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Controller synthesis & Ordinal Automata[*]

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Abstract. Ordinal automata are used to model physical systems with Zeno behavior. Using automata and games techniques we solve a control problem formulated and left open by Demri and Nowak in 2005. It involves partial observability and a new synchronization between the controller and the environment.

1 Introduction

Controller synthesis. The synthesis of controller is today one of the most important challenges in computer science. Since [RW89] different formalisms have been considered to model (un)controllable and (un)observable actions. The problem is well understood for finite systems admitting infinite behavior (indexed by ω) [PR89]. Recent developments concern extensions to e.g. infinite state systems or timed systems [BDMP03].

Transforming control problems into two-player games have provided efficient solutions [Tho95]. In this setting the controller is modeled by a player and the environment by her opponent. Determining whether a controller exists falls down to determine the winner and computing a winning strategy is equivalent to synthesizing a controller.

Ordinal automata. A Büchi or Muller automaton, after reading an ω-sequence, simply accepts or rejects, depending on the states visited infinitely often. In an ordinal automaton there is a limit transition to a new state, also depending on the states visited infinitely often and the run goes on from this state. This allows to model a system preforming ω actions in a finite time and reaching a limit state.

Systems with Zeno behaviors. When modeling physical systems we face the problem that different components can have different time scales. For example the controller of an anti-lock braking system (ABS) is supposed to react much quicker than the physical environment. In the opposite one can consider physical systems admitting Zeno behavior —infinitely many actions in a finite amount of time— whereas the controller is a computer with constant clock frequency. A simple example is a bouncing ball. Another one is the physical description of an electronic circuit which evolves much quicker than its logical description in

VHDL. The speeds are so different that one can consider that the former one evolves infinitely quicker than the latter one.

Following this idea Demri and Nowak [DN05] have proposed to model physical systems by ordinal automata, thus admitting ordinal sequences as behavior (typically of length ω^k). They define a logic LTL(ω^k) as an extension of LTL to express properties of such systems. The controller should be a usual automaton whose execution is an ω-sequence. The synchronization between controller and environment is the following: environment makes ω^{k-1} steps "alone", then controller and environment makes one step together, and so on.

Particularly in the context of timed systems, different techniques have been proposed to forbid or restrict Zeno behaviors, see introduction of [AFH+03] for an overview. Our claim is that we want to allow Zeno behavior, to model them and express properties about them, and finally to control such systems.

Our contribution. The main contribution of our article is a solution to the control problem stated and left open in [DN05]. Given a physical system modeled by an ordinal automaton and a formula ψ of LTL(ω^k) we want to determine whether a controller exists and synthesize one. The technique used is to transform the control problem into a game problem. Because of the unobservable actions and also because of the different time scales, the controller can not fully observe the current state of the system. For that reason we construct a game of imperfect information. Another difficulty is that the length of the interaction is greater than ω, but fortunately one can summarize ω^{k-1} steps done by the environment "alone". Several games and automata techniques are used.

Related work. It is known that games of imperfect information have higher computational complexity [Rei84]. Zeno behavior have already been considered in the literature. In [BP00] languages of ordinal words accepted by timed automata are studied. In the framework of hybrid systems [AM98,Bou99] or cellular automata on continuous time and space [DL05] it is known that allowing Zeno behaviors gives rise to highly undecidable problems. In [DN05] Demri and Nowak solve the satisfiability and the model-checking problem for LTL(ω^k): given an ordinal automaton reading ω^k-sequences and a formula ψ, determine whether every run of the automaton satisfies ψ. For this they use a "succinct" form of ordinal automata to have better complexity bounds.

Plan of the paper In the next section we present the temporal logic LTL(ω^k), ordinal automata and the control problem. We show a translation to first order logic. In Section 3 we solve our main problem. We first explain how to translate it to a game and why the controller has imperfect information about the system. An example is provided in Section 4.

2   Reasoning about transfinite sequences

We assume basic knowledge about ordinals less than ω^ω, see e.g. [Ros82]. An *ordinal* is a well and totally ordered set. It is either 0 or a successor ordinal of the

form $\beta + 1$ or a limit ordinal. The first limit ordinal is denoted ω. For all ordinal α, $\beta < \alpha \Leftrightarrow \beta \in \alpha$ and $\alpha = \{\beta : \beta < a\}$. In this article we restrict ourselves to ordinals less or equal than ω^ω. By the Cantor Normal Form theorem, for all $\alpha < \omega^\omega$ there exists unique integers $p, n_1, \ldots, n_p$ and $k_1, \ldots, k_p$ such that $k_1 > k_2 > \cdots > k_p$ and $\alpha = \omega^{k_1} n_1 + \omega^{k_2} n_2 + \cdots + \omega^{k_p} n_p$. Recall e.g. that $2\omega = \omega$ and $\omega + \omega^2 = \omega^2$. An ordinal α is said to be closed under addition whenever $\beta, \beta' < \alpha$ implies $\beta + \beta' < \alpha$. In particular for every $\alpha \le \omega^\omega$, α is closed under addition iff α is equal to ω^β for some $\beta \le \omega$ or $\alpha = 0$. In the following we will consider a logic whose models are ω^k sequences for some $k < \omega$.

2.1 Temporal Logic

We recall the definition of the logic $LTL(\alpha)$ introduced in [DN05]. For every ordinal α closed under addition, the models of $LTL(\alpha)$ are precisely sequences of the form $\sigma : \alpha \to 2^{AP}$ for some countably infinite set AP of atomic propositions. The formulas of $LTL(\alpha)$ are defined as follows: $\phi ::= p \mid \neg \phi \mid \phi_1 \wedge \phi_2 \mid X^\beta \phi \mid \phi_1 U^{\beta'} \phi_2$, where $p \in AP$, $\beta < \alpha$ and $\beta' \le \alpha$. The satisfaction relation is inductively defined below where σ is a model for $LTL(\alpha)$ and $\beta < \alpha$:

- $\sigma, \beta \models p$ iff $p \in \sigma(\beta)$,
- $\sigma, \beta \models \phi_1 \wedge \phi_2$ iff $\sigma, \beta \models \phi_1$ and $\sigma, \beta \models \phi_2$, $\sigma, \beta \models \neg \phi$ iff not $\sigma, \beta \models \phi$,
- $\sigma, \beta \models X^{\beta'} \phi$ iff $\sigma, \beta + \beta' \models \phi$,
- $\sigma, \beta \models \phi_1 U^{\beta'} \phi_2$ iff there is $\gamma < \beta'$ such that $\sigma, \beta + \gamma \models \phi_2$ and for every $\gamma' < \gamma$, $\sigma, \beta + \gamma' \models \phi_1$.

