) *holds.*

Approximation scheme. For a fixed integer $m \ge 1$ and $\tau := T/m$ we define the approximation u^τ for the solution u by an implicit time discretization of equation (1.1) as follows:

$$\begin{aligned} u^\tau(t_0) &= u_0\,, \\ u^\tau(t_{i+1}) &= u^\tau(t_i) + \tau\, A^\tau_{t_i}\big(u^\tau(t_{i+1})\big) \\ &\quad + \sum_{k=1}^{d_1} B^\tau_{k,t_i}\big(u^\tau(t_i)\big)\,\big(W^k(t_{i+1}) - W^k(t_i)\big) \quad \text{for } 0 \le i < m, \end{aligned} \tag{2.14}$$

where $t_i := i\tau$ and

$$A^\tau_{t_i}(v) = \frac{1}{\tau}\int_{t_i}^{t_{i+1}} A(s,v)\,ds\,, \tag{2.15}$$

$$B^\tau_{k,0}(v) = 0, \quad B^\tau_{k,t_{i+1}}(v) = \frac{1}{\tau}\int_{t_i}^{t_{i+1}} B_k(s,v)\,ds \tag{2.16}$$

for $i = 0, 1, 2, ..., m$.

A random vector $u^\tau := \{u^\tau(t_i) : i = 0, 1, 2, ..., m\}$ is called a solution to scheme (2.14) if $u^\tau(t_i)$ is a V-valued $\mathcal{F}_{t_i}$-measurable random variable such that $E|u^\tau(t_i)|_V^2 < \infty$ and (2.14) hold for every $i = 0, \cdots, m-1$.

We use the notation

$$\kappa_1(t) := i\tau \ \text{ for } t \in [i\tau, (i+1)\tau[, \text{ and } \ \kappa_2(t) := (i+1)\tau \ \text{ for } t \in]i\tau, (i+1)\tau] \tag{2.17}$$

for integers $i \ge 0$, and set

$$A_t(v) = A_{t_i}(v), \quad B_{k,t}(v) = B_{t_i}(v)$$

for $t \in [t_i, t_{i+1}[$, $i = 0, 1, 2, ...m-1$ and $v \in V$.

Another possible choice is

$$A^\tau_{t_i}(u) = A(t_{i+1}, u) \quad \text{and} \quad B^\tau_{k,t_i}(u) = B_k(t_i, u) \quad \text{for } i = 0, 1, \cdots, m-1. \tag{2.18}$$

The following theorem establishes the existence and uniqueness of u^τ for large enough m, and provides estimates in V and in H. We remark that in practice (2.14) should also be solved numerically. This is possible for example by Galerkin's approximations and by finite elements methods. In the continuation of this paper we consider explicit and implicit time discretization schemes together with simultaneous 'space discretizations', and we estimate the error of the corresponding approximations for (1.1).

Theorem 2.6. *Assume that A and B satisfy the monotonicity, coercivity, linear growth and hemicontinuity conditions (i)–(iv). Assume also that (C4) holds. Let A^τ and B^τ be defined either by* (2.15) *and* (2.16)*, or by* (2.18)*. Then there exist an integer m_0 and a constant C, such that for $m \geq m_0$ equation (2.14) has a unique solution $\{u^\tau(t_i) : i = 0, 1, ..., m\}$, and*

$$E \max_{0\le i\le m} \left|u^\tau(i\tau)\right|_H^2 + E\sum_{i=1}^m \left|u^\tau(i\tau)\right|_V^2 \tau \le C\,. \tag{2.19}$$

Proof. For the sake of simplicity, we only give the proof in the case A^τ and B^τ are defined by (2.15) and (2.16). This theorem with estimate

$$\max_{0\le i\le m} E\left|u^\tau(i\tau)\right|_H^2 + E\sum_{i=1}^m \left|u^\tau(i\tau)\right|_V^2 \tau \le C \tag{2.20}$$

in place of (2.19) is proved in [5] for a slightly different implicit scheme. For the above implicit scheme the same proof can be repeated without essential changes. For the convenience of the reader we recall from [5] that the existence and uniqueness of the solution $\{u^\tau(t_i) : i = 0, 1, 2, ..., m\}$ to (2.14)–(2.16) is based on the following proposition (Proposition 3.4 from [5]): Let $D : V \to V^*$ be a mapping such that

(a) D is monotone, i.e., for every $x, y \in V$, $\langle D(x) - D(y), x - y\rangle \ge 0$;

(b) D is hemicontinuous, i.e., $\lim_{\varepsilon\to 0}\langle D(x+\varepsilon y), z\rangle = \langle D(x), z\rangle$ for every $x, y, z \in V$;

(c) there exist positive constants K, C_1 and C_2, such that

$$|D(x)|_{V^*} \le K\,(1 + |x|_V), \quad \langle D(x), x\rangle \ge C_1\,|x|_V^2 - C_2\,, \quad \forall x \in V.$$

Then for every $y \in V^*$, there exists $x \in V$ such that $D(x) = y$ and

$$|x|_V^2 \le \frac{C_1 + 2\,C_2}{C_1} + \frac{1}{C_1^2}\,|y|_{V^*}^2\,.$$

If there exists a positive constant C_3 such that

$$\langle D(x_1) - D(x_2), x_1 - x_2\rangle \ge C_3\,|x_1 - x_2|_{V^*}^2\,, \quad \forall x_1, x_2 \in V\,, \tag{2.21}$$

then for any $y \in V^*$, the equation $D(x) = y$ has a unique solution $x \in V$.
Note that for each $i = 1, 2, ...m-1$ equation (2.14) for $x := u^\tau(t_{i+1})$ can be rewritten as $Dx = y$ with

$$D := I - \tau A^\tau_{t_i}, \quad y := u^\tau(t_i) + \sum_{k=1}^{d_1} B^\tau_{k,t_i}\big(u^\tau(t_i)\big)\left(W^k(t_{i+1}) - W^k(t_i)\right)$$

where I denotes the identity on V. It is easy to verify that due to conditions (i)–(iv) and (C4) the operator D satisfies the conditions (a), (b) and (c) for sufficiently large m. Thus a solution $\{u^\tau(t_i) : i = 0, 1, ..., m\}$ can be obtained by recursion on i for all m greater than some m_0. To show the uniqueness we need only verify (2.21). By (2.15) and by the monotonicity condition (i) we have

$$\langle D(x_1) - D(x_2), x_1 - x_2\rangle = |x_1 - x_2|_H^2 - \int_{t_i}^{t_{i+1}} \langle A(s, x_1) - A(s, x_2), x_1 - x_2\rangle\, ds$$

$$\geq |x_1 - x_2|_H^2 - K\tau|x_1 - x_2|_H^2 = (1 - K\tau)|x_1 - x_2|_H^2,$$

where the constant K is from (1.2). Hence it is clear that (2.21) holds if m is sufficiently large.

Now we show (2.20). From the definition of $u^\tau(t_{i+1})$ we have

$$|u^\tau(t_j)|_H^2 = |u_0|_H^2 + \mathcal{I}(t_j) + \mathcal{J}(t_j) + \mathcal{K}(t_j) - \sum_{i=1}^{j} |A_{t_i}^\tau(u^\tau(i\tau))|_H^2 \tau \tag{2.22}$$

for $t_j = j\tau$, $j = 0, 1, 2, ...m$, where

$$\begin{aligned}
\mathcal{I}(t_j) &:= 2\int_0^{t_j} \langle u^\tau(\kappa_2(s)), A(s, u^\tau(\kappa_2(s)))\rangle \, ds, \\
\mathcal{J}(t_j) &:= \sum_{1\leq i<j} |\sum_k B_{k,t_i}^\tau(u^\tau(i\tau))(W^k(t_{i+1}) - W^k(t_i))|_H^2, \\
\mathcal{K}(t_j) &:= 2\sum_k \int_0^{t_j} \big(u^\tau(\kappa_1(s)), B_{k,s}^\tau(u^\tau(\kappa_1(s)))\big) \, dW^k(s),
\end{aligned}$$

and κ_1, κ_2 are piece-wise constant functions defined by (2.17). By Itô's formula for every $k, l = 1, 2, ..., d_1$

$$\begin{aligned}
&(W^k(t_{i+1}) - W^k(t_i))(W^l(t_{i+1}) - W^l(t_i)) \\
&= \delta_{kl}(t_{i+1} - t_i) + M^{kl}(t_{i+1}) - M^{kl}(t_i),
\end{aligned}$$

where $\delta_{kl} = 1$ for $k = l$ and 0 otherwise, and

$$M^{kl}(t) := \int_0^t \big(W^k(s) - W^k(\kappa_1(s))\big) \, dW^l(s) + \int_0^t \big(W^l(s) - W^l(\kappa_1(s))\big) \, dW^k(s).$$

