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Jean-Philippe Anker, Bruno Schapira, Bartosz Trojan

HEAT KERNEL AND GREEN FUNCTION ESTIMATES ON AFFINE BUILDINGS OF TYPE $\tilde{A}_r$

JEAN-PHILIPPE ANKER, BRUNO SCHAPIRA, AND BARTOSZ TROJAN

Abstract. We obtain a global estimate of the transition density $p^n(0,x)$ associated to a nearest neighbor random walk, called here "simple", on affine buildings of type $\widetilde{A}_r$. Then we deduce a global estimate of the Green function. This is the analogue of a result on Riemannian symmetric spaces of the noncompact type.

Key words: random walk, affine building, transition density, global estimate, Green's function.

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1. Introduction

This work is meant as a first attempt to understand the full behavior of random walks on affine buildings of higher rank. Such a study was carried out by Lalley (see [9], [10] and the report in [19]) for rather general random walks on rank one buildings i.e. homogeneous trees, and by the first author (in the joint works [2], [3], [4]) for the heat diffusion on a general Riemannian symmetric space of the noncompact type, which is a continuous counterpart of the present discrete setting. Apart from its own interest, this information is pivotal for further study. For instance, in potential theory, it is used to estimate the Green's function and to describe the Martin boundary. It will also be used in [18] to study the asymptotic behavior of normalized bridges.

Our paper deals with a simple higher rank case. We consider buildings of type $\widetilde{A}_r$, which are known to be most simple among affine buildings, and a particular random walk to the nearest neighbors, that we shall call "simple". This random walk, actually its Fourier transform, satisfies a "magic" combinatorial formula, which is technically very helpful. Our main result is a global upper bound for the transition density $p^n(x,y)$, which is also a lower bound, at least when $n-d(x,y)$ is large enough. As a consequence, we get the same upper and lower bound for the Green function, away from the diagonal. In rank one, we recover in a simpler way the main result of Lalley [9], specialized to the simple random walk (see [5] for more details).

Our method consists in analyzing carefully the transition density, using the inverse Fourier transform. Recall that Fourier analysis was developed in the seventies by Macdonald [11] for p–adic like buildings. It was resumed recently, first by Cartwright [7] for affine buildings of type $\widetilde{A}_r$ and next by Parkinson ([13], [14]) in the general case. Notice that these authors used it already to study isotropic random walks ([8], [13], [15]). They obtained in particular local and central limit theorem i.e. the asymptotics of the transition densities $p^n(x,y)$ when $n \to +\infty$ and x, y remain fixed.

Our paper is organized as follows. In Section 2, we recall the setting of our study and specify the basic objects involved: affine buildings (of type $\widetilde{A}_r$), the (inverse) Fourier transform and the "simple" random walk on these spaces. Section 3 is devoted to the rank 2 case, which is typical of the higher rank case and which is easier to deal with first. In this case, our method works in fact for any isotropic nearest neighbor random walk. Moreover we obtain the same upper and lower bound for $p^n(x,0)$ in the full range $|x| \leq n$. Section 4 deals with the general case. The result is similar, except that the lower bound is not shown to hold in the range $n - C \leq |x| \leq n$, where C is some positive constant (possibly large). In Section 5, we deduce sharp estimates (same upper and lower bound) for the Green function, at or above the bottom of the l^2 spectrum.

2. Preliminaries

2.1. **Root system.** Let R be a root system of type A_r in a real vector space $\mathfrak{a}$. Let $\mathfrak{a}_{\mathbb{C}} = \mathfrak{a} + i\mathfrak{a}$ be the complexification of $\mathfrak{a}$, equipped with its inner product $\langle \cdot, \cdot \rangle$. We shall briefly introduce some standard notation (for more details see e.g [6]). First let R^+ be a choice of positive roots. Let $\{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots. We denote by $\mathfrak{a}_+$ the associated positive Weyl chamber and by $\overline{\mathfrak{a}_+}$ its closure. Let $\{\lambda_1, \ldots, \lambda_r\}$ be the set of fundamental weights. Let $P = \sum_{i=1}^r \mathbb{Z}\lambda_i$ be the lattice of weights. Let P^+ be the subset of dominant weights, i.e. which lie in $\overline{\mathfrak{a}_+}$ and let P^{++} be the subset of strictly dominant weights i.e., those which lie in $\mathfrak{a}_+$. Let $Q = \sum_{i=1}^r \mathbb{Z}\alpha_i$ be the lattice of roots. The lattice P is the set of vertices of a simplicial complex, which is called the Coxeter complex. We denote by W_0 the Weyl group and by $\widetilde{W}$ the extended affine Weyl group (see e.g [13]). For $\alpha \in R$, let $\alpha^\vee = \frac{2}{|\alpha|^2}\alpha$ be the coroot associated to α. We have

$$\frac{1}{2} \sum_{\alpha \in R^+} \alpha^\vee = \sum_{i=1}^r \lambda_i.$$

Let $q \geq 2$ be an integer. We can define q_w for all $w \in \tilde{W}$. Then if $t_\lambda \in \widetilde{W}$ is the translation by λ, we have (see e.g. [14])

$$q_{t_\lambda} = q^{\sum_{\alpha \in R^+} \langle \lambda, \alpha \rangle}.$$

If $\lambda \in P^+$, we denote by $W_{0\lambda}$ the stabilizer of λ under the action of W_0. If $\lambda = \sum_{i=1}^r n_i \lambda_i$, with $n_i \in \mathbb{N}$ for all i, we denote by $|\lambda| = \sum_{i=1}^r n_i$ the length of λ. Eventually, the function π is defined on P by

$$\pi(\lambda) = \prod_{\alpha \in R^+} \langle \alpha^\vee, \lambda \rangle.$$

2.2. **The symmetric Macdonald polynomials.** The Weyl denominator Δ and the functions $\mathbf{c}$ and b are defined respectively, for $z \in \mathfrak{a}_{\mathbb{C}}$, by

$$\Delta(z) = \prod_{\alpha \in R^+} (e^{\frac{\langle \alpha^\vee, z\rangle}{2}} - e^{-\frac{\langle \alpha^\vee, z\rangle}{2}}),$$

$$\mathbf{c}(z) = \prod_{\alpha \in R^+} \frac{1 - q^{-1} e^{-\langle \alpha^\vee, z\rangle}}{1 - e^{-\langle \alpha^\vee, z\rangle}},$$

$$\frac{1}{\mathbf{c}(z)} = \Delta(z) b(z) e^{-\sum_{i=1}^r \langle \lambda_i, z\rangle}.$$

In particular, $|b(i\theta + s)|$ is bounded above and below by a fixed strictly positive constant, for $(\theta, s) \in i\mathfrak{a} \times \overline{\mathfrak{a}_+}$. The symmetric Macdonald polynomial is defined (see [12], [14]) for $\lambda \in P^+$ and $z \in \mathfrak{a}_{\mathbb{C}}$ by

$$P_\lambda(z) = \frac{q_{t_\lambda}^{-\frac{1}{2}}}{W_0(q^{-1})} \sum_{w \in W_0} \mathbf{c}(w \cdot z) e^{\langle \lambda, w\cdot z\rangle}. \tag{1}$$

where $\cdot$ denotes the action of W_0 on $\mathfrak{a}_{\mathbb{C}}$. Moreover, for the root systems of type A_r, we have

$$P_{\lambda_i}(z) = q_{t_{\lambda_i}}^{\frac{1}{2}} \frac{1}{N_{\lambda_i}} \sum_{\lambda \in W_0 \cdot \lambda_i} e^{\langle \lambda, z\rangle}. \tag{2}$$

We define the function h for $z \in \mathfrak{a}_{\mathbb{C}}$ by

$$h(z) = \sum_{i=1}^r \sum_{\lambda \in W_0 \cdot \lambda_i} e^{\langle \lambda, z\rangle}.$$

2.3. **Affine building and averaging operators.** An affine building (see [16] or [13]) of type $\tilde{A}_r$ is a nonempty simplicial complex containing subcomplexes called apartments such that:

- Each apartment is isomorphic (see [13]) to the Coxeter complex.
- Given two chambers (simplices of maximal dimension) there is an apartment containing both.
- Given two apartments that contain at least a common chamber, there exists a unique isomorphism between them, which fixes pointwise their intersection.

The building will be assumed to be regular. By definition this means that given any chamber C and any face F (simplex of codimension 1) of C, the cardinality of the set of chambers different from C and containing F is independent of C and F, and is equal to q. We denote by $\mathcal{X}$ the set of vertices (simplices of dimension 1) of the building. Observe for instance, that if $r = 1$, the building is a tree such that each vertex has $q + 1$ neighbors.

