Controllability of the moments for Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation

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Abstract

Recalling the proprieties of the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation, we prove the controllability of moments result for the linear part of KZK equation. Then we prove the local controllability result for the full KZK equation applying a known method of perturbation for the nonlinear inverse problem.

1   Introduction

Having a goal to prove a controllability of moments result for the KZK equation, we firstly say several words about this equation. The full mathematical results of the analysis of KZK equation can be found in [33]. For explaining our controllability result, we need to have a well-posed direct linear problem, and we use the proprieties of the KZK equation, briefly described in the section 2. To go from the linear inverse problem to the nonlinear one we would like to try our usual controllability method (see [30, 31, 32]), the main points of which are explained in the section 3. In the section 6 we imply it.

The KZK equation, named after Khokhlov, Zabolotskaya and Kuznetsov, was originally derived as a tool for the description of nonlinear acoustic beams (cf for instance [10, 40]). It is used in acoustical problems as mathematical model that describes the pulse finite amplitude sound beam nonlinear propagation in the thermo-viscous medium, see for example [1, 21, 8, 9, 26]. Later it has been used in several other fields and in particular in the description of long waves in ferromagnetic media [34].

The KZK equation, as it have been demonstrated in [8], accurately describes the entire process of self-demodulation throughout the near field and into the far field, both on and off the axis of the beam (in water and glycerin). The term "self-demodulation", which was coined in the 1960s by Berktay, refers to

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the nonlinear generation of a low-frequency signal by a pulsed, high-frequency sound beam.

As it is known [9], the use of intense ultrasound in medical and industrial applications has increased considerably in recent years. Both plane and focused sources are used widely in either continuous wave or pules mode, and at intensities which lead to nonlinear effects such as harmonic generation and shock formation. Typical ultrasonic sources generate strong diffraction phenomena, which combine with finite amplitude effects to produce waveforms that vary from point to point within the sound beam. Nonlinear effects have become especially important at acoustic intensities employed in many current therapeutic and surgical procedures. In addition, biological media can introduce significant absorption of sound, which must also be considered. The KZK equation, as a nonlinear equation with effects of diffraction and of absorption, which can provide shock formation, is the mathematical model of these phenomena.

The non-linear phenomena found a recent application in the field of the ultrasonic medical imagery known under the name of "harmonic imagery". In medical imagery where the echographic bars concentrates energy in a very narrow beam, the approach most commonly employed is the resolution of the equation KZK which describes focused beams.

The KZK equation is not an integrable equation at variance Kadomtsev-Petviashvili (KP) equation known to be integrable. Numerically in [10] has been obtained the existence of a shock wave in the case of propagation of the beam in nondissipative media and a quasi shock wave for the dissipative media. The last phenomenon corresponds to the approximation of the beam's front to the shock wave but the solution has the tentative to be global. We obtained the proof of the existence of the shock wave for the problem without viscosity. We have established the global existence in time of the propagation in viscous media only for rather small initial data. The announcement of the results can be found in [11, 12, 13] and all details in [33].

We would like also to add a remark about the derivation of the equation.

In [33] one proves a large time validity of the approximation for two cases: for non viscous thermoellastic media and viscous thermoellastic media.

More precisely we prove existence and stability of solutions described by the KZK equation with the following properties

1. they are concentrated near the axis x_1;

2. they propagate along the x_1 direction;

3. they are generated either by initial condition or by a forcing on the boundary $x_1 = 0$.

This corresponds to the description of the quasi one d propagation of a signal in an homogenous but nonlinear isentropic media.

Therefore it is assumed that its variation in the direction

$$x' = (x_2, x_3, \ldots, x_n)$$

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perpendicular to the x_1 axis is much larger that its variation along the axis x_1.

The KZK equation contains the linear, diffusive and diffractive terms. It usually has the following form for some positive constants β and γ:

$$\partial^2_{\tau,z}U - \frac{1}{2}\partial^2_\tau U^2 - \beta\partial^3_\tau U - \gamma\Delta_y U = 0.$$

The derivation of the KZK equation takes into account the viscosity and the size of the nonlinear terms. One starts from an isentropic Navier Stokes system:

$$\partial_t\rho + \nabla(\rho u) = 0, \quad \rho[\partial_t u + (u\cdot\nabla)u] = -\nabla p(\rho) + b\Delta u, \tag{1}$$

the pressure is given by a state law $p = p(\rho)$, where ρ is density.

With the hypothesis of potential motion one introduces constant states

$$\rho = \rho_0, \quad u = u_0.$$

Next one assumes that the fluctuation of density (around the constant state ρ_0), of velocity (around u_0, which can be taken equal to zero with galilean), are of the same order ϵ:

$$\rho_\epsilon = \rho_0 + \epsilon\tilde{\rho}_\epsilon, \quad u_\epsilon = \epsilon\tilde{u}_\epsilon, \quad b = \epsilon\nu,$$

here ϵ is a dimensionless parameter which characterizes the smallness of the perturbation. For instance in water with a initial power of the order of $0.3\,\text{Vt/cm}^2$ $\epsilon = 10^{-5}$.

The approximate state equation is

$$p = p(\rho_\epsilon) = c^2\epsilon\tilde{\rho}_\epsilon + \frac{(\gamma-1)c^2}{2\rho_0}\epsilon^2\tilde{\rho}^2_\epsilon. \tag{2}$$

Next one reminds the direction of propagation of the beam say along the axis x_1, and therefore considers the following profiles:

$$\rho_\epsilon = \rho_0 + \epsilon I(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x'), \quad u_\epsilon = \epsilon(v + \epsilon v_1, \sqrt{\epsilon}\vec{w})(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x'), \tag{3}$$

In (3) the argument of the functions will be denoted by (τ, z, y) and c is taken equal to the sound speed $c = \sqrt{p'(\rho_0)}$. Inserting the functions $\rho_\epsilon = \rho_0 + \epsilon I$, u_ϵ in the Navier-Stokes system (1) one obtains:

1 For the conservation of mass:

$$\partial_t\rho_\epsilon + \nabla(\rho_\epsilon u_\epsilon) = \epsilon(\partial_\tau I - \frac{\rho_0}{c}\partial_\tau v) +$$

$$+\epsilon^2\left(-\frac{\rho_0}{c}\partial_\tau v_1 + \rho_0(\partial_z v + \nabla_y\cdot w) - \frac{1}{\rho_0}\partial_\tau I^2\right) + O(\epsilon^3) = 0. \tag{4}$$

2 For the conservation of momentum in the x_1 direction:

$$\rho_\epsilon\epsilon(\partial_t u_{\epsilon,1} + u_\epsilon\nabla u_{\epsilon,1}) + \partial_{x_1}p(\rho_\epsilon) - \epsilon^2\nu\Delta u_{\epsilon,1} = \epsilon(\rho_0\partial_\tau v - c\partial_\tau I) +$$

$$+\epsilon^2\left(\rho_0\partial_\tau v_1 + c^2\partial_z I - \frac{\gamma-1}{2\rho_0}c\partial_\tau I^2 - \frac{\nu}{c\rho_0}\partial^2_\tau I\right) + O(\epsilon^3) = 0. \tag{5}$$

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And finally for the orthogonal (to the axis x_1) component of the momentum one has:

$$\rho_\epsilon \epsilon(\partial_t u'_\epsilon + u_\epsilon \nabla u'_\epsilon) + \partial_{x'} p(\rho_\epsilon) - \epsilon^2 \nu \Delta u'_\epsilon = \epsilon^{\frac{3}{2}}(\rho_0 \partial_\tau \vec{w} + c^2 \nabla_y I) +$$
$$+O(\epsilon^{\frac{5}{2}}) = 0. \tag{6}$$

Or in other words:

$$v(\tau, z, y) = \frac{c}{\rho_0} I(\tau, z, y), \tag{7}$$

$$\partial_\tau \vec{w}(\tau, z, y) = -\frac{c^2}{\rho_0} \nabla_y I(\tau, z, y), \tag{8}$$

$$\partial_\tau v_1 = \frac{\gamma - 1}{2\rho_0^2} c \partial_\tau I^2 + \frac{\nu}{c\rho_0^2} \partial_\tau^2 I - \frac{c^2}{\rho_0} \partial_z I, \tag{9}$$

$$c\partial_{\tau z}^2 I - \frac{(\gamma + 1)}{4\rho_0} \partial_\tau^2 I^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^3 I - \frac{c^2}{2} \Delta_y I = 0. \tag{10}$$

The KZK equation (10) is written for the perturbation of density, but the same equation with only different constants can be also derived for the pressure and the velocity. The passage between these KZK equations is possible thanks to (2), (7) and (8). For example the equation for the pressure has the form

$$\partial_{\tau z}^2 p - \frac{\beta}{2\rho_0 c^3} \partial_\tau^2 p^2 - \frac{\delta}{2c^3} \partial_\tau^3 p - \frac{c}{2} \triangle_y p = 0.$$

At this point one can state a theorem with hypothesis specified in [33].