Thus we get

$$\mathcal{J}(t_j) = \mathcal{J}_1(t_j) + \mathcal{J}_2(t_j),$$

with

$$\mathcal{J}_1(t_j) := \sum_{1\leq i<j} \sum_k |B_{k,t_i}^\tau(u^\tau(t_i))|_H^2 \tau$$

$$\mathcal{J}_2(t_j) := \int_0^{t_j} \sum_{k,l} (B_{k,s}^\tau(u^\tau(\kappa_1(s))), B_{l,s}^\tau(u^\tau(\kappa_1(s)))) \, dM^{kl}(s).$$

By the Davis inequality we have

$$\begin{aligned}
&E \max_{j\leq m} |\mathcal{J}_2(t_j)| = \\
&\leq 3\sum_{k,l} E\left\{\int_0^T |B_{k,s}^\tau(u^\tau(\kappa_1(s)))|_H^2 |B_{l,s}^\tau(u^\tau(\kappa_1(s)))|_H^2 \, d\langle M^{kl}\rangle(s)\right\}^{1/2} \\
&\leq C_1 \sum_{k,l} E\left\{\int_0^T |B_{k,s}^\tau(u^\tau(\kappa_1(s)))|_H^4 |W^l(s) - W^l(\kappa_1(s))|^2 \, ds\right\}^{1/2} \\
&\leq C_1 \sum_{k,l} E\Big[\max_j \big|B_{k,t_j}^\tau(u^\tau(t_j))\big|_H \sqrt{\tau} \\
&\qquad \times \left\{\frac{1}{\tau}\int_0^T |B_{k,s}^\tau(u^\tau(\kappa_1(s)))|_H^2 \big|W^l(s) - W^l(\kappa_1(s))\big|^2 ds\right\}^{1/2}\Big]
\end{aligned}$$

$$\le d_1 C_1 \sum_k \tau E \max_j \left| B^\tau_{k,t_j}(u^\tau(t_j)) \right|_H^2$$

$$+ C_1 \tau^{-1} \sum_{k,l} E \int_0^T |B^\tau_{k,s}(u^\tau(\kappa_1(s)))|_H^2 \left| W^l(s) - W^l(\kappa_1(s)) \right|^2 ds$$

$$\le C_2 \Big(1 + E \sum_{j \le 1} |u^\tau(j\tau)|_V^2 \tau \Big),$$

where C_1 and C_2 are constants, independent of τ. Here we use that by Jensen's inequality for every k

$$\sum_{1 \le i < j} |B^\tau_{k,t_i}(u^\tau(i\tau))|_H^2 \tau \le \int_0^{t_j} |B_k(s, u^\tau(\kappa_2(s))|_H^2 \, ds,$$

and that the coercivity condition (ii) and the growth condition on (iii) imply the growth condition (2.6) on B with some constant L_1 and random variable K_1 satisfying $EK_1 < \infty$. Hence by taking into account the coercivity condition we obtain

$$E \max_{j \le m} \left[\mathcal{I}(t_j) + \mathcal{J}(t_j) \right]$$

$$\le E \max_{j \le m} \int_0^{t_j} \Big[2 \left\langle u^\tau(\kappa_2(s)) \, , \, A(s, u^\tau(\kappa_2(s))) \right\rangle + \sum_k |B_k(s, u^\tau(\kappa_2(s))|_H^2 \Big] ds$$

$$+ E \max_{j \le m} |\mathcal{J}_2(t_j)|$$

$$\le C \Big(1 + \max_{j \le m} E |u^\tau(j\tau)|_H^2 + E \sum_{j=1}^m |u^\tau(j\tau)|_V^2 \tau \Big) \tag{2.23}$$

with a constant C independent of τ. By using the Davis inequality again we obtain

$$E \max_{j \le m} \left| \mathcal{K}(t_j) \right| \le 6 \, E \left\{ \int_0^T \sum_k \left| \left(u^\tau(\kappa_1(s)), B^\tau_{k,s}\big(u^\tau(\kappa_1(s))\big) \right|^2 ds \right\}^{1/2}$$

$$\le 6 \, E \left[\max_{j \le m} \left| u^\tau(j\tau) \right|_H \left\{ \int_0^T \sum_k |B^\tau_{k,s}(u^\tau(\kappa_1(s)))|_H^2 \, ds \right\}^{1/2} \right]$$

$$\le \tfrac{1}{2} \, E \max_{j \le m} |u^\tau(j\tau)|_H^2 + 18 \, E \int_0^T \sum_k \left| B^\tau_{k,s}(u^\tau(\kappa_1(s))) \right|_H^2 ds$$

$$\le \tfrac{1}{2} \, E \max_{j \le m} |u^\tau(j\tau)|_H^2 + C \left(1 + E \sum_{j \le m} |u^\tau(j\tau)|_V^2 \tau \right) \tag{2.24}$$

with a constant C independent of τ. From (2.20)–(2.24) we get

$$E \max_{j \le m} |u^\tau(j\tau)|_H^2 \le E|u_0|^2 + E \max_{j \le m} \big(\mathcal{I}(t_j) + \mathcal{J}(t_j) \big) + E \max_{j \le m} |\mathcal{K}(t_j)|$$

$$\le \tfrac{1}{2} \, E \max_{j \le m} |u^\tau(j\tau)|_H^2 + C \, (1 + \max_{j \le m} E|u^\tau(j\tau)|_H^2 + E \sum_{j \le m} |u^\tau(j\tau)|_V^2 \tau)$$

$$\le \tfrac{1}{2} \, E \max_{j \le m} |u^\tau(j\tau)|_H^2 + C \, (1 + L) < \infty$$

by virtue of (2.20), which proves the estimate (2.19). □

3. Convergence results

In order to obtain a speed of convergence, we require further properties from $B(t,v)$ and from the solution u of (1.1).

We assume that there exists a constant $\nu \in]0,1/2]$ such that:

(T1) The coefficient B satisfies the following *time-regularity*: There exists a constant C and a random variable $\eta \geq 0$ with finite first moment, such that almost surely

$$\sum_{k=1}^{d_1} |B_k(t,v) - B_k(s,v)|_H^2 \leq |t-s|^{2\nu}(\eta + C|v|_V^2) \tag{3.1}$$

for all $s \in [0,T]$ and $v \in V$.

(T2) The solution u to equation (1.1) satisfies the following *regularity* property: almost surely $u(t) \in V$ for all $t \in [0,T]$, and there exists a constant $C > 0$ such that

$$E|u(t) - u(s)|_V^2 \leq C\,|t-s|^{2\nu} \tag{3.2}$$

for all $s,t \in [0,T]$.

Remark 3.1. *Clearly,* (3.2) *implies*

$$\sup_{t\in[0,T]} E|u(t)|_V^2 < \infty. \tag{3.3}$$

Finally, in order to prove a convergence result in the H norm uniformly in time, we also have to require the following uniform estimate on the V-norm of u:

(T3) There exists a random variable ξ such that $E\xi^2 < \infty$ and

$$\sup_{t\leq T} |u(t)|_V \leq \xi \quad \text{(a.s.)}.$$

In order to establish the rate of convergence of the approximations we first suppose that the coefficients A and B satisfy the Lipschitz property.

Theorem 3.2. *Suppose that the conditions (C1)-(C4), (T1) and (T2) hold. Let A^τ and B^τ be defined by* (2.15) *and* (2.16). *Then there exist a constant C and an integer $m_0 \geq 1$ such that*

$$\sup_{0\leq l\leq m} E|u(l\tau) - u^\tau(l\tau)|_H^2 + E\sum_{j=0}^{m} |u(j\tau) - u^\tau(j\tau)|_V^2 \tau \leq C\,\tau^{2\nu} \tag{3.4}$$

for all integers $m \geq m_0$.

The following proposition plays a key role in the proof.

Proposition 3.3. *Assume assumptions (i) through (iv) from the Introduction and let A^τ and B^τ be defined by* (2.15) *and* (2.16). *Suppose, moreover condition (C4). Then*

$$|u(t_l) - u^\tau(t_l)|_H^2 = 2\int_0^{t_l} \big\langle u(\kappa_2(s)) - u^\tau(\kappa_2(s)), A(s,u(s)) - A(s,u^\tau(\kappa_2(s)))\big\rangle\, ds$$

$$+\sum_{i=0}^{l-1}\Big|\int_{t_i}^{t_{i+1}}\sum_{k=1}^{d_1}\big[B_k(s,u(s))-B^\tau_{k,s}(u^\tau(t_i))\big]\,dW^k(s)\Big|_H^2$$
$$+2\sum_{k=1}^{d_1}\int_0^{t_l}\big(B_k(s,u(s))-B^\tau_{k,s}(u^\tau(t_i))\,,\,u(\kappa_1(s))-u^\tau(\kappa_1(s))\big)\,dW^k(s)$$
$$-\sum_{i=0}^{l-1}\Big|\int_{t_i}^{t_{i+1}}\big[A(s,u(s))-A(s,u^\tau(t_{i+1}))\,\big]\,ds\Big|_H^2 \tag{3.5}$$

holds for every $l=1,2,...,m$.