We fix a vertex 0 called the origin, an apartment A_0 containing 0, and a chamber C_0 of A_0 containing 0. We can identify the set of vertices of A_0 with the elements of the weight lattice P. Then we can identify P^+ with a subset of A_0 containing the vertices of C_0. This subset is denoted by A_0^+. Now given $x \in \mathcal{X}$, there exists an apartment containing C_0 and x. There is also an isomorphism between this apartment and A_0 fixing C_0. The image of x by this isomorphism has a unique conjugate by the Weyl group W_0 which lies in A_0^+. This conjugate is called the radial part or coordinate of x and is identified with an element of P^+.

For $\lambda \in P^+$, we denote by $V_\lambda(0)$ the sphere of radius λ around 0, which by definition is the set of vertices in the building with radial part equal to λ. Its cardinality is denoted by $N_\lambda = |V_\lambda(O)|$. Then we have (see [7])

$$N_\lambda = \frac{W_0(q^{-1})}{W_{0\lambda}(q^{-1})} q_{t_\lambda},$$

where $V(q^{-1}) = \sum_{w \in V} q_w^{-1}$, for all subgroups V of W_0. For $x \in V_\lambda(O)$ we set $|x| = |\lambda|$ and $\overline{x} = \lambda$. We set also $x_i = \langle \alpha_i, \lambda \rangle$ for all $i \leq r$.

By $\mathcal{A}$ we denote the algebra of averaging symmetric operators on $\mathcal{X}$. It was proved in [7] that $\mathcal{A}$ is a commutative algebra generated by the operators

$$\triangle_j f(x) = \frac{1}{N_{\lambda_j}} \sum_{y \in V_{\lambda_j}(O)} f(y), \quad j = 1, \dots, r,$$

where f is a complex-valued function on $\mathcal{X}$. Consider $l^2(\mathcal{X})$ with a natural scalar product

$$\langle f, g \rangle = \sum_{x \in \mathcal{X}} f(x)\overline{g(x)}.$$

Then the closure $\overline{\mathcal{A}}$ is a commutative C^*-algebra. Moreover, $\overline{\mathcal{A}}$ is isometrically isomorphic to the algebra of W_0-invariant continuous functions on

$$U = \{\theta \in \mathfrak{a} \mid \text{for all } \alpha \in R,\ \langle \alpha, \theta \rangle \ \leq \pi\},$$

and the Gelfand map is given by

$$\widehat{\triangle_j} = P_{\lambda_j}.$$

We observe here that U is W_0-invariant and a fundamental domain for the action of the lattice $2\pi Q$ on $\mathfrak{a}$. Eventually, for $A \in \overline{\mathcal{A}}$ we have the inversion formula

$$A\delta_y(x) = \frac{W_0(q^{-1})}{|W_0|} \int_U \widehat{A}(\theta) \overline{\widehat{\triangle_j}(\theta)} \frac{d\theta}{|\mathbf{c}(\theta)|^2},$$

for $x, y \in \mathcal{X}$ and $y \in V_{\lambda_j}(x)$.

2.4. **The simple random walk.** This is defined as the Markov chain on $\mathcal{X}$, with transition probabilities given by

$$p(x,y) = \begin{cases} q_{t_{\lambda_i}}^{-\frac{1}{2}} \rho & \text{if } y \in V_{\lambda_i}(x) \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\rho = \frac{1}{\sum_{i=1}^r q_{t_{\lambda_i}}^{-\frac{1}{2}} N_{\lambda_i}}.$$

Let also $\tilde{\rho} = \rho h(0)$ be the associated spectral radius. For example for the tree, i.e. the $\tilde{A}_1$ case, we have $\tilde{\rho} = \frac{2\sqrt{q}}{q+1}$. In the case $\tilde{A}_2$ we have

$$\tilde{\rho} = \frac{3q}{q^2 + q + 1},$$

and in the case $\tilde{A}_3$,

$$\tilde{\rho} = \frac{14q^2}{(1+q^2)[(q^2+q+1)+2q^{\frac{1}{2}}(q+1)]}.$$

2.5. **The function F_0.** It is defined on P^+ by

$$F_0(\lambda) = P_\lambda(0).$$

The following Proposition is the analogue of a result obtained in [1] and generalized in [17].

Proposition 2.1. *In P^+,*

$$F_0(\lambda) \asymp {}^1 q_{t_\lambda}^{-\frac{1}{2}} \prod_{\alpha \in R^+} (1 + \langle \alpha^\vee, \lambda \rangle). \tag{3}$$

Moreover,

$$F_0(\lambda) \sim {}^2 const \cdot \pi(\lambda) q_{t_\lambda}^{-\frac{1}{2}}. \tag{4}$$

when $\langle \alpha, \lambda \rangle \to +\infty$, for all $\alpha \in R^+$.

Proof. First we multiply (1) by $\pi(i\theta)$ removing the singularities of the **c**-function, and then we apply the operator $\pi(-i\partial)|_{\theta=0}$. Since the left hand side is equal to $F_0(\lambda)$ up to a constant, we get

$$q_{t_\lambda}^{\frac{1}{2}} F_0(\lambda) = p(\lambda),$$

where p is a polynomial in coordinates of λ with highest order term proportional to $\pi(\lambda)$. This proves (4) and (3) away from the walls. Next we extend our estimate along the walls by using a local Harnack principle. This is obtained immediately by using that F_0 is an eigenfunction of the averaging operators:

$$\sum_{y \in V_{\lambda_i}(x)} p(x,y) F_0(\overline{y}) = \rho |W_0 \cdot \lambda_i| F_0(\overline{x}),$$

for all $x \in \mathcal{X}$ and all $i \leq r$. □

3. Heat kernel estimates: the case $\tilde{A}_2$

Let $n \in \mathbb{N}$ and $x \in V_\lambda(O)$. Let $\alpha_0 = \alpha_1 + \alpha_2$. We set $\delta = \frac{1}{n+2}(\lambda + \lambda_1 + \lambda_2)$. For $i = 0, 1, 2$ we put $\delta_i = \langle \delta, \alpha_i \rangle$. Let

$$\phi(\delta) = \min\{u \in \overline{\mathfrak{a}_+} \mid \log h(u) - \langle \delta, u \rangle\}.$$

The main goal of this section is to prove

Theorem 3.1. *The following estimate*

$$p^n(0,x) \asymp \frac{1}{n^3} \rho^n e^{n\phi(\delta)} F_0(\overline{x}) \frac{1}{\sqrt{n^2(1-\delta_0)(1-\delta_1)(1-\delta_2)}}, \tag{5}$$

holds uniformly on the set $\{|x| \leq n-1\}$.

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[1] we say that $f \asymp g$, when there exists a constant $C > 0$ such that, $\frac{1}{C} g(\lambda) \leq f(\lambda) \leq C g(\lambda)$ for all λ.

[2] we say that $f \sim g$, when $\frac{f}{g} \to 1$.

We will see that the function $e^{\phi(\delta)}$ is bounded. Thus the exponent n in the theorem can be replaced by $n+2$, which appears more naturally in the proof. The next theorem gives a more precise statement of the estimate at the boundary of the domain. We adopt the following notation for the binomial coefficients:

$$C_k^n = \frac{n!}{k!(n-k)!}.$$

Theorem 3.2. *Let $K>0$. Then*

$$p^n(0,x) \asymp n^d(\rho q^{-1})^n C^{n-d}_{x_1\vee x_2-d} \tag{6}$$

uniformly in the set $\{n \geq |x| \geq n-K\}$, where $d = n-|x|$.

Note 3.1. *When $n-x_1\vee x_2 \leq K'$ for some fixed constant $K'>0$, then the estimate becomes*

$$p^n(0,x) \asymp (\rho q^{-1})^n n^{(n-x_1\vee x_2)+d}.$$

Remark 3.1. Here is the corresponding result for the tree (cf [5]). In this case we can give an explicit formula of the function ϕ appearing in the estimate. In fact we have

$$p^n(0,x) \asymp \frac{|x|}{n\sqrt{n-|x|}}\rho^n e^{n\phi(\delta)} q^{-\frac{|x|}{2}},$$

where

$$\phi(\delta) = \frac{1}{2}\{(1+\delta)\log(1+\delta) + (1-\delta)\log(1-\delta)\}.$$

3.1. **Proof: the beginning.** If $\theta = \theta_1\alpha_1 + \theta_2\alpha_2$, we set $|\theta|_\infty = \max\{|\theta_1|, |\theta_2|\}$. We say that a weight is away from a wall, when its distance to the wall is larger than some fixed constant (which will be determined in the proof). We denote by C a constant whose value may change from line to line.

