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Abstract

In this paper, we study additive coalescents. Using their representation as fragmentation processes, we prove that the law of a large class of eternal additive coalescents is absolutely continuous with respect to the law of the standard additive coalescent on any bounded time interval.

Key Words. Additive coalescent, fragmentation process

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1 Introduction

The paper deals with additive coalescent processes, a class of Markov processes which have been introduced first by Evans and Pitman [11]. In the simple situation of a system initially composed of a finite number k of clusters with masses $m_1, m_2, \ldots, m_k$, the dynamics are such that each pair of clusters (m_i, m_j) merges into a unique cluster with mass $m_i + m_j$ at rate $m_i + m_j$, independently of the other pairs. In the sequel, we always assume that we start with a total mass equal to 1 (i.e. $m_1 + \ldots + m_k = 1$). This induces no loss of generality since we can then deduce the law of any additive coalescent process through a time renormalization. Hence, an additive coalescent lives on the compact set

$$\mathcal{S}^{\downarrow} = \{x = (x_i)_{i\geq 1}, x_1 \geq x_2 \geq \ldots \geq 0, \sum_{i\geq 1} x_i \leq 1\},$$

endowed with the topology of uniform convergence.

Evans and Pitman [11] proved that we can define an additive coalescent on the whole real line for a system starting at time $t = -\infty$ with an infinite number of infinitesimally small clusters. Such a process will be called an eternal coalescent process. More precisely, if we denote by $(C^n(t), t \geq 0)$ the additive coalescent starting from the configuration $(1/n, 1/n, \ldots, 1/n)$, they proved that the sequence of processes $(C^n(t+\frac{1}{2}\ln n), t \geq -\frac{1}{2}\ln n)$ converges in distribution on the space of càdlàg paths with values in the set $\mathcal{S}^{\downarrow}$ toward some process $(C^{\infty}(t), t \in \mathbb{R})$, which is called the standard additive coalescent. We stress that this process is defined for all time $t \in \mathbb{R}$. A remarkable property of the standard additive coalescent is that, up to time-reversal, its becomes a fragmentation process. Namely, the process $(F(t), t \geq 0)$ defined by $F(t) = C^{\infty}(-\ln t)$ is a self-similar fragmentation process with index of self similarity $\alpha = 1/2$, with no erosion and with dislocation measure ν given by

$$\nu(x_1 \in dy) = (2\pi y^3(1-y)^3)^{-1/2} dy \quad \text{for } y \in]1/2, 1[, \quad \nu(x_3 > 0) = 0.$$

We refer to Bertoin [7] for the definition of erosion, dislocation measure, and index of self similarity of a fragmentation process and a proof. Just recall that in a fragmentation process, distinct fragments evolve independently of each others.

Aldous and Pitman [1] constructed this fragmentation process $(F(t), t \geq 0)$ by cutting the skeleton of the continuum Brownian random tree according to a Poisson point process. In another paper [2], they gave a generalization of this result: consider for each $n \in \mathbb{N}$ a decreasing sequence $r_{n,1} \geq \ldots \geq r_{n,n} \geq 0$ with sum 1, set $\sigma_n^2 = \sum_{i=1}^n r_{n,i}^2$ and suppose that

$$\lim_{n\to\infty} \sigma_n = 0 \text{ and } \lim_{n\to\infty} \frac{r_{n,i}}{\sigma_n} = \theta_i \text{ for all } i \in \mathbb{N}.$$

Assume further that $\sum_i \theta_i^2 < 1$ or $\sum_i \theta_i = \infty$. Then, it is proved in [2] that if $M^n = (M^n(t), t \geq 0)$ denotes the additive coalescent process starting with n clusters with mass $r_{n,1} \geq \ldots \geq r_{n,n}$, then $(M^{(n)}(t - \ln \sigma_n), t \geq \ln \sigma_n)$ has a limit distribution as $n \to \infty$, which can be obtained by cutting a specific inhomogeneous random tree with a point Poisson process. Furthermore, any extreme eternal additive coalescent can be obtained this way up to a deterministic time translation.

Bertoin [4] gave another construction of the limit of the process $(M^{(n)}(t - \ln \sigma_n), t \geq \ln \sigma_n)$ in the following way. Let b_θ be the bridge with exchangeable increments defined for $s \in [0,1]$ by

$$b_\theta(s) = \sigma b_s + \sum_{i=1}^{\infty} \theta_i (\mathbb{1}_{\{s \geq V_i\}} - s),$$

where $(b_s, s \in [0,1])$ is a standard Brownian bridge, $(V_i)_{i\geq 1}$ is an i.i.d. sequence of uniform random variable on [0,1] independent of b and $\sigma = 1 - \sum_i \theta_i^2$. Let $\varepsilon_\theta = (\varepsilon_\theta(s), s \in [0,1])$ be the excursion obtained from b_θ by Vervaat's transform, i.e. $\varepsilon_\theta(s) = b_\theta(s + m \mod 1) - b_\theta(m)$, where m is the point of [0,1] where b_θ reaches its minimum. For all $t \geq 0$, consider

$$\varepsilon_\theta^{(t)}(s) = ts - \varepsilon_\theta(s), \qquad S_\theta^{(t)}(s) = \sup_{0 \leq u \leq s} \varepsilon_\theta^{(t)}(u),$$

and define $F^\theta(t)$ as the sequence of the lengths of the constancy intervals of the process $(S_\theta^{(t)}(s), 0 \leq s \leq 1)$. Then the limit of the process $(M^{(n)}(t - \ln \sigma_n), t \geq \ln \sigma_n)$ has the law of $(F^\theta(e^{-t}), t \in \mathbb{R})$. Miermont [13] studied the same process in the special case where ε_θ is the normalized excursion above the minimum of a spectrally negative Lévy process. More precisely let $(X_t, t \geq 0)$ be a Lévy process with no positive jump, with unbounded variation and with positive and finite mean. Let $\overline{X}(t) = \sup_{0 \leq s \leq t} X_t$ and denote by $\varepsilon_X = (\varepsilon_X(s), s \in [0,1])$ the normalized excursion with duration 1 of the reflected process $\overline{X} - X$. We now define in the same way as for b_θ, the processes $\varepsilon_X^{(t)}(s)$, $S_X^{(t)}(s)$ and $F^X(t)$. Then, the process $(F^X(e^{-t}), t \in \mathbb{R})$ is a mixture of some eternal additive coalescents (see [13] for more details). Furthermore, $(F^X(t), t \geq 0)$

is a fragmentation process in the sense that distinct fragments evolve independently of each other (however, it is not necessarily homogeneous in time). It is quite remarkable that the Lévy property of X ensures the branching property of F^X. We stress that there exist other eternal additive coalescents for which this property fails. Notice that when the Lévy process X is the standard Brownian motion B, the process $(F^B(e^{-t}), t \in \mathbb{R})$ is then the standard additive coalescent and $(F^B(t), t \geq 0)$ is a self-similar and time-homogeneous fragmentation process.