Closure under addition of α guarantees that $\beta + \beta'$ and $\beta + \gamma$ above are strictly smaller than α. Usual LTL is expressively equivalent to $LTL(\omega)$: X is equivalent to X^1 and U is equivalent to U^ω, conversely X^n and U^n can be expressed in LTL. Standard abbreviations are also extended: $F^\beta \phi \stackrel{\text{def}}{=} \top U^\beta \phi$ and $G^\beta \phi \stackrel{\text{def}}{=} \neg F^\beta \neg \phi$. Using Cantor Normal Form it is easy to effectively encode an $LTL(\omega^k)$ formula for $k < \omega$. We provide below properties dealing with limit states that can be easily expressed in $LTL(\omega^k)$ ($k \ge 2$).
"p holds in the states indexed by limit ordinals strictly less than ω^k":

$$G^{\omega^k}(X^\omega p \wedge \cdots \wedge X^{\omega^{k-1}} p).$$

For $1 \le k' \le k - 2$, "if p holds infinitely often in states indexed by ordinals of the form $\omega^{k'} \times n$, $n \ge 1$, then q holds in the state indexed by $\omega^{k'+1}$":

$$(G^{\omega^{k'+1}} F^{\omega^{k'+1}} X^{\omega^{k'}} p) \Rightarrow (X^{\omega^{k'+1}} q).$$

2.2 Translation to First Order Logic

In [DN05] it is proved that $LTL(\omega^\omega)$ (hence also $LTL(\omega^k)$) can be translated to the monadic second order theory of $\langle \omega^\omega, < \rangle$, which gives a non-elementary decision procedure for satisfiability [BS73]. We improve this result by showing that $LTL(\omega^\omega)$ can be translated even to the first order theory (FO) of $\langle \omega^\omega, < \rangle$.

Proposition 1. *For every* $\text{LTL}(\omega^\omega)$ *formula there exists an equivalent first order formula over* $\langle \omega^\omega, < \rangle$.

It is open whether the converse also holds, extending Kamp's theorem [Kam68].

Proof (sketch). The main point is the definition of a formula $+_\beta(x, y)$ for some $\beta < \omega^\omega$ such that $\langle \omega^\omega, < \rangle \models_v +_\beta(x, y)$ with $v : \{x, y\} \to \omega^\omega$ iff $v(y) = v(x) + \beta$. The relation $\models_v$ is the standard satisfaction relation under the valuation v. The formulas of the form $+_\beta(x, y)$ with $\beta < \omega^\omega$ are inductively defined as:

1. $+_0(x, y) \stackrel{\text{def}}{=} (x = y)$,
2. $+_1(x, y) \stackrel{\text{def}}{=} (x < y) \land \forall\, z\ (z > x \Rightarrow y \leq z)$,
3. $+_{\omega^k n + \beta}(x, y) \stackrel{\text{def}}{=} \exists\, z\ +_{\omega^k}(x, z) \land +_{\omega^k(n-1)+\beta}(z, y)\ (n \geq 1,\ k \geq 0)$,
4. $+_{\omega^k}(x, y) \stackrel{\text{def}}{=} (x < y) \land \forall z(x \leq z < y \Rightarrow \exists z'(+_{\omega^{k-1}}(z, z') \land z' < y)) \land$
   $\forall y'[((x < y') \land \forall z(x \leq z < y' \Rightarrow \exists z'(+_{\omega^{k-1}}(z, z') \land z' < y'))) \Rightarrow y \leq y']$
   $(k \geq 1)$.

For $k = 1$, the latter formula is written in the following way. The ordinal y such that $+_\omega(x, y)$ holds is greater than x, greater than every finite step successors of x, and y is the least ordinal satisfying this two conditions. By induction one can show that $y > x + n$ for every $n < \omega$. Analogously for $k > 1$, the formula implies that $y > x + \omega^{k-1} n$ for every $n < \omega$. $\qquad\square$

The first order theory of $\langle \omega^\omega, + \rangle$ has a non-elementary decision procedure [Mau96]. We are not aware of the exact complexity of the more restricted first order theory of $\langle \omega^\omega, < \rangle$. We use ordinal automata, both to model physical systems and to represent specifications.

2.3 Ordinal Automata

Since Büchi in the 1960s and Choueka in the 1970s, different forms of ordinal automata have been proposed. A particular class of ordinal automata is well suited to solve our problem. See [Bed98] for the equivalence between different definitions. Ordinal automata has two kinds of transitions: usual one-step transition for successor ordinals and limit transitions for limit ordinals where the state reached is determined by the set of states visited again and again "before" that ordinal. An ordinal automaton is a tuple $(Q, \Sigma, \delta, E, I, F)$ where:

- Q is a finite set of states,
- Σ is a finite alphabet,
- $\delta \subseteq Q \times \Sigma \times Q$ is a one-step transition relation,
- $E \subseteq 2^Q \times Q$ is a limit transition relation,
- $I \subseteq Q$ is a finite set of initial states,
- $F \subseteq Q$ is a finite set of final states.

We write $q \xrightarrow{a} q'$ whenever $\langle q, a, q' \rangle \in \delta$ and $P \to q$ whenever $\langle P, q \rangle \in E$. A *path* of length $\alpha + 1$ is an $(\alpha + 1)$-sequence $r : \alpha + 1 \to Q$ labeled by an

α-sequence $\sigma : \alpha \to \Sigma$ such that for every $\beta \in \alpha$, $r(\beta) \xrightarrow{\sigma(\beta)} r(\beta + 1)$ and for every limit ordinal $\beta \in \alpha + 1$, there is $P \to r(\beta) \in E$ s.t. $P = \mathit{cofinal}(\beta, r)$ with $\mathit{cofinal}(\beta, r) \stackrel{\text{def}}{=} \{q \in Q : \text{for every } \gamma \in \beta, \text{ there is } \gamma' \text{ such that } \gamma < \gamma' < \beta \text{ and } r(\gamma') = q\}$. The set $\mathit{cofinal}(\beta, r)$ is the set of states visited again and again arbitrary close to β (hence infinitely often).

If moreover $r(0) \in I$, it is a *run*. If moreover $r(\alpha) \in F$, it is accepting.