Theorem 1 *Let I be a smooth solution of the KZK equation (10), define the functions v, w and v_1 by the known I. Define the function $\overline{U}_\epsilon = (\overline{\rho}_\epsilon, \overline{u}_\epsilon)$ by the formula:*

$$(\overline{\rho}_\epsilon, \overline{u}_\epsilon)(x_1, x', t) = (\rho_0 + \epsilon I, \epsilon(v + \epsilon v_1, \sqrt{\epsilon}\vec{w}))(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x').$$

Then there exist constants $C \geq 0$ and $T_0 = O(1)$, such that for any finite time $0 < t < T_0 \frac{1}{\epsilon} \ln \frac{1}{\epsilon}$ and $\epsilon > 0$, there exists a smooth solution $U_\epsilon = (R_\epsilon, U_\epsilon)(x, t)$ of the isentropic Navier-Stokes equation such that one has for some $s \geq 0$:

$$\|\overline{U}_\epsilon - U_\epsilon\|_{H^s} \leq \epsilon^{\frac{5}{2}} e^{\epsilon C t}.$$

It is interesting to notice that for the non viscous case, i.e., for the isentropic compressible Euler system, the KZK like equation with $\beta = 0$ have been obtained using the scaling of nonlinear diffractive geometric optic theory in [15, p. 1233] (in 2d) in the framework of nonlinear diffractive geometric optic with rectification. The analyse of this work, also as [39] where one studies the short wave approximation for general symmetric hyperbolic systems can be found in [33].

The scaling of Sanchez [34] for Landau-Lifshitz-Maxwell equations in $\mathcal{R}^3$ is very different(see [33]) .

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Remark 1 *There are mathematical works [23], [24] for KZK type equation*

$$\alpha u_{z\tau} = (f(u_\tau))_\tau + \beta u_{\tau\tau\tau} + \gamma u_\tau + \triangle_x u,$$

where $u_\tau = u_\tau(z, x, \tau)$ is the acoustic pressure, $(z, x) \in \mathcal{R}^d \times \mathcal{R}$, $d = 1, 2$ are space variables and τ is the retarded time. The equation is studied with the hypothesis that the nonlinearity f has bounded derivative which allows to proof the global existence for the case when the coefficients are rapidly oscillating functions of z. So this problem is not related with our "acoustical" problem for the KZK equation where as we will see later there is a blow-up result illustrating the existence of a shock wave.

2 Some basic results on the solutions of KZK equation

2.1 Existence uniqueness and stability of solutions of the KZK equation

Following the mathematical tradition in this section and in the next one the unknown will be denoted by u, and the variables $(x, y) \in \mathcal{R}_x \times (\Omega \subseteq \mathcal{R}^{n-1})$. When $\Omega \neq \mathcal{R}^{n-1}$ it is assumed that the solution satisfies on its boundary the Neumann boundary condition. Multiplying u by a positive scalar one reduces the problem to an equation involving only two constants β and γ

$$(u_t - u u_x - \beta u_{xx})_x - \gamma \Delta_y u = 0 \quad \text{in } \mathcal{R}_x/(L\mathcal{Z}) \times \Omega. \tag{11}$$

For sake of simplicity and because this also corresponds to practical situations [10, 40] we consider solutions which are periodic with respect to the variable x and which are of mean value zero:

$$u(x + L, y, t) = u(x, y, t), \quad \int_0^L u(x, y, t) dx = 0. \tag{12}$$

Observe that the conditions (12) are compatible with the flow and that the second one is "natural" because we consider fluctuations.

For these functions the norm of the space H^s ($s \in R$, $s \geq 0$) is denoted by

$$\|u\|_{H^s} = \left(\int_{R^{n-1}} \sum_{k=-\infty}^{+\infty} (1 + k^2 + \eta^2)^s |\hat{u}(k, \eta)|^2 d\eta \right)^{\frac{1}{2}}.$$

If we introduce the operator $\Lambda = (1 - \Delta)^{\frac{1}{2}}$ as $\widehat{(\Lambda u)}(\zeta) = (1 + |\zeta|^2)^{\frac{1}{2}} \hat{u}(\zeta)$, then

$$\Lambda^s = (1 - \Delta)^{\frac{s}{2}}, \quad \|u\|_{H^s} = \|\Lambda^s u\|_{L_2}. \tag{13}$$

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We define the inverse of the derivative ∂_x^{-1} as an operator acting in the space of periodic functions with mean value zero this gives the formula:

$$\partial_x^{-1} f = \int_0^x f(s)ds + \int_0^L \frac{s}{L} f(s)ds. \tag{14}$$

This form of the operator ∂_x^{-1} preserves the both qualities: the periodicity and having the mean value zero.

In this situation equation (11) is equivalent to the equation

$$u_t - uu_x - \beta u_{xx} - \gamma \partial_x^{-1} \Delta_y u = 0 \quad \text{in } \mathcal{R}_x/(L\mathcal{Z}) \times \Omega. \tag{15}$$

Finally when $\gamma = 0$ equation (11) reduces to the Burgers-Hopf equation for which existence smoothness and uniqueness of solution are well known. For $\gamma = \beta = 0$ it reduces to the Burgers equation

$$\partial_t u - \partial_x \frac{u^2}{2} = 0,$$

which after a finite time exhibits singularities. After this "blow-up" time the solution can be uniquely continued into a weak solution satisfying an elementary entropy condition (in the present case with $\gamma \neq 0$ it seems that this construction cannot be adapted to equation (11) with $\beta = 0$ and $\gamma \neq 0$).

We would like also to notice that the J. Bourgain-type method and introduction the Bourgain spaces as in [36, 27, 28] and others are not useful for the KZK problem because of absence of the terms with an odd derivative as for example u_{xxx} in (15). The presence only of the second derivative make impossible the main estimations and equalities of this method.

2.1.1 A priori estimates for smooth solutions

According to the standard approach we first establish a priori estimates for smooth solutions which are in particular a consequence of the relation:

$$\int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x^{-1}(\Delta_y u) u \, dx dy = -\int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x^{-1}(\nabla_y u) \nabla_y u \, dx dy$$

$$= \int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x^{-1}(\nabla_y u) \partial_x (\partial_x^{-1}(\nabla_y u)) dx dy = 0. \tag{16}$$

The L_2 norm and the H^s in $(\mathcal{R}_x^+/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ are denoted by $|u|$ and by $\|u\|_s$.

Proposition 1 *The following estimates are valid for solutions of the integrated KZK equation (15):*

$$\frac{1}{2}\frac{d}{dt}|u(\cdot,\cdot,t)|^2 + \beta|\partial_x u(\cdot,\cdot,t)|^2 = 0, \tag{17}$$

$$For \ s > [\frac{n}{2}] + 1 \quad \frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta\|\partial_x u\|_s^2 \leq C(s)\|u\|_s^3 \tag{18}$$

$$and \ \frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta C(L)\|u\|_s^2 \leq C(s)\|u\|_s^3. \tag{19}$$

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The estimates (18), (19) are valid for $s > [\frac{n}{2}]+1$ which is the necessary condition because of application of the Sobolev theorem.