In this paper, we study the relationship between the laws $\mathbb{P}^{(X)}$ of $(F^X(t), t \geq 0)$ and $\mathbb{P}^{(B)}$ of $(F^B(t), t \geq 0)$. We prove that, for certain Lévy processes $(X_t, t \geq 0)$, the law $\mathbb{P}^{(X)}$ is absolutely continuous with respect to $\mathbb{P}^{(B)}$ and we compute explicitly the density. Our main result is the following:

Theorem 1.1. *Let $(\Gamma(t), t \geq 0)$ be a subordinator with no drift. Assume that $\mathbb{E}(\Gamma_1) < \infty$ and take any $c \geq \mathbb{E}(\Gamma_1)$. We define $X_t = B_t - \Gamma_t + ct$, where B denotes a Brownian motion independent of Γ. Let $(p_t(u), u \in \mathbb{R})$ and $(q_t(u), u \in \mathbb{R})$ stand for the respective density of B_t and X_t. In particular $p_t(u) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{u^2}{2t})$. Let $\mathcal{S}_1$ be the space of positive sequences with sum 1. We consider the function $\mathbf{h} : \mathbb{R}_+ \times \mathcal{S}_1$ defined by*

$$\mathbf{h}(t, \mathbf{x}) = e^{tc} \frac{p_1(0)}{q_1(0)} \prod_{i=1}^{\infty} \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)} \qquad \text{with } \mathbf{x} = (x_i)_{i \geq 1}.$$

Then, for all $t \geq 0$, the function $\mathbf{h}(t, \cdot)$ is bounded on $\mathcal{S}_1$ and has the following properties:

- *$\mathbf{h}(t, F(t))$ is a $\mathbb{P}^{(B)}$-martingale,*
- *for every $t \geq 0$, the law of the process $(F^X(s), 0 \leq s \leq t)$ is absolutely continuous with respect to that of $(F^B(s), 0 \leq s \leq t)$ with density $\mathbf{h}(t, F^B(t))$.*

Let us notice that $\mathbf{h}(t, \cdot)$ is a multiplicative function, i.e. it can be written as the product of functions, each of them depending only on the size of a single fragment. In the sequel we will use the notation

$$h(t, x) = e^{tcx} \left(\frac{p_1(0)}{q_1(0)} \right)^x \frac{q_x(-tx)}{p_x(-tx)} \qquad \text{for } x \in]0, 1] \text{ and } t \geq 0,$$

so we have $\mathbf{h}(t, \mathbf{x}) = \prod_i h(t, x_i)$. This multiplicative form of $\mathbf{h}(t, \cdot)$ implies that the process F^X has the branching property (i.e. distinct fragments evolve independently of each other) since F^B has it. Indeed, for every multiplicative bounded continuous function $\mathbf{f} : \mathcal{S}^\downarrow \mapsto \mathbb{R}_+$, for all $t' > t > 0$ and $\mathbf{x} \in \mathcal{S}^\downarrow$, we have, since $\mathbf{h}(t, F^B(t))$ is a $\mathbb{P}^{(B)}$-martingale,

$$\mathbb{E}^{(X)} \Big(\mathbf{f}(F(t')) \,\big|\, F(t) = \mathbf{x} \Big) = \frac{1}{\mathbf{h}(t, \mathbf{x})} \mathbb{E}^{(B)} \Big(\mathbf{h}(t', F(t')) \mathbf{f}(F(t')) \,\big|\, F(t) = \mathbf{x} \Big).$$

Using the branching property of F^B and the multiplicative form of $\mathbf{h}(t, \cdot)$, we get

$$\mathbb{E}^{(X)} \Big(\mathbf{f}(F(t')) \,\big|\, F(t) = \mathbf{x} \Big) = \frac{1}{\mathbf{h}(t, \mathbf{x})} \prod_i \mathbb{E}^{(B)} \Big(\mathbf{h}(t', F(t')) \mathbf{f}(F(t')) \,\big|\, F(t) = (x_i, 0, \ldots) \Big).$$

And finally we deduce

$$\begin{aligned} \mathbb{E}^{(X)} \Big(\mathbf{f}(F(t')) \,\big|\, F(t) = \mathbf{x} \Big) &= \frac{1}{\mathbf{h}(t, \mathbf{x})} \prod_i h(t, x_i) \mathbb{E}^{(X)} \Big(\mathbf{f}(F(t')) \,\big|\, F(t) = (x_i, 0, \ldots) \Big) \\ &= \prod_i \mathbb{E}^{(X)} \Big(\mathbf{f}(F(t')) \,\big|\, F(t) = (x_i, 0, \ldots) \Big). \end{aligned}$$

Let $M_{\mathbf{x}}$ (resp. M_{x_i}) be the random measure on]0,1[defined by $M_{\mathbf{x}} = \sum_i \delta_{s_i}$ where the sequence $(s_i)_{i\geq 1}$ has the law of $F(t')$ conditioned on $F(t) = \mathbf{x}$ (resp. $F(t) = (x_i, 0, \ldots)$). Hence we have, for every bounded continuous function $g : \mathbb{R} \mapsto \mathbb{R}$,

$$\mathbb{E}\Big(\exp(- < g, M_{\mathbf{x}} >)\Big) = \prod_{i=1}^{\infty} \mathbb{E}\Big(\exp(- < g, M_{x_i} >)\Big),$$

which proves that $M_{\mathbf{x}}$ has the law of $\sum_i M_{x_i}$ where the random measures $(M_{x_i})_{i\geq 1}$ are independent. Hence the process F^X has the branching property. Notice also that other multiplicative martingales have already been studied in the case of branching random walks [9, 10, 14, 12].

This paper will be divided in two sections. The first section is devoted to the proof of this theorem and in the next one, we will use the fact that $\mathbf{h}(t, F^B(t))$ is a $\mathbb{P}^{(B)}$-martingale to describe an integro-differential equation solved by the function h.

2 Proof of Theorem 1.1

The assumptions and notation in Theorem 1.1 are implicitly enforced throughout this section.

2.1 Absolute continuity

In order to prove Theorem 1.1, we will first prove the absolute continuity of the law $\mathbb{P}_t^{(X)}$ of $F^X(t)$ with respect to the law $\mathbb{P}_t^{(B)}$ of $F^X(t)$ for a fixed time $t > 0$ and for a finite number of fragments. We begin first by a definition:

Definition 2.1. *Let $x = (x_1, x_2, \ldots)$ be a sequence of positive numbers with sum 1. We call the random variable $y = (x_{j_1}, x_{j_2}, \ldots)$ a size biased rearrangement of x if we have:*

$$\forall i \in \mathbb{N},\ \mathbb{P}(j_1 = i) = x_i,$$

and by induction

$$\forall i \in \mathbb{N}\backslash\{i_1, \ldots, i_k\},\ \mathbb{P}(j_{k+1} = i \mid j_1 = i_1, \ldots, j_k = i_k) = \frac{x_i}{1 - \sum_{l=1}^{k} x_{i_l}}.$$

Notice that for every Lévy process X satisfying hypotheses of Theorem 1.1, we have $\sum_{i=1}^{\infty} F_i(t) = 1 \quad \mathbb{P}_t^{(X)}$-a.s. (it is clear by the construction from an excursion of X since X has unbounded variation, cf [13], Section 3.2). Hence the above definition can be applied to $F^X(t)$.