Example 1. We present here an example of ordinal automaton $\mathcal{A}$ with limit transitions $\{0\} \to 1$ and $\{0, 1\} \to 2$. One can show that $\mathrm{L}(\mathcal{A})$ contains only ω^2-sequences and $\mathrm{L}(\mathcal{A}) = (a^\omega \cdot b)^\omega$.
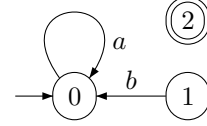
For all $k < \omega$ there exists an ordinal automaton accepting exactly the sequences of length ω^k, using $k + 1$ states. But if an ordinal automaton accepts a sequence of length ω^ω, then it must also accept longer sequences. That is a second reason, beside closure under addition, why we restrict ourselves to ordinals less than ω^ω.

Level An ordinal automaton $\mathcal{A} = \langle Q, \Sigma, \delta, E, I, F \rangle$ is of *level* $k \geq 1$ iff there is a map $l : Q \to \{0, \ldots, k\}$ such that:

- for every $q \in F$, $l(q) = k$;
- $q \xrightarrow{a} q' \in \delta$ implies $l(q') = 0$ and $l(q) < k$;
- $P \to q \in E$ implies $l(q) \geq 1$, for every $q' \in P$, $l(q') < l(q)$, and there is $q' \in P$ such that $l(q') = l(q) - 1$.

The idea is that a state of level i is reached at positions $\beta + \omega^i.j$, $j < \omega$. Since [VW86], different techniques for translating logic formulas to automata are widely used.

Proposition 2 ([DN05]). *For all* $\mathrm{LTL}(\omega^k)$ *formula, there exists an equivalent ordinal automaton.*

This result can be obtain by translating an $\mathrm{LTL}(\omega^k)$ formula into an equivalent first order formula (or even monadic second order) and applying results from [BS73]. In [DN05] a succinct version of ordinal automata is defined to improve the complexity of the translation from non-elementary to polynomial (resp. exponential) space when integers in the formulas are encoded in unary (resp. binary).

2.4 Control Problem

Before we recall the control problem from [DN05] we need some preliminary definitions. In order for the physical system to evolve much faster than the controller we need a particular synchronization between them.

Synchronous product. We define below the synchronous product of two ordinal automata having possibly different alphabets. They synchronize only on the common actions. This is used later to model unobservable actions. Let $\Sigma_i = 2^{Act_i}$ for $i = 1, 2$, a letter from Σ_i is a set of actions. Given two ordinal automata $\mathcal{A}_i = \langle Q_i, \Sigma_i, \delta_i, E_i, I_i, F_i \rangle$, for $i = 1, 2$, their synchronous product is defined as $\mathcal{A}_1 \times \mathcal{A}_2 = \langle Q, \Sigma, \delta, E, I, F \rangle$ where:

- $Q = Q_1 \times Q_2, \qquad \Sigma = 2^{Act_1 \cup Act_2}$.
- $\langle q_1, q_2 \rangle \xrightarrow{a} \langle q_1', q_2' \rangle \in \delta$ iff $q_1 \xrightarrow{a \cap Act_1} q_1'$ and $q_2 \xrightarrow{a \cap Act_2} q_2'$.
- $P \to \langle q_1, q_2 \rangle \in E$ iff there exists $P_1 \to q_1 \in E_1$ and $P_2 \to q_2 \in E_2$ such that $\{q : \langle q, q' \rangle \in P\} = P_1$ and $\{q' : \langle q, q' \rangle \in P\} = P_2$.
- $I = I_1 \times I_2, \qquad F = F_1 \times F_2$.

Lifting. In order to synchronize the system with a controller working on ω-sequences, we need to transform the controller so that its product with $\mathcal{S}$ only constraints states on positions $\omega^{k-1} \times n$, $n < \omega$. The other positions are not constrained.

Let $\mathcal{A} = \langle Q, \Sigma, \delta, E, I, F, l \rangle$ be an automaton of level 1. We define its lifting $lift_k(\mathcal{A})$ at level $k \geq 2$ to be the automaton $\langle Q', \Sigma, \delta', E', I', F', l' \rangle$ by:

- $Q' = \{0, \ldots, k\} \times Q, \qquad I' = \{k-1\} \times I, \qquad F' = \{k\} \times F$
- $l'(\langle i, q' \rangle) = i$,
- $\delta' = \{\langle k-1, q \rangle \xrightarrow{a} \langle 0, q' \rangle \; : \; q \xrightarrow{a} q' \in \delta\} \cup$
  $\quad \{\langle i, q \rangle \xrightarrow{a} \langle 0, q \rangle \; : \; 0 \leq i \leq k-2, \, a \in \Sigma, \, q \notin F\}$,
- $E' = \{\{\langle 0, q \rangle, \ldots, \langle i-1, q \rangle\} \to \langle i, q \rangle \; : \; 1 \leq i < k, \, q \in Q\} \cup \{\{\langle 0, q_1 \rangle, \ldots, \langle k-1, q_1 \rangle, \ldots, \langle 0, q_n \rangle, \ldots, \langle k-1, q_n \rangle\} \to \langle k, q \rangle \; | \; \{q_1, \ldots q_n\} \to q \in E\}$.

Example 2. We present below an example of ordinal automaton $\mathcal{A}$ with limit transition $\{q_0, q_1\} \to q_2$ and the corresponding automaton $lift_2(\mathcal{A})$ with limit transitions $\{\langle 0, q_0 \rangle\} \to \langle 1, q_0 \rangle$, $\{\langle 0, q_1 \rangle\} \to \langle 1, q_1 \rangle$, and
$\{\langle 0, q_0 \rangle, \langle 1, q_0 \rangle, \langle 0, q_1 \rangle, \langle 1, q_1 \rangle, \} \to \langle 2, q_2 \rangle$. We omit useless transitions.
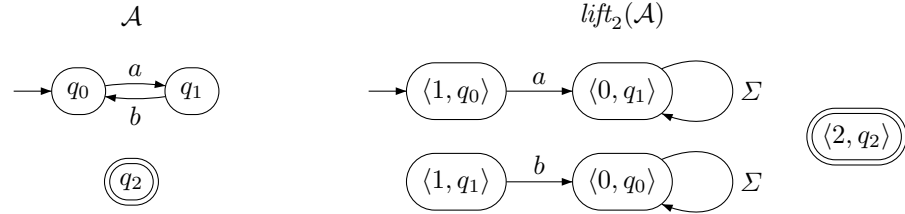
Proposition 3 ([DN05]). *For all $w \in \Sigma^{\omega^k}$, $w \in \mathrm{L}(lift_k(\mathcal{A}))$ iff the word $w' \in \Sigma^\omega$, defined by $w'(i) = w(\omega^{k-1} \times i)$, is in $\mathrm{L}(\mathcal{A})$.*

A physical system $\mathcal{S}$ is modeled as a structure

$$\langle \mathcal{A}_{\mathcal{S}}, Act_C, Act_O, Act \rangle$$

where $\mathcal{A}_\mathcal{S}$ is an ordinal automaton of level k with alphabet 2^{Act} where Act is a finite set of actions, $Act_O \subseteq Act$ is the set of observable actions, $Act_C \subseteq Act_O$ is the set of controllable actions. The set $Act \backslash Act_C$ of uncontrollable actions is denoted by Act_{nc}. A specification of the system $\mathcal{S}$ is naturally an LTL(ω^k) formula ψ. A controller $\mathcal{C}$ for the pair $\langle \mathcal{S}, \psi \rangle$ is a system whose complete executions are ω-sequences (typically ordinal automata of level 1) verifying the properties below.