Proof. To obtain the relation (17) multiply (15) by u, and integrate by part. It shows that for $\beta = 0$ we have the conservation law for the norm of u in $L_2(\mathcal{R}_x^+/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$. If $\beta > 0$ we also have according to the physical phenomena [10] the dissipation of energy.

For the clarity the proof of (18) is done firstly in 3 space variables, with $\Omega = \mathcal{R}^2$ and s an integer (i.e. in the present case $s = 3$) and after we give the proof in general case. In $2d$ in particular when $\Omega = S^1$ the proof is even simpler. The proof in the whole is similar except for the relation (19) which holds only in the periodic case and not on the whole line. (In this later case the H^s norm of $\partial_x u$ does not control the H^s norm of u).

For the proof of general case $s \in \mathcal{R}$ one has used the representation of the norm in H^s with the help of the operator Λ by (13) and the technique demonstrated in [20] and [35] for periodic and nonperiodic cases, which allows to deduce

$$\frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta\|\partial_x u\|_s^2 \leq C\|\nabla_{x,y}u\|_{L_\infty}\|u\|_s^2,$$

and this implies the necessity of our restriction for s:

$$\text{if } s > [\frac{n}{2}]+1 \text{ then } H^{s-1} \subset L_\infty.$$

For more details see [33].

Finally to prove (19) one uses the fact that u is of x mean value 0 and therefore it is (cf: (14)) related to $\partial_x u$ by the formula

$$u = \partial_x^{-1}\partial_x u = \int_0^x \partial_x u(s,y)ds + \int_0^L \frac{s}{L}\partial_x u(s,y)ds, \tag{20}$$

which implies the relation

$$\|u\|_{H^s} \leq C\|\frac{\partial u}{\partial x}\|_{H^s}. \quad \square$$

2.1.2 Existence and uniqueness for smooth solutions

The following theorem is an easy consequence of the a priori estimates.

Theorem 2 *For the following Cauchy problem*

$$u_t - uu_x - \beta u_{xx} - \gamma\partial_x^{-1}(\Delta_y u) = 0 \,, u(x,y,0) = u_0 \tag{21}$$

considered in $(\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}$, i.e. in the class of x periodic functions with mean value 0 with the operator ∂_x^{-1} defined by the formula (14) and finally with $\beta \geq 0$ one has the following results.

1 For $s > [\frac{n}{2}] + 1$ ($s = 3$ for instance in dimension 3) there exists a constant $C(s, L)$ such that for any initial data $u_0 \in H^s$ the problem (21) has on an interval $[0, T[$ with

$$T \geq \frac{1}{C(s, L)\|u_0\|_{H^s}} \tag{22}$$

a solution in $C([0, T[, H^s) \cap C^1([0, T[, H^{s-2})$.

2 Let T^* be the biggest time on which such solution is defined then one has

$$\int_0^{T^*} \sup_{x,y}(|\partial_x u(x, y, t)| + |\nabla_y u(x, y, t)|)dt = \infty. \tag{23}$$

3 If $\beta > 0$ there exists a constant C_1 such that

$$\|u_0\|_s \leq C_1 \Rightarrow T^* = \infty. \tag{24}$$

4 For two solutions u and v of KZK equation, assume that $u \in L_\infty([0, T[; H^s)$, $v \in L^2([0, T[; L_2)$. Then one has the following stability uniqueness result:

$$|u(., t) - v(., t)|_{L^2} \leq e^{\int_0^t \sup_{x,y}|\partial_x u(x,y,s)|ds}|u(., 0) - v(., 0)|_{L^2}. \tag{25}$$

Remark 2 *The estimate (25) is of strong-weak form, as in [14] only the L_∞ norm of u_x is needed.*

Remark 3 *When there is no viscosity all the corresponding statements of the theorem 2 remain valid for $0 > t > -C$ with a convenient C.*

Remark 4 *As (11) is envisaged for $u(t, x, y)$ with $x \in \mathcal{R}/(L\mathcal{Z})$, the KZK equation can be also written for $u(t, x, y) = v(t, -x, y)$ in the equivalent form*

$$(v_t + vv_x - \beta v_{xx})_x + \gamma \triangle_y v = 0.$$

So it is important to keep invariant the sign $-\beta v_{xxx}$, $\beta \geq 0$, but all other signs can be changed.

Proof. To construct a solution one can proceed by regularization, by a fractional step method, or by any other type of approximation. In particular it was done for the general case with the help of Kato theory from [16, 17, 18, 19]. Since we intend to analyze the numerical methods, the fractional step is favored and once again the only case $n = 3$ and $s = 3$ with periodic solutions is analyzed. The idea of this kind of proof can be found in [38] and firstly have been introduced by Marchuk and Yanenko. Furthermore as for a priori estimates result we cite in [33] two proofs: one with the analysis of the fractional step method for the case $n = 3$ and $s = 3$ and an other proof for general case.

2.2 Blow-up and singularities

The first remark is that for $\nu = 0$ (or $\beta = 0$) and for function independent of y the KZK equation

$$(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0 \quad \text{in } \mathcal{R}_{t_+} \times \mathcal{R}_x \times \Omega \tag{26}$$

becomes Burgers equation which is known to exhibit singularities. On the other hand the derivation and the approximation results of the following section show that any solution of the KZK equation has in its neighborhood a solution of the isentropic Euler equation [33]. Once again it is known that such solution even with smooth initial data may exhibit singularities (cf. [14] or [37]). These observations are reflected by the fact that for $\beta = 0$ and $\gamma > 0$ the equation (26) may generate singularities.

We prove the geometric blow-up result using the method of S. Alinhac, which is based on the fact that the studied equation degenerates to the Burgers equation. In fact Alinhac's method is the generalized method of characteristics for the Burgers equation adapted to the multidimensional case. As we can see the equation (26) possess all this main properties, and gives us the reason to apply it.

For instance one has the theorem:

Theorem 3 *The equation*

$$(u_t - uu_x)_x - \gamma \Delta_y u = 0 \quad \text{in } \mathcal{R}_{t_+} \times \mathcal{R}_x \times \Omega \tag{27}$$

with Neumann boundary condition on $\partial \Omega$ has no global in time smooth solution if

$$\sup_{x,y} \partial_x u(x, y, 0)$$

is large enough with respect to γ.

Remark 5 *As we can see from [10] the result of the theorem perfectly confirms the numerical results. In practically from numerical results one observes that for $\beta \to 0$ the KZK equation has a quasi shock approaching to the shock wave, into which it degenerates for $\beta = 0$.*

Proof. The proof follows the ideas of S. Alinhac ([2], [3] and [4]). First the blow-up is observed for $\gamma = 0$ and related to a singularity in the projection of an unfolded "blow-up system". Second the properties of this unfolded blow-up system are shown to be stable under small perturbations. One uses a Nash-Moser theorem with tamed estimates and this is the reason why will exists a T^* such that:

$$\lim_{t \to T^*} (T^* - t) \sup_{x,y} \partial_x u(x, y, t) > 0.$$

Remark 6 *The Nash-Moser theory and the definition of the tamed estimates can be found in [7].*

Remark 7 *An equation of the type (27) is introduced by Alinhac to analyze the blow-up of multidimensional (in $\mathcal{R}^{2+1}$) nonlinear wave equation by following the wave cone*

$$\partial_t^2 u - \triangle_x u + \sum_{0 \le i,j,k \le 2} g_{ij}^k \partial_k u \partial_{ij}^2 u = 0,$$

where

$$x_0 = t, \quad x = (x_1, x_2), \quad g_{ij}^k = g_{ji}^k,$$

with small smooth initial data (see [5]). In fact this corresponds to the same scaling as the KZK equation because from this wave equation with some changes of variable and approximate manipulations Alinhac obtains (see [3, 5, 6])

$$\partial_{xt}^2 u + (\partial_x u)(\partial_x^2 u) + \epsilon \partial_y^2 u = 0.$$

Moreover, the Euler system, with $\rho = \rho_0 + \tilde{\rho}$ and $\nabla q(\rho) = \frac{1}{\rho}\nabla p(\rho)$, can be written as

$$\partial_t \tilde{\rho} + \nabla.u + u\nabla\tilde{\rho} + \tilde{\rho}\nabla.u = 0,$$
$$\partial_t u + q'(\rho_0)\nabla\tilde{\rho} + (u, \nabla)u + q'(\tilde{\rho})\tilde{\rho}\nabla\tilde{\rho} = 0,$$

or

$$\partial_t \tilde{\rho} + \nabla.u = F(u, \nabla.u, \tilde{\rho}, \nabla\tilde{\rho}),$$
$$\partial_t u + q'(\rho_0)\nabla\tilde{\rho} = G(u, \nabla.u, \tilde{\rho}, \nabla\tilde{\rho}).$$

If we now derive the first equation on t and take ∇ of the last, then we take their difference, we obtain the wave equation of Alinhac's form :

$$\partial_t^2 \tilde{\rho} - q'(\rho_0)\triangle\tilde{\rho} + (\nabla G - \partial_t F)(u, \nabla.u, \tilde{\rho}, \nabla\tilde{\rho}) = 0.$$

The similar wave equation can be also obtained for u.
    This is the reason for the analogy.