The following lemma gives the distribution of the first n fragments of $F^X(t)$, chosen with a size-biased pick:

Lemma 2.2. *Let $(\tilde{F}_1^X(t), \tilde{F}_2^X(t), \ldots)$ be a size biased rearrangement of $F^X(t)$. Then for all $n \in \mathbb{N}$, for all $x_1, \ldots, x_n \in \mathbb{R}_+$ such that $S = \sum_{i=1}^{n} x_i < 1$, we have*

$$\mathbb{P}_t^{(X)}(\tilde{F}_1^X \in dx_1, \ldots, \tilde{F}_n^X \in dx_n) = \frac{t^n}{q_1(0)} q_{1-S}(St) \prod_{i=1}^{n} \frac{q_{x_i}(-tx_i)}{1 - \sum_{k=1}^{i} x_k} dx_1 \ldots dx_n.$$

Proof. On the one hand, Miermont [13] gave a description of the law of $F^X(t)$: let $T^{(t)}$ be a subordinator with Lévy measure $z^{-1} q_z(-tz) \mathbb{1}_{z>0} dz$. Then $F^X(t)$ has the law of the sequence of the jumps of $T^{(t)}$ before time t conditioned on $T_t^{(t)} = 1$.

One the other hand, consider a subordinator T on the time interval $[0,u]$ conditioned by $T_u = y$ and pick a jump of T by size-biased sampling. Then, its distribution has density

$$\frac{zuh(z)f_u(y-z)}{yf_u(y)}dz,$$

where h is the density of the Lévy measure of T and f_u is the density of T_u (see Theorem 2.1 of [15]). Then, in the present case, we have

$$u = t, \quad y = 1, \quad h(z) = z^{-1}q_z(-tz), \quad f_u(z) = \frac{u}{z}q_z(u - zt) \quad \text{(cf. Lemma 9 of [13]).}$$

Hence we get

$$\mathbb{P}_t^{(X)}(\tilde{F}_1^X \in dz) = \frac{tq_z(-tz)q_{1-z}(zt)}{(1-z)q_1(0)}dz.$$

This proves the lemma in the case $n = 1$. The proof for $n \geq 2$ uses an induction. Assume that we have proved the case $n-1$ and let us prove the case n. We have

$$\mathbb{P}_t^{(X)}(\tilde{F}_1^X \in dx_1, \ldots, \tilde{F}_n^X \in dx_n) = \\ \mathbb{P}_t^{(X)}(\tilde{F}_1^X \in dx_1, \ldots, \tilde{F}_{n-1}^X \in dx_{n-1})\mathbb{P}_t^{(X)}(\tilde{F}_n^X \in dx_n \mid \tilde{F}_1^X \in dx_1, \ldots, \tilde{F}_{n-1}^X \in dx_{n-1}).$$

Furthermore, Perman, Pitman and Yor [15] have proved that the n-th size biased picked jump Δ_n of a subordinator before time u conditioned by $T_u = y$ and $\Delta_1 = x_1, \ldots, \Delta_{n-1} = x_{n-1}$ has the law of a size biased picked jump of the subordinator T before time u conditioned by $T_u = y - x_1 - \ldots - x_{n-1}$. Hence we get:

$$\mathbb{P}_t^{(X)}(\tilde{F}_1^X \in dx_1, \ldots, \tilde{F}_n^X \in dx_n) = \\ \left(\frac{t^{n-1}}{q_1(0)}q_{1-S_{n-1}}(S_{n-1}t)\prod_{i=1}^{n-1}\frac{q_{x_i}(-tx_i)}{1-S_i}\right)\frac{tq_{x_n}(-tx_n)q_{1-S_n}(S_nt)}{(1-S_n)q_{1-S_{n-1}}(S_{n-1}t)}dx_1\ldots dx_n,$$

where $S_i = \sum_{k=1}^{i} x_k$. And so the lemma is proved by induction.

□

Since the lemma is clearly also true for $\mathbb{P}^{(B)}$ (take $\Gamma = c = 0$), we get:

Corollary 2.3. *Let $(F(t), t \geq 0)$ be a fragmentation process. Let $(\tilde{F}_1(t), \tilde{F}_2(t), \ldots)$ be a size biased rearrangement of $F(t)$. Then for all $n \in \mathbb{N}$, for all $x_1, \ldots, x_n \in \mathbb{R}_+$ such that $S = \sum_{i=1}^{n} x_i < 1$, we have*

$$\frac{\mathbb{P}_t^{(X)}(\tilde{F}_1 \in dx_1, \ldots, \tilde{F}_n \in dx_n)}{\mathbb{P}_t^{(B)}(\tilde{F}_1 \in dx_1, \ldots, \tilde{F}_n \in dx_n)} = h_n(t, x_1, \ldots, x_n),$$

$$\text{with } h_n(t, x_1, \ldots, x_n) = \frac{p_1(0)}{q_1(0)}\frac{q_{1-S}(St)}{p_{1-S}(St)}\prod_{i=1}^{n}\frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)}.$$

To establish that the law of $F^X(t)$ is absolutely continuous with respect to the law of $F^B(t)$ with density $\mathbf{h}(t, \cdot)$, it remains to check that the function h_n converges as n tends to infinity to $\mathbf{h}$ $\mathbb{P}_t^{(B)}$-a.s. and in $L^1(\mathbb{P}_t^{(B)})$. In this direction, we first prove two lemmas:

Lemma 2.4. *We have* $\frac{q_y(-ty)}{p_y(-ty)} < 1$ *for all* $y > 0$ *sufficiently small. As a consequence, if* $(x_i)_{i\geq 1}$ *is a sequence of positive numbers with* $\lim_{i\to\infty} x_i = 0$*, then the product* $\prod_{i=1}^n \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)}$ *converges as* n *tends to infinity.*

Proof. Since $X_t = B_t - \Gamma_t + tc$, notice that we have

$$\forall s > 0,\ \forall u \in \mathbb{R}, \quad q_s(u) = \mathbb{E}\Big(p_s(u + \Gamma_s - cs)\Big).$$

Hence if we replace $p_s(u)$ by its expression $\frac{1}{\sqrt{2\pi s}}\exp(-\frac{u^2}{2s})$, we get

$$\frac{q_s(u)}{p_s(u)} = \exp\left(cu - \frac{c^2 s}{2}\right)\mathbb{E}\left[\exp\left(-\frac{\Gamma_s^2}{2s} - \Gamma_s\left(\frac{u}{s} - c\right)\right)\right]. \tag{1}$$

i.e., for all $y > 0$, for all $t \geq 0$,

$$\frac{q_y(-ty)}{p_y(-ty)} = \exp\left(-y(ct + \frac{c^2}{2})\right)\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y} + \Gamma_y(t+c)\right)\right].$$

Using the inequality $(c-a)(c-b) \geq -\left(\frac{b-a}{2}\right)^2$, we have

$$-\frac{\Gamma_y^2}{2y} + \Gamma_y(t+c) \leq \frac{y(t+c)^2}{2}$$

and we deduce

$$\frac{q_y(-ty)}{p_y(-ty)} \leq e^{\frac{t^2 y}{2}}.$$

Fix $c' \in]0, c[$, let f be the function defined by $f(y) = \mathbb{P}(\Gamma_y \leq c'y)$. Since Γ_t is a subordinator with no drift, we have $\lim_{y\to 0} f(y) = 1$ (indeed, $\Gamma_y = o(y)$ a.s., see [3]). On the event $\{\Gamma_y \leq c'y\}$, we have

$$\begin{aligned}
\exp\left(-y(ct + \frac{c^2}{2})\right)\exp\left(-\frac{\Gamma_y^2}{2y} + \Gamma_y(t+c)\right) &\leq \exp(-y(\frac{1}{2}(c-c')^2 + t(c-c'))) \\
&\leq \exp(-\varepsilon y),
\end{aligned}$$

with $\varepsilon = \frac{1}{2}(c-c')^2$. Hence, we get the upper bound

$$\frac{q_y(-ty)}{p_y(-ty)} \leq e^{-\varepsilon y} f(y) + (1 - f(y))e^{\frac{yt^2}{2}}.$$