(obs) Only observable actions are present in the controller. Hence, thanks to the synchronization mode, in the product system between $\mathcal{S}$ and $\mathcal{C}$, unobservable actions do not change the $\mathcal{C}$-component of the current state. So the alphabet of $\mathcal{C}$ is 2^{Act_O}. Moreover for every state q of $\mathcal{C}$ there is a transition $q \xrightarrow{\emptyset} q$.

(unc) From any state of $\mathcal{C}$, uncontrollable actions can always be executed: $\forall q \cdot \forall a \subseteq Act_O \setminus Act_C$, there is a transition $q \xrightarrow{b} q'$ in $\mathcal{C}$ such that $b \cap Act_{nc} = a$.

(prod) Finally, the system $\mathcal{S}$ controlled by $\mathcal{C}$ satisfies ψ. Because $\mathcal{S}$ and $\mathcal{C}$ work on sequences of different length, the controlled system is in fact equal to $lift_k(\mathcal{C}) \times \mathcal{S}$. So $lift_k(\mathcal{C}) \times \mathcal{S} \models \psi$ should hold. This is equivalent to the emptiness of the language of the product automaton $lift_k(\mathcal{C}) \times \mathcal{S} \times \mathcal{A}_{\neg\psi}$.

We say that $\mathcal{C}$ is a controller for $\mathcal{S}$ (without mentioning ψ) if $\mathcal{C}$ fulfills the first two conditions. The notion of final state is not relevant for the controller or the physical system. To conform with previous definitions we require that every $(\omega + 1)$-run of the controller and $(\omega^k + 1)$-run of $\mathcal{S}$ end in a final state.

The *control problem for* LTL(ω^k) is defined as follows:

input: a system $\mathcal{S} = \langle \mathcal{A}_\mathcal{S}, Act_C, Act_O, Act \rangle$ with ordinal automaton $\mathcal{A}_\mathcal{S}$ of level k and an LTL(ω^k) formula ψ over atomic formulas in Act.

output: an ordinal automaton $\mathcal{C}$ of level 1 satisfying the conditions (obs), (unc) and (prod) above if there exists one. Otherwise the answer "no controller exists".

3 Solving the Control Problem

Given a physical system $\mathcal{S}$ modeled by an ordinal automaton $\mathcal{A}_\mathcal{S}$ of level k and an LTL(ω^k)-formula ψ, we are looking for a controller $\mathcal{C}$ such that $lift_k(\mathcal{C}) \times \mathcal{A}_\mathcal{S} \models \psi$ and $\mathcal{C}$ has the expected properties about uncontrollable and unobservable actions.

From Control Problem to Game. Let $\mathcal{B} = lift_k(\mathcal{C}) \times \mathcal{A}_\mathcal{S} \times \mathcal{A}_{\neg\psi}$. At a given point in a run of $\mathcal{B}$ the controller is in a state q. From q and for all $o \subseteq Act_o \cap Act_{nc}$ it must have at least one transition labeled by $o \cup c$ for some $c \subseteq Act_c$. The most general form of a controller (possibly with infinite memory) is a function $f : (2^{Act_o})^* \times (2^{Act_o \cap Act_{nc}}) \rightarrow 2^{Act_c}$, because the current state of the controller shall only depend on the past observable actions. This function is exactly a strategy in a game that we will define. A controller for $\langle \mathcal{S}, \psi \rangle$ is such that every run according to f is winning.

Let $\mathcal{A} = \mathcal{A}_\mathcal{S} \times \mathcal{A}_{\neg\psi}$. It is also an ordinal automaton of level $k : \mathcal{A} = \langle Q, \Sigma, \delta, E, I, F, l \rangle$. We are looking for a controller $\mathcal{C}$ such that the language

of $lift_k(\mathcal{C}) \times \mathcal{A}$ is empty. We will consider a game where the environment tries to build an accepting run of $\mathcal{A}$, whereas the controller tries to avoid that, using the controlled actions. In fact the environment plays both for the system $\mathcal{S}$ and for the automaton of $\neg\psi$, as we will see later.

3.1 Some Definitions from Game Theory

We recall some definitions about games. See for example [Tho95,GTW02] for an introduction. An *arena*, or *game graph*, is a triple (V_0, V_1, G), where $V = V_0 \cup V_1$ is the set of vertices and $G \subseteq V \times V$ is the set of edges. The vertices of V_0 belongs to Player 0, those of V_1 to Player 1 ($V_0 \cap V_1 = \emptyset$). A *play* from $v_0 \in V$ proceeds as follows: if $v \in V_0$, Player 0 chooses a successor v_1 of v_0, else Player 1 does. Again from $v_1 \in V_i$, Player i chooses a successor v_2 of v_1, and so on.

A play $\pi = v_0, v_1, v_2, \ldots$ is a finite or infinite sequence of vertices such that $\forall i, (v_i, v_{i+1}) \in G$. If the play is finite, the convention is that the player who belongs the last vertex loses (he is stuck). If the play is infinite, the winner is determined by a *winning set*, $Win \subseteq V^\omega$: Player 0 wins an infinite play π if and only if $\pi \in Win$. Usually Win is an ω-regular set, defined by a Büchi, Rabin, parity or Muller automaton. One speaks also of *winning condition*. A *game* (V_0, V_1, G, Win) is an arena together with a winning condition and possibly an initial vertex $v_0 \in V$.

For a game or an automaton, a Büchi condition is given by a set $F \subseteq V$ of "final" vertices and $\pi \in Win$ if and only if $\forall i > 0, \exists j > i, \pi_i \in F$. A Muller condition is given by $\mathcal{F} \subseteq 2^V$, $\mathcal{F} = \{F_1, \cdots, F_n\}$, and $\pi \in Win$ if and only if the set of states visited infinitely often along π is equal to one of the F_i's.