Proof. Using (2.14) we have for any $i=0,\cdots,m-1$

$$|u(t_{i+1})-u^\tau(t_{i+1})|_H^2-|u(t_i)-u^\tau(t_i)|_H^2=$$
$$2\int_{t_i}^{t_{i+1}}\big\langle u(t_{i+1})-u^\tau(t_{i+1}),A(s,u(s))-A(s,u^\tau(t_{i+1}))\big\rangle\,ds$$
$$+\,2\sum_{k=1}^{d_1}\Big(\int_{t_i}^{t_{i+1}}\big[B_k(s,u(s))-B^\tau_{k,s}(u^\tau(t_i))\big]\,dW^k(s)\,,\,u(t_{i+1})-u^\tau(t_{i+1})\Big)$$
$$-\Big|\int_{t_i}^{t_{i+1}}\big[A(s,u(s))-A(s,u^\tau(t_{i+1}))\,\big]\,ds$$
$$+\sum_{k=1}^{d_1}\int_{t_i}^{t_{i+1}}\big[B_k(s,u(s))-B^\tau_{k,s}(u^\tau(t_i))\big]\,dW^k(s)\Big|_H^2$$
$$=2\int_{t_i}^{t_{i+1}}\big\langle u(t_{i+1})-u^\tau(t_{i+1}),A(s,u(s))-A(s,u^\tau(t_{i+1}))\big\rangle\,ds$$
$$+\Big|\sum_{k=1}^{d_1}\int_{t_i}^{t_{i+1}}\big[B_k(s,u(s))-B^\tau_{k,s}(u^\tau(t_i))\big]\,dW^k(s)\Big|_H^2$$
$$+\,2\sum_{k=1}^{d_1}\Big(\int_{t_i}^{t_{i+1}}\big[B_k(s,u(s))-B^\tau_{k,s}(u^\tau(t_i))\big]\,dW^k(s)\,,\,u(t_i)-u^\tau(t_i)\Big)$$
$$-\Big|\int_{t_i}^{t_{i+1}}\big[A(s,u(s))-A(s,u^\tau(t_{i+1}))\,\big]\,ds\Big|_H^2$$

Summing up for $i=1,\cdots,l-1$, we obtain (3.5). □

Proof of Theorem 3.2.
Taking expectations in both sided of (3.5) and using the strong monotonicity condition (C1), we deduce that for $l=1,\cdots,m$,

$$E|u(t_l)-u^\tau(t_l)|_H^2$$

$$\le E\int_0^{t_l}2\big\langle u(\kappa_2(s))-u^\tau(\kappa_2(s)),A(s,u(\kappa_2(s)))-A(s,u^\tau(\kappa_2(s)))\big\rangle ds$$

$$
\begin{aligned}
&+ \sum_{k=1}^{d_1} E \int_0^{t_{l-1}} |B_k(s,u(\kappa_2(s))) - B_k(s,u^\tau(\kappa_2(s)))|_H^2 \, ds + \sum_{k=1}^{3} R_k \\
\leq & -\lambda\, E \int_0^{t_l} |u(\kappa_2(s)) - u^\tau(\kappa_2(s))|_V^2 \, ds + L\tau E|u(t_l) - u^\tau(t_l)|_H^2 \\
&+ L\tau \sum_{i=1}^{l-1} E|u(t_i) - u^\tau(t_i)|_H^2 \, ds + \sum_{k=1}^{3} R_k \,, \qquad (3.6)
\end{aligned}
$$

where

$$
\begin{aligned}
R_1 =& E \int_0^{t_l} 2 \left\langle u(\kappa_2(s)) - u^\tau(\kappa_2(s)), A(s,u(s)) - A(s,u(\kappa_2(s))) \right\rangle ds \,, \\
R_2 =& \sum_{k=1}^{d_1} E \int_0^{\tau} |B_k(s,u(s))|_H^2 \, ds \,, \\
R_3 =& \sum_{k=1}^{d_1} \sum_{i=1}^{l-1} E \Big[\int_{t_i}^{t_{i+1}} ds \, \Big| B_k(s,u(s)) - \frac{1}{\tau} \int_{t_{i-1}}^{t_i} B_k(t,u^\tau(t_i)) \, dt \Big|_H^2 \\
& \qquad - \int_{t_{i-1}}^{t_i} |B_k(t,u(t_i)) - B_k(t,u^\tau(t_i))|_H^2 \, dt \Big].
\end{aligned}
$$

The Lipschitz property of A imposed in (2.3), (3.2) and Schwarz's inequality imply

$$
\begin{aligned}
|R_1| \leq & L_2\, E \int_0^{t_l} |u(\kappa_2(s)) - u^\tau(\kappa_2(s))|_V \, |u(s) - u(\kappa_2(s))|_V \, ds \,, \\
\leq & L_2 \left(E \int_0^{t_l} |u(\kappa_2(s)) - u^\tau(\kappa_2(s))|_V^2 ds \right)^{\frac{1}{2}} \left(E \int_0^{t_l} |u(s) - u(\kappa_2(s))|_V^2 ds \right)^{\frac{1}{2}} \\
\leq & \frac{\lambda}{3} E \int_0^{t_l} |u(\kappa_2(s)) - u^\tau(\kappa_2(s))|_V^2 ds + C\tau^{2\nu}. \qquad (3.7)
\end{aligned}
$$

A similar computation based on (2.2) yields

$$
\begin{aligned}
|R_3| \leq & \sum_{k=1}^{d_1} \sum_{i=1}^{l-1} E \int_{t_{i-1}}^{t_i} dt \frac{1}{\tau} \int_{t_i}^{t_{i+1}} ds \Big(|B_k(s,u(s)) - B_k(t,u^\tau(t_i))|_H^2 \\
& \qquad - |B_k(t,u(t_i))) - B_k(t,u^\tau(t_i)))|_H^2 \Big) \\
\leq & \frac{\lambda}{3} E \int_0^{t_{l-1}} |u(\kappa_2(t)) - u^\tau(\kappa_2(t))|_V^2 dt + C\, R_3'
\end{aligned}
$$

where

$$
R_3' = \sum_{k=1}^{d_1} E \frac{1}{\tau} \int_{t_1}^{t_l} ds \int_{\kappa_1(s)-\tau}^{\kappa_1(s)} dt \, |B_k(s,u(s)) - B_k(t,u(\kappa_2(t)))|_H^2.
$$

Hence, using (2.2), (3.1) and (3.2) we have

$$
R_3' \leq \sum_{k=1}^{d_1} E \frac{1}{\tau} \int_{t_1}^{t_l} ds \int_{\kappa_1(s)-\tau}^{\kappa_1(s)} dt \Big[|B_k(s,u(s)) - B_k(t,u(s))|_H^2
$$

$$+ |B_k(t,u(s)) - B_k(t,u(t))|_H^2 + |B_k(t,u(t)) - B_k(t,u(\kappa_2(t)))|_H^2\Big]$$

$$\leq E\int_{t_1}^{t_l} \tau^{2\nu}\,|u(s)|_V^2\,ds + CE\frac{1}{\tau}\int_0^{t_{l-1}} dt \int_{\kappa_2(t)}^{\kappa_2(t)+\tau} ds\Big[|u(s)-u(t)|_V^2$$

$$+ |u(t)-u(\kappa_2(t)|_V^2\Big] \leq C\,\tau^{2\nu}.$$

Hence

$$|R_3| \leq C\,\tau^{2\nu} + \frac{\lambda}{3}E\int_0^{t_l} |u(\kappa_2(s)) - u^\tau(\kappa_2(s))|_V^2 ds\,. \tag{3.8}$$

Furthermore (2.6) and (3.3) imply

$$|R_2| \leq C\tau \tag{3.9}$$

with a constant C independent of τ. By inequalities (3.6)–(3.9), for sufficiently large m,

$$E|u(t_l) - u^\tau(t_l)|_H^2 + \frac{\lambda}{3}E\int_0^{t_l} |u(\kappa_2(s)) - u^\tau(\kappa_2(s))|_V^2 ds$$

$$\leq \sum_{i=1}^{l-1} L\,\tau\, E|u(t_i) - u^\tau(t_i)|_H^2 + C\tau^{2\nu}. \tag{3.10}$$

Since $\sup_m \sum_{i=1}^m L\,\tau < +\infty$, a discrete version of Gronwall's lemma yields that there exists $C>0$ such that for m large enough

$$\sup_{0\leq l\leq m} E|u(t_l) - u^\tau(t_l)|_H^2 \leq C\tau^{2\nu}.$$

This in turn with (3.2) implies

$$E\int_0^T |u(s) - u^\tau(\kappa_2(s))|_V^2 ds \leq C\tau^{2\nu},$$

which completes the proof of the theorem. □

Assume now that the solution u of equation (1.1) satisfies also the condition (T3) Then we can improve the estimate (3.4) in the previous theorem.

Theorem 3.4. *Let (C1)-(C4) and (T1)–(T3) hold, and let A^τ and B^τ be defined by* (2.15) *and* (2.16). *Then for all sufficiently large m*

$$E\max_{0\leq j\leq m} |u(j\tau) - u^\tau(j\tau)|_H^2 + E\sum_{j=0}^m |u(j\tau) - u^\tau(j\tau)|_V^2\tau \leq C\,\tau^{2\nu} \tag{3.11}$$

holds, where C is a constant independent of τ.