We begin by some elementary transformations of $p^n(O,x)$. First, by using (2), we get

$$\begin{aligned} p^n(0,x) &= C\int_U \left(\frac{1}{2}P_{\lambda_1}(i\theta) + \frac{1}{2}P_{\lambda_2}(i\theta)\right)^n P_\lambda(i\theta)\frac{d\theta}{|\mathbf{c}(i\theta)|^2} \\ &= C\rho^n \int_U h^n(i\theta)P_\lambda(i\theta)\frac{d\theta}{|\mathbf{c}(i\theta)|^2}. \end{aligned}$$

Next by (1) and the W_0-invariance of h, we get

$$p^n(0,x) = C\rho^n q_{t_\lambda}^{-\frac{1}{2}} \int_U h^n(i\theta)e^{-i\langle\theta,\lambda+\lambda_1+\lambda_2\rangle}\Delta(i\theta)b(i\theta)d\theta. \tag{7}$$

Now we make two elementary observations. First

$$h+2 = (1+e^{\lambda_1})(1+e^{-\lambda_2})(1+e^{\lambda_2-\lambda_1}). \tag{8}$$

Moreover, we have the following lemma, whose proof is left to the reader (see also Section 4 for a more general result).

Lemma 3.1.

$$\pi(\partial)[h^{n+3}] = (n+3)(n+2)(n+1)\left[\frac{n+3}{n+1}h+2\right]h^n\Delta.$$

The idea now is to use Lemma 3.1 and integrate by parts in (7). This will be done in two different ways, depending on if $|x| \leq \frac{n}{2}$ or if $|x| > \frac{n}{2}$. We notice that the factor $\frac{1}{2}$ plays no role here, and could be replaced by $1-\eta$ for any $\eta \in (0,1)$.

3.2. **The case when $|x| \leq \frac{n}{2}$.** We consider the function

$$q^n(x) = \frac{n+3}{(n+1)\rho} p^{n+1}(0,x) + 2p^n(0,x).$$

By Lemma 3.1, after an integration by parts we get the expression

$$q^n(x) = C \frac{\rho^n q_{t_\lambda}^{-\frac{1}{2}}}{(n+3)(n+2)(n+1)} \int_U h^{n+3}(i\theta)\pi(\partial)\left[e^{-i\langle\theta,\lambda+\lambda_1+\lambda_2\rangle} b(i\theta)\right] d\theta.$$

Now we make the change of variables $i\theta \to i\theta + s$ for some $s \in \overline{\mathfrak{a}_+}$ whose value will be specified in the sequel, and we find

$$q^n(x) = C \frac{\rho^n q_{t_\lambda}^{-\frac{1}{2}} e^{-\langle\lambda+\lambda_1+\lambda_2,s\rangle}}{(n+3)(n+2)(n+1)} \int_U h^{n+3}(i\theta+s)\pi(\partial)[e^{-i\langle\theta,\lambda+\lambda_1+\lambda_2\rangle} b(i\theta+s)] d\theta.$$

Moreover $b(i\theta+s)$ and all its derivatives are bounded functions of $(\theta,s) \in U \times \overline{\mathfrak{a}_+}$. Thus for x or λ sufficiently away from the walls,

$$\begin{aligned} q^n(x) &= C \tfrac{\rho^n q_{t_\lambda}^{-\frac{1}{2}} e^{-\langle\lambda+\lambda_1+\lambda_2,s\rangle}\pi(\lambda+\lambda_1+\lambda_2)}{(n+3)(n+2)(n+1)} \\ &\times \textstyle\int_U h^{n+3}(i\theta+s) e^{-i\langle\theta,\lambda+\lambda_1+\lambda_2\rangle} b(i\theta+s) d\theta + \text{lower order terms.} \end{aligned}$$

3.3. **The case when $|x| > \frac{n}{2}$.** In this case q^n and p^n are not anymore comparable. However, when λ is sufficiently away from the wall $\{\alpha_1 - \alpha_2 = 0\}$, then it results from (8) and our choice of s in the next subsection (see also Remark 3.2), that at least for n sufficiently large, the function $\theta \mapsto \frac{1}{\frac{n+3}{n+1}h(i\theta+s)+2}$ does not vanish on U. Thus after an integration by parts in (7), we see that for λ away from the walls $\{\alpha_2 = 0\}$ and $\{\alpha_1 - \alpha_2 = 0\}$,

$$\begin{aligned} p^n(0,x) &= C \frac{\rho^n q_{t_\lambda}^{-\frac{1}{2}} e^{-\langle\lambda+\lambda_1+\lambda_2,s\rangle}\pi(\lambda+\lambda_1+\lambda_2)}{(n+3)(n+2)(n+1)} \\ &\times \int_U h^{n+3}(i\theta+s) e^{-i\langle\theta,\lambda+\lambda_1+\lambda_2\rangle}(b_1+b_2)(i\theta+s) d\theta, \end{aligned}$$

where $b_1 = \frac{b}{\frac{n+3}{n+1}h+2}$ and $b_2 = \frac{e^{i(\lambda+\lambda_1+\lambda_2)}}{\pi(\lambda+\lambda_1+\lambda_2)}\pi(\partial)[b_1 e^{-i(\lambda+\lambda_1+\lambda_2)}] - b_1$ is the remainder.

3.4. **Choice of s and the stationary phase method.** In the preceding subsections we have seen that $q^n(x)$, in the range $|x| \leq \frac{n}{2}$, and $p^n(0,x)$, in the range $|x| > \frac{n}{2}$, were comparable for λ away from the walls to

$$C(n,\lambda) \int_U \frac{h^{n+2}(i\theta+s)}{h^{n+2}(s)} e^{-i\langle\theta,\lambda+\lambda_1+\lambda_2\rangle} \tilde{b}(i\theta+s) d\theta, \tag{9}$$

with

$$C(n,\lambda) = \frac{\rho^n q_{t_\lambda}^{-\frac{1}{2}} e^{-\langle\lambda+\lambda_1+\lambda_2,s\rangle} h^{n+2}(s)\pi(\lambda)}{n^3},$$

and $\tilde{b}$ equal to hb or $h(b_1+b_2)$ respectively if $|x| \leq \frac{n}{2}$ or if $|x| > \frac{n}{2}$. Now we choose the shift s according to the stationary phase method, i.e. as a solution in $\overline{\mathfrak{a}_+}$ of the equation

$$\frac{\nabla h}{h}(s) = \delta. \tag{10}$$

Solving such an equation is classical. We consider the function $\phi^\delta : u \mapsto \log h(u) - \langle \delta, u \rangle$ on $\mathfrak{a}$. Since $|\delta| < 1$ (by hypothesis $|x| < n$), this function tends to infinity when $|u| \to +\infty$. Moreover, h is W_0-invariant, and $\langle \delta, u \rangle$ is maximal when $u \in \overline{\mathfrak{a}_+}$. Thus ϕ^δ attains its minimum in $\overline{\mathfrak{a}_+}$ at some point $s = s(\delta)$, which satisfies equation (10). Without loss of generality, we will assume in the sequel that $\langle \lambda_1, s \rangle \geq \langle \lambda_2, s \rangle$. The following lemma collects some properties of s.