Since $f(y) \to 1$ as $y \to 0$, we deduce

$$e^{-\varepsilon y} f(y) + (1 - f(y))e^{\frac{yt^2}{2}} = 1 - \varepsilon y + o(y).$$

Thus, we have $\frac{q_y(-ty)}{p_y(-ty)} < 1$ for y small enough, and so the product converges for every sequence $(x_i)_{i\geq 0}$ which tends to 0. □

We prove now a second lemma:

Lemma 2.5. *We have*

$$\lim_{s\to 1^-} \frac{q_{1-s}(st)}{p_{1-s}(st)} = e^{tc}.$$

Proof. We use again Identity (1) established in the proof of Lemma 2.4. We get:

$$\frac{q_{1-s}(st)}{p_{1-s}(st)} = \exp\left(tsc - \frac{c^2}{2}(1-s)\right)\mathbb{E}\left[\exp\left(-\frac{\Gamma_{1-s}^2}{2(1-s)} - \Gamma_{1-s}(\frac{ts}{1-s} - c)\right)\right].$$

For s close enough to 1, $\frac{ts}{1-s} - c \geq 0$, hence we get

$$\mathbb{E}\left[\exp\left(-\frac{\Gamma_{1-s}^2}{2(1-s)} - \Gamma_{1-s}(\frac{ts}{1-s} - c)\right)\right] \leq 1$$

and we deduce

$$\limsup_{s\to 1^-} \frac{q_{1-s}(st)}{p_{1-s}(st)} \leq e^{tc}.$$

For the lower bound, we write

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\frac{\Gamma_{1-s}^2}{2(1-s)} - \Gamma_{1-s}(\frac{ts}{1-s} - c)\right)\right] \\
&\qquad \geq \mathbb{E}\left[\exp\left(-\frac{\Gamma_{1-s}}{2(1-s)} - \Gamma_{1-s}(\frac{ts}{1-s} - c)\right)\mathbb{1}_{\{\Gamma_{1-s}\leq 1\}}\right] \\
&\qquad \geq \mathbb{E}\left[\exp\left(-\Gamma_{1-s}\frac{1+2ts}{2(1-s)}\right)\mathbb{1}_{\{\Gamma_{1-s}\leq 1\}}\right] \\
&\qquad \geq \mathbb{E}\left[\exp\left(-\Gamma_{1-s}\frac{1+2ts}{2(1-s)}\right)\right] - \mathbb{P}(\Gamma_{1-s}\geq 1).
\end{aligned}$$

Since Γ_t is a subordinator with no drift, $\lim_{u\to 0}\frac{\Gamma_u}{u} = 0$ a.s., and we have for all $K > 0$,

$$\lim_{u\to 0^+} \mathbb{E}\left[\exp\left(-K\frac{\Gamma_u}{u}\right)\right] = 1.$$

Hence, we get

$$\liminf_{s\to 1^-} \frac{q_{1-s}(st)}{p_{1-s}(st)} \geq e^{tc}.$$

□

We are now able to prove the absolute continuity of $\mathbb{P}_t^{(X)}$ with respect to $\mathbb{P}_t^{(B)}$. Since $S_n = \sum_{i=1}^n x_i$ converges $\mathbb{P}_t^{(B)}$-a.s. to 1, Lemma 2.4 and 2.5 imply that $H_n = h_n(t, \tilde{F}_1(t), \ldots, \tilde{F}_n(t))$ converges to $H = \mathbf{h}(t, F(t))$ $\mathbb{P}^{(B)}$-a.s.

Let us now prove that H_n is uniformly bounded, which implies the L^1 convergence. We have already proved that there exists $\varepsilon > 0$ such that:

$$\forall x \in]0, \varepsilon[, \quad \frac{q_x(-tx)}{p_x(-tx)} \leq 1.$$

Besides, it is well known that, if $X_t = B_t - \Gamma_t + ct$, its density $(t,u) \to q_t(u)$ is continuous on $\mathbb{R}_+^* \times \mathbb{R}$. Hence, on $[\varepsilon, 1]$, the function $x \to \frac{q_x(-tx)}{p_x(-tx)}$ is continuous and we can find an upper bound $A > 0$ of this function . As there are at most $\frac{1}{\varepsilon}$ fragments of $F(t)$ larger than ε, we deduce the upper bound:

$$\prod_{i=1}^{\infty} \frac{q_{F_i}(-tF_i)}{p_{F_i}(-tF_i)} \leq A^{\frac{1}{\varepsilon}}.$$

Likewise, the function $S \to \frac{q_{1-S}(St)}{p_{1-S}(St)}$ is continuous on $[0,1[$ and has a limit at 1, so it is bounded by some $D > 0$ on $[0,1]$. Hence we get

$$H_n \leq A^{\frac{1}{\varepsilon}} D \frac{p_1(0)}{q_1(0)} \quad \mathbb{P}^{(B)}\text{-a.s.}$$

So H_n converges to $H \quad \mathbb{P}^{(B)}$-a.s. and in $L^1(\mathbb{P}^{(B)})$. Furthermore, by construction, H_n is a $\mathbb{P}^{(B)}$-martingale, hence we get for all $n \in \mathbb{N}$,

$$\mathbb{E}^{(B)}(H \mid \tilde{F}_1, \ldots, \tilde{F}_n) = H_n,$$

and so, for every bounded continuous function $f : \mathcal{S}_1 \to \mathbb{R}$, we have

$$\mathbb{E}^{(X)}\Big[f(F(t))\Big] = \mathbb{E}^{(B)}\Big[f(F(t))\mathbf{h}(t, F(t))\Big].$$

Hence, we have proved that, for a fixed time $t \geq 0$, the law of $F^X(t)$ is absolutely continuous with respect to that of $F^B(t)$ with density $\mathbf{h}(t, F^B(t))$. Furthermore, Miermont [13] has proved that the processes $(F^X(e^{-t}), t \in \mathbb{R})$ and $(F^B(e^{-t}), t \in \mathbb{R})$ are both eternal additive coalescents (with different entrance laws). Hence, they have the same semi-group of transition and we get the absolute continuity of the law of the process $(F^X(s), 0 \leq s \leq t)$ with respect to that of $(F^B(s), 0 \leq s \leq t)$ with density $\mathbf{h}(t, F^B(t))$.

2.2 Sufficient condition for equivalence

We can now wonder whether the measure $\mathbb{P}^{(X)}$ is equivalent to the measure $\mathbb{P}^{(B)}$, that is whether $\mathbf{h}(t, F(t))$ is strictly positive $\mathbb{P}^{(B)}$-a.s. A sufficient condition is given by the following proposition.

Proposition 2.6. *Let ϕ be the Laplace exponent of the subordinator Γ, i.e.*

$$\forall s \geq 0, \forall q \geq 0, \quad \mathbb{E}(\exp(-q\Gamma_s)) = \exp(-s\phi(q)).$$

Assume that there exists $\delta > 0$ such that

$$\lim_{x \to \infty} \phi(x) x^{\delta - 1} = 0, \tag{2}$$

then the function $\mathbf{h}(t, F(t))$ defined in Theorem 1.1 is strictly positive $\mathbb{P}^{(B)}$-a.s.