Proof. For $k=1,\cdots,d_1$, set

$$F_k(t) = B_k(t,u(t)) - B_{k,t}^\tau(u^\tau(\kappa_1(t)))$$

$$m(t) = \sum_{k=1}^{d_1}\int_0^t F_k(s)\,dW^k(s) \quad\text{and}\quad G(s) = m(s) - m(\kappa_1(s))$$

Then by Itô's formula

$$|m(t_{i+1})-m(t_i)|_H^2 = 2\int_{t_i}^{t_{i+1}} \sum_k (G(s)\,,\,F_k(s))\,dW^k(s) + \sum_{k=1}^{d_1}\int_{t_i}^{t_{i+1}} |F_k(s)|_H^2\,ds$$

for $i=0,...,m-1$. Hence by using (3.5) we deduce that for $l=1,\cdots,m$

$$|u(t_l)-u^\tau(t_l)|_H^2 \le I_1(t_l)+I_2(t_l)+2M_1(t_l)+2M_2(t_l) \tag{3.12}$$

with

$$I_1(t) := 2\int_0^t \langle u(\kappa_2(s))-u^\tau(\kappa_2(s))\,,\,A(s,u(s))-A(s,u^\tau(\kappa_2(s)))\rangle\,ds,$$

$$I_2(t) := \sum_{k=1}^{d_1}\int_0^t |B_k(s,u(s))-B^\tau_{k,s}(u^\tau(\kappa_1(s)))|_H^2\,ds,$$

$$M_1(t) := \sum_{k=1}^{d_1}\int_0^t \big(G(s)\,,\,F_k(s)\big)\,dW^k(s),$$

$$M_2(t) := \sum_{k=1}^{d_1}\int_0^t \big(F_k(s)\,,\,u(\kappa_1(s))-u^\tau(\kappa_1(s))\big)\,dW^k(s).$$

By (C3)

$$\sup_{0\le l\le m}|I_1(t_l)| \le \int_0^T |u(\kappa_2(s))-u^\tau(\kappa_2(s))|_V^2\,ds + L_2\int_0^T |u(s)-u^\tau(\kappa_2(s))|_V^2\,ds$$
$$\le (1+2L_2)\sum_{i=1}^m |u(t_i)-u^\tau(t_i)|_V^2\tau + 2L_2\int_0^T |u(s)-u(\kappa_2(s))|_V^2\,ds.$$

Hence by Theorem 3.2 and by condition (T2)

$$E\sup_{0\le l\le m}|I_1(t_l)| \le C\tau^{2\nu}, \tag{3.13}$$

where C is a constant independent of τ.

Using Jensen's inequality, (2.6) and condition (T1) we have for $s\le\tau$

$$\sum_k |F_k(s)|_H^2 = \sum_k |B_k(s,u(s))|_H^2 \le 2\,L_1\,|u(s)|_H^2 + 2\,K_1, \tag{3.14}$$

while for $s\in[t_i,t_{i+1}]$, $1\le i\le m$, one has for some constant C independent of τ

$$\sum_k |F_k(s)|_H^2 \le \frac{1}{\tau}\sum_k\int_{t_{i-1}}^{t_i} |B_k(s,u(s))-B_k(r,u^\tau(t_i))|_H^2\,dr$$
$$\le 3\frac{1}{\tau}\sum_k\int_{t_{i-1}}^{t_i}\Big[|B_k(s,u(s))-B_k(r,u(s))|_H^2$$
$$+|B_k(r,u(s))-B_k(r,u(t_i))|_H^2+|B_k(r,u(t_i))-B_k(u^\tau(t_i))|_H^2\Big]\,dr$$
$$\le C\Big[\tau^{2\nu}\left(\eta+|u(s)|_V^2\right)+|u(s)-u(t_i)|_V^2+|u(t_i)-u^\tau(t_i)|_V^2\Big]. \tag{3.15}$$

Thus, (3.14) and (3.15) yield

$$\sup_{0\le l\le m} |I_2(t_l)| \le \; C\int_0^\tau |u(s)|_V^2\,ds + C\tau + C\,\tau^{2\nu} \int_0^T \big(\eta + C\,|u(s)|_V^2\big)\,ds$$
$$+ \,C \int_0^T |u(s)-u(\kappa_2(s))|_V^2\,ds + C\,\sum_{i=1}^m |u(t_i)-u^\tau(t_i)|_V^2\,\tau.$$

Hence by Theorem 3.2 and by condition (T2)

$$E \sup_{0\le l\le m} |I_2(t_l)| \le C\tau^{2\nu}, \tag{3.16}$$

where C is a constant independent of τ. Since $\sup_{0\le s\le T}\sum_k |F_k(s)|_H^2$ need not be measurable, we denote by Γ the set of random variables ζ satisfying

$$\sup_{0\le s\le T}\sum_k |F_k(s)|_H^2 \le \zeta \quad \text{(a.s.)}.$$

For $\zeta\in\Gamma$, the Davis inequality, and the simple inequality $ab\le \frac{\tau}{2}a^2+\frac{1}{2\tau}b^2$ yield

$$E \sup_{1\le l\le m} |M_1(t_l)| \le 3\,E\left(\int_0^T \sum_{k=1}^{d_1} \big|\big(F_k(s)\,,\,G(s)\big)\big|^2\,ds\right)^{\frac12}$$
$$\le 3\,E\left(\zeta^{1/2}\ \left[\int_0^T |G(s)|_H^2\,ds\right]^{\frac12}\right)$$
$$\le \frac{3}{2}\tau \inf_{\zeta\in\Gamma} E\zeta + \frac{3}{2\tau} E\int_0^T |G(s)|_H^2\,ds. \tag{3.17}$$

By (2.6) and (3.15) we deduce

$$\sup_{0\le s\le T}\sum_k |F_k(s)|_H^2 \le C\,\tau^{2\nu}\,\big(\sup_{0\le s\le T} |u(s)|_V^2 + \eta + 1\big) + C\,\max_{1\le i\le m} |u(t_i)-u^\tau(t_i)|_V^2$$
$$\le C\,\Big(1+\xi+\max_{1\le i\le m} |u(t_i)-u^\tau(t_i)|_V^2\Big),$$

where ξ is the random variable from condition (T3) and C is a constant, independent of τ. Hence Theorem 3.2 yield

$$\tau \inf_{\zeta\in\Gamma} E\zeta \le \tau\, C\,(E\eta + E\xi) + C\,\tau \sum_{i=1}^m E|u(t_i)-u^\tau(t_i)|_V^2 \le C_1\,\tau^{2\nu}, \tag{3.18}$$

where C_1 is a constant, independent of τ. Similarly, due to conditions (T1)-(T2) and Theorem 3.2

$$E\sum_k \int_0^T |F_k(s)|_H^2\,ds \le C\,\tau^{2\nu}\Big(1+E\int_0^T |u(s)|_V^2\,ds\Big)$$
$$+\,C\,\tau^{2\nu} + C\,\tau\,E\sum_{i=1}^m |u(t_i)-u^\tau(t_i)|_V^2 \le C\,\tau^{2\nu} \tag{3.19}$$

with a constant C, independent of τ. Furthermore, the isometry of stochastic integrals and (3.22) yield

$$\frac{1}{\tau} E \int_0^T |G(t)|_H^2\, dt \le \frac{1}{\tau} E \int_0^T \Big| \int_{\kappa_1(t)}^t \sum_k F_k(s)\, dW^k(s) \Big|_H^2 dt$$
$$\le \frac{1}{\tau} E \int_0^T dt \int_{\kappa_1(t)}^t \sum_k |F_k(s)|_H^2\, ds \le C\, \tau^{2\nu}. \tag{3.20}$$

Thus from (3.17) by (3.18) and (3.20) we have

$$E \sup_{1\le l\le m} |M_1(t_l)| \le C \tau^{2\nu} \tag{3.21}$$

Finally, the Davis inequality implies

$$E \sup_{1\le l\le m} |M_2(t_l)|_H \le 3\, E \left(\int_0^T \sum_k |\big(F_k(s)\,,\, u(\kappa_1(s)) - u^\tau(\kappa_1(s))\big)|^2\, ds \right)^{\frac{1}{2}}$$
$$\le \frac{1}{4} E \sup_{1\le l\le m} |u(\kappa_1(s)) - u^\tau(\kappa_1(s))\big)|_H^2 + 18\, E \int_0^{t_j} |F_k(s)|_H^2\, ds. \tag{3.22}$$

Thus, from (3.12) by inequalities (3.13), (3.16), (3.21) and (3.22) we obtain

$$\frac{1}{2} E \sup_{1\le l\le m} |u(t_l) - u^\tau(t_l)|_H^2 \le C\, \tau^{2\nu},$$

with a constant C, independent of τ, which with (3.4) completes the proof of the theorem. □

We now prove that if the coefficient A does not satisfy the Lipschitz property (C3) but only the coercivity and growth conditions (2.7)-(2.9), then the order of convergence is divided by two.