Lemma 3.2. (1) *The condition $\langle \lambda_1, s \rangle \geq \langle \lambda_2, s \rangle$ implies $\delta_1 \geq \delta_2$.*
(2) *The function ϕ^δ is strictly convex, thus s is the unique point where it attains its minimum, and $\delta \mapsto s$ is continuous on $\{|\delta| < 1\}$.*
(3) *When $|\delta| \to 1$, $|s| \to +\infty$, where $|s|$ is the Euclidean norm of s. More precisely, when $|\delta| \to 1$,*

$$e^{\langle \lambda_2 - \lambda_1, s \rangle} = \frac{1 - \delta_1}{\delta_1} + O(e^{-\langle \lambda_2, s \rangle}),$$

$$e^{-\langle \lambda_2, s \rangle} \asymp 1 - |\delta|.$$

(4) *We have $s = 0$ if, and only if, $\delta = 0$. When $\delta_1 - \delta_2 \to 0$, then*

$$\langle \lambda_2 - \lambda_1, s \rangle \asymp \delta_1 - \delta_2.$$

Proof. Equation (10) is equivalent to the system

$$\left\{ \begin{array}{ll} \sinh \langle \lambda_1, s \rangle + \sinh \langle \lambda_1 - \lambda_2, s \rangle = & \delta_1 h \\ \sinh \langle \lambda_2, s \rangle - \sinh \langle \lambda_1 - \lambda_2, s \rangle = & \delta_2 h. \end{array} \right. \tag{11}$$

The first statement follows since sinh is increasing and has the same sign than its argument. The convexity of ϕ^δ comes from a more general result: assume that $h = \sum_\lambda e^\lambda$ is a sum of exponentials. Then the second order derivative $d^2\phi^\delta_u$ of ϕ^δ at some point $u \in \mathfrak{a}$ is given by

$$d^2\phi^\delta_u = \frac{hd^2h - |\nabla h|^2}{h^2}(u) = \frac{1}{h^2(u)} \sum_{\lambda \neq \lambda'} [\lambda - \lambda']^2 e^{\langle \lambda + \lambda', u \rangle},$$

which implies that $d^2\phi^\delta_u$ is positive definite. Thus ϕ^δ is strictly convex. The second point follows immediately. By adding the both equations of (11), and because $\sinh < \cosh$, we see that $|s| \to \infty$ when $|\delta| \to 1$. Multiplying by $e^{-\langle \lambda_1, s \rangle}$ in both side of the first equation gives immediately the asymptotic of $e^{\langle \lambda_2 - \lambda_1, s \rangle}$. Doing the same in the sum of the two equations gives the asymptotic of $e^{-\langle \lambda_2, s \rangle}$. By doing now the difference between them, we obtain in the same way the last point. The assertion that $s = 0$ if, and only if, $\delta = 0$ is straightforward. This concludes the proof of the lemma. □

Remark 3.2.
(1) As announced, the estimates of the lemma show that the function $e^{\phi(\delta)} = h(s)e^{-\langle \delta, s \rangle}$ in (5) is bounded.
(2) The last point of the lemma implies that $(1 + e^{\langle \lambda_2 - \lambda_1, i\theta + s \rangle})$ is larger (up to a constant) than $\delta_1 - \delta_2$ for any $\theta \in U$. Thus, thanks to Formula (8), we see that for n and $\langle \alpha_1 - \alpha_2, \lambda \rangle$ large enough, $\frac{n+3}{n+1}h + 2$ does not vanish in U. This justifies our assumption of section 3.3.

We consider now the phase function

$$F^\delta(\theta) = \log h(i\theta + s) - \log h(s) - i \langle \delta, \theta \rangle \tag{12}$$

which is well defined at least in a small neighborhood of 0, independent of s (or δ). By our choice of s, $F^\delta(0) = 0$ and $\nabla F^\delta(0) = 0$. The next lemma gives a much more precise result on the behavior of F^δ near 0. Let $\Re F^\delta$ and $\Im F^\delta$ denote respectively the real and imaginary part of F^δ.

Lemma 3.3. *There exists two constants $\epsilon > 0$ and $C > 0$ such that,*

$$\Re F^\delta(\theta) \asymp -q_\delta(\theta), \text{ and } |\Im F^\delta(\theta)| \le C|\theta|_\infty q_\delta(\theta),$$

uniformly for $|\theta|_\infty \le \epsilon$ and $|\delta| < 1$, where

$$q_\delta(\theta) = e^{\langle \lambda_2 - \lambda_1, s\rangle} \langle \lambda_1 - \lambda_2, \theta\rangle^2 + e^{-\langle \lambda_2, s\rangle} \langle \lambda_2, \theta\rangle^2 .$$

Proof. Since $F^\delta(0) = 0$ and $\nabla F^\delta(0) = 0$, we see that for all $\theta \in U$, there exists $\theta' \in U$ such that $|\theta'|_\infty \le |\theta|_\infty$ and

$$F^\delta(\theta) = d^2 F^\delta_{\theta'}(\theta),$$

where $d^2 F^\delta_{\theta'}$ is the second order derivative of F^δ in θ'. We can compute it, as we did for ϕ^δ in Lemma 3.2:

$$\begin{aligned} d^2 F^\delta_{\theta'}(\theta) &= \frac{-1}{h(i\theta'+s)^2} \sum_{\lambda,\lambda'} \langle \lambda - \lambda', \theta\rangle^2 e^{\left\langle \lambda+\lambda', i\theta'+s\right\rangle} \\ &= \frac{-1}{|h(i\theta'+s)|^4} \sum_{\lambda,\lambda',\mu,\mu'} \langle \lambda - \lambda', \theta\rangle^2 e^{\left\langle \lambda+\lambda'+\mu+\mu', s\right\rangle} e^{i\left\langle \lambda+\lambda'-\mu-\mu', \theta'\right\rangle}. \end{aligned}$$

Next observe that $|h(i\theta'+s)e^{-\langle \lambda_1, s\rangle}|$ is bounded above and below by strictly positive constants, when $|\theta'|_\infty$ is small. So if we multiply up and down the right member of the last equality by $e^{-4\langle \lambda_1, s\rangle}$, and then take successively the real part and the imaginary part, we get the two assertions of the lemma. □

We denote by $J(n,\lambda)$ the integral appearing in (9). According to the notation of the preceding lemma, let $\epsilon' < \epsilon$ (taken small in the next proposition). We divide $J(n,\lambda)$ into the sum of the integral, let say $J_1(n,\lambda)$, over $[-\epsilon', \epsilon']^2$ and the integral $J_2(n,\lambda)$ over $U \smallsetminus [-\epsilon', \epsilon']^2$, where $[-\epsilon', \epsilon']^2$ denotes the set $\{|\theta|_\infty \le \epsilon'\}$.

Proposition 3.1. *There exists $\epsilon' > 0$ (independent of δ and n), such that the following estimates hold for n large enough and λ away from the walls*

$$|J_1(n,\lambda)| \asymp \frac{1}{\sqrt{n^2(1-\delta_1)(1-|\delta|)}},$$

$$|J_2(n,\lambda)| \le \frac{C}{\sqrt{n(1-\delta_1)}} e^{-cn(1-|\delta|)},$$

where C and c are two strictly positive constants.

Proof. We have

$$J_1(n,\lambda) = \int_{|\theta|_\infty \le \epsilon'} e^{(n+2)F^\delta(i\theta+s)} \tilde{b}(i\theta+s)d\theta.$$

Thus essentially the first statement of the proposition is given by Lemma 3.3, and a change of variable. In fact we just need in addition a control of $\tilde{b}(i\theta+s)$ for small θ's. In the case where $|x| \le \frac{n}{2}$, this is immediate, since $|b|$ is bounded above and below by strictly positive constants, and by continuity of $\delta \mapsto s$, $|h|$ also, if ϵ' is sufficiently small. In the other case, where $|x| > \frac{n}{2}$, by (8), we see that $|h+2|$ stays away from 0 if $|\theta|_\infty$ is small. Thus for λ sufficiently away from the walls, b_2

becomes negligible in front of b_1. Then since $(hb_1)(s)$ is real and bounded (above and below) on $\overline{\mathfrak{a}_+}$, this gives the desired estimate of $|J_1(n,\lambda)|$. The estimate of J_2 is more complicated, since $|hb_2|$ may become larger than $|hb_1|$ and even explode, for instance when $\langle \lambda_2 - \lambda_1, \theta\rangle = \pm\pi$, and $\delta_1 - \delta_2$ tends to 0 (cf. (8)). Fortunately, as we will see, this is compensated by the exponential decay of $e^{(n+2)\Re F^{\delta}(i\theta+s)}$. Indeed

$$\frac{|h(i\theta+s)|}{h(s)} = e^{\frac{1}{2}\log(1-(1-\frac{|h(i\theta+s)|^2}{h^2(s)}))} \leq e^{-\frac{1}{2}[1-\frac{|h(i\theta+s)|^2}{h^2(s)}]}.$$

But

$$1 - \frac{|h(i\theta+s)|^2}{h^2(s)} = \frac{1}{h^2(s)}\sum_{\lambda,\lambda'} e^{\langle \lambda+\lambda', s\rangle}(1-\cos\langle \lambda-\lambda', \theta\rangle).$$