The details of the proof can be found in [33].

3   Controllability method

We describe now the controllability method of resolving the nonlinear inverse problems using the well posedness of the linear one. All details on applications and developments of this method can be find in [30, 31, 32].
    We suppose that we have some linear problem of the following type:

$$u_t - (Au)(t) = h(x,t)f(t), \tag{28}$$

$$u|_{t=0} = 0, \quad u|_{\partial\Omega} = 0 \text{ (if } \partial\Omega \neq \emptyset) \tag{29}$$

$$\int_{\Omega_x} u(x,t)\omega(x)dx = \chi(t). \tag{30}$$

Here A is some linear operator. The functions $h(x,t)$, $\omega(x)$ and $\chi(t)$ (also the operator A) are given in the such way that the direct problem (28), (29) is well-posed for all control functions f from some space denoted $Z \subseteq L_2[0,T]$.

More precisely, we suppose that

$$F(x,t) = h(x,t)f(t) \in Y, \tag{31}$$

and Y is at least a Banach space, as well as X and under the "well-posedness" we understand that

for all $\quad F \in Y$ there exists a unique solution $u \in X$ such that

$\|u\|_X \leq C\|F\|_Y$ ($C > 0$ is a constant independent of F). (32)

Using the fact (32), if we introduce the solution space of the direct problem (28), (29)

$$H = \{v \in X | \exists F \in Y : v \text{ is the solution of the direct problem (28), (29)}\},$$

we obtain that the operator

$$L = d/dt - A \tag{33}$$

induces an isometric isomorphism of H on $LH = Y$ with norm $\|u\|_H = \|Lu\|_Y$ and, therefore, the space $(H, \|\cdot\|_H)$ is Banach. Moreover, by the *a priori* estimate (32) H is continuously embedded in X.

We remark now that the condition (30) can be rewritten as the inner product in $L_2(\Omega_x)$:

$$(u(\cdot,t), \omega(\cdot))_{L_2(\Omega_x)} = \chi(t).$$

We introduce the linear operator l of overdetermination:

$$l(u) = (u(\cdot,t), \omega(\cdot))_{L_2(\Omega_x)}, \; l : Y \mapsto Z \subseteq L_2(0,T)$$

which we apply to the equation (28), supposing the fact that the both parts of the equation belong to Y:

$$\varphi(t) = l(h(\cdot,t))f(t) \in Z,$$

$$l(u'_t) = \frac{d}{dt}l(u) = \chi'(t) \in Z,$$

$$-l(Au) = -(u, A^*\omega)_{L_2(\Omega_x)} \in Z.$$

Using the linear operator

$$\hat{A} : Z \to Z, \quad \hat{A}\varphi = -(u, A^*\omega)_{L_2(\Omega_x)}, \tag{34}$$

where $u(t)$ is the solution of the linear problem (28)-(30), which can be obtained for $f = \frac{\varphi}{l(h)}$ (we always assume that $|l(h)| \geq \delta > 0$). So we obtain roughly the same constrains for the functions h, χ and ω as in the theorem 5:

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- $\omega \in \overset{\circ}{H}^1 (\Omega_x)$,

- $\chi \in H^1(0, T)$, $\chi(0) = 0$,

- $h(x, t)$ be a function such that $h \in L_2(\Omega)$, $\|h(\cdot, \cdot, t)\|_{L_2(\Omega_x)}$ is bounded on $[0, T]$,

- $\left| \int\limits_{\Omega_x} h(x, t)\omega(x)dx \right| \geq \delta > 0$ for almost all $t \in [0, T]$.

Finally we have the operator equation

$$\chi'(t) = \varphi(t) - \hat{A}\varphi(t) = (I - \hat{A})\varphi(t), \tag{35}$$

which is equivalent to the inverse problem (28)-(30). Indeed, we must show that if φ is a solution of equation (35) with some χ, then the solution of the direct problem (28)-(30) u obtained from the given $f(t) = \frac{\varphi}{l(h)}$, satisfies condition (30) with the same function $\chi(t)$.

Assume the converse:

$$\int_{\Omega_x} u(x, t)\omega(x)dx = \chi_1(t) \in H^1(0, T).$$

Since $u|_{t=0} = 0$, then $\chi_1(0) = 0$. Deriving the operator equation for these φ and χ_1, we find that φ also satisfies the equation

$$\varphi - A\varphi = \chi_1'. \tag{36}$$

We subtract (35) from (36), and we obtain

$$\chi'(t) = \chi_1'(t), \quad \chi_1(0) = \chi(0) = 0.$$

Then $\chi(t) = \chi_1(t)$, $t \in [0, T]$, which contradicts the original assumption and it proves the equivalence.

We suppose now that the inverse linear problem is also well-posed (there exits a unique solution $f \in Z$) and

$$\|\hat{A}\|_{\mathcal{L}(Z,Z)} < 1. \tag{37}$$

Under the inverse problem (28)-(30) here we understand that our goal is to choose, by varying the element f, the function $u(t) = u(t; f)$ satisfying condition (30), among all the solutions of the direct problem (28), (29).

Theorem 4 *Suppose that the linear problems (28)-(30) and (28), (29) are well posed, and (37) holds. Let $G : X \to Y$ be a nonlinear, strictly Fréchet-differentiable operator satisfying the conditions $G'(0) = 0$, $G(0) = 0$.*

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Then the nonlinear inverse problem

$$u'_t = [Au] + Gu + F(x,t), \tag{38}$$
$$u|_{t=0} = 0, \quad u|_{\partial\Omega} = 0 \ \ (if \ \partial\Omega \neq \emptyset)$$
$$\int_{\Omega_x} u(x,t)\omega(x)dx = \chi(t)$$

where $F(x,t)$ is the same as in (31), has a unique solution f in a neighborhood of zero in Z for sufficiently small (in norm) χ.

Remark 8 *In the theorem the nonlinearity $G(u)$ can be rather general. Precisely, if (Ω, Σ, μ) is a space with measure with t, $s \in \Omega$ then the following types of operators satisfies the conditions of theorem on the operator G:*

1. *Nemytski's operator: $u(t) \mapsto g(t, u(t))$,*

2. *Urysohn's operator: $u(t) \mapsto \int_{\Omega} K(t, s, u(s))d\mu(s)$,*

3. *Hammerstein's operator: $u(t) \mapsto \int_{\Omega} K(t, s)g(s, u(s))d\mu(s)$.*

Proof.

Since H is subset of X, it follows that the mapping $G : H \to Y$ is also strictly differentiable in the sense of Fréchet. Taking also into account the equality $G'(0) = 0$, the fact that $L : H \to Y$ is an isomorphism (see (33)), and using the inverse function theorem, we see that the mapping $\xi : u \mapsto Lu - Gu$ is a local diffeomorphism of class C^1 in a neighborhood of zero of U' in H onto a neighborhood of zero of V' in Y.

Suppose that $\eta = \xi^{-1} : V' \to U'$ is the mapping inverse to this local diffeomorphism, i.e., $\eta : F \longmapsto u$, where u is a solution of equation (38) and η is strictly differentiable on V'.