Theorem 3.5. *Let A and B satisfy the conditions (C1), (C2) and (C4). Suppose that conditions (T1) and (T2) hold, and let A^τ and B^τ be defined by* (2.15) *and* (2.16). *Then there exists a constant C, independent of τ, such that for all sufficiently large m*

$$\sup_{0\le j\le m} E|u(j\tau) - u^\tau(j\tau)|_H^2 + E \sum_{j=1}^m |u(j\tau) - u^\tau(j\tau)|_V^2\, \tau \le C\, \tau^\nu\, . \tag{3.23}$$

Proof. Using (3.5), taking expectations and using (C1) with $u(s)$ and $u^\tau(\kappa_2(s))$, we obtain for every $l = 1 \cdots, m$

$$E|u(t_l) - u^\tau(t_l)|_H^2 \le -\lambda E \int_0^{t_l} |u(s) - u^\tau(\kappa_2(s)|_V^2\, ds$$
$$+ E \int_0^{t_l} K_1\, |u(s) - u^\tau(\kappa_2(s)|_H^2\, ds + \sum_{k=1}^3 \bar{R}_i\, , \tag{3.24}$$

where

$$\bar{R}_1 = \sum_{j=1}^r 2E \int_0^{t_l} \big\langle u(\kappa_2(s)) - u(s)\,,\, A(s, u(s)) - A(s, u^\tau(\kappa_2(s)))\big\rangle\, ds\, ,$$

$$\bar{R}_2 = \sum_{k=1}^{d_1} E \int_0^{\tau} |B_k(s,u(s))|_H^2 \, ds \,,$$

$$\bar{R}_3 = \sum_{k=1}^{d_1} \sum_{i=1}^{l-1} E \frac{1}{\tau} \int_{t_i}^{t_{i+1}} ds \int_{t_{i-1}}^{t_i} dt \Big[|B_k(s,u(s)) - B_k(t,u^\tau(t_i)))|_H^2 - |B_k(t,u(t)) - B_k(t,u^\tau(t_i))|_H^2 \Big] \,.$$

Using (2.7), (3.2), (3.3) and Schwarz's inequality, we deduce

$$\begin{aligned}
|\bar{R}_1| &\leq C\, E \int_0^{t_l} |u(\kappa_2(s)) - u(s)|_V \big[|u(s)|_V + |u^\tau(\kappa_2(s))|_V + K_2 \big] \, ds \\
&\leq C \left(E \int_0^{t_l} |u(s) - u(\kappa_2(s))|_V^2 \, ds \right)^{\frac{1}{2}} \left(E \int_0^{t_l} \Big(|u(s)|_V^2 + |u(\kappa_2(s))|_V^2 \Big) \, ds \right)^{\frac{1}{2}} \\
&\quad + C \left(E \int_0^{t_l} |u(s) - u(\kappa_2(s))|_V^2 \, ds \right)^{\frac{1}{2}} \\
&\leq C \tau^\nu \,.
\end{aligned} \tag{3.25}$$

Furthermore, Schwarz's inequality, (C2) and computations similar to that proving (3.8) yield for any $\delta > 0$ small enough

$$\begin{aligned}
|\bar{R}_3| &\leq \delta \sum_{k=1}^{d_1} E \int_0^{t_{l-1}} |B_k(t,u(t)) - B_k(t,u^\tau(\kappa_2(t)))|_H^2 \, dt \\
&\quad + C \sum_{k=1}^{d_1} \sum_{i=1}^{l-1} \frac{1}{\tau} E \int_{t_i}^{t_{i+1}} ds \int_{t_{i-1}}^{t_i} dt \, |B_k(s,u(s))) - B_k(t,u(t))|_H^2 \\
&\leq \frac{\lambda}{2} E \int_0^{t_{l-1}} |u(s) - u^\tau(\kappa_2(s))|_V^2 \, ds + C \tau^{2\nu} \,.
\end{aligned}$$

This inequality and (3.25) imply that

$$\begin{aligned}
E|u(t_l) - u^\tau(t_l)|_H^2 + \frac{\lambda}{2} E \int_0^{t_l} |u(s) - u^\tau(\kappa_2(s))|_V^2 ds \\
\leq K_1 \int_0^{t_l} E|u(s) - u^\tau(\kappa_2(s))|_H^2 \, ds + C\, \tau^\nu.
\end{aligned}$$

Hence for any $t \in [0,T]$,

$$\begin{aligned}
E|u(t) - u^\tau(\kappa_2(t)|_H^2 &\leq 2\, E|u(\kappa_2(t)) - u^\tau(\kappa_2(t))|_H^2 + 2\, E|u(t) - u(\kappa_2(t))|_H^2 \\
&\leq 2\, K_1 \int_0^{\kappa_2(t)} E|u(s) - u^\tau(\kappa_2(s))|_H^2 ds + C\, \tau^\nu + 2\, E|u(t) - u(\kappa_2(t))|_H^2 \\
&\leq 2\, K_1 \int_0^t E|u(s) - u^\tau(\kappa_2(s))|_H^2 \, ds + C\, \tau^\nu + 2\, E|u(t) - u(\kappa_2(t))|_H^2 \\
&\quad + C\, \tau \Big[\sup_s E\big(|u(s)|_H^2 + u^\tau(\kappa_2(s))|_H^2 \big) \Big].
\end{aligned}$$

Itô's formula and (2.9) imply that for any $t \in [0,T]$,

$$E|u(t)-u(\kappa_2(t))|_H^2 = E\int_t^{\kappa_2(t)} \Big[2\langle A(s,u(s))\,,\,u(s)\rangle + \sum_{k=1}^{d_1} |B_k(s,u(s))|_H^2\Big]\,ds$$

$$\leq K_1\, E\int_t^{\kappa_2(t)} |u(s)|_H^2\,ds \leq K_1\,\tau \sup_{0\leq s\leq T} E|u(s)|_H^2.$$

Hence (2.13) and (2.19) imply that

$$E|u(t)-u^\tau(\kappa_2(t))|_H^2 \leq 2\,K_1\int_0^t E|u(s)-u^\tau(\kappa_2(s))|_H^2\,ds + C\,\tau^\nu$$

and Gronwall's lemma yields

$$\sup_{0\leq t\leq T} E|u(t)-u^\tau(\kappa_2(t))|_H^2 \leq C\tau^\nu. \tag{3.26}$$

Therefore,

$$E\int_0^T |u(t)-u^\tau(\kappa_2(t)|_V^2\,dt < C\tau^\nu \tag{3.27}$$

follows by (3.24). Finally taking into account that by (T2) there exists a constant C such that

$$E|u(t)-u(\kappa_2(t)|_V^2 \leq C\tau^{2\nu}\text{for all } t\in[0,T],$$

from (3.26) and (3.27) we obtain (3.23). □

Using the above result one can easily obtain the following theorem in the same way as Theorem 3.2 is obtained from Theorem 3.4.

Theorem 3.6. *Let A and B satisfy the conditions (C1), (C2) and (C4). Suppose that conditions (T1)–(T3) hold and let A^τ and B^τ be defined by* (2.15) *and* (2.16). *Then there exists a constant C such that for m large enough,*

$$E\max_{0\leq j\leq m} |u(j\tau)-u^\tau(j\tau)|_H^2 + E\sum_{j=0}^m |u(j\tau)-u^\tau(j\tau)|_V^2\tau \leq C\,\tau^\nu\,. \tag{3.28}$$

Remark 3.7. *By analyzing their proof, it is not difficult to see that Theorems 3.2, 3.4, 3.5 and 3.6 remain true, if instead of* (2.15) *and* (2.16)*, one uses* (2.18) *in the definition of the implicit scheme, and requires furthermore that A satisfies the following time-regularity similar to (T1): there exist a constant $C\geq 0$ and a random variable $\eta\geq 0$ with finite expectation, such that almost surely*

$$|A(t,u)-A(s,u)|_{V^*}^2 \leq |t-s|^{2\nu}\big(\eta + C\|u\|_V^2\big)$$

for $0\leq s\leq t\leq T$ and $u\in V$.

4. Examples

4.1. **Quasilinear stochastic PDEs.** Let us consider the stochastic partial differential equation

$$du(t,x) = \big(Lu(t) + F(t,x,\nabla u(t,x),u(t,x)\big)\,dt + \sum_{k=1}^{d_1}\big(M_k u(t,x) + G_k(t,x,u(t,x))\big)\,dW^k(t), \tag{4.1}$$

for $t \in (0, T]$, $x \in \mathbb{R}^d$ with initial condition

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^d, \tag{4.2}$$

where W is a d_1-dimensional Wiener martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, F and G_k are Borel functions of $(\omega, t, x, p, r) \in \Omega \times [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ and of $(\omega, t, x, r) \in \Omega \times [0, \infty) \times \mathbb{R}^d \times \mathbb{R}$, respectively, and L, M_k are differential operators of the form

$$L(t)v(x) = \sum_{|\alpha|\leq 1, |\beta|\leq 1} D^\alpha(a^{\alpha\beta}(t, x)D^\beta v(x)), \quad M_k(t)v(x) = \sum_{|\alpha|\leq 1} b_k^\alpha(t, x)D^\alpha v(x), \tag{4.3}$$

with functions $a^{\alpha\beta}$ and b_k^α of $(\omega, t, x) \in \Omega \times [0, \infty) \times \mathbb{R}^d$, for all multi-indices $\alpha = (\alpha_1, ..., \alpha_d)$, $\beta = (\beta_1, ..., \beta_d)$ of length $|\alpha| = \sum_i \alpha_i \leq 1$, $|\beta| \leq 1$.