Multiplying again the numerator and denominator by $e^{-2\langle \lambda_1, s\rangle}$, we obtain

$$1 - \frac{|h(i\theta+s)|^2}{h^2(s)} \geq c\sum_{\lambda,\lambda'} e^{\langle \lambda+\lambda'-2\lambda_1, s\rangle}\left\langle \lambda-\lambda', \theta\right\rangle^2,$$

for some constant $c>0$. Observe now that since $s=0 \Leftrightarrow \delta=0$ and s is continuous, then s stays away from 0 when $|x| > \frac{n}{2}$. In particular $\langle \lambda_1, s\rangle$ and $\langle \lambda_2, s\rangle$ stay also away from 0. Therefore with (8) we see that $|b_1|$ or $|b_2|$ can explode only when $1+e^{\lambda_2-\lambda_1}$ is small, i.e when $\langle \lambda_2-\lambda_1, \theta\rangle \pm \pi$ and $\delta_1-\delta_2$ are small. But in this case

$$(n+2)e^{\langle \lambda_2-\lambda_1, s\rangle}\left\langle \lambda_2-\lambda_1, \theta\right\rangle^2 \asymp (n+2)(1-\delta_1) \asymp n,$$

since $\delta_1-\delta_2$ small implies that $1-\delta_1$ is away from 0. As a consequence

$$e^{-\frac{c}{2}(n+2)\sum_{\lambda,\lambda'} e^{\langle \lambda+\lambda'-2\lambda_1, s\rangle}\langle \lambda-\lambda', \theta\rangle^2}(|hb_1(i\theta+s)| + |hb_2(i\theta+s)|)$$

is bounded in U. The estimate of $|J_2(n,\lambda)|$ follows with Lemma 3.2 and a change of variable. □

By the preceding proposition, when $n(1-|\delta|)\to\infty$, J_2 becomes negligible in front of J_1. Thus we can find a constant $K>0$ such that $C(n,\lambda)|J(n,\lambda)|$ has the estimate (5) when λ is away from the walls and $n(1-|\delta|)\geq K$. Moreover the preceding proposition implies also that $C(n,\lambda)|J(n,\lambda)|$ is bounded by the expression in (5). Now the rest of the proof can be decomposed into two steps. First we prove the lower estimate when $n(1-|\delta|)\leq K$. Then we extend our estimate along the walls by using a local Harnack principle, and we prove by the way that $q^n(x)$ is comparable to $p^n(0,x)$ when $|x|\leq \frac{n}{2}$.

3.5. **Lower bound when $n(1-|\delta|)\leq K$.** We present two proofs. The first is analytical, and the second is purely combinatorial. One advantage of the second proof is that it is valid in the whole range $n(1-|\delta|)\leq K$, whereas the first is only valid when λ is away from the walls $\{\alpha_2=0\}$ and $\{\alpha_1-\alpha_2=0\}$. Another advantage of the combinatorial approach is that it provides a more elementary proof of the upper bound.

3.5.1. *Analytical proof.* We begin by

Lemma 3.4. *When $(n+2)(1-\delta_1)\to+\infty$ and $(n+2)(\delta_1-\delta_2)\to+\infty$, then $\tilde{b}(i\theta+s)\to 1$, uniformly in U.*

Proof. Two cases may cause problems. Either when $\delta_1 - \delta_2$ tends to 0. But in this case $\langle \alpha, \lambda \rangle \asymp n$ for all $\alpha \in \mathbb{R}^+$, since the line $\lambda_1 = \lambda_2$ does not cross walls in the range $n(1 - |\delta|) \le K$ (at least if n is large enough). Thus when $n(\delta_1 - \delta_2) \to +\infty$, $|b_2|$ tends to 0. The other case is when $(1 - \delta_1) \to 0$, because we are not sure a priori that the function b tends to 1. But this becomes true if $n(1 - \delta_1) \to +\infty$. Now the proof of the lemma is straightforward. □

By the preceding lemma there exists a constant $K' > 0$ such that if $(n+2)(1 - \delta_1) \ge K'$ and $(n+2)(\delta_1 - \delta_2) \ge K'$, then $J(n, \lambda)$ is comparable to

$$\frac{1}{h^{n+2}(s)} \int_U h^{n+2}(i\theta + s) e^{-i\langle \lambda + \lambda_1 + \lambda_2, \theta \rangle} d\theta.$$

We denote by $I(n, \lambda)$ the integral in the last expression. We compute it by developing $h^{n+2}(i\theta + s)$ and using that the integral of $e^{i\langle \mu, \theta \rangle}$ is null whenever μ is a non zero weight. We get, if $|U|$ denotes the Lebesgue measure of U,

$$I(n, \lambda) = |U| \sum \frac{(n+2)!}{n_1! n_2! \dots n_6!},$$

where the sum is over the family of integers $(n_1, \dots, n_6)$ such that $n_1 + \dots + n_6 = n + 2$, $n_1 - n_2 + n_5 - n_6 = x_1 + 1$ and $n_3 - n_4 - n_5 + n_6 = x_2 + 1$. In particular we can choose $n_1 = x_1 + 1 - d$, $n_3 = x_2 + 1 + d$, $n_5 = d$, $n_2 = n_4 = n_6 = 0$, with $d = n - x_1 - x_2$. Thus, by Stirling's formula, we get

$$\begin{aligned} I(n, \lambda) &\ge c_1 \frac{(n+2)^{n+2}}{x_1^{x_1+1-d} x_2^{x_2+1+d}} \frac{1}{\sqrt{x_2}} \\ &\ge c_2 n^d \left(\frac{n+2}{x_1}\right)^{x_1+1-d} \left(\frac{n+2}{x_2}\right)^{x_2+1+d} \frac{1}{\sqrt{n(1-\delta_1)}} \\ &\ge c_3 n^d \frac{1}{\delta_1^{x_1+1-d}} \frac{1}{(1-\delta_1)^{x_2+1+d}} \frac{1}{\sqrt{n(1-\delta_1)}}, \end{aligned}$$

where c_1, c_2 and c_3 are strictly positive constants. The passage from the second to the third line is justified by the inequality $\frac{(n+2)(1-\delta_1)}{x_2} \ge 1$. On the other side, with Lemma 3.2 we get

$$h(s)^{n+2} e^{-\langle \lambda + \lambda_1 + \lambda_2, s \rangle} \asymp \left(\frac{\delta_1}{1-\delta_1}\right)^{x_2+1+d} \left(1 + \frac{1-\delta_1}{\delta_1}\right)^{n+2} \frac{1}{(1-|\delta|)^{(n+2)(1-|\delta|)}}.$$

Then

$$h(s)^{n+2} e^{-\langle \lambda + \lambda_1 + \lambda_2, s \rangle} \asymp \left(\frac{\delta_1}{1-\delta_1}\right)^{x_2+1+d} \frac{1}{\delta_1^{n+2}} n^d.$$

Thus we have proved the lower bound when $(n+2)(1 - \delta_1) \ge K'$ (and λ is away from the walls).

3.5.2. *Combinatorial proof.* We have just seen above that

$$(h(s) e^{-\langle \delta, s \rangle})^n \frac{1}{\sqrt{n(1-\delta_1)}} \asymp C_{x_1+1-d}^{n+2-d} \, n^d,$$

for x and n such that $(n+2)(1 - |\delta|) \le K$. Now observe that in this range

$$C_{x_1+1-d}^{n+2-d} \asymp \frac{n}{x_2+1} C_{x_1-d}^{n-d},$$

and $q_{t_\lambda} \asymp q^{2n}$. Thus we are lead to prove the estimate of Theorem 3.2. In fact it is more convenient to prove the corresponding result for the radial random walk. If $\overline{p}$ denotes its transition kernel, then by definition $\overline{p}^n(0,\lambda) = p^n(0,x)N_\lambda$ for all $n \in \mathbb{N}$, all $\lambda \in P^+$, and all $x \in V_\lambda(0)$. Since N_λ is comparable to q_{t_λ} we have to prove that

$$\overline{p}^n(0,\lambda) \geq c(q\rho)^n C_{x_1-d}^{n-d}\, n^d,$$

for some constant $c > 0$. But for any $\lambda \in P^+$, $q\rho = \overline{p}(\lambda, \lambda+\lambda_1)$, and when $\lambda \in P^{++}$, we have also $q\rho = \overline{p}(\lambda, \lambda + \lambda_2)$. Thus it suffices to prove that, if $n + 2 - |\lambda| \leq K$, then the number of paths from 0 to λ in P^+, is comparable to $C_{x_1-d}^{n-d}\, n^d$. This is elementary, and can be seen as follows. Consider the sequence of increments of the radial random walk up to time n, $(\epsilon_1, \dots, \epsilon_n)$, with $\epsilon_i \in \{\pm\lambda_1, \pm\lambda_2, \pm(\lambda_1 - \lambda_2)\}$ for all $i \leq n$. Then choose in arbitrary order, $x_1 - d$ terms equal to λ_1, $x_2 + d$ terms equal to λ_2, and d terms equal to $\lambda_1 - \lambda_2$. The number of ways to do it is comparable to $C_{x_1-d}^{n-d}\, n^d$, which proves the lower bound.