We stress that the condition 2 is very weak. For instance, let π be the Lévy measure of the subordinator and $I(x) = \int_0^x \overline{\pi}(t)dt$ where $\overline{\pi}(t)$ denotes $\pi(]t, \infty[)$. It is well known that $\phi(x)$ behaves like $xI(1/x)$ as x tends to infinity (see [3] Section III). Thus, the condition 2 is equivalent to $I(x) = o(x^\delta)$ as x tends to 0 (recall that we always have $I(x) = o(1)$).

Proof. Let $t > 0$. We must check that $\prod_{i=1}^\infty \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)}$ is $\mathbb{P}_t^{(B)}$-almost surely strictly positive. Using (1), we have:

$$\frac{q_y(-ty)}{p_y(-ty)} = \exp\left(-y(ct + \frac{c^2}{2})\right) \mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y} + \Gamma_y(t + c)\right)\right].$$

Since we have $\sum_{i=1}^\infty x_i = 1 \quad \mathbb{P}_t^{(B)}$-a.s., we get

$$\prod_{i=1}^\infty \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)} \geq \exp\left(-ct + \frac{c^2}{2}\right) \prod_{i=1}^\infty \mathbb{E}\left[\exp\left(-\frac{\Gamma_{x_i}^2}{2x_i} + c\Gamma_{x_i}\right)\right].$$

Hence we have to find a lower bound for $\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y}+c\Gamma_y\right)\right]$. Since $c \geq \mathbb{E}(\Gamma_1)$, we have

$$\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y}+c\Gamma_y\right)\right] \geq \mathbb{E}\left[\exp\left(\frac{\Gamma_y}{y}(\mathbb{E}(\Gamma_y)-\frac{\Gamma_y}{2})\right)\right].$$

Set $A=\mathbb{E}(\Gamma_1)$ and let us fix $K>0$. Notice that the event $\mathbb{E}(\Gamma_y)-\frac{\Gamma_y}{2} \geq -Ky$ is equivalent to the event $\Gamma_y \leq (2A+K)y$ and by Markov inequality, we have

$$\mathbb{P}(\Gamma_y \geq (2A+K)y) \leq \frac{A}{2A+K}.$$

Hence we get

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y}+c\Gamma_y\right)\right] &\geq \mathbb{E}\left[\exp\left(\frac{\Gamma_y}{y}(\mathbb{E}(\Gamma_y)-\frac{\Gamma_y}{2})\mathbb{1}_{\{\Gamma_y\leq(2A+K)y\}}\right)\right]\\ &\geq \mathbb{E}\left(\exp(-K\Gamma_y)\mathbb{1}_{\{\Gamma_y\leq(2A+K)y\}}\right)\\ &\geq \mathbb{E}\left(\exp(-K\Gamma_y)\right)-\mathbb{E}\left(\exp(-K\Gamma_y)\mathbb{1}_{\{\Gamma_y>(2A+K)y\}}\right)\\ &\geq \exp(-\phi(K)y)-\frac{A}{2A+K}.\end{aligned}$$

This inequality holds for all $K>0$. Hence, with $\varepsilon>0$ and $K=y^{-\frac{1}{2}-\varepsilon}$, we get

$$\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y}+c\Gamma_y\right)\right] \geq \exp\left(-\phi(y^{-\frac{1}{2}-\varepsilon})y\right)-Ay^{\frac{1}{2}+\varepsilon}.$$

Furthermore, the product $\prod_{i=1}^{\infty}\mathbb{E}\left[\exp\left(-\frac{\Gamma_{x_i}^2}{2x_i}+c\Gamma_{x_i}\right)\right]$ is strictly positive if the series

$$\sum_{i=1}^{\infty} 1-\mathbb{E}\left[\exp\left(-\frac{\Gamma_{x_i}^2}{2x_i}+c\Gamma_{x_i}\right)\right]$$

converges. Hence, a sufficient condition is

$$\exists\, \varepsilon>0 \text{ such that } \sum_{i=1}^{\infty}\left(1-\exp\left(-\phi(x_i^{-\frac{1}{2}-\varepsilon})x_i\right)+x_i^{\frac{1}{2}+\varepsilon}\right)<\infty \quad \mathbb{P}_t^{(B)}\text{-a.s.}$$

Recall that the distribution of the Brownian fragmentation at time t is equal to the distribution of the jumps of a stable subordinator T with index $1/2$ before time t conditioned on $T_t=1$ (see [1]). Hence, it is well known that we have for all $\varepsilon>0$

$$\sum_{i=1}^{\infty} x_i^{\frac{1}{2}+\varepsilon}<\infty \quad \mathbb{P}_t^{(B)}\text{-a.s.} \quad \text{(see Formula (9) of [1]).}$$

Thus, we have equivalence between $\mathbb{P}_t^{(B)}$ and $\mathbb{P}_t^{(X)}$ as soon as there exist two strictly positive numbers ε,ε' such that, for x small enough

$$\phi(x^{-\frac{1}{2}-\varepsilon})x \leq x^{\frac{1}{2}+\varepsilon'}.$$

One can easily check that this condition is equivalent to (2). □

In Theorem 1.1, we have supposed that X_t can be written as $B_t + \Gamma_t - ct$, with $c \geq \mathbb{E}(\Gamma_1)$ and Γ_t subordinator. We can wonder whether the theorem applies for a larger class of Lévy processes. Notice first that the process X must fulfill the conditions of Miermont's paper [13] recalled in the introduction, i.e. X has no positive jumps, unbounded variation and finite and positive mean. Hence, a possible extension of the Theorem would be for example for $X_t = \sigma^2 B_t + \Gamma_t - ct$, with $\sigma > 0$, $\sigma \neq 1$. In fact, it is clear that Theorem 1.1 fails in this case. Let just consider for example $X_t = 2B_t$. Using Proposition 3 of [13], we get that

$$(F^X(2t), t \geq 0) \overset{\text{law}}{=} (F^B(t), t \geq 0).$$

But, it is well known that we have

$$\lim_{n \to \infty} n^2 F_n^{\downarrow}(t) = t\sqrt{2/\pi} \quad \mathbb{P}^{(B)}\text{-a.s.} \quad \text{(see [6])}$$

Hence, the laws $\mathbb{P}_t^{(B)}$ and $\mathbb{P}_{2t}^{(B)}$ are mutually singular.

3 An integro-differential equation

Since $\mathbf{h}(t, F(t))$ is the density of $\mathbb{P}^{(X)}$ with respect to $\mathbb{P}^{(B)}$ on the sigma-field $\mathcal{F}_t = \sigma(F(s), s \leq t)$, it is a $\mathbb{P}^{(B)}$-martingale. Hence, in this section, we will compute the infinitesimal generator of a fragmentation to deduce a remarkable integro-differential equation.

3.1 The infinitesimal generator of a fragmentation process

In this section, we recall a result obtained by Bertoin and Rouault in an unpublished paper [8].