Here, and later on D^α denotes $D_1^{\alpha_1}...D_d^{\alpha_d}$ for any multi-indices $\alpha = (\alpha_1, ..., \alpha_d) \in \{0, 1, 2, ...\}^d$, where $D_i = \frac{\partial}{\partial x_i}$ and D_i^0 is the identity operator.

We use the notation $\nabla_p := (\partial/\partial p_1, ..., \partial/\partial p_d)$. For $r \geq 0$ let $W_2^r(\mathbb{R}^d)$ denote the space of Borel functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ whose derivatives up to order r are square integrable functions. The norm $|\varphi|_r$ of φ in W_2^r is defined by

$$|\varphi|_r^2 = \sum_{|\gamma|\leq r} \int_{\mathbb{R}^d} |D^\gamma \varphi(x)|^2 \, dx.$$

In particular, $W_0^2(\mathbb{R}^d) = L_2(\mathbb{R}^d)$ and $|\varphi|_0 := |\varphi|_{L_2(\mathbb{R}^d)}$. Let us use the notation $\mathcal{P}$ for the σ-algebra of predictable subsets of $\Omega \times [0, \infty)$, and $\mathcal{B}(\mathbb{R}^d)$ for the Borel σ-algebra on $\mathbb{R}^d$.

We fix an integer $l \geq 0$ and assume that the following conditions hold.

Assumption (A1) (Stochastic parabolicity). *There exists a constant $\lambda > 0$ such that*

$$\sum_{|\alpha|=1, |\beta|=1} \left(a^{\alpha\beta}(t, x) - \tfrac{1}{2} \sum_{k=1}^{d_1} b_k^\alpha b_k^\beta(t, x) \right) z^\alpha \, z^\beta \geq \lambda \sum_{|\alpha|=1} |z^\alpha|^2 \tag{4.4}$$

for all $\omega \in \Omega$, $t \in [0, T]$, $x \in \mathbb{R}^d$ and $z = (z^1, ..., z^d) \in \mathbb{R}^d$, where $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2}...z_d^{\alpha_d}$ for $z \in \mathbb{R}^d$ and multi-indices $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$.

Assumption (A2) (Smoothness of the linear term). *The derivatives of $a^{\alpha\beta}$ and b_k^α up to order l are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable real functions such that for a constant K*

$$|D^\gamma a^{\alpha\beta}(t, x)| \leq K, \quad |D^\gamma b_k^\alpha(t, x)| \leq K, \quad \text{for all } |\alpha| \leq 1, \, |\beta| \leq 1, \, k = 1, \cdots, d_1, \tag{4.5}$$

for all $\omega \in \Omega$, $t \in [0, T]$, $x \in \mathbb{R}^d$ and multi-indices γ with $|\gamma| \leq l$.

Assumption (A3) (Smoothness of the initial condition). *Let u_0 be a W_2^l-valued $\mathcal{F}_0$-measurable random variable such that*

$$E|u_0|_l^2 < \infty. \tag{4.6}$$

Assumption (A4) (Smoothness of the nonlinear term). *The function F and their first order partial derivatives in p and r are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$-measurable* *functions, and g_k and its first order derivatives in r are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R})$ -measurable functions for every $k=1,..,d_1$. There exists a constant K such that*

$$|\nabla_p F(t,x,p,r)| + |\tfrac{\partial}{\partial r} F(t,x,p,r)| + \sum_{k=1}^{d_1} |\tfrac{\partial}{\partial r} G_k(r,x)| \le K \tag{4.7}$$

for all $\omega\in\Omega$, $t\in[0,T]$, $x\in\mathbb{R}^d$, $p\in\mathbb{R}^d$ and $r\in\mathbb{R}$. There exists a random variable ξ with finite first moment, such that

$$|F(t,\cdot,0,0)|_0^2 + \sum_{k=1}^{d_1} |G_k(t,\cdot,0)|_0^2 \le \xi \tag{4.8}$$

for all $\omega\in\Omega$ and $t\in[0,T]$.

Definition 4.1. *An $L_2(\mathbb{R}^d)$-valued continuous $\mathcal{F}_t$-adapted process $u=\{u(t):t\in[0,T]\}$ is called a generalized solution to the Cauchy problem (4.1)-(4.2) on $[0,T]$ if almost surely $u(t)\in W_2^1(\mathbb{R}^d)$ for almost every t,*

$$\int_0^T |u(t)|_1^2\, dt < \infty,$$

and

$$\begin{aligned} d(u(t),\varphi) =& \Big\{ \sum_{|\alpha|\le 1, |\beta|\le 1} (-1)^{|\alpha|} \big(a^{\alpha\beta}\, D^\beta u(t)\,,\, D^\alpha\varphi\big) + \big(F(t,\nabla u(t),u(t))\,,\,\varphi\big)\Big\}\, dt \\ &+ \sum_{k=1}^{d_1} \Big\{ \sum_{|\alpha|\le 1} \big(b_k^\alpha D^\alpha u(t)\,,\,\varphi\big) + \big(G_k(t,u(t))\,,\,\varphi\big)\Big\}\, dW^k(t) \end{aligned}$$

holds on $[0,T]$ for every $\varphi\in C_0^\infty(\mathbb{R}^d)$, where (v,φ) denotes the inner product of v and φ in $L_2(\mathbb{R}^d)$.

Set $H=L_2(\mathbb{R}^d)$, $V=W_2^1(\mathbb{R}^d)$ and consider the normal triplet $V\hookrightarrow H\hookrightarrow V^*$ based on the inner product in $L_2(\mathbb{R}^d)$, which determines the duality $\langle\,,\,\rangle$ between V and $V^*=W_2^{-1}(\mathbb{R}^d)$. By (4.5), (4.7) and (4.8) there exist a constant C and a random variable ξ with finite first moment, such that

$$\Big|\sum_{|\alpha|\le 1, |\beta|\le 1} (-1)^{|\alpha|}\big(a^{\alpha\beta}(t)\, D^\beta v\,,\, D^\alpha\varphi\big)\Big| \le C|v|_1|\varphi|_1, \quad \sum_{k=1}^{d_1} |(b_k^\alpha(t) D^\alpha v\,,\,\varphi)|^2 \le C|v|_0^2|\varphi|_0^2,$$

$$|\big(F(t,\nabla v,v)\,,\,\varphi\big)|^2 \le C|v|_1^2|\varphi|_1^2 + \xi, \quad \sum_{k=1}^{d_1} |(G_k(t,u(t))\,,\,\varphi)|^2 \le C|v|_1^2|\varphi|_0^2 + \xi$$

for all ω, $t\in[0,T]$ and $v,\varphi\in V$. Therefore the operators $A(t)$, $B_k(t)$ defined by

$$\begin{aligned} \langle A(t,v),\varphi\rangle &= \sum_{|\alpha|\le 1, |\beta|\le 1} (-1)^{|\alpha|}\big(a^{\alpha\beta}(t)\, D^\beta v\,,\, D^\alpha\varphi\big) + \big(F(t,\nabla v,v)\,,\,\varphi\big), \\ (B_k(t,v)\,,\,\varphi) &= \big(b_k^\alpha(t) D^\alpha v\,,\,\varphi\big) + \big(G_k(t,v)\,,\,\varphi\big), \quad v,\varphi\in V \end{aligned} \tag{4.9}$$

are mappings from V into V^* and H, respectively, for each k and ω,t, such that the growth conditions (2.6) and (2.7) hold. Thus we can cast the Cauchy problem (4.1)– (4.2) into the evolution equation (1.1), and it is an easy exercise to show that

Assumptions (A1), (A2) with $l = 0$ and Assumption (A4) ensure that conditions (C1) and (C2) hold. Hence Corollary 2.5 gives the following result.

Theorem 4.2. *Let Assumptions (A1)-(A4) hold with $l = 0$. Then problem* (4.1)-(4.2) *admits a unique generalized solution u on $[0, T]$. Moreover,*

$$E\Big(\sup_{t\in[0,T]} |u(t)|_0^2\Big) + E\int_0^T |u(t)|_1^2\,dt < \infty. \tag{4.10}$$

Next we formulate a result on the regularity of the generalized solution. We need the following assumptions.

