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Abstract. — We are interested in a model of rotating fluids, describing the motion of the ocean in the equatorial zone. This model is known as the Saint-Venant, or shallow-water type system, to which a rotation term is added whose amplitude is linear with respect to the latitude; in particular it vanishes at the equator. After a physical introduction to the model, we describe the various waves involved and study in detail the resonances associated with those waves. We then exhibit the formal limit system (as the rotation becomes large), obtained as usual by filtering out the waves, and prove its wellposedness. Finally we prove three types of convergence results: a weak convergence result towards a linear, geostrophic equation, a strong convergence result of the filtered solutions towards the unique strong solution to the limit system, and finally a "hybrid" strong convergence result of the filtered solutions towards a weak solution to the limit system. In particular we obtain that there are no confined equatorial waves in the mean motion as the rotation becomes large.

Résumé. — On s'intéresse à un modèle de fluides en rotation rapide, décrivant le mouvement de l'océan dans la zone équatoriale. Ce modèle est connu sous le nom de Saint-Venant, ou système "shallow water", auquel on ajoute un terme de rotation dont l'amplitude est linéaire en la latitude ; en particulier il s'annule à l'équateur. Après une introduction physique au modèle, on décrit les différentes ondes en jeu et l'on étudie en détail les résonances associées à ces ondes. On exhibe ensuite un système limite formel (dans la limite d'une forte rotation), obtenu comme d'habitude en filtrant les ondes, et l'on démontre qu'il est bien posé. Enfin on démontre trois types de résultats de convergence : un théorème de convergence faible vers un système géostrophique linéaire, un théorème de convergence forte des solutions filtrées vers la solution unique du système limite, et enfin un résultat "hybride" de convergence forte des solutions filtrées vers une solution faible du système limite. En particulier on démontre l'absence d'ondes équatoriales confinées dans le mouvement moyen, quand la rotation augmente.

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CHAPTER 1

INTRODUCTION

The aim of this paper is to obtain a description of geophysical flows, especially oceanic flows, in the equatorial zone. For the scales considered, i.e., on domains extending over many thousands of kilometers, the forces with dominating influence are the gravity and the Coriolis force. The question is therefore to understand how they counterbalance eachother to impose the so-called geostrophic constraint on the mean motion, and to describe the oscillations which are generated around this geostrophic equilibrium.

At mid-latitudes, on "small" geographical zones, the variations of the Coriolis force due to the curvature of the Earth are usually neglected, which leads to a singular perturbation problem with constant coefficients. The corresponding asymptotics, called asymptotics of rotating fluids, have been studied by a number of authors. We refer for instance to the review by R. Temam and M. Ziane [**34**], or to the work by J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier [**4**].

In order to a get a more realistic description, which allows for instance to exhibit the specificity of the equatorial zone, one has to study more intricate models, taking into account especially the interaction between the fluid and the atmosphere (free surface), and the geometry of the Earth (variations of the local vertical component of the Earth rotation). The mathematical modelling of these various phenomena, as well as their respective importance according to the scales considered, have been studied in a rather systematic way by A. Majda [**24**], and R. Klein and A. Majda [**19**].

Here we will focus on *quasigeostrophic, oceanic* flows, meaning that we will consider horizontal length scales of order 1000km and vertical length scales of order 5km, so that the aspect ratio is very small and the shallow-water approximation is relevant (see for instance the works by D. Bresch, B. Desjardins and C.K. Lin [**3**] or by J.-F. Gerbeau and B. Perthame [**13**]). In this framework, the asymptotics of homogeneous rotating fluids have been studied by D. Bresch and B. Desjardins [**2**].

For the description of *equatorial* flows, one has to take further into account the variations of the Coriolis force, and especially the fact that it cancels at equator. The inhomogeneity of the Coriolis force has already been studied by B. Desjardins and E. Grenier [**6**] and by the authors [**10**] for an incompressible fluid with rigid lid upper boundary (see also [**7**] for a study of the wellposedness and weak asymptotics of a non viscous model). The question here is then to understand the combination of the effects due to the free surface, and of the effects due to the variations of the Coriolis force.

Note that, for the sake of simplicity, we will not discuss the effects of the interaction with the boundaries, describing neither the vertical boundary layers, known as Ekman layers (see for instance the paper by D. Gérard-Varet [**12**]), nor the lateral boundary layers, known as Munk and Stommel layers (see for instance [**6**]). We will indeed consider a purely horizontal model, assuming periodicity with respect to the longitude (omitting the stopping conditions on the continents) and and infinite domain for the latitude (using the exponential decay of the equatorial waves to neglect the boundary).

1.1. Physical phenomena observed in the equatorial zone of the earth

The rotation of the earth has a dominating influence on the way the atmosphere and the ocean respond to imposed changes. The dynamic effect is caused (see [**14**], [**28**]) by the Coriolis acceleration, which is equal to the product of the Coriolis parameter f and the horizontal velocity.