We have $F(x,t) = h(x,t)\frac{\varphi}{l(h(\cdot,t))}$ (see (31)).

Consider

$$P(\varphi) = \eta \left[h(x,t)\frac{\varphi}{l(h(\cdot,t))} \right], \quad P : Z \to H. \tag{39}$$

Since the mapping

$$\Lambda : \varphi \longmapsto h(x,t)\frac{\varphi}{l(h(\cdot,t))}, \quad \Lambda : Z \to Y \tag{40}$$

is linear and continuous, it follows that P is strictly differentiable in the sense of Fréchet in a neighborhood of zero in the space Z as a mapping into H.

Further, we look for a solution of the nonlinear problem (38), (29), (30) as

$$u = P(\varphi),$$

where u is a solution of (38), (29), (30) with $F(x,t) = h(x,t)\frac{\varphi}{l(h(\cdot,t))}$ on the right-hand side of (38).

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Both sides of equations (38) belong to Y; therefore to these equations we can apply the linear overdetermination operator $l \in \mathcal{L}(Y, Z)$:

$$\int_{\Omega_x} u_t' \omega(x) dx = \int_{\Omega_x} u(x,t) A^* \omega(x) dx + \int_{\Omega_x} Gu(x,t)\omega(x) dx + \int_{\Omega_x} h(x,t) f(t) \omega(x) dx,$$

Let us introduce the mapping

$$M : Z \to Z, \quad M : \varphi \longmapsto \left\{ \varphi + \int_{\Omega_x} [P(\varphi)] A^* \omega(x) dx + \int_{\Omega_x} G[P(\varphi)] d\omega(x) x \right\}. \tag{41}$$

Then the system (38), (29), (30) can be written as

$$M\varphi(t) = \chi'(t). \tag{42}$$

The proof that the nonlinear inverse system (38), (29), (30) is equivalent to the operator equation (42), is analogous to the the proof of the linear one. For this it is sufficient to show that if φ is a solution of equation (42) with some χ, then $u = P(\varphi)$ (obtained from φ), where u is a solution of the direct nonlinear problem (38), (29) with given $F(t) = h(x,t) \frac{\varphi}{l(h(\cdot,t))}$, satisfies condition (30) with the same function $\chi(t)$.

Let us show that M is strictly differentiable in the sense of Fréchet in a neighborhood of zero in Z and $M'(0) = I - \hat{A}$ (see (34)).

Note that the mapping $\varphi \longmapsto \int_{\Omega_x} G[P(\varphi)]\omega(x) dx$ is strictly differentiable in the sense of Fréchet in a neighborhood of zero by the theorem on the differentiability of a composite function and its derivative at zero is zero, since $G'(0) = 0$. Further, the mapping $\varphi \longmapsto \int_{\Omega_x} P(\varphi) A^* \omega(x) dx$ strictly differentiable in a neighborhood of zero, since $v(\cdot) \longmapsto \int_{\Omega_x} v(x,t) A^* \omega dx$ is a linear continuous operator from X to Z and, especially, from H to Z.

Moreover,

$$\int_{\Omega_x} P'(0)\varphi A^* \omega dx = -\hat{A}\varphi. \tag{43}$$

Therefore, M is strictly differentiable in a neighborhood of zero in E and $M'(0) = I - \hat{A}$ is the operator of the linear problem (28)- (30) (see (35)). Since the linear problem is well defined, there exist $(M'(0))^{-1}$ and $\left\| (M'(0))^{-1} \right\| \leq 1/(1 - \left\| \hat{A} \right\|)$. By the inverse function theorem, there exist open neighborhoods of zero of U and V in Z such that M induces a diffeomorphism of class C^1 of U onto V.

Let $\tilde{V} = \left\{ \chi \in H^1(0,T), \ \chi(0) = 0 | \chi'(t) \in V \right\}$. Then $\tilde{V}$ is an open neighborhood of zero in $H^1(0,T) \subset Z$ and for all $\chi \in \tilde{V}$ there exists a unique $\varphi \in U$ such that $M\varphi = \chi'$.

Thus, we have proved the local unique solvability of the operator equation equivalent to problem (38), (29), (30) which concludes the proof of theorem 4. $\square$

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4   The direct problem for linearized KZK equation

As it can be easily seen, thanks to proved estimates for full KZK equation and to the proof of the theorem 2 of existence and uniqueness of the solution, the problem

$$u_t - \beta u_{xx} - \gamma \partial_x^{-1} \triangle_y u = F(x, y, t), \tag{44}$$

$$u|_{t=0} = u_0, \quad u(x+L, y, t) = u(x, y, t), \quad \int_0^L u dx = 0 \tag{45}$$

has a unique global solution in H^s for all $s \geq 0$. In particular for the homogeneous equation it follows from the estimate

$$\frac{d}{dt} \|u\|_{H^s}^2 \leq \beta C(L) \|u\|_{H^s}^2,$$

which takes place for all $s \geq 0$, and it follows also from the fact that the operator $\partial_x^{-1} \triangle_y$ is generator of a unitary C_0-group in L_2 with mean value zero [29, p.41] and this unitary group $e^{-t\partial_x^{-1}\triangle_y}$ preserves the H^s norm. For nonhomogeneous problem we can use the theorem from [29, p.107], supposing $F \in C^1([0, T], H^s)$ $(T \leq \infty)$. Then for the solution $u \in C([0, T], H^s) \cap C^1([0, T], H^{s-2})$ $(s - 2 \geq 0)$ of the problem (44) the Cauchy formula holds

$$u = S(t)u_0 + \int_0^t S(t-s)F ds,$$

which gives the estimate

$$\|u\|_{C^1([0,T], H^{s-2})} \leq C(\|u_0\|_{H^s} + \|F\|_{C^1([0,T], H^s)}) \tag{46}$$

for some $s \geq 0$.

On the other hand it can be easily shown with the estimate

$$\frac{d}{dt} \|u\|_{L_2(\Omega)} + \frac{\beta}{C^2(\Omega)} \|u\|_{L_2(\Omega)} \leq \|F(\cdot, \cdot, t)\|_{L_2(\Omega)}$$

(see   (60)) and the Galerkin method as in [25] that for all $F(x, y, t) \in L_2((0, T), H^2(\Omega))$ and $u_0 \in H^2(\Omega)$ there exists a unique solution of (44) $u \in W_{2,0}^{0,1}(Q_T)$ such that (see [25, pp.167,189])

$$\|u\|_{H^{0,1}(Q_T)} \leq C(\|u_0\|_{H^2(\Omega)} + \|F\|_{L_2((0,T), H^2(\Omega))}). \tag{47}$$

To obtain the result of nonlinear controllability for the KZK equation it is natural to use the fact that for all $F(x, y, t) \in L_2((0, T); H^s(\Omega))$ and $u_0 \in H^s(\Omega)$ there exists unique solution of (44) $u \in W_{2,0}^{s-2,1}(Q_T)$ such that

$$\|u\|_{H^{s-2,1}(Q_T)} \leq C(\|u_0\|_{H^s(\Omega)} + \|F\|_{L_2((0,T); H^s(\Omega))}). \tag{48}$$

5 The inverse problem for linearized KZK equation

We consider the controllability problem for the equation (44) in a domain $Q_T = [0, T] \times \Omega_{x,y}$, where $\Omega_{x,y}$ can be bounded domain: $\Omega_{x,y} = [0, L] \times \Omega_y$ with some $\Omega_y \subset \mathcal{R}^{n-1}$; or can be unbounded domain $\Omega = \Omega_{x,y} = \mathcal{R}/(L\mathcal{Z}) \times \Omega_y$ with $\Omega_y \subseteq \mathcal{R}^{n-1}$. The boundary of Ω is denoted $\partial\Omega$. If $\Omega = \Omega_{x,y}$ is not bounded, than the constant of Poincaré-Friedrichs in the proof of the theorem 5 must be replaced by a constant of the periodicity on x $C(L)$. So we envisage the controllability problem (44) in a domain $Q_T = [0, T] \times \Omega_{x,y}$ with an additional condition, called the condition of overdetermination,

$$\int_0^L \int_\Omega u(x, y, t)\omega(x, y)dxdy = \chi(t), \tag{49}$$

with homogeneous boundary conditions and mean value zero on x in the case of bounded domain

$$u|_{t=0} = 0, \quad u|_{\partial\Omega} = 0, \quad \int_0^L u dx = 0, \tag{50}$$

and with

$$u|_{t=0} = 0, \quad u(x + L, y, t) = u(x, y, t), \quad \int_0^L u dx = 0, \tag{51}$$

for unbounded domain (if it is bounded on y we always suppose that $u|_{\partial\Omega_y} = 0$).