A *strategy* for Player 0 is a (partial) function $f_0 : V^*V_0 \mapsto V$ such that for every prefix $v_0, v_1, v_2, \cdots v_i$ of a play, where $v_i \in V_0$, $f(v_0 v_1 v_2 \cdots v_i)$ is a vertex v_{i+1} such that $(v_i, v_{i+1}) \in G$. A play π is played according to a strategy f_0 if $\forall i, v_i \in V_0 \Rightarrow v_{i+1} = f(v_0 v_1 v_2 \cdots v_i)$. A strategy for Player 1 is defined analogously. A strategy of Player 0 is *winning* if every play according to it is winning for Player 0. An important case in practice is when the strategy is *positional*: it depends only on the current vertex, not on the past of the play, *i.e.*, for all $v_0, v_1, v_2, \cdots v_i$, $f(v_0 v_1 v_2 \cdots v_i) = f(v_i)$.

From [Mar75] we know that every zero-sum two-player turn based game of complete information with Borel winning condition (including ω-regular and many more) is determined: from a given initial configuration, one of the players has a winning strategy.

In the case of incomplete information, the players do not in general know exactly the current position of the game. They only know that the position belongs to a certain set of uncertainty. The move chosen by a player (by his strategy) shall depend on this set, but not on the precise position of the play. As we will see in some cases one can transform such a game into a game of complete information, where a vertex represents a set of positions of the original game.

3.2 A Solution With Incomplete Information

Summarizing ω^{k-1} steps. From the definition of $lift_k$ we see that the controller can act only every ω^{k-1} steps of the environment. Our aim is to summarize ω^{k-1} steps of the environment in a single step. One can compute a relation $\mathcal{R} \subseteq Q \times 2^Q \times Q$ such that $(q, P, q') \in \mathcal{R}$ iff there exists in $\mathcal{A}$ a path from q to q' of length $\omega^{k-1} + 1$ where the set of states seen along this path is exactly P. Note that to determine $\mathcal{R}$, one has to look for cycles in $\mathcal{A}$ and states that are seen infinitely often, but in $\mathcal{R}$ itself we only need to know states that are ever visited. The reason is that (considering $cofinal(\omega^k, r)$) it is not relevant to know that some state is visited infinitely often between e.g. $\omega^{k-1}3$ and $\omega^{k-1}4$ and no more visited after $\omega^{k-1}4$. Relation $\mathcal{R}$ can be computed in time $2^{\mathcal{O}(|Q|)}$ [Car02].

Game. We introduce a game ($\mathcal{G}$) modeling the interaction between the controller (Cont) and the environment (Env). It is not possible in general for Cont to know exactly the current state of the system for several reasons.

- Cont cannot know the ω^{k-1} steps done by the environment without control.
- As Env act, by choosing $v \subseteq Act_{nc}$, Cont can only observe the actions that are in Act_o.
- Moreover $\mathcal{A}$ is not necessarily deterministic. In particular it is possible that $\mathcal{A}_{\neg\psi}$ is not deterministic and Env has to "choose" which subformulas of $\neg\psi$ he wants to make true.
- Also Cont cannot know exactly the initial state chosen by Env.

In the game $\mathcal{G}$ Cont has partial information: a position of the game is a subset Q_i of Q, such that Cont knows that the current state of the system is in Q_i, but does not know which state exactly. The game is defined by the following steps:

1. $i = 0$ and the initial position is $Q_0 = I$, the set of initial states of $\mathcal{A}$
2. Env chooses $o_i \subseteq Act_o \cap Act_{nc}$,
3. Cont chooses $c_i \subseteq Act_c$,
4. there is a one step transition to

$$Q'_i = \{q' \in Q : \exists u \subseteq Act \setminus Act_o, \exists q \in Q_i, q \xrightarrow{c_i \cup o_i \cup u} q'\},$$

5. there is a jump to Q_{i+1}, summarizing ω^{k-1} steps

$$Q_{i+1} = \{q \in Q : \exists q' \in Q'_i, \exists (q', P, q) \in \mathcal{R}\},$$

6. $i = i + 1$, continue at point 2.

In this game the knowledge of Cont about the current state is exactly what a controller can compute in the original problem, based on the observable actions. A play is essentially a sequence $Q_0, Q'_0, Q_1, Q'_1, \ldots$ (a more precise definition of the game graph is given below) and now it is more intricate to determine the winner. The sequence $Q_0, Q'_0, Q_1, Q'_1, \ldots$ represents the point of view of the controller, and we call it an abstract play. After the game is played a *referee* has to

choose inside this abstract play a concrete path (if it exists one) $q_0, q_0', q_1, q_1', \ldots$ such that $q_i \in Q_i, q_i' \in Q_i'$ and compatible to the sequence of c_i's and o_i's. That is to say one has to choose $q_0 \in Q_0$, a sequence of elements $u_i \in Act \backslash Act_o$ such that $q_i \xrightarrow{c_i \cup o_i \cup u_i} q_i'$ and elements $(q_i', P_i, q_i) \in \mathcal{R}$. The sequence $q_0, q_0', P_0, q_1, q_1', P_1, \ldots$ summarizes a run in $\mathcal{A}$ and we can determine if it is accepting, in which case Env wins the play. Note that for the acceptance condition of $\mathcal{A}$ it is relevant to know whether some $q \in Q$ appears in infinitely many P_i's. Therefore the set of winning plays of Env can be defined by a *non deterministic* Muller automaton searching a concrete path, as we will see below, after we make some comments.

The advantage that Env plays "abstractly" the game, and one selects a concrete path only afterward is not unfair. Again we want a controller that is secure, and we worry if the environment *could have* won. And in the case that the controller does not have a winning strategy, it does not necessarily mean that the environment has one, but it means that there is a risk that the environment wins. This is related to the fact that games of incomplete information are not determined in general: it is possible that no player has a winning strategy.

We now describe the automaton defining the set of winning plays and then the arena in more details. Note that the sequence $Q_0, Q_0', Q_1, Q_1', \ldots$ above is uniquely determined by the sequence $o_0, c_0, o_1, c_1, \ldots$ of actions chosen by Cont and Env. The state space of the automaton $\mathcal{A}_{Win}$ recognizing the winning plays for Env is $Q \times 2^Q$. For all $P \neq \emptyset$ there is a transition $(q, P) \xrightarrow{c \cup o} (q', \emptyset)$ if and only if $\exists u \subseteq Act \backslash Act_o, \ \exists q \xrightarrow{c \cup o \cup u} q'$ in $\mathcal{A}$ and there is a transition $(q', \emptyset) \xrightarrow{\epsilon} (q, P)$ if and only if $\exists (q', P, q) \in \mathcal{R}$.