We denote by $\mathcal{D}$ the space of functions $f : [0,1] \mapsto]0,1]$ of class $\mathcal{C}^1$ and with $f(0) = 1$. For $f \in \mathcal{D}$ and $\mathbf{x} \in \mathcal{S}^{\downarrow}$, we set

$$\mathbf{f}(\mathbf{x}) = \prod_{i=1}^{\infty} f(x_i).$$

For $\alpha \in \mathbb{R}_+$ and ν measure on $\mathcal{S}^{\downarrow}$ such that $\int_{\mathcal{S}^{\downarrow}} (1 - x_1)\nu(d\mathbf{x}) < \infty$, we define the operator

$$G_\alpha \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \sum_{i=1}^{\infty} x_i^\alpha \int \nu(d\mathbf{y}) \left(\frac{\mathbf{f}(x_i \mathbf{y})}{f(x_i)} - 1 \right) \qquad \text{for } f \in \mathcal{D} \text{ and } \mathbf{x} \in \mathcal{S}^{\downarrow}.$$

Proposition 3.1. *Let $(X(t), t \geq 0)$ be a self-similar fragmentation with index of self-similarity $\alpha > 0$, dislocation measure ν and no erosion. Then, for every function $f \in \mathcal{D}$, the process*

$$\mathbf{f}(X(t)) - \int_0^t G_\alpha \mathbf{f}(X(s)) ds$$

is a martingale.

Proof. We will first prove the following lemma

Lemma 3.2. *For $f \in \mathcal{D}, \mathbf{y} \in \mathcal{S}^{\downarrow}, r \in [0,1]$, we have*

$$\left| \frac{\mathbf{f}(r\mathbf{y})}{f(r)} - 1 \right| \leq 2C_f e^{C_f} r(1 - y_1),$$

with $C_f = \left|\left| \frac{f'}{f^2} \right|\right|_\infty$.

Notice that, since f is $\mathcal{C}^1$ on $[0,1]$ and strictly positive, C_f is always finite.

Proof. First, we write

$$|\ln f(ry_1) - \ln f(r)| \le \left|\left|\frac{f'}{f}\right|\right|_\infty (1-y_1)r \le C_f(1-y_1)r.$$

We deduce then

$$\frac{\mathbf{f}(r\mathbf{y})}{f(r)} - 1 \le \frac{f(ry_1)}{f(r)} - 1 \le e^{C_f(1-y_1)r} - 1 \le C_f e^{C_f}(1-y_1)r.$$

Besides we have

$$\ln \frac{1}{f(x_1)} \le \frac{1}{f(x_i)} - 1 \le C_f x_i, \quad \text{which implies} \quad \mathbf{f}(\mathbf{x}) \ge f(x_1)\exp(-C_f \sum_{i=2}^{\infty} x_i).$$

Hence we get

$$\frac{\mathbf{f}(r\mathbf{y})}{f(r)} \ge \frac{f(ry_1)}{f(r)} \exp(-C_f(1-y_1)r) \ge \exp(-2C_f(1-y_1)r),$$

and we deduce

$$1 - \frac{\mathbf{f}(r\mathbf{y})}{f(r)} \le 2C_f e^{C_f}(1-y_1)r.$$

□

We can now prove Proposition 3.1. We denote by $\mathcal{T}$ the set of times where some dislocation occurs (which is a countable set). Hence we can write

$$\mathbf{f}(X(t)) - \mathbf{f}(X(0)) = \sum_{s\in[0,t]\cap\mathcal{T}} \Big(\mathbf{f}(X(s)) - \mathbf{f}(X(s-))\Big),$$

as soon as

$$\sum_{s\in[0,t]\cap\mathcal{T}} \Big|\mathbf{f}(X(s)) - \mathbf{f}(X(s-))\Big| < \infty$$

For $s \in \mathcal{T}$, if the i-th fragment $X_i(s-)$ is involved in the dislocation, we set $k_s = i$ and we denote by Δ_s the element of $\mathcal{S}^\downarrow$ according to $X(s-)$ has been broken. Hence, we have

$$\sum_{s\in[0,t]\cap\mathcal{T}} \Big|\mathbf{f}(X(s)) - \mathbf{f}(X(s-))\Big| = \sum_{s\in\mathcal{T}\cap[0,t]} \mathbf{f}(X(s-)) \left(\sum_{i=1}^{\infty} \mathbb{1}_{k_s=i} \Big|\frac{\mathbf{f}(X_i(s-)\Delta_s)}{f(X_i(s-))} - 1\Big|\right).$$

Hence, since a fragment of mass r has a rate of dislocation $\nu_r(dx) = r^\alpha \nu(dx)$, the predictable compensator is

$$\int_0^t ds\, \mathbf{f}(X(s-)) \int_{\mathcal{S}^\downarrow} \nu(d\mathbf{y}) \sum_{i=1}^{\infty} X_i^\alpha(s-) \Big|\frac{\mathbf{f}(X_i(s-)\mathbf{y})}{f(X_i(s-))} - 1\Big|$$

$$\le 2C_f e^{C_f} \int_0^t \sum_{i=1}^{\infty} X_i(s-) \int_{\mathcal{S}^\downarrow} (1-y_1)\nu(d\mathbf{y}) ds.$$

$$\le 2C_f e^{C_f} t \int_{\mathcal{S}^\downarrow} (1-y_1)\nu(d\mathbf{y})$$

Hence

$$\sum_{s\in[0,t]\cap\mathcal{T}} \left|\mathbf{f}(X(s)) - \mathbf{f}(X(s-))\right| < \infty \quad \text{a.s.},$$

and thus we have

$$\mathbf{f}(X(t)) - \mathbf{f}(X(0)) = \sum_{s\in[0,t]\cap\mathcal{T}} \Big(\mathbf{f}(X(s)) - \mathbf{f}(X(s-))\Big),$$

i.e.

$$\mathbf{f}(X(t)) - \mathbf{f}(X(0)) = \sum_{s\in\mathcal{T}\cap[0,t]} \mathbf{f}(X(s-)) \left(\sum_{i=1}^{\infty} \mathbb{1}_{k_s=i} \left(\frac{\mathbf{f}(X_i(s-)\Delta_s)}{f(X_i(s-))} - 1\right)\right),$$

whose predictable compensator is

$$\int_0^t ds\, \mathbf{f}(X(s-)) \int_{\mathcal{S}^{\downarrow}} \nu(d\mathbf{y}) \sum_{i=1}^{\infty} X_i^{\alpha}(s-) \left(\frac{\mathbf{f}(X_i(s-)\mathbf{y})}{f(X_i(s-))} - 1\right) = \int_0^t G_\alpha \mathbf{f}(X(s))ds.$$

□

3.2 Application to $\mathbf{h}(t, F(t))$

Let $F(t)$ be a fragmentation process and $q_t(x)$ be the density of a Lévy process fulfilling the hypotheses of Theorem 1.1. We have proved in the first section that the function

$$H_t = \mathbf{h}(t, F(t)) = e^{tc}\frac{p_1(0)}{q_1(0)} \prod_{i=1}^{\infty} \frac{q_{F_i(t)}(-tF_i(t))}{p_{F_i(t)}(-tF_i(t))}$$

is a $\mathbb{P}^{(B)}$-martingale (since it is equal to $\frac{d\mathbb{P}^{(X)}}{d\mathbb{P}^{(B)}}|\mathcal{F}_t$). We set

$$g(t,x) = e^{tcx}\frac{q_x(-tx)}{p_x(-tx)} \quad \text{for } x \in]0,1], t \geq 0 \quad \text{and } g(t,0) = 1.$$

$$\text{Set now} \quad \mathbf{g}(t,\mathbf{x}) = \prod_{i=1}^{\infty} g(t, x_i(t)) \quad \text{for } \mathbf{x} \in \mathcal{S}^{\downarrow}, t \geq 0.$$