We suppose in what follows

$$F(x, y, t) = h(x, y, t)f(t). \tag{52}$$

Here the functions h, ω, χ are imposed, and f - is an unknown function, which we call the control.

Remark 9 *The problem (44), (50), (49) can be easily, thanks to its linearity, generalized on nonhomogeneous case of following form*

$$u_t - \beta u_{xx} - \gamma \partial_x^{-1} \triangle_y u = h(x, y, t)f(t) + g(x, y, t)$$

$$u|_{t=0} = u_0(x, y), \quad u|_{\partial\Omega} = u_1(x, y, t), \quad \int_0^L u dx = 0,$$

$$\int_0^L \int_\Omega u(x, y, t)\omega(x, y)dxdy = \chi(t)$$

if the known functions g, u_0, u_1 are sufficiently smooth and the matching condition

$$\int_0^L \int_\Omega u_0(x, y)\omega(x, y)dxdy = \chi(0)$$

is satisfied.

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Definition 1 *The function $f \in L_2(0,T)$ is called by the solution of the inverse problem (44), (50) (or (51)), (49) if the solution u of the problem (44), (50) (or (51)) with this f satisfies the condition of overdetermination (49) almost everywhere on $[0,T]$.*

We define in $L_2(0,T)$ an equivalent norm by the expression

$$\|f\|^2_{L_2(0,T)} = \int_0^T e^{-\alpha t}|f(t)|^2 dt, \tag{53}$$

where $\alpha > 0$ is some number the choice of which will be done later.

Theorem 5 *Suppose that $\omega \in H^2(\Omega_{xy})\cap \overset{\circ}{H^1}(\Omega_{xy})$, $\chi \in H^1(0,T)$, $\chi(0) = 0$. Further, let $h(x,t)$ be a function such that $h \in L_2(Q_T)$, $\|h(\cdot,\cdot,t)\|_{L_2(\Omega_{xy})}$ is bounded on $[0,T]$ and $\left|\int\limits_{\Omega_{xy}} h(x,y,t)\omega(x,y)dxdy\right| \geq \delta > 0$ for almost all $t \in [0,T]$. Then there exists a unique solution of the problem (44), (50) (or (51)), (49) and the stability estimate holds:*

$$\delta \|f\|_{L_2(0,T)} \leq \|\chi'\|_{L_2(0,T)}/(1-m), \tag{54}$$

where

$$m = C_1 \left(\beta\|\omega_{xx}\|_{L_2(\Omega_{xy})} + \gamma\|\partial_x^{-1}\Delta_y\omega\|_{L_2(\Omega_{xy})}\right) \frac{e^{\frac{T}{2}}}{(\alpha + 2/C^2(\Omega))^{\frac{1}{2}}},$$

$C_1 > 0$ *such that*

$$\frac{\|h(\cdot,\cdot,t)\|_{L_2(\Omega_{xy})}}{(h,\omega)_{L_2(\Omega_{xy})}} \leq C_1,$$

$C(\Omega)$ *is a constant of Poincaré-Friedrichs (in the case of unbounded domain $C(\Omega)$ is replaced by $C(L)$) , $\alpha > 0$ is chosen from the condition $m < 1$.*

Proof. It follows from equation (44) that

$$\left(u_t - \beta u_{xx} - \gamma\partial_x^{-1}\Delta_y u, \omega\right)_{L_2(\Omega_{xy})} = (h,\omega)_{L_2(\Omega_{xy})} f \tag{55}$$

for almost all $t \in [0,T]$.

We set

$$\varphi(t) = (h,\omega)_{L_2(\Omega_{xy})} f.$$

By virtue of the assumptions of theorem 5, f can be uniquely determined on the basis of φ. Let us assume that the solution of the problem exists and derive an operator equation. As $(u,\omega)_{L_2(\Omega_{xy})} \in H^1(0,T)$, then

$$(u_t,\omega)_{L_2(\Omega_{xy})} = d/dt(u,\omega)_{L_2(\Omega_{xy})} = \chi'(t).$$

Since $u \in W^{2,1}_{2,0}(Q_T)$, $\omega \in H^2(\Omega)\cap \overset{\circ}{H^1}(\Omega)$, it follows that

$$\left(-\gamma\partial_x^{-1}\Delta_y u, \omega\right)_{L_2(\Omega_{xy})} = \left(u, \gamma\partial_x^{-1}\Delta_y\omega\right)_{L_2(\Omega_{xy})},$$

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$$(-\beta u_{xx}, \omega)_{L_2(\Omega_{xy})} = -(u, \beta \omega_{xx})_{L_2(\Omega_{xy})}$$

for $t \in [0, T]$.

Then relation (55) implies that

$$\chi'(t) + \int_0^L \int_{\Omega_y} u(x, y, t)(-\beta \omega_{xx} + \gamma \partial_x^{-1} \Delta_y \omega)(x, y) dx dy = \varphi(t).$$

We denote $\Psi(t) = \chi'(t)$, $\Psi \in L_2(0, T)$.

We introduce a linear operator

$$A : L_2(0, T) \to L_2(0, T),$$

$$(A\varphi)(t) = \int_0^L \int_{\Omega_y} u(x, y, t)(-\beta \omega_{xx} + \gamma \partial_x^{-1} \Delta_y \omega)(x, y) dx dy, \quad (56)$$

which map φ according to the following way : $\varphi \to f \to u \to A\varphi$, where the solution of inverse problem (44), (50), (49) is found by the given f, and f is found by the formula

$$f(t) = \frac{\varphi(t)}{(h, \omega)_{L_2(\Omega_{xy})}}.$$

Consequently, we obtain the operator equation

$$\varphi - A\varphi = \Psi. \tag{57}$$

Let us prove that $A \in \mathcal{L}(L_2(0, T))$ and $\|A\| < 1$.

We estimate $\|A\|$ for all $\varphi \in L_2(0, T)$

$$\|A\varphi\|_{L_2(0,T)}^2 = \int_0^T e^{-\alpha t} \left(\int_0^L \int_{\Omega_y} u(x, y, t)(-\beta \omega_{xx} + \gamma \partial_x^{-1} \Delta_y \omega)(x, y) dx dy \right)^2 dt. \tag{58}$$

By Cauchy-Schwarz-Bunyakovkii inequality we have for $t \in [0, T]$

$$\left(\int_0^L \int_{\Omega_y} u(x, y, t)(-\beta \omega_{xx} + \gamma \partial_x^{-1} \Delta_y \omega)(x, y) dx dy \right)^2 \leq$$

$$\leq \left(\beta \|\omega_{xx}\|_{L_2(\Omega_{xy})} + \gamma \|\partial_x^{-1} \Delta_y \omega\|_{L_2(\Omega_{xy})} \right)^2 \|u(\cdot, \cdot, t)\|_{L_2(\Omega_{xy})}^2.$$

We substitute it into (58), and obtain:

$$\|A\varphi\|_{L_2(0,T)}^2 \leq \left(\beta \|\omega_{xx}\|_{L_2(\Omega_{xy})} + \gamma \|\partial_x^{-1} \Delta_y \omega\|_{L_2(\Omega_{xy})} \right)^2 \int_0^T e^{-\alpha t} \|u(\cdot, \cdot, t)\|_{L_2(\Omega_{xy})}^2 dt. \tag{59}$$

We find the estimate for $\|u(\cdot, \cdot, t)\|_{L_2(\Omega_{xy})}$ (see [25, p.167], [22, p.47]). Taking the inner product in $L_2(\Omega_{xy})$ with u the equation (44), we have, noting in what follows $\Omega = \Omega_{xy}$,