In fact we can also prove the upper bound by combinatorial arguments. Indeed, there must be at least $x_1 - d$ terms equal to λ_1 in the sequence, and at most $x_1 + d$. Otherwise the random walk could not reach the point λ at time n. Now if λ_1 appears $k \in \{x_1 - d, \dots, x_1 + d\}$ times, then for the same reason there must be also at least $n - d - k$ terms equal to λ_2. But when k is fixed, the number of sequences satisfying these conditions is bounded (up to a constant) by $C_k^{n-d}\, n^d$. Moreover, since $x_1 \geq \frac{n-d}{2}$ (and $d \leq K$), there exists a constant $c > 0$ such that $C_k^{n-d} \leq cC_{x_1-d}^{n-d}$, for all $k \in \{x_1 - d, \dots, x_1 + d\}$. This proves the desired upper bound.

3.6. **Local Harnack principle and estimate along the walls.** Let us assume that the estimate proved until now is valid when $\overline{x}$ is at distance at least D from the walls $\{\alpha_2 = 0\}$ and $\{\alpha_1 - \alpha_2 = 0\}$. From the heat equation satisfied by p, we know that there exists a constant $C > 0$, such that

$$p^n(O,y) \leq Cp^{n+1}(O,x), \tag{13}$$

for all neighbors x and y. Let now $x \in \mathcal{X}$ be such that $\overline{x}$ is at distance less than D of the wall $\{\alpha_2 = 0\}$ for instance. By repeated applications of (13) we see that there exist vertices y_1 and y_2 such that $\overline{y_1} = \overline{x} + D(\lambda_2 - \lambda_1)$, $\overline{y_2} = \overline{x} + D\lambda_2$, and

$$cp^{n-D}(0,y_1) \leq p^n(0,x) \leq Cp^{n+D}(0,y_2),$$

where $c > 0$ and $C > 0$ are other constants. Therefore, we only need to show that our upper estimate of $p^{n+D}(0,y_2)$ is comparable to our lower estimate of $p^{n-D}(0,y_1)$. This yields to prove that $e^{n[\phi(\delta_{y_2})-\phi(\delta_{y_1})]}$ is bounded, where $\delta_{y_i} = \frac{\overline{y_i}+\lambda_1+\lambda_2}{n+2}$. But by an elementary calculus, we see that

$$\nabla\phi(\delta) = -s.$$

Hence

$$\langle \nabla\phi(\delta_{y_1}), \delta_{y_2} - \delta_{y_1} \rangle \leq 0.$$

This implies well that $e^{n[\phi(\delta_{y_2})-\phi(\delta_{y_1})]}$ is bounded. This proves that our estimate extend near the walls. The only missing part now is to see that $q^n(x)$ and $p^n(0,x)$ are comparable when $|x| \leq \frac{n}{2}$. This results also from (13). Indeed it implies that

$$p^n(0,x) \leq q^n(x) \leq Cp^{n+3}(0,x),$$

and our estimates show that p^{n+3} is comparable to p^n when $|x| \leq \frac{n}{2}$.

The proof of the theorems 3.1 and 3.2 is now finished. The extension of (6) when $|x| = n$ is straightforward.

Remark 3.3. For the rank 2 case, this proof may in fact be applied for any isotropic nearest neighbor random walk. Let us detail. Such random walk has transition density given by $p(x, y) = p_i$ if $y \in V_{\lambda_i}(x)$, for $i = 1, 2$, and $p(x, y) = 0$ if x and y are not neighbors. Now in rank 2, $q_{t_{\lambda_1}} = q_{t_{\lambda_2}}$ and $N_{\lambda_1} = N_{\lambda_2}$. Thus we have the formula similar to (7)

$$p^n(0, x) = C\rho^n q_{t_\lambda}^{-\frac{1}{2}} \int_U h^n(i\theta) e^{-i\langle\theta,\lambda+\lambda_1+\lambda_2\rangle} \Delta(i\theta) b(i\theta) d\theta,$$

but with $\rho^{-1} = q_{t_{\lambda_1}}^{-\frac{1}{2}} N_{\lambda_1}$ and $h = p_1 \sum_{\lambda\in W_0\lambda_1} e^\lambda + p_2 \sum_{\lambda\in W_0\lambda_2} e^\lambda$. In particular observe that the spectral radius $\tilde\rho := \rho h(0)$ is the same for all these random walks. Now remark that for some constants $c, c' > 0$, we have

$$h + c + cc'^3 = c(1 + c'e^{\lambda_1})(1 + c'e^{\lambda_2-\lambda_1})(1 + c'e^{-\lambda_2}).$$

Namely $c' = p_2/p_1$ and $c = p_1^2/p_2$. So all the preceding proof can be applied, and Theorem 3.1 can in fact be deduced for all the isotropic nearest neighbor random walks.

4. Heat kernel estimate for general buildings of type $\tilde A_r$

Given $x \in V_\lambda(0)$ and $n \geq 0$, we set $\delta = \frac{\lambda+\sum_{i=1}^r \lambda_i}{n+r}$, and for $i \leq r$, let $\delta_i := \langle\alpha_i, \delta\rangle$. We define again ϕ by

$$\phi(\delta) = \min\{u \in \overline{\mathfrak{a}_+} \mid \log h(u) - \langle u, \delta\rangle\}.$$

We have the following result

Theorem 4.1. *There exists a constant $K > 0$, such that the following estimate holds uniformly in the set $\{|x| \leq n - K\}$*

$$p^n(0, x) \asymp \frac{1}{n^{|R^+|}} \rho^n e^{n\phi(\delta)} F_0(\overline{x}) \frac{1}{\sqrt{n^r \prod_{\alpha\in R^+}(1 - \langle\alpha, \delta\rangle)}}. \tag{14}$$

Moreover, the upper estimate holds in the whole domain $\{|x| < n\}$.

Proof. The proof follows exactly the same lines as in rank 2. After elementary computations we get

$$p^n(0, x) = C\rho^n q_{t_\lambda}^{-\frac{1}{2}} \int_U h^n(i\theta) e^{-\langle i\theta,\lambda+\sum_{i=1}^r \lambda_i\rangle} \Delta(i\theta) b(i\theta) d\theta.$$

The next result is the analogue of Lemma 3.1

Lemma 4.1. *The following formula holds*

$$\pi(\partial)\left[h^{n+|R^+|}\right] = (n + |R^+|)\dots(n + 1) r_n(h) h^n \Delta,$$

where r_n is a polynomial of the form

$$r_n(h) = \sum_{k=r}^{|R^+|} c_k (h + 2)^{k-r} \frac{h^{|R^+|-k}}{(n + 1)\dots(n + |R^+| - k)},$$

with constants $c_k \in \mathbb{R}$.

Proof. Let us first show that

$$h+2=\prod_{\lambda\in W_0\lambda_1}(1+e^{\lambda})=\prod_{i=1}^{r+1}(1+e^{\lambda_i-\lambda_{i-1}}), \tag{15}$$

with the convention $\lambda_{-1}=\lambda_{r+1}=0$. We prove it by using a well known symbolic description of the fundamental weights and of their conjugates by W_0. We associate to λ_1 the symbol x_1, to λ_2 the symbol x_1+x_2, and so on until λ_r to which we associate the symbol $x_1+\cdots+x_r$, with the rule $x_1+\cdots+x_{r+1}=0$. For instance $x_2+\cdots+x_{r+1}$ represents the weight $-\lambda_1$. Then we have a nice description of the conjugates of a fundamental weight

$$W_0\lambda_k=\{x_{i_1}+\cdots+x_{i_k}\mid i_i\neq i_2\cdots\neq i_k\},$$

for all $k\le r$. With this notation it is now elementary to see that

$$h+2=\prod_{i=1}^{r}(1+e^{x_i}),$$

which gives (15). Next, for any subset I of R^+, we define π^I on P^+ by

$$\pi^I(\lambda)=\prod_{\alpha\in I}\langle\alpha^\vee,\lambda\rangle\,.$$

We have

$$\pi(\partial)\left[h^{n+|R^+|}\right]=\sum_{k=1}^{|R^+|}(n+|R^+|)\dots(n+|R^+|-k+1)h^{n+|R^+|-k}f_k,$$

where

$$f_k=\sum_{I_1\cup I_2\cup\cdots\cup I_k=R^+}\left[\pi^{I_1}(\partial)(h+2)\right]\dots\left[\pi^{I_k}(\partial)(h+2)\right],$$

for all $k\le|R^+|$. The polynomials f_k are skew-invariant. Thus they are divisible by Δ. Moreover, f_k is of degree less or equal to k. It implies that $f_k=0$ when $k<r$, since Δ is of degree r. Thus we may assume $k\ge r$. Now we observe that there is only r roots α in R^+ such that $\langle\alpha,\lambda_1\rangle\neq 0$. The same holds for all the conjugates of λ_1. Therefore, by (15), f_k is divisible by all the factors $(1+e^{w\lambda_1})^{k-r}$. Hence, it is also divisible by $(h+2)^{k-r}$. Since h is W_0-invariant, we obtain that f_k is proportional to $(h+2)^{k-r}\Delta$. This concludes the proof of the lemma. □