So we have, as $\sum_i F_i(t) = 1$ $\mathbb{P}^{(B)}$-a.s.,

$$H_t = \frac{p_1(0)}{q_1(0)}\mathbf{g}(t, F(t)) \quad \text{for all } t \geq 0.$$

It is well known that if $q_t(u)$ is the density of a Lévy process $X_t = B_t - \Gamma_t + ct$, the function $(t,u) \mapsto q_t(u)$ is $\mathcal{C}^\infty$ on $\mathbb{R}_+^* \times \mathbb{R}$. Hence $(t,x) \mapsto g(t,x)$ is also $\mathcal{C}^\infty$ on $\mathbb{R}_+ \times]0,1]$ and in particular, for all $x \in [0,1]$, the function $t \to g(t,x)$ is $\mathcal{C}^1$ and so $\partial_t g(t,x)$ is well defined. The next proposition gives a integro-differential equation solved by the function g when g has some properties of regularity at points $(t,0)$, $t \in \mathbb{R}_+$.

Proposition 3.3. *1. Assume that for all $t \geq 0$, $\partial_x g(t,0)$ exists and the function $(t,x) \to \partial_x g(t,x)$ is continuous at $(t,0)$. Then g solves the equation:*

$$\begin{cases} \partial_t g(t,x) + \sqrt{x}\displaystyle\int_0^1 \frac{dy}{\sqrt{8\pi y^3(1-y)^3}}\Big(g(t,xy)g(t,x(1-y)) - g(t,x)\Big) = 0 \\ g(0,x) = \frac{q_x(0)}{p_x(0)}. \end{cases}$$

2. If the Lévy measure of the subordinator Γ is finite, then the above conditions on g hold.

Proof. Let us first notice that the hypotheses of the proposition imply that the integral

$$\int_0^1 \frac{dy}{\sqrt{8\pi y^3(1-y)^3}}\Big(g(t,xy)g(t,x(1-y))-g(t,x)\Big)$$

is well defined and is continuous in x and in t. Indeed, this integral is equal to

$$2\int_0^{\frac{1}{2}} \frac{dy}{\sqrt{8\pi y^3(1-y)^3}}\Big(g(t,xy)g(t,x(1-y))-g(t,x)\Big).$$

And for all $y \in]0,1/2[, x \in]0,1], t \in \mathbb{R}_+$, there exist $c, c' \in [0,x]$ such that

$$\frac{g(t,xy)g(t,x(1-y))-g(t,x)}{y} = x(g(t,x)\partial_x g(t,c) - g(t,xy)\partial_x g(t,c')).$$

Thanks to the hypothesis that the function $(t,x) \to \partial_x g(t,x)$ is continuous on $\mathbb{R}_+ \times [0,1]$, $|x(g(t,x)\partial_x g(t,c) - g(t,xy)\partial_x g(t,c'))|$ is uniformly bounded on $[0,T]\times[0,1]\times[0,\frac{1}{2}]$ and so by application of the theorem of dominated convergence, the integral is continuous in t on $\mathbb{R}_+$ and in x on [0,1].

We begin by proving the first point of the proposition. Recall that, according to Proposition 3.1, the generator of the Brownian fragmentation is

$$G_{\frac{1}{2}}\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x})\sum_{i=1}^{\infty}\sqrt{x_i}\int \nu(d\mathbf{y})\left(\frac{\mathbf{f}(x_i\mathbf{y})}{f(x_i)}-1\right),$$

with

$$\nu(y_1 \in du) = (2\pi u^3(1-u)^3)^{-1/2}du \quad \text{for } u \in]1/2,1[, \quad \nu(y_1+y_2 \neq 1) = 0 \quad \text{(cf. [5]).}$$

Hence,

$$M_t = \mathbf{g}(t,F(t)) - \mathbf{g}(0,F(0)) - \int_0^t G_{\frac{1}{2}}\mathbf{g}(s,F(s)) + \partial_t\mathbf{g}(s,F(s))ds$$

is a $\mathbb{P}^{(B)}$-martingale. Since $\mathbf{g}(t,F(t))$ is already a $\mathbb{P}^{(B)}$-martingale, we get

$$G_{\frac{1}{2}}\mathbf{g}(s,F(s)) + \partial_t\mathbf{g}(s,F(s)) = 0 \quad \mathbb{P}^{(B)}\text{-a.s.} \quad \text{for almost every } s>0,$$

i.e. for almost every $s>0$

$$\mathbf{g}(s,F(s))\sum_{i=1}^{\infty}\left[F_i^{1/2}(s)\int_{\mathcal{S}^{\downarrow}}\nu(d\mathbf{y})\left(\frac{\mathbf{g}(s,F_i(s)\mathbf{y})}{g(s,F_i(s))}-1\right)+\frac{\partial_t g(s,F_i(s))}{g(s,F_i(s))}\right] = 0 \quad \mathbb{P}^{(B)}\text{-a.s.}$$

With $F(s) = (x_1, x_2, \ldots)$, we get

$$\sum_{i=1}^{\infty}\left[x_i^{1/2}\int_{\mathcal{S}^{\downarrow}}\nu(d\mathbf{y})\left(\frac{\mathbf{g}(s,x_i\mathbf{y})}{g(s,x_i)}-1\right)+\frac{\partial_t g(s,x_i)}{g(s,x_i)}\right] = 0 \quad \mathbb{P}_s^{(B)}\text{-a.s.}$$

Notice also that this series is absolutely convergent. Indeed, thanks to Lemma 3.2, we have

$$\left|x_i^{1/2}\int_{\mathcal{S}^{\downarrow}}\nu(d\mathbf{y})\left(\frac{\mathbf{g}(s,x_i\mathbf{y})}{g(s,x_i)}-1\right)\right| \le C_{g,s}x_i\int_{\mathcal{S}^{\downarrow}}(1-y_1)\nu(d\mathbf{y}),$$

where $C_{g,s}$ is a positive constant (which depends on g and s), and, besides we have

$$g(t,x) = \exp\left(-x\frac{c^2}{2}\right)\mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)\right].$$

Thus, by application of the theorem of dominated convergence, it is easy to prove that the function $t \to \mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)\right]$ is derivable with derivative

$$\partial_t \mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)\right] = \mathbb{E}\left[\Gamma_x \exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)\right].$$

Notice also that this quantity is continuous in x on [0,1].

Hence we have

$$\forall x_i \in]0,1[, \forall s > 0, \quad \frac{\partial_t g(s,x_i)}{g(s,x_i)} > 0.$$

Thus we deduce

$$\sum_{i=1}^{\infty} \frac{\partial_t g(s,x_i)}{g(s,x_i)} < \infty \quad \mathbb{P}_s^{(B)}\text{-a.s.}$$

Let define

$$k(t,x) = \partial_t g(t,x) + \sqrt{x}\int_0^1 \frac{dy}{\sqrt{8\pi y^3(1-y)^3}}\Big(g(t,xy)g(t,x(1-y)) - g(t,x)\Big).$$

Hence we have

$$\sum_{i=1}^{\infty} k(s,x_i) = 0 \quad \mathbb{P}_s^{(B)}\text{-a.s.} \quad \text{for almost every } s > 0, \tag{3}$$

and

$$\sum_{i=1}^{\infty} |k(s,x_i)| < \infty \quad \mathbb{P}_s^{(B)}\text{-a.s.} \quad \text{for almost every } s > 0. \tag{4}$$

Furthermore, $x \to k(t,x)$ is continuous on $[0,1]$, hence, thanks to the following lemma, we get for almost every $s > 0$, $k(s,x) = 0$ for $x \in [0,1]$. And, since $s \to k(s,x)$ is continuous on $\mathbb{R}_+$, we deduce $k \equiv 0$ on $\mathbb{R}_+ \times [0,1]$.