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L_2(\Omega)}^2 + \beta \|u_x\|_{L_2(\Omega)}^2 \leq \|u\|_{L_2(\Omega)} \|f(t) h(\cdot, \cdot, t)\|_{L_2(\Omega)}.$$

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Since the domain Ω is bounded in $\mathcal{R}^{n+1}$, we can apply the equality of Poincaré-Friedrichs, from which it is obviously following that

$$\frac{1}{C^2(\Omega)}\|u\|_{L_2(\Omega)}^2 \leq \|u_x\|_{L_2(\Omega)}^2$$

and

$$\|u\|_{L_2(\Omega)}\frac{d}{dt}\|u\|_{L_2(\Omega)} + \frac{\beta}{C^2(\Omega)}\|u\|_{L_2(\Omega)}^2 \leq \|u\|_{L_2(\Omega)}\|f(t)h(\cdot,\cdot,t)\|_{L_2(\Omega)}.$$

And then

$$\frac{d}{dt}\|u\|_{L_2(\Omega)} + \frac{\beta}{C^2(\Omega)}\|u\|_{L_2(\Omega)} \leq \|f(t)h(\cdot,\cdot,t)\|_{L_2(\Omega)}. \tag{60}$$

The last inequality can be rewritten with the help of integrable factor in the way

$$\frac{d}{dt}\left(e^{\frac{\beta}{C^2(\Omega)}t}\|u\|_{L_2(\Omega)}\right) \leq e^{\frac{\beta}{C^2(\Omega)}t}|f(t)|\|h(\cdot,\cdot,t)\|_{L_2(\Omega)}.$$

Since the initial data has been chosen equivalent to zero, in the end we obtain

$$\|u\|_{L_2(\Omega)} \leq e^{-\frac{\beta}{C^2(\Omega)}t}\int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}|f(\tau)|\|h(\cdot,\cdot,\tau)\|_{L_2(\Omega)}d\tau. \tag{61}$$

Let us transform the right-hand part of inequality in such way that it depends on the function $\varphi(t)$. For it we multiply and divide on $\int_\Omega |h(x,y,t)\omega(x,y)|dxdy$, using the assumption of the theorem about separability from zero of this integral, and using the fact of the existence of a positive constant C_1 such that $\|h(\cdot,\cdot,t)\|_{L_2(\Omega_{xy})}/(h,\omega)_{L_2(\Omega_{xy})} \leq C_1$:

$$\int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}|f(\tau)|\|h(\cdot,\cdot,\tau)\|_{L_2(\Omega)}d\tau =$$

$$= \int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}\frac{\|h(\cdot,\cdot,\tau)\|_{L_2(\Omega)}}{\int_\Omega |h(x,y,\tau)\omega(x,y)|dxdy}\left(|f(\tau)|\int_\Omega |h(x,y,\tau)\omega(x,y)|dxdy\right)d\tau$$

$$\leq C_1\int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}|\varphi(\tau)|d\tau.$$

I.e. (61) takes the form

$$\|u\|_{L_2(\Omega)} \leq C_1 e^{-\frac{\beta}{C^2(\Omega)}t}\int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}|\varphi(\tau)|d\tau. \tag{62}$$

Let us envisage the integral in the right-hand side of the inequality (62)

$$\int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}|\varphi(\tau)|d\tau \leq \left(\int_0^t e^{(\frac{2\beta}{C^2(\Omega)}+\alpha)\tau}d\tau\right)^{\frac{1}{2}}\|\varphi\|_{L_2(0,T)}. \tag{63}$$

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Returning now to (59), with the help of (62), (63), we obtain that

$$\|A\varphi\|_{L_2(0,T)}^2 \leq N^2 \frac{C_1^2}{\frac{2\beta}{C^2(\Omega)} + \alpha} \|\varphi\|_{L_2(0,T)}^2 \int_0^T \left(e^{\frac{2\beta}{C^2(\Omega)}t} - e^{-\alpha t}\right) dt \leq$$

$$\leq N^2 C \frac{C_1^2}{\frac{2\beta}{C^2(\Omega)} + \alpha} e^T \|\varphi\|_{L_2(0,T)}^2,$$

where $N = \left(\beta\|\omega_{xx}\|_{L_2(\Omega_{xy})} + \gamma\|\partial_x^{-1}\Delta_y\omega\|_{L_2(\Omega_{xy})}\right)^2$.

From where we conclude that $A \in \mathcal{L}(L_2(0,T))$ and there exists $\alpha > 0$ such that $\|A\| < 1$.

The condition $\|A\| < 1$ guarantees the one-valued solvability in $L_2(0,T)$ of the operator equation (57).

Let us prove that (57) is equivalent to the inverse problem (44), (50), (49). Indeed, let φ is a solution of the equation (57) with given in condition of the theorem function χ. We unequivocally define $f = \varphi/(h,\omega)_{L_2(\Omega)}$. By virtue of the assumptions of theorem 5 $f \in L_2(0,T)$. Let us show that the solution $u \in W_{2,0}^{2,1}(Q_T)$ founded by f of the direct problem (44), (50) satisfies the condition of overdetermination (49).

Assume the converse:

$$\int_0^L \int_\Omega u(x,y,t)\omega(x,y)dxdy = \chi_1(t) \in H^1(0,T).$$

Since $u|_{t=0} = 0$, then $\chi_1(0) = 0$. Deriving the operator equation for these φ and χ_1, we find that φ also satisfies the equation

$$\varphi - A\varphi = \chi_1'. \tag{64}$$

We subtract (57) from (64), and we obtain

$$\chi'(t) = \chi_1'(t), \quad \chi_1(0) = \chi(0) = 0.$$

Then $\chi(t) = \chi_1(t)$, $t \in [0,T]$, which contradicts the original assumption.

Let us prove that the solution of the problem (44), (50), (49) is unique. Assume the converse. Then repeating the derivation of the operator equation (57) for the difference $u - u_1$, we obtain that φ satisfies the homogeneous equation.

By virtue of the uniqueness of solution of the operator equation (57) we obtain $f = 0$. By virtue of the uniqueness of solution of the direct problem $u - u_1 = 0$.

Let us show now the stability estimate (54).

Indeed, if we envisage the relation between φ and f, then we can notice that

$$\|\varphi\|_{L_2(0,T)}^2 = \int_0^T e^{-\alpha t} \left|\int_\Omega h(x,y,t)\omega(x,y)dxdy\right|^2 |f(t)|^2 dt \geq$$
$$\geq \delta^2 \|f\|_{L_2(0,T)}^2.$$

Since $\|A\varphi\| \le m\|\varphi\|$ for all $\varphi \in L_2(0,T)$, i.e.,

$$\|A\| = m,$$

and $\varphi - A\varphi = \chi'$, we obtain that

$$(1-m)\|\varphi\|_{L_2(0,T)} = \|\chi'\|_{L_2(0,T)},$$

from where it follows (54). This completes the proof of the theorem 5.□

We note that if we suppose in assumptions of the theorem 5 the additional regularity on (x,y) we obtain the following theorem.