For the rest of the proof we need to divide $\overline{\mathfrak{a}_+}$ in a few parts. If $\{i_1,\dots,i_r\}=\{1,\dots,r\}$, we set

$$\Lambda(i_1,\dots,i_r):=\{s\in\overline{\mathfrak{a}_+}\mid\langle\lambda_{i_1},s\rangle\ \ge\cdots\ge\ \langle\lambda_{i_r},s\rangle\}.$$

In fact only a few of these sets are enough to cover $\overline{\mathfrak{a}_+}$

Lemma 4.2. *The set $\Lambda(i_1,\dots,i_r)$ has a non empty intersection with $\mathfrak{a}_+$ if, and only if, for all $2\le j\le r$, there exists $k<j$, such that $i_j=i_k\pm1$.*

Proof. Let $s=s_1\alpha_1+\cdots+s_r\alpha_r$ be the decomposition of $s\in\mathfrak{a}_+$ in the basis of the simple roots. Assume that the hypothesis on $i_1,\dots,i_r$ of the lemma does not hold. It means that for some i, we have either $\langle\lambda_i,s\rangle\ge\langle\lambda_{i+2},s\rangle\ge\langle\lambda_{i+1},s\rangle$, or $\langle\lambda_{i+2},s\rangle\ge\langle\lambda_i,s\rangle\ge\langle\lambda_{i+1},s\rangle$. Assume that we are in the first case (the second is similar). Then $s_i\ge s_{i+2}\ge s_{i+1}$. But on the other hand, since $s\in\mathfrak{a}_+$, $\left\langle\alpha_{i+1}^\vee,s\right\rangle>0$. This is absurd because $\left\langle\alpha_{i+1}^\vee,s\right\rangle=s_{i+1}-\frac{1}{2}(s_i+s_{i+2})$. □

Now for the same reason than in rank 2 the equation $\nabla h(s) = \delta h(s)$ has a unique solution in $\overline{\mathfrak{a}_+}$. In the sequel we will assume that it lies in $\Lambda(1,\dots,r)$. The proof works the same in the other cases.

Lemma 4.3. (1) *For all* $1 \le i \le r$,
$$e^{\langle \lambda_{i+1} - \lambda_i, s\rangle} \asymp (1 - \langle \alpha_1 + \cdots + \alpha_i, \delta\rangle).$$
(2) *For all* $2 \le i \le r$, *when* $\langle \lambda_i - \lambda_{i-1}, s\rangle \to 0$,
$$\langle \lambda_i - \lambda_{i-1}, s\rangle \asymp \delta_i - \delta_{i-1}.$$

Proof. Let us prove the first claim by induction on $i \le r$. The equation $\langle \alpha_1, \nabla h(s)\rangle = \delta_1 h(s)$ may be rewritten as
$$\sum_\mu \sinh \langle \lambda_1 + \mu, s\rangle = \delta_1 (\sum_\mu \cosh \langle \lambda_1 + \mu, s\rangle + e^{\langle \lambda_2, s\rangle} + \sum_\nu e^{\langle \nu, s\rangle}),$$
where the last sum is over weights ν such that $\langle \nu, s\rangle \le \langle \lambda_2, s\rangle$. Multiplying the left and right members of the last equality by $e^{-\langle \lambda_1, s\rangle}$ gives immediately $e^{\langle \lambda_2 - \lambda_1, s\rangle} \asymp 1 - \delta_1$. Let now $i \le r$. We write
$$\langle \alpha_1 + \cdots + \alpha_i, \nabla h(s)\rangle = \langle \alpha_1 + \cdots + \alpha_i, \delta\rangle\, h(s).$$
The exponential $e^{\lambda_1 + \lambda_{i+1} - \lambda_i}$ appears in the right member of the last equality, whereas it does not in the left member. Then we conclude by the same argument as before. The last statement is proved in a similar way, by using the equations $\langle \alpha_{i+1} - \alpha_i, \nabla h(s)\rangle = (\delta_{i+1} - \delta_i) h(s)$. This concludes the proof of the lemma. □

The end of the proof of Theorem 4.1 is now completely similar to the rank 2 case, if one avoids Section 3.5. Let us just notice that the quadratic form appearing in Lemma 3.3 is equal in general to
$$q_\delta(\theta) = \sum_{i=1}^r e^{\langle \lambda_{i+1} - \lambda_i, s\rangle} \langle \lambda_{i+1} - \lambda_i, \theta\rangle^2 .$$
We leave the other details of the proof to the reader. □

5. Green's function estimate

5.1. **statement of the result.** Green's function is defined for $x, y \in \mathcal{X}$ by
$$G(x, y|z) = \sum_{n \ge |x|} p^n(x, y) z^n,$$
for all $z \in \mathbb{C}$ such that $|z| \le \frac{1}{\rho}$. We set
$$G(x, z) := G(0, x|z).$$
We will give a sharp estimate of this function when z is real positive. As usual we always use the implicit notation $x \in V_l a(0)$ relating x and λ.

Theorem 5.1. (1) *Let* $z \in (0, \tilde\rho^{-1})$. *Then*
$$G(x, z) \asymp \frac{1}{|\lambda|^{|R^+| + \frac{r-1}{2}}} e^{-\langle \lambda, s_0\rangle} F_0(\lambda),$$
where $s_0 \in \overline{\mathfrak{a}_+}$ *is uniquely determined by the conditions:* $h(s_0) = (\rho z)^{-1}$ *and* $\nabla h(s_0)$ *is proportional to* λ.

(2) *We have*

$$G(x,\tilde{\rho}^{-1}) \asymp \frac{1}{|\lambda|^{2|R^+|+r-2}} F_0(\lambda).$$

5.2. **Proof.**

5.2.1. *The case $z < \tilde{\rho}^{-1}$.* First we need some preliminary results. We set $g = \nabla \log h$, and for any $\delta \in \overline{\mathfrak{a}_+}$, we define $s = s(\delta)$ as the unique point in $\overline{\mathfrak{a}_+}$ such that $g(s) = \delta$. We have the following

Lemma 5.1. *The function g is locally invertible. Moreover, its differential dg_s at any point $s \in \overline{\mathfrak{a}_+}$ satisfies $\langle u, dg_s(u)\rangle > 0$, for all $u \in \mathfrak{a}$.*

Proof. We compute the differential of g at some point s

$$dg_s(u) = \frac{1}{h^2(s)} \sum_{\lambda,\lambda'} [\frac{\langle \lambda, u\rangle}{2}\lambda + \frac{\langle \lambda', u\rangle}{2}\lambda' - \langle \lambda, u\rangle \lambda' - \langle \lambda', u\rangle \lambda] e^{\langle \lambda+\lambda', s\rangle}.$$

Thus

$$\langle u, dg_s(u)\rangle = \frac{1}{2h^2(s)} \sum_{\lambda,\lambda'} [\langle \lambda, u\rangle - \langle \lambda', u\rangle]^2 e^{\langle \lambda+\lambda', s\rangle} > 0,$$

for all $u \in \mathfrak{a}$. In particular dg_s is invertible at each point s. We conclude by using the local inversion theorem. □

For $t > |\lambda|$ we set $\delta_t = \frac{\lambda}{t}$, and $s_t = g^{-1}(\delta_t)$. Now we define the function Ψ in $(|\lambda|, \infty)$ by

$$\Psi(t) = t[\log h(s_t) - \langle \delta_t, s_t\rangle + \log(\rho z)].$$

We have

$$\Psi'(t) = \log h(s_t) + \log(\rho z),$$

and

$$\Psi''(t) = -\left\langle \delta_t, dg_{\delta_t}^{-1}(\frac{\delta_t}{t})\right\rangle.$$