The automaton $\mathcal{A}_{Win}$ non-deterministically guesses a run in $\mathcal{A}$ conforming to the sequence $o_0, c_0, o_1, c_1, \ldots$ The acceptance condition of $\mathcal{A}_{Win}$ is the same as those of $\mathcal{A}$: it can be seen as a Muller condition depending on the states appearing infinitely often in a run. It is given by a set of sets $\mathcal{F} \subseteq 2^Q$. The usual way to handle such a non-deterministic Muller automaton is to transform it into a non-deterministic Büchi automaton [GTW02, Ch. 1]. The Büchi automaton $\mathcal{B}_{Win}$ simulates $\mathcal{A}_{Win}$ and guesses at some point which subset of states are going to be visited infinitely often and that other states are no longer visited. The state space of $\mathcal{B}_{Win}$ is $Q \cup Q \times \mathcal{F} \times (Q \cup \{q_f\})$. It checks in turn that each state of the chosen acceptance component $F \in \mathcal{F}$ is visited infinitely often and it is not necessary to remember the whole $(q, P) \in Q \times 2^Q$ of $\mathcal{A}_{Win}$. Using e.g. Safra's construction [GTW02, Ch. 3] one can transform the Büchi automaton $\mathcal{B}_{Win}$ into a *deterministic* Rabin automaton $\mathcal{C}_{Win}$. Then the Index Appearance Record allows to have a deterministic parity automaton $\mathcal{D}_{Win}$ [GTW02, p.86] [Löd98].

For defining the arena, we see that Cont and Env essentially choose the actions c_i and o_i:

$$V_{Env} = 2^{Act_c}, \quad V_{Cont} = 2^{Act_o \cap Act_{nc}}, \quad G = (V_{Env} \times V_{Cont}) \cup (V_{Cont} \times V_{Env})$$

Now the product of the arena (V_{Env}, V_{Cont}, G) by the parity automaton $\mathcal{D}_{Win}$ gives rise to a parity game on a finite graph. One can determine the winner and compute a positional winning strategy [GTW02, Ch.6,7] [JPZ06]. Due to the

synchronization between the arena and $\mathcal{D}_{Win}$, the set V_{Env} can be merged to a single vertex: it is not needed to remember the move of Cont because its effect on $\mathcal{D}_{Win}$ is sufficient. In fact the successive sets $Q_0, Q'_0, Q_1, Q'_1, \ldots$ of the above description are computed by $\mathcal{D}_{Win}$ (thanks to Safra's construction already in $\mathcal{C}_{Win}$).

Theorem 1. *The control problem defined in Section 2.4 can be solved in* 2EX-PTIME. *Moreover if a controller exists, then there is one with finite memory of double exponential size.*

The complexity is measured in the number $|Q|$ of states of $\mathcal{A} = \mathcal{A}_{\mathcal{S}} \times \mathcal{A}_{\neg\psi}$. Recall that the usual control problem is 2EXPTIME-complete [PR89] in the size of the system and the length of the formula.

See Appendix for the proof. The idea is to prove the following facts. If the game $\mathcal{G}$ is won by Cont then a controller for $\langle \mathcal{S}, \psi \rangle$ exists, and it can be constructed. Conversely if a controller for $\langle \mathcal{S}, \psi \rangle$ exists then $\mathcal{G}$ is won by Cont. By construction a strategy for Cont in $\mathcal{G}$ is a finite state automaton with expected properties about (un)observable and (un)controllable actions. Moreover if that strategy is winning, it defines a controller for $\langle \mathcal{S}, \psi \rangle$: every run of $lift_k(\mathcal{C}) \times \mathcal{S}$ fulfills ψ. Conversely, if a controller for $\langle \mathcal{S}, \psi \rangle$ exists, possibly with infinite memory, then this controller provides a winning strategy for Cont in $\mathcal{G}$. From the analysis above we know that if there is a controller for $\langle \mathcal{S}, \psi \rangle$, then there is one with finite memory, and one can compute it.

4 Example

We illustrate our construction by a (slightly modified) example from [DN05]. The system is a bouncing ball with three actions *lift-up*, *bounce* and *stop*, where only *lift-up* is controllable, and only *stop* and *lift-up* are observable. The law of the ball is described by the following LTL(ω^2) formula:

$$\phi = \mathtt{G}^{\omega^2}(lift\text{-}up \implies \mathtt{X}^1(\mathtt{G}^\omega bounce \wedge \mathtt{X}^\omega stop)) \ .$$
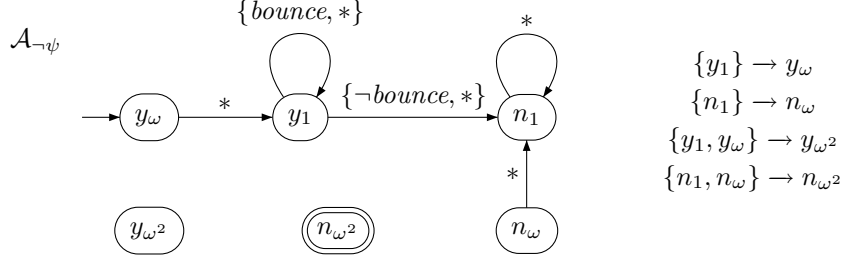
Informally, ϕ states that when the ball is lifted-up, it bounces an infinite number of times in a finite time and then stops. Equivalently the behavior of the system is modeled by the following ordinal automaton of level 2.
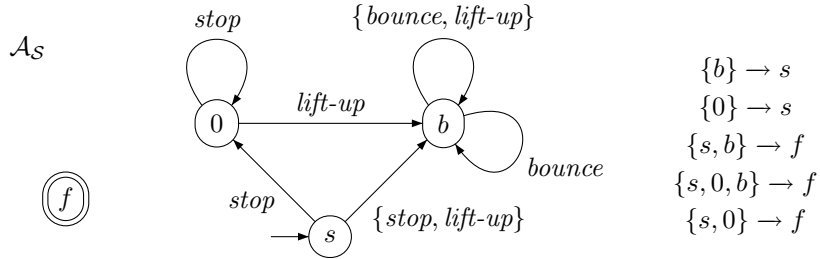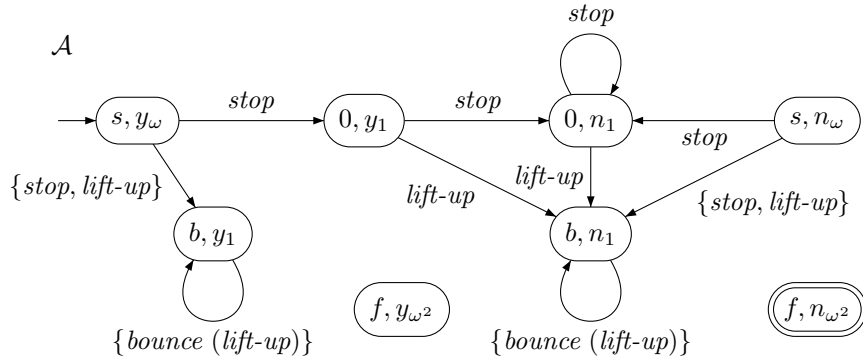