Assumption (A5) *The first order derivatives of G_k in x are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R})$ -measurable functions, and there exist a constant L, a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$ -measurable function K of (ω, t, x) and a random variable ξ with finite first moment, such that*

$$\sum_{k=1}^{d_1} |D^\alpha G_k(t,x,r)| \le L|r| + K(t,x), \quad |K(t)|_0^2 \le \xi$$

for all multi-indices α with $|\alpha| = 1$, for all $\omega\in\Omega$, $t\in[0,T]$, $x\in\mathbb{R}^d$ and $r\in\mathbb{R}$.

Assumption (A6) *The first order derivatives of F in x are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R})$-measurable functions, and there exist a constant L, a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$ -measurable function K of (ω, t, x) and a random variable ξ with finite first moment, such that*

$$|\nabla_x F(t,x,p,r)| \le L(|p| + |r|) + K(t,x), \quad |K(t)|_0^2 \le \xi$$

for all ω, t, x, p, r.

Assumption (A7) *There exist $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$ -measurable functions g_k such that*

$$G_k(t,x,r) = g_k(t,x) \quad \text{for all } k = 1,2,...,d_1,\ t,x,r,$$

and the derivatives in x of g_k up to order l are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$ -measurable functions such that

$$\sum_{k=1}^{d_1} |g_k(t)|_l^2 \le \xi,$$

for all (ω, t), where ξ is a random variable with finite first moment.

Theorem 4.3. *Let Assume (A1)-(A4) with $l = 1$. Then for the generalized solution u of* (4.1)-(4.2) *the following statements hold:*

(i) Suppose (A5). Then u is a $W_2^1(\mathbb{R}^d)$-valued continuous process and

$$E\Big(\sup_{t\le T} |u(t)|_1^2\Big) + E\int_0^T |u(t)|_2^2\,dt < \infty\,; \tag{4.11}$$

(ii) Suppose (A6) and (A7) with $l = 2$. Then u is a $W_2^2(\mathbb{R}^d)$-valued continuous process and

$$E\Big(\sup_{t\le T} |u(t)|_2^2\Big) + E\int_0^T |u(t)|_3^2\,dt < \infty\,. \tag{4.12}$$

Proof. Define

$$\psi(t,x) = F(t,x,\nabla u(t,x), u(t,x)), \quad \phi_k(t,x) = G_k(t,x,u(t,x))$$

for $t \in [0,T]$, $\omega \in \Omega$ and $x \in \mathbb{R}^d$, where u is the generalized solution of (4.1)-(4.2). Then due to (4.10)

$$E\int_0^T |\psi(t)|_0^2\,dt < \infty, \quad E\sum_k \int_0^T |\phi_k(t)|_1^2\,dt < \infty.$$

Therefore, the Cauchy problem

$$\begin{aligned} dv(t,x) &= (Lv(t,x) + \psi(t,x))\,dt \\ &\quad + \sum_{k=1}^{d_1} (M_k v(t,x) + \phi_k(t,x))\,dW^k(t), \quad t \in (0,T], \quad x \in \mathbb{R}^d, \quad (4.13) \\ v(0,x) &= u_0(x), \quad x \in \mathbb{R}^d \quad (4.14) \end{aligned}$$

has a unique generalized solution v on $[0,T]$. Moreover, by Theorem 1.1 from [6], v is a W_2^1-valued continuous $\mathcal{F}_t$-adapted process and

$$E\Big(\sup_{t\le T} |v(t)|_1^2\Big) + E\int_0^T |v(t)|_2^2\,dt < \infty.$$

Since u is a generalized solution to (4.13)–(4.14), by virtue of the uniqueness of the generalized solution we have $u = v$, which proves (i). Assume now (A6) and (A7). Then obviously (A5) holds, and therefore due to (4.11)

$$E\int_0^T |\psi(t)|_1^2\,dt < \infty, \quad E\sum_k \int_0^T |\phi_k(t)|_2^2\,dt < \infty.$$

Thus by Theorem 1.1 of [6] the generalized solution $v = u$ of (4.13)–(4.14) is a $W_2^2(\mathbb{R}^d)$-valued continuous process such that (4.12) holds. The proof of the theorem is complete. □

Corollary 4.4. *Let (A1)-(A4) hold with $l = 2$. Assume also (A6) and (A7). Then there exists a constant C such that for the generalized solution u of* (4.1)–(4.2) *we have*

$$E|u(t) - u(s)|_1^2 \le C|t-s| \quad \text{for all } s,t \in [0,T].$$

Proof. By the theorem on Itô's formula from [7] (or see [2]) from almost surely

$$u(t) = u_0 + \int_0^t (Lu(s) + \psi(s))\,ds + \sum_{k=1}^{d_1} \int_0^t (M_k u(s) + g_k(s)\,dW^k(s)$$

holds, as an equality in $L_2(\mathbb{R}^d)$, for all $t \in [0,T]$, where

$$\psi(s,\cdot) := F(s, \cdot \nabla u(s,\cdot), u(s,\cdot)).$$

Due to (ii) from Theorem 4.3

$$\begin{aligned} E\Big|\int_s^t \big(Lu(r) + \psi(r)\big)\,dr\Big|_1^2 &\le E\Big(\int_s^t |Lu(r) + \psi(r)|_1\,dr\Big)^2 \\ &\le |t-s|\,E\int_s^t |Lu(r) + \psi(r)|_1^2\,dr \\ &\le C\,|t-s|\,\Big(E\int_0^T |u(t)|_3^2\,dt + E\int_0^T |\psi(t)|_1^2\,dt\Big) \le C|t-s| \end{aligned}$$

for all $s, t \in [0, T]$, where C is a constant. Furthermore, by Doob's inequality

$$E\left|\int_s^t M_k u(r) + g_k(r)\, dW^k(r)\right|_1^2 \le 4\int_s^t E|M_k u(r) + g_k(r)|_1^2\, dr$$
$$\le C_1|t-s|\Big[1 + E\Big(\sup_{t\le T}|u(t)|_2^2\Big)\Big] \le C_2|t-s|$$

for all $s, t \in [0, T]$, where C_1 and C_2 are constants. Hence

$$E|u(t)-u(s)|_1^2 \le 2E\Big|\int_s^t \big(Lu(r) + \psi(r)\big)\, dr\Big|_1^2$$
$$+ 2E\Big|\sum_{k=1}^{d_1}\int_s^t (M_k u(r) + g_k(r))dW^k(r)\Big|_1^2 \le C|t-s|,$$

and the proof of the corollary is complete. □

The implicit scheme (2.14) applied to problem (4.1)-(4.2) reads as follows.

$$\begin{aligned} u^\tau(t_0) &= u_0\,, \\ u^\tau(t_{i+1}) &= u^\tau(t_i) + \big(L^\tau_{t_i} u^\tau(t_{i+1}) + F^\tau_{t_i}(u^\tau(t_{i+1}))\big)\,\tau \\ &\quad + \sum_{k=1}^{d_1}\big(M^\tau_{k,t_i} u^\tau(t_i) + G^\tau_{k,t_i}(u^\tau(t_i))\big)\,\big(W^k(t_{i+1}) - W^k(t_i)\big)\,, \end{aligned} \tag{4.15}$$

for $0 \le i < m$, where

$$L^\tau_{t_i} v := \sum_{|\alpha|\le 1, |\beta|\le 1} D^\alpha(a^{\alpha\beta}_{t_i}(x) D^\beta v), \quad M^\tau_{k,t_i} := \sum_{|\alpha|\le 1} b^\alpha_{k,t_i} D^\alpha v,$$

$$a^{\alpha\beta}_{t_i}(x) := \frac{1}{\tau}\int_{t_i}^{t_{i+1}} a^{\alpha\beta}(s,x)\, ds, \tag{4.16}$$

$$b^\alpha_{k,0}(x) = 0, \quad b^\alpha_{k,t_{i+1}}(x) = \frac{1}{\tau}\int_{t_i}^{t_{i+1}} b_k(s,x)\, ds, \tag{4.17}$$

$$F^\tau_{t_i}(x,p,r) := \frac{1}{\tau}\int_{t_i}^{t_{i+1}} F(s,x,p,r)\, ds,$$

$$G^\tau_{k,0}(x,r) := 0, \quad G^\tau_{k,t_{i+1}}(x,r) := \int_{t_i}^{t_{i+1}} G_k(s,x,r)\, ds.$$

Definition 4.5. *A random vector $\{u^\tau(t_i) : i = 0, 1, 2, ..., m\}$ is a called a generalized solution of the scheme (4.15) if $u^\tau(t_0) = u_0$, $u^\tau(t_i)$ is a $W^1_2(\mathbb{R}^d)$-valued $\mathcal{F}_{t_i}$-measurable random variable such that*

$$E|u^\tau(t_i)|_1^2 < \infty$$

and almost surely

$$(u^\tau(t_i), \varphi) = \sum_{|\alpha|\le 1, |\beta|\le 1} (-1)^{|\alpha|}(a^{\alpha\beta}_{t_i} D^\beta u^\tau(t_i), D^\alpha\varphi)\tau + (F^\tau_{t_{i-1}}(\nabla u^\tau_{t_{i-1}}, u^\tau_{t_{i-1}}), \varphi)\tau$$

$$+\sum_k(\sum_{|\alpha|\le 1} b^{\alpha}_{t_{i-1}} D^{\alpha} u^{\tau}_{t_{i-1}} + G_{k,t_{i-1}}(u^{\tau}_{t_{i-1}}),\varphi)(W^k(t_i)-W^k(t_{i-1}))$$

for $i = 1, 2, ..., m$ and all $\varphi \in C_0^{\infty}(\mathbb{R}^d)$, where $(\cdot,\cdot)$ is the inner product in $L_2(\mathbb{R}^d)$.