In particular, by Lemma 5.1, $\Psi''(t) < 0$ for all $t > |\lambda|$. Thus Ψ is strictly concave, and attains its maximum at a unique point t_0, which satisfies the equation $h(s_{t_0}) = (\rho z)^{-1}$. For simplify we set $s_0 := s_{t_0}$ and $\delta_0 = \delta_{t_0}$. Observe that they depend only on $\frac{x}{|x|}$. We have

Lemma 5.2. *There exist constants $c > 0$ and $C > 0$ such that for all $x \in \mathcal{X}$,*

$$c \le |s_0| \le C \qquad \textit{and} \qquad c \le |\delta_0| \le 1 - c.$$

Proof. The proof is straightforward. First $h(s_0) = (\rho z)^{-1}$. Moreover h is continuous, $h(0) < (\rho z)^{-1}$, and $h(s) \to \infty$, when $|s| \to \infty$. Eventually $s \to 0$ when $|\delta| \to 0$, and $|s| \to \infty$ when $|\delta| \to 1$. □

In the sequel it will be convenient to introduce also the function Φ defined for any $|\delta| < 1$ by

$$\Phi(\delta) = \log h(s) - \langle \delta, s\rangle + \log(\rho z).$$

We have

$$\nabla \Phi(\delta) = -s, \tag{16}$$

and for all $u \in \mathfrak{a}$,

$$d^2\Phi_\delta(u) = -\left\langle u, dg_\delta^{-1}(u)\right\rangle, \tag{17}$$

where $d^2\Phi_\delta$ is the second order differential of Φ at the point δ. Writing the Taylor development of Φ at order 2, we get

$$\Phi(\delta) = -\langle \delta, s_0 \rangle - \left\langle \delta - \delta_0, dg_{\delta_0}^{-1}(\delta - \delta_0) \right\rangle + \mathcal{O}(|\delta - \delta_0|^2).$$

Thus there exists $\epsilon > 0$, $c > 0$ and $C > 0$ such that

$$-C|\delta - \delta_0|^2 \leq \Phi(\delta) + \langle \delta, s_0 \rangle \leq -c|\delta - \delta_0|^2, \tag{18}$$

for all δ such that $|\delta - \delta_0| \leq \epsilon$. Taking smaller ϵ if necessary, we can also assume that there exists a constant $c > 0$ such that for δ in the preceding range, $c \leq |\delta| \leq 1 - c$. We can now prove

Lemma 5.3. *We have the estimate*

$$\sum_{|\delta_n - \delta_0| \leq \epsilon} p^n(0, x) z^n \asymp \frac{1}{|\lambda|^{|R^+| + \frac{r-1}{2}}} e^{-\langle \lambda, s_0 \rangle} F_0(\lambda).$$

Proof. First by Theorem 4.1, we see that for all n such that $|\delta_n - \delta_0| \leq \epsilon$,

$$p^n(0, x) z^n \asymp \frac{1}{|\lambda|^{|R^+| + \frac{r}{2}}} e^{\Psi(n)} F_0(\lambda).$$

But $\Psi(n) = n\Phi(\delta_n)$. Thus by (18)

$$e^{-\langle \lambda, s_0 \rangle - Cn|\delta - \delta_0|^2} \leq e^{\Psi(n)} \leq e^{-\langle \lambda, s_0 \rangle - cn|\delta - \delta_0|^2},$$

for all n such that $|\delta_n - \delta_0| \leq \epsilon$. Now we write

$$\sum_{|\delta_n - \delta_0| \leq \epsilon} e^{\Psi(n)} = \sum_{p=0}^{+\infty} \sum_{\frac{\epsilon}{2^{p+1}} \leq |\delta_n - \delta_0| \leq \frac{\epsilon}{2^p}} e^{\Psi(n)}.$$

Next for all $p \geq 0$,

$$\left| \left\{ n \mid \frac{\epsilon}{2^{p+1}} \leq |\delta_n - \delta_0| \leq \frac{\epsilon}{2^p} \right\} \right| \asymp \frac{\epsilon|\lambda|}{2^p}.$$

Thus

$$\sum_{|\delta_n - \delta_0| \leq \epsilon} e^{\Psi(n)} \leq \text{const} \cdot |\lambda|^{\frac{1}{2}} e^{-\langle \lambda, s_0 \rangle} \sum_{p=0}^{+\infty} \frac{|\lambda|^{\frac{1}{2}}}{2^p} e^{-c|\lambda|/2^{2p}}.$$

Moreover it is elementary to see that the last sum is bounded. Therefore we get as expected

$$\sum_{|\delta_n - \delta_0| \leq \epsilon} e^{\Psi(n)} \leq C|\lambda|^{\frac{1}{2}} e^{-\langle \lambda, s_0 \rangle}.$$

for some constant $C > 0$. By the same argument we can prove the lower estimate

$$\sum_{|\delta_n - \delta_0| \leq \epsilon} e^{\Psi(n)} \geq c|\lambda|^{\frac{1}{2}} e^{-\langle \lambda, s_0 \rangle},$$

for some constant $c > 0$. This concludes the proof of the lemma. □

The proof is now almost finished. Since Ψ is concave, there exists $c > 0$ such that

$$\Psi(n) \leq -\langle \lambda, s_0 \rangle - cn,$$

for all n such that $|\delta_n - \delta_0| \geq \epsilon$. Thus

$$\sum_{|\delta_n - \delta_0| \geq \epsilon,\ |\delta_n| \geq \epsilon} e^{\Psi(n)} \leq C|\lambda| e^{-\langle \lambda, s_0 \rangle - c|\lambda|}. \tag{19}$$

In fact in the preceding sum we must assume that $n > |x|$, because the estimate of $p^{|x|}(0,x)$ is not contained in Theorem 4.1. But by the Harnack principle (given by the heat equation), there exists a constant $C > 0$ such that for all x,

$$p^{|x|}(0,x) \leq Cp^{|x|+2}(0,x),$$

which gives also a control of the term $p^{|x|}(0,x)$. Taking now smaller ϵ if necessary, we can assume that there exists $c > 0$ such that,

$$\rho z h(s) \leq 1 - c,$$

when $|\delta| \leq \epsilon$. Then, by Theorem 4.1,

$$\sum_{|\delta_n| \leq \epsilon} e^{\Psi(n)} \leq C \sum_{n \geq \frac{|\lambda|}{\epsilon}} (1-c)^n \leq C(1-c)^{\frac{|\lambda|}{\epsilon}}.$$

Taking again smaller ϵ if necessary, we get

$$\sum_{|\delta_n| \leq \epsilon} e^{\Psi(n)} \leq Ce^{-\langle \lambda, s_0 \rangle - c|\lambda|}. \tag{20}$$

Together with Lemma 5.3, and (19), this proves Theorem 5.1 in the case $z < \tilde{\rho}^{-1}$.

5.2.2. *The case $z = \tilde{\rho}^{-1}$.* First by Lemma 5.1 dg_0 is invertible. Thus $|s| \asymp |\delta|$ near 0. It implies with (16) that there exists a constant $c > 0$ such that

$$\Phi(\delta) \leq -c|\delta|^2,$$

for all $|\delta| < 1$. Therefore for all $\epsilon > 0$, we get a constant $c' > 0$ such that

$$\sum_{|\delta_n| \geq \epsilon} p^n(0,x)\tilde{\rho}^{-n} \leq e^{-c'|x|} F_0(\lambda).$$

Moreover, by (16) and (17), $\nabla\Phi(0) = 0$ and $d^2\Phi_0$ is definite negative. Hence we get

$$\Phi(\delta) \asymp -|\delta|^2,$$

near 0. It follows that

$$\sum_{|\delta_n| \leq \epsilon} p^n(0,x)\tilde{\rho}^{-n} \asymp F_0(\lambda) \int_{\frac{|x|}{\epsilon}}^{+\infty} \frac{1}{t^m} e^{-\frac{|x|^2}{t}} dt,$$

where $m = |R^+| + \frac{r}{2}$. Next we do the change of variable $t \to \frac{|x|^2}{t}$ and we find the desired estimate. This concludes the proof of Theorem 5.1.

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J.-Ph. Anker and Br. Schapira
Université d'Orléans, Fédération Denis Poisson, Laboratoire MAPMO
B.P. 6759, 45067 Orléans cedex 2, France.
mails: jean-philippe.anker@univ-orleans.fr and bruno.schapira@univ-orleans.fr

Br. Schapira
Université Pierre et Marie Curie, Laboratoire de Probabilités et Modèles Aléatoires,
4 place Jussieu, F-75252 Paris cedex 05, France.

B. Trojan
School of Mathematics and Statistics
Carslaw Building (F07), University of Sydney NSW 2006, Australia.
mail: B.Trojan@maths.usyd.edu.au