Theorem 6 *Suppose that for $s > [\frac{n}{2}] + 1$*

$$\omega \in H^{2s-2}(\Omega_{xy}) \cap \overset{\circ}{H}{}^{s-1}(\Omega_{xy}),$$

$\chi \in H^1(0,T)$, $\chi(0) = 0$ and, let $h(x,t)$ be a function such that $h \in (L_2((0,T);H^{s-2}(\Omega_{xy}))$, $\|h(\cdot,\cdot,t)\|_{H^{s-2}(\Omega_{xy})}$ is bounded on $[0,T]$ and

$$(h(x,y,t),\omega(x,y))_{H^{s-2}(\Omega_{xy})} \ge \delta > 0$$

for almost all $t \in [0,T]$. Then there exists a unique solution of the problem (44), (50), (or (51)) and

$$(u(x,y,t),\omega(x,y))_{H^{s-2}(\Omega_{xy})} = \chi(t), \tag{65}$$

which is equivalent, thanks to the smoothness of ω, to

$$\int_{\Omega_{x,y}} u(x,y,t)\Lambda^{2(s-2)}\omega(x,y)dxdy = \chi(t),$$

and then the stability estimate holds:

$$\delta\,\|f\|_{L_2(0,T)} \le \|\chi'\|_{L_2(0,T)}/(1-m),$$

where

$$m = C_1\left(\beta\|\omega_{xx}\|_{H^{s-2}(\Omega_{xy})} + \gamma\|\partial_x^{-1}\Delta_y\omega\|_{H^{s-2}(\Omega_{xy})}\right)\frac{e^{\frac{T}{2}}}{(\alpha + 2/C^2(\Omega))^{\frac{1}{2}}},$$

$C_1 > 0$ such that

$$\frac{\|h(\cdot,\cdot,t)\|_{H^{s-2}(\Omega_{xy})}}{(h,\omega)_{H^{s-2}(\Omega_{xy})}} \le C_1,$$

$C(\Omega)$ is a constant of Poincaré-Friedrichs,(or it is the periodicity constant $C(L)$) $\alpha > 0$ is chosen from the condition $m < 1$.

Remark 10 *To prove the theorem it is sufficient to replace the norm in $L_2(\Omega)$ in theorem 5 by $\|\cdot\|_{H^{s-2}} = \|\Lambda^{s-2}\cdot\|_{L_2}$.*

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6 The controllability for nonlinear KZK equation

We consider now the inverse problem in the domain $(0, T) \times \Omega$ with $S_T = \partial\Omega \times [0, T]$:

$$u_t - uu_x - \beta u_{xx} - \gamma \partial_x^{-1} \triangle_y u = F(x, y, t), \tag{66}$$

$$u|_{t=0} = 0, \quad u|_{S_T} = 0, \quad \int_0^L u dx = 0 \text{ (if } S_T = \emptyset \text{ then}$$

$$u(x + L, y, t) = u(x, y, t)), \tag{67}$$

$$(u, w(x, y))_{H^s} = \chi(t), \tag{68}$$

with F from (52) and $F \in L_2((0, T), H^{s+3}(\Omega))$.

Having the result of previous section for linear part of KZK in the form of the theorem 6, we would like to use the method of two times application of inverse function theorem demonstrated in the section 3.

Remark 11 *Unfortunately we cannot use the result of the theorem 5 because a simple reason: for construction of the space of solutions H for linear direct problem for (44) we need an isomorphism $Lu = F \in L_2(Q_T)$, but we have it only for $u \in W_{2,0}^{0,1}$ which is insufficient to control the nonlinearity $\|uu_x\|_{L_2}$.*

From [7, p. 100] we have the estimate

$$\|\Phi(u)\|_{L_2(\Omega)} = \|uu_x\|_{L_2(\Omega)} \le C\|u\|_{H^1(\Omega)}^2 \text{ for } u \in H^{s'}, \quad s' > [\frac{n}{2}] + 1,$$

which requests to have the solutions more regular on (x, y).

But we cannot neither use the theorem 6 with $F(x, y, t) \in L_2((0, T), H^s(\Omega))$ as soon as (by virtue of the a priori estimate (48) we have $u \in H^1((0, T); H^{s-2}(\Omega)))$

$$\Phi(u) = uu_x, \quad \Phi : H^1((0, T); H^{s-2}(\Omega)) \to L_2((0, T); H^s(\Omega))$$

should be strictly differentiable, which, it is seems, is impossible to prove.

So the idea is to use the theorem 6 for more smooth right-hand side, $F \in L_2((0, T), H^{s+3}(\Omega))$, and take the condition of overdetermination as the inner product in H^s. We know from (48) that for all $F \in L_2((0, T), H^{s+3}(\Omega))$ there exits a unique solution of (44) $u \in H^1((0, T); H^{s+1}(\Omega))$.

This allows us to introduce the operator

$$L = \partial_t - \beta \partial_x^2 - \gamma \partial_x^{-1} \triangle_y, \tag{69}$$

and the space of the solutions of linear direct problem

$$H = \{v \in H^1((0, T); H^{s+1}(\Omega)) | \exists F \in L_2((0, T); H^{s+3}(\Omega)) : v \text{ is a solution of problem}$$
$$\text{(44), (50) with} \quad u_0 = 0\} \tag{70}$$

with the norm $\|v\|_H = \|Lv\|_{L_2((0,T);H^{s+3}(\Omega))}$. Then

$$L : H \to L_2((0,T);H^{s+3}(\Omega)) \text{ is an isometric isomorphism.} \qquad (71)$$

Note that $H = \{v \in H^1((0,T);H^{s+1}(\Omega))|v|_{t=0} = 0,\ v|_{S_T} = 0\}$, and $\|\cdot\|_H$ is equivalent to $\|\cdot\|_{W_{2,0}^{s+1,1}(Q_T)}$.

This implies that $(H,\|\cdot\|_H)$ is complete, the embedding $H \subset W_{2,0}^{s+1,1}(Q_T)$ is continuous.

Now we can easily see that the operator defined as

$$\Phi(u) = uu_x, \quad \Phi : H^1((0,T);H^{s+1}(\Omega)) \to L_2((0,T);H^s(\Omega))$$

is strictly differentiable. Indeed, we have

$$\|\Phi(u)\|_{L_2((0,T);H^s(\Omega))} = \|uu_x\|_{L_2((0,T);H^s(\Omega))} \leq C\|u\|^2_{H^1((0,T);H^{s+1}(\Omega))}. \qquad (72)$$

Besides, $\|u\|^2_{H^1((0,T);H^{s+1}(\Omega))}$ is bounded in the unit ball of $H^1((0,T);H^{s+1}(\Omega))$ and so the quadratic application uu_x on the unit ball in $H^1((0,T);H^{s+1}(\Omega))$, then $\Phi(u)$ is infinitely differentiable in the sense of Frécher (even analytic in all the space). So the nonlinear operator Φ is strictly Frécher- differentiable on $H^1((0,T);H^{s+1}(\Omega))$.

If we take now a subset of $L_2((0,T);H^s(\Omega))$ the propriety doesn't change, so we can say that Φ is also strictly differentiable from $L_2((0,T);H^{s+3}(\Omega))$ on $H^1((0,T);H^{s+1}(\Omega))$ in the sense of (72).

Proposition 2 *The operator*

$$G : H^1((0,T);H^{s+1}(\Omega)) \to L_2(0,T), \quad [G(u)](t) = (\Phi(u(x,y,t)),\omega(x,y))_{H^s}$$

is strictly Frécher differentiable on $H^1((0,T);H^{s+1}(\Omega))$ for $s > [\frac{n}{2}] + 1$.

Proof. First we show that G maps $H^1((0,T);H^{s+1}(\Omega))$ into $L_2(0,T)$.

$$\int_0^T |G(u)|^2 dt = \int_0^T (uu_x,\omega)^2_{H^s} dt \leq \int_0^T \|uu_x\|_{H^s}\|\omega\|^2_{H^s} dt \leq$$

$$\leq \|\omega\|^2_{H^s} \int_0^T \|u\|^4_{H^{s+1}} dt < \infty.$$

G is strictly differentiable as the composition of the linear operator and the strictly differentiable one. $\square$

So if we take $h \in L_2((0,T);H^{s+3}(\Omega))$, $\omega \in H^{2s}\cap \overset{\circ}{H}{}^{2s-1}(\Omega)$ and all other assumptions of the theorem 6 we obtain that the linear direct and inverse problem associated to (66)-(68) are well posed. It means that we can prove the analog of the theorem 4.

Theorem 7 *Suppose that holds all assumptions of the Theorem 6. Then the problem (66)-(65) with F from (52) has a unique solution f in a neighborhood of zero in $L_2(0,T)$ for $\chi \in {}_{\circ}H^1(0,T)$ with sufficiently small norm.*

Here we set ${}_{\circ}H^1(0,T) = \{\chi \in H^1(0,T)|\chi(0) = 0\}$ with the norm $\|\cdot\|_{H^1(0,T)}$.

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