From this definition it is clear that, using the operators A, B_k defined by (4.9), we can cast the scheme (4.15) into the abstract scheme (2.14). Thus by applying Theorem 2.6 we get the following theorem.

Theorem 4.6. *Let (A1)-(A4) hold with $l = 0$. Then there exists an integer m_0 such that (4.15) has a unique generalized solution $\{u^{\tau}(t_i) : i = 0, 1, ..., m\}$ for every $m \ge m_0$. Moreover, there exists a constant C such that*

$$E\max_{0\le i\le m}|u^{\tau}(t_i)|_0^2 + E\sum_{i=1}^{m}|u^{\tau}(t_i)|_1^2 \le C$$

for all integers $m \ge m_0$.

To ensure condition (T1) to hold we impose the following assumption.

Assumption (H) *There exists a constant C and a random variable ξ with finite first moment such that for $k = 1, 2, ..., d_1$*

$$|D^{\gamma}(b^{\alpha}_k(t,x) - b^{\alpha}_k(s,x))| \le C|t-s|^{1/2} \quad \text{for all } \omega\in\Omega, x\in\mathbb{R}^d \text{ and } \ |\gamma|\le l,$$
$$|g_k(s) - g_k(s)|_l^2 \le \xi|t-s|$$

for all $s, t \in [0, T]$.

Now applying Theorem 3.4 we obtain the following result.

Theorem 4.7. *Let (A1)-A(4) and (A6)-(A7) hold with $l = 2$. Assume (H) with $l = 0$. Then (4.1)–(4.2) and (4.15) have a unique generalized solution u and $u^{\tau} = \{u^{\tau}(t_i) : i = 0, 1, 2, ..., m\}$, respectively, for all integers m larger than some integer m_0. Moreover, for all integers $m > m_0$*

$$E\max_{0\le i\le m}|u(i\tau) - u^{\tau}(i\tau)|_0^2 + E\sum_{i=1}^{m}|u(i\tau)-u^{\tau}(i\tau)|_1^2\tau \le C\tau, \tag{4.18}$$

where C is a constant, independent of τ.

Proof. By Theorems 4.2 and 4.6 the problems (4.1)–(4.2) and (4.15) have a unique solution u and u^{τ}, respectively. It is an easy exercise to verify that Assumption (H) ensures that condition (T1) holds. By virtue of Corollary 4.4 condition (T2) is valid with $\nu = 1/2$. Condition (T3) clearly holds by statement (i) of Theorem 4.3. Now we can apply Theorem 3.4 , which gives (4.18).

□

4.2. **Linear stochastic PDEs.** Let Assumptions (A1)-(A3) and (A7) hold and impose also the following condition on F.

Assumption (A8) *There exist a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$ -measurable function f such that*

$$F(t,x,p,r) = f(t,x), \quad \text{for all } t,x,p,r,$$

and the derivatives in x of f up to order l are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$ -measurable functions such that

$$|f(t)|_l^2 \le \xi,$$

for all (ω, t), where ξ is a random variable with finite first moment.

Now equation (4.13) has become the linear stochastic PDE

$$du(t,x) = (Lu(t,x) + f(t,x))\,dt + \sum_{k=1}^{d_1}(M_k u(t,x) + g_k(t,x))\,dW^k(t), \tag{4.19}$$

and by Theorem 3.4 we have the following result.

Theorem 4.8. *Let $r \geq 0$ be an integer. Let Assumptions (A1)–(A3) and (A7)–(A8) hold with $l := r+2$, and let Assumption (H) hold with $l = r$. Then there is an integer m_0 such that* (4.19)–(4.2) *and* (4.15) *have a unique generalized solution u and $u^\tau = \{u^\tau(t_i) : i = 0, 1, 2, ..., m\}$, respectively, for all integers $m > m_0$. Moreover,*

$$E \max_{0 \leq i \leq m} |u(i\tau) - u^\tau(i\tau)|_r^2 + E\sum_{i=1}^{m} |u(i\tau) - u^\tau(i\tau)|_{r+1}^2 \tau \leq C\tau \tag{4.20}$$

holds for all $m > m_0$, where C is a constant independent of τ.

Proof. For $r = 0$ the statement of this theorem follows immediately from Theorem 4.7. For $r > 0$ set $H = W_2^r(\mathbb{R}^d)$ and $V = W_2^{r+1}(\mathbb{R}^d)$ and consider the normal triplet $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ based on the inner product $(\cdot\,,\cdot) := (\cdot\,,\cdot)_r$ in $W_2^r(\mathbb{R}^d)$, which determines the duality $\langle \cdot\,,\,\cdot \rangle$ between V and V^*. Using Assumptions (A3), (A7) and (A8) with $l = r$, one can easily show that there exist a constant C and a random variable ξ such that $E\xi^2 < \infty$ and

$$\Big| \sum_{|\alpha|\leq 1, |\beta| \leq 1} (-1)^{|\alpha|}\big(a^{\alpha\beta}\, D^\beta v\,,\, D^\alpha \varphi\big)_r \Big| \leq C|v|_{r+1}|\varphi|_{r+1},$$

$$\sum_{k=1}^{d_1} |(b_k^\alpha D^\alpha v\,,\,\varphi)_r|^2 \leq C|v|_{r+1}^2|\varphi|_r^2,$$

$$|\big(f(t)\,,\,\varphi\big)_r|^2 \leq \xi|\varphi|_r^2, \quad \sum_{k=1}^{d_1} |(g_k(t)\,,\,\varphi)_r|^2 \leq \xi|\varphi|_r^2$$

for all ω, $t \in [0, T]$ and $v, \varphi \in W_2^r(\mathbb{R}^d)$. Therefore the operators $A(t, \cdot)$, $B_k(t, \cdot)$ defined by

$$\langle A(t,v), \varphi \rangle = \sum_{|\alpha|\leq 1, |\beta|\leq 1} (-1)^{|\alpha|}\big(a^{\alpha\beta}\, D^\beta v\,,\, D^\alpha\varphi\big)_r + \big(f(t)\,,\,\varphi\big)_r,$$

$$(B_k(t,v)\,,\,\varphi) = \big(b_k^\alpha D^\alpha v\,,\,\varphi\big)_r + \big(g_k(t)\,,\,\varphi\big)_r, \quad v, \varphi \in V \tag{4.21}$$

are mappings from V into V^* and H, respectively, for each k and ω, t, such that the growth conditions (2.6) and (2.7) hold. Thus we can cast the Cauchy problem (4.19)–(4.2) into the evolution equation (1.1), and it is an easy to verify that conditions (C1)–(C4) hold. Thus this evolution equation admits a unique solution u, which clearly a generalized solution to (4.19)– (4.2). Due to assumptions (A1)–(A3) and (A7)–(A8) by Theorem 1.1 of [6] u is a $W^{r+2}(\mathbb{R}^d)$-valued stochastic process such that

$$E \sup_{t \leq T} |u(t)|_{r+2}^2 + E\int_0^T |u(t)|_{r+3}^2\, dt < \infty.$$

Hence it is obvious that (T3) holds, and it is easy to verify (T2) with $\nu = \frac{1}{2}$ like it is done in the proof of Corollary 4.4. Finally, it is an easy exercise to show that (T1) holds. Now we can finish the proof of the theorem by applying Theorem 3.4. □

From the previous theorem we obtain the following corollary by Sobolev's embedding from W_2^r to $\mathcal{C}^q$.

Corollary 4.9. *Let q be any non-negative number and assume that the assumptions of Theorem 4.8 hold with $r > q + \frac{d}{2}$. Then there exist modifications $\bar{u}$ and $\bar{u}^\tau$ of u and u^τ, respectively, such that the derivatives $D^\gamma \bar{u}$ and $D^\gamma \bar{u}^\tau$ in x up to order q are functions continuous in x. Moreover, there exists a constant C independent of τ such that*

$$E \max_{0\le i\le m} \sup_{x\in\mathbb{R}^d} \sum_{|\gamma|\le q} |D^\gamma\big(\bar{u}(i\tau,x) - \bar{u}^\tau(i\tau,x)\big)|^2 + E\sum_{i=1}^m \sup_{x\in\mathbb{R}^d} \sum_{|\gamma|\le q+1} \big|D^\gamma\big(\bar{u}(i\tau,x) - \bar{u}^\tau(i\tau,x)\big|^2 \tau \le C\tau. \tag{4.22}$$

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