□

Lemma 3.4. *Fix $t > 0$. Let $\mathbb{P}_t^{(B)}$ denote the law of the Brownian fragmentation at time t. Let $k : [0,1] \mapsto \mathbb{R}$ be a continuous function, such that*

$$\sum_{i=1}^{\infty} k(x_i) = 0 \quad \mathbb{P}_t^{(B)}\text{-a.s. and} \sum_{i=1}^{\infty} |k(x_i)| < \infty \quad \mathbb{P}_t^{(B)}\text{-a.s.}$$

Then $k \equiv 0$ on [0,1].

Proof. Let $F(t) = (F_1(t), F_2(t)\ldots)$ be a Brownian fragmentation at time t where the sequence $(F_i(t))_{i\geq 1}$ is ordered by a size-biased pick. We denote by $\mathcal{S}$ the set of positive sequence with sum less than 1. Since $F(t)$ has the law of the size biased reordering of the jumps of a stable subordinator T (with index 1/2) before time t, conditioned by $T_t = 1$ (see [1]), it is obvious that we have

$$\forall x \in]0, 1-S[, \quad \mathbb{P}_t^{(B)}(F_1 \in dx \mid (F_i)_{i\geq 3}) > 0,$$

where $S=\sum_{i\geq 3} F_i$. Let $\mathbb{Q}_t$ be the measure on $\mathcal{S}$ defined by

$$\forall A \subset \mathcal{S}, \quad \mathbb{Q}_t(A) = \mathbb{P}_t^{(B)}((F_i)_{i\geq 3} \in A)$$

and λ the Lebesgue measure on $[0,1]$. Hence we have, for all $y \in \mathcal{S}$ - $\mathbb{Q}_t$-a.s.

$$\forall x \in]0,S[, \quad k(x) + k(1-S-x) + \sum_{i=1}^{\infty} k(y_i) = 0 \quad \lambda\text{-a.s.},$$

where $S = \sum_i y_i$. We choose now $y \in \mathcal{S}$ such that this equality holds for almost every $x \in]0,S[$. Thus, we get that there exists a constant $C = C(y)$ such that

$$k(x) + k(1-S-x) = C, \qquad \text{for all } x \in]0,S[\quad \lambda\text{-a.s.}$$

Since k is continuous, this equality holds in fact for all $x \in [0,S]$. Furthermore, we have also

$$\forall s \in]0,1[, \quad \mathbb{Q}_t(S \in ds) > 0.$$

Hence, this implies the existence for almost every $s \in]0,1[$ of a constant C_s such that

$$k(x) + k(1-s-x) = C_s \qquad \text{for all } x \in]0,s[.$$

Thanks to the continuity of k, we can deduce that this property holds in fact for all $s \in [0,1]$. Hence we have

$$\forall x,y \in [0,1]^2, \text{ such that } x+y \leq 1, \ k(x+y) = k(x) + k(y).$$

So k is a linear function and since $\sum_{i=1}^{\infty} x_i = 1 \ \mathbb{P}_t^{(B)}$-a.s., we get $k \equiv 0$ on [0,1]. □

We prove now the point 2 of Proposition 3.3.

Proof. Assume that the Lévy measure of Γ is finite. It is obvious that g has the same regularity that the function $\frac{q_x(-tx)}{p_x(-tx)}$. Recall now that we have

$$\frac{q_x(-tx)}{p_x(-tx)} = \exp\left(-x(ct + \frac{c^2}{2})\right) \mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)\right].$$

Hence a sufficient condition for g to fulfill the hypotheses of Proposition 3.3 is

- $u_t(x) = \mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)\right]$ is derivable at 0.
- $w(t,x) = u_t'(x)$ is continuous at $(t,0)$ for $t \in \mathbb{R}_+$.

We write $u_t(x) = a_t(x,x)$ with

$$a_t(y,z) = \mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2z} + \Gamma_y(t+c)\right)\right].$$

Since the function $(y,z) \to \frac{y^2}{2z^2}\exp\left(-\frac{y^2}{2z} + y(t+c)\right)$ is bounded on $\mathbb{R}_+ \times [0,1]$, we get

$$\partial_z a_t(y,z) = \mathbb{E}\left[\frac{\Gamma_y^2}{2z^2}\exp\left(-\frac{\Gamma_y^2}{2z} + \Gamma_y(t+c)\right)\right] \qquad \text{for } z \in]0,1].$$

Recall that the generator of a subordinator with no drift and Lévy measure π is given for every bounded function f C^1 with bounded derivative by

$$\forall y \in \mathbb{R}_+,\ Lf(y) = \int_0^\infty (f(y+s) - f(y))\pi(ds), \quad \text{(c.f. Section 31 of [16]).}$$

Hence, we get for all $z_0 > 0$,

$$\begin{aligned}\partial_y a_t(y, z_0) &= \mathbb{E}(La_t(\Gamma_y, z_0)) \\ &= \mathbb{E}\left[\int_0^\infty \left(\exp\left(-\frac{(\Gamma_y + s)^2}{2z_0} + (\Gamma_y + s)(t+c)\right) - \exp\left(-\frac{\Gamma_y^2}{2z_0} + \Gamma_y(t+c)\right)\right)\pi(ds)\right].\end{aligned}$$

And we deduce

$$\begin{aligned}u_t'(x) &= \mathbb{E}\left[\tfrac{\Gamma_x^2}{2x^2}\exp\left(-\tfrac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)\right] \\ &\quad + \mathbb{E}\left[\int_0^\infty \left(\exp\left(-\frac{(\Gamma_x + y)^2}{2x} + (\Gamma_x + y)(t+c)\right) - \exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)\right)\pi(dy)\right],\end{aligned}$$

We must prove that $(t, x) \to u_t'(x)$ is continuous at $(t, 0)$ for $t \geq 0$. For every Lévy measure π, the first term has limit 0 as (t', x) tends to $(t, 0)$ (by dominated convergence). For the second term, notice that we have for all $x \in]0, 1]$,

$$\Big| \exp\left(-\frac{(\Gamma_x + y)^2}{2x} + (\Gamma_x + y)(t+c)\right) - \exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right) \Big| \leq 2\exp\left(\frac{(t+c)^2 x}{2}\right),$$

and for all $y > 0$, $\exp\left(-\frac{(\Gamma_x+y)^2}{2x} + (\Gamma_x + y)(t+c)\right) - \exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)$ converges almost surely to -1 as (t', x) tends to $(t, 0)$. Hence, if $\pi(\mathbb{R}_+) < \infty$, we deduce that the $\lim_{(t',x)\to(t,0)} u_t'(x)$ exists (and is equal to $-\pi(\mathbb{R}_+)$). □

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