The specification is given by the LTL(ω^2) formula:

$$\psi = \mathtt{G}^{\omega^2}\mathtt{X}^1 bounce$$

Informally, ψ states that the ball should almost always be bouncing. In the following picture of the automaton $\mathcal{A}_{\neg\psi}$, the star ($*$) stands for any subset of actions of *Act*.

$\mathcal{A}_{\neg\psi}$

$$\{y_1\} \rightarrow y_\omega$$
$$\{n_1\} \rightarrow n_\omega$$
$$\{y_1, y_\omega\} \rightarrow y_{\omega^2}$$
$$\{n_1, n_\omega\} \rightarrow n_{\omega^2}$$

The automaton $\mathcal{A} = \mathcal{A}_\mathcal{S} \times \mathcal{A}_{\neg\psi}$ is then

$\mathcal{A}$

We omit here the limit transitions. In the relation $\mathcal{R} \subseteq Q \times 2^Q \times Q$ the relevant elements are

$$(\langle b, y_1 \rangle, \{\langle b, y_1 \rangle\}, \langle s, y_\omega \rangle) \qquad (\langle 0, y_1 \rangle, \{\langle 0, n_1 \rangle\}, \langle s, n_\omega \rangle)$$
$$(\langle b, n_1 \rangle, \{\langle b, n_1 \rangle\}, \langle s, n_\omega \rangle) \qquad (\langle 0, n_1 \rangle, \{\langle 0, n_1 \rangle\}, \langle s, n_\omega \rangle)$$
$$(\langle 0, n_1 \rangle, \{\langle 0, n_1 \rangle, \langle b, n_1 \rangle\}, \langle s, n_\omega \rangle)$$

If we construct the automaton $\mathcal{A}_{Win}$, we see that its (Muller) acceptance condition can be reduced to a Büchi condition. In the next figure the automaton $\mathcal{D}_{Win}$ is simplified, and some unnecessary transitions are omitted.

$\mathcal{D}_{Win}$ Game graph
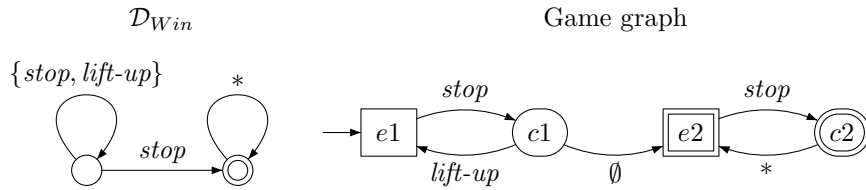
The winning strategy for Cont is: from $c1$ always go to $e1$. The corresponding controller for $\langle \mathcal{S}, \psi \rangle$ has essentially two loops on its initial state: one labeled $\{stop, lift\text{-}up\}$ and one labeled $\{lift\text{-}up\}$.

5   Perspectives

It is open whether the upper bounds of Theorem 1 are tight, and whether one can find LTL-fragments or restrictions on the physical system such that the complexity of the control problem is lower.

We would like to extend the previous results in two directions: to timed systems and to other linear orderings. Given a timed automaton, it is possible to determine whether it has Zeno behaviors. Our motivation is to extend the semantics such that after ω transitions there is a limit transition to a new control state and the new clock values are the limit of the former ones (see [BP00]).

A Zeno behavior is not necessarily an ordinal sequence, it can be a more general linear ordering (see [BC05]). One should extend the results to this more general class of automata.

References

AFH⁺03. L. de Alfaro, M. Faëlla, T. A. Henzinger, R. Majumdar, and M. Stoelinga. The element of surprise in timed games. *CONCUR'03*, LNCS 2761, pp. 142–156. 2003.

AM98.   E. Asarin and O. Maler. Achilles and the tortoise climbing up the arithmetical hierarchy. *JCSS* 57(3), pp. 389–398. 1998.

BC05.   A. Bès and O. Carton. A Kleene theorem for languages of words indexed by linear orderings. *DLT'05*, LNCS 3572, pp. 158–167. 2005.

BDMP03. P. Bouyer, D. D'Souza, P. Madhusudan, and A. Petit. Timed control with partial observability. *CAV'03*, LNCS 2725, pp. 180–192. 2003.

Bed98.   N. Bedon. *Langages reconnaissables de mots indexés par des ordinaux*. PhD thesis, Université de Marne-la-Vallée. 1998.

Bou99.   O. Bournez. Achilles and the tortoise climbing up the hyper-arithmetical hierarchy. *TCS*, 210(1):21–71. 1999.

BP00.   B. Bérard and C. Picaronny. Accepting Zeno words: A way toward timed refinements. *Acta Informatica*, 37(1):45–81. 2000.

BS73.   J. R. Buchi and D. Siefkes. *The monadic second order theory of all countable ordinals*, *Lect. Notes in Math.* 328 Springer. 1973.

Car02.   O. Carton. Accessibility in automata on scattered linear orderings. *MFCS'02*, LNCS 2420, pp. 155–164. 2002.

DL05.   J. Durand-Lose. Abstract geometrical computation for black hole computation (extended abstract). In *Machines, computations, and universality*, LNCS 3354, pp. 176–187. 2005.

DN05.   S. Demri and D. Nowak. Reasoning about transfinite sequences (extended abstract). *ATVA'05*, LNCS 3707, pp. 248–262. 2005.

GTW02.   E. Grädel, W. Thomas, and T. Wilke, editors. *Automata, Logics, and Infinite Games: A Guide to Current Research*, *LNCS* 2500. 2002.

GW94.   P. Godefroid and P. Wolper. A partial approach to model checking. *Inform. and Comput.*, 110(2):305–326. 1994.

JPZ06.   M. Jurdzinski, M. Paterson, and U. Zwick. A deterministic subexponential algorithm for solving parity games. *SODA*, pp. 117–123, 2006.

Kam68.   H. Kamp. *Tense Logic and the Theory of Linear Order*. PhD thesis, University of California at Los Angeles, 1968.

Löd98.   C. Löding. Methods for the transformation of omega-automata: Complexity and connection to second order logic. Master's thesis, Christian-Albrechts-University of Kiel, 1998.

Mar75.   D. A. Martin. Borel Determinacy. *Annals of Math.*, 102:363–371, 1975.

Mau96.   Françoise Maurin. Exact complexity bounds for ordinal addition. *Theor. Comput. Sci.*, 165(2):247–273, 1996.

PR89.   A. Pnueli and R. Rosner. On the synthesis of a reactive module. In *POPL'89*, pp. 179–190. ACM, 1989.

Rei84.   J. H. Reif. The complexity of two-player games of incomplete information. *J. Comput. System Sci.*, 29(2):274–301. 1984.

Ros82.   J. G. Rosenstein. *Linear orderings*. Academic Press Inc. 1982.

RW89.   P. J. Ramadge and W. M. Wonham. The control of discrete event systems. *Proceedings of IEEE 77(1)*, pp. 81–98. 1989.

Tho95.   W. Thomas. On the synthesis of strategies in infinite games. *STACS'95*, LNCS 900, pp. 1–13. 1995.

VW86.   M. Y. Vardi and P. Wolper. An automata-theoretic approach to automatic program verification. LICS'86, pp. 332–344. 1986.

Appendix

Correctness. We claim that the game $\mathcal{G}$ is won by Cont iff a controller for $\langle \mathcal{S}, \psi \rangle$ exists.

If $\mathcal{G}$ is won by Cont, we can compute a positional winning strategy for Cont. It consists for each position of Cont to have exactly one outgoing edge. Now one can remove these intermediate states and get a finite automaton (of size $|\mathcal{D}_{Win}|$) where the transitions are labeled by letter in 2^{Act_o}. This automaton is a controller $\mathcal{C}$ for $\langle \mathcal{S}, \psi \rangle$. It fulfills condition (obs) of Section 2.4 clearly by construction, and condition (unc) because Cont chooses only controllable actions. Moreover the language accepted by $\mathcal{C}$ is disjoint from those of $\mathcal{D}_{Win}$ and thus from those of $\mathcal{C}_{Win}$, $\mathcal{B}_{Win}$ and $\mathcal{A}_{Win}$. Finally the language of $lift_k(\mathcal{C}) \times \mathcal{S} \times \mathcal{A}_{\neg\psi}$ is empty.

Conversely suppose that there exists a controller $\mathcal{C}$ for $\langle \mathcal{S}, \psi \rangle$, possibly with infinite memory. The emptiness of $lift_k(\mathcal{C}) \times \mathcal{A}$ is equivalent to The emptiness of $\mathcal{C} \times \mathcal{A}_{Win}$ and of $\mathcal{C} \times \mathcal{D}_{Win}$. It follows that $\mathcal{C}$ defines a winning strategy in the game $\mathcal{G}$.

Complexity. The sizes, in number of states, are as follows:

$$|\mathcal{A}_{Win}| = |Q|$$
$$|\mathcal{B}_{Win}| = \mathcal{O}\left(|Q|^2 . |\mathcal{F}|\right) = \mathcal{O}\left(|Q|^2 . 2^{|Q|}\right)$$
$$|\mathcal{C}_{Win}| = 2^{\mathcal{O}(|\mathcal{B}_{Win}| . \log(|\mathcal{B}_{Win}|))} = 2^{\mathcal{O}\left(|Q|^3 . 2^{|Q|}\right)}$$

But the number of Rabin pairs of the acceptance condition of $\mathcal{C}_{Win}$ is in $\mathcal{O}\left(|\mathcal{B}_{Win}|\right)$.

$$|\mathcal{D}_{Win}| = |\mathcal{C}_{Win}| . 2^{\mathcal{O}(|\mathcal{B}_{Win}| . \log(|\mathcal{B}_{Win}|))} \quad \text{hence} \quad |\mathcal{D}_{Win}| = 2^{\mathcal{O}\left(|Q|^6 . 4^{|Q|}\right)}$$

The size of $\mathcal{D}_{Win}$ is exponential only in the number of Rabin pairs of the acceptance condition of $\mathcal{C}_{Win}$. The number of priorities of the parity automaton $\mathcal{D}_{Win}$ is in $\mathcal{O}(|\mathcal{B}_{Win}|)$. Now the number of vertices of the game graph is

$$n = |\mathcal{D}_{Win}| . (|V_{Cont}| + 1) = 2^{\mathcal{O}\left(|Q|^3 . 2^{|Q|}\right)} . 2^{Act_o \cap Act_{nc}}$$

the number of edges is

$$m = |\mathcal{D}_{Win}| . |V_{Cont}| . (|V_{Env}| + 1)$$

and the number of priorities

$$d = \mathcal{O}(|\mathcal{B}_{Win}|) \ .$$

The number of priorities of the parity game is very low compared to the number of states. In such a case the best known deterministic algorithm for solving parity games is polynomial in the size of the graph, and exponential in the number of priorities, see [JPZ06] and references therein. The time complexity is in:

$$\mathcal{O}\left(d.m.\left(\frac{2n}{d}\right)^{d/2}\right)$$

which is here in

$$|V_{Env}| \left(2^{\mathcal{O}\left(|Q|^6.4^{|Q|}\right)}|V_{Cont}|\right)^{\mathcal{O}\left(|Q|^2.2^{|Q|}\right)} =$$
$$|V_{Env}|2^{\mathcal{O}\left(|Q|^8.8^{|Q|}\right)}|V_{Cont}|^{\mathcal{O}\left(|Q|^2.2^{|Q|}\right)} =$$
$$2^{|Act_c|}2^{\mathcal{O}\left(|Q|^8.8^{|Q|}\right)}2^{\mathcal{O}\left(|Act_o \cap Act_{nc}|.|Q|^2.2^{|Q|}\right)}$$

The result of the algorithm is a positional winning strategy for the winner. In other words it is a finite graph also with n vertices. In the case that Cont wins the game, it defines directly a controller for $\langle \mathcal{S}, \psi \rangle$ with at most n states. More precisely the transitions of the controller are labeled by letters from 2^{Act_o} and we do not need the intermediate states representing the moves of Env, so the controller has at most $|\mathcal{D}_{Win}|$ states and $|\mathcal{D}_{Win}|.2^{|Act_o \cap Act_{nc}|}$ transitions.