An important feature of the response of a rotating fluid to gravity is that it does not adjust to a state of rest, but rather to an equilibrium which contains more potential energy than does the rest state. The steady equilibrium solution is a geostrophic balance, i.e. a balance between the Coriolis acceleration and the pressure gradient divided by density. The equation determining this steady solution contains a length scale a, called the Rossby radius of deformation, which is equal to $c/|f|$ where c is the wave speed in the absence of rotation effects. If f tends to zero, then a tends to infinity, indicating that for length scales small compared with a, rotation effects are small, whereas for scales comparable to or larger than a, rotation effects are important. Added to that mean, geostrophic motion, are time oscillations which correspond to the so-called ageostrophic motion. The use of a constant-f approximation to describe motion on the earth is adequate to handle the adjustment process at mid-latitudes : Kelvin [**35**] stated that his wave solutions (also known as Poincaré waves) are applicable *"in any narrow lake or portion of the sea covering not more than a few degrees of the earth's surface, if for* $\frac{1}{2}f$ *we take the component of the earth's angular velocity round a vertical through the locality, that is to say*

$$\frac{1}{2}f = \Omega \sin \phi,$$

where Ω *denotes the earth's angular velocity and* ϕ *the latitude."*

The adjustment processes are somewhat special when the Coriolis acceleration vanishes : the equatorial zone is actually found to be a waveguide: as explained in [**14**], there is an equatorial Kelvin wave, and there are equatorially trapped waves, which are the equivalent of the Poincaré waves in a uniformly rotating system. There is also an important new class of waves with much slower frequencies, called planetary or quasi-geostrophic waves. These owe their existence to the variations in the undisturbed potential vorticity and thus exist at all latitudes. However, the ray paths along which they propagate bend, as do the paths of gravity waves, because of the variation of Coriolis parameter with latitude, and it is this bending that tends to confine the waves to the equatorial waveguide.

1.2. A mathematical model for the ocean in the equatorial zone

In order to explore the qualitative features of the equatorial flow, we restrict our attention here to a very simplified model of oceanography. More precisely, we consider the ocean as an incompressible viscous fluid with free surface submitted to gravitation, and further make the following classical assumptions :

$(H1)$ the density of the fluid is homogeneous,

$(H2)$ the pressure law is given by the hydrostatic approximation,

$(H3)$ the motion is essentially horizontal and does not depend on the vertical coordinate,

leading to the so-called shallow water approximation.

We therefore consider a so-called viscous Saint-Venant model, which describes vertically averaged flows in three dimensional shallow domains in terms of the horizontal mean velocity field u and the depth variation h due to the free surface. Taking into account the Coriolis force, a particular model reads as

$$\begin{aligned} &\partial_t h + \nabla\cdot (hu) = 0 \\ &\partial_t(hu) + \nabla\cdot (hu \otimes u) + f(hu)^\perp + \frac{1}{\mathrm{Fr}^2} h\nabla h - h\nabla K(h) - A(h,u) = 0 \end{aligned} \tag{1.2.1}$$

where f denotes the vertical component of the earth rotation, Fr the Froude number, and K and A are the capillarity and viscosity operators. We have written $u^\perp$ for the vector $(u_2, -u_1)$.

Note that, from a theoretical point of view, it is not clear that the use of the shallow water approximation is relevant in this context since the Coriolis force is known to generate vertical oscillations which are completely neglected in such an approach. Nevertheless, this very simplified model is commonly used by physicists [**14, 29**] and we will see that its study already gives a description of the horizontal motion corresponding to experimental observations.

Of course, in order that the curvature of the earth can be neglected, and that latitude and longitude can be considered as cartesian coordinates, we should consider only a thin strip around the equator. This means that we should study (1.2.1) on a bounded domain, and supplement it with boundary conditions. Nevertheless, as we expect the Coriolis force to confine equatorial waves, we will perform our study on $\mathbf{R} \times \mathbf{T}$ where $\mathbf{T}$ is the one-dimensional torus $\mathbf{R}_{/2\pi\mathbf{Z}}$, and check a posteriori that oscillating modes vanish far from the equator, so that it is reasonable to conjecture that they should not be disturbed by boundary conditions.

1.3. Some orders of magnitude in the equatorial zone

For motions near the equator, the approximations

$$\sin\phi \sim \phi, \quad \cos\phi \sim 1$$

may be used, giving what is called the equatorial betaplane approximation. Half of the earth's surface lies at latitudes of less than 30^o and the maximum percentage error in the above approximation in that range of latitudes is only 14%. In this approximation, f is given by

$$f = \beta x_1,$$

where x_1 is distance northward from the equator, taking values in the range

$$x_1 \in [-3000\,\mathrm{km}\,,\ 3000\,\mathrm{km}],$$

and β is a constant given by

$$\beta = \frac{2\Omega}{r} = 2.3 \times 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}.$$

A formal analysis of the linearized versions of the equations shows then that rotation effects do not allow the motion in each plane $x_1 = \mathit{const}$ to be independent because a geostrophic balance between the eastward velocity and the north-south pressure gradient is required. Equatorial waves actually decay in a distance of order a_e, the so-called equatorial radius of deformation,

$$a_e = \left(\frac{c}{2\beta}\right)^{1/2}$$

where c is the square root of gH, H being interpreted as the equivalent depth. For baroclinic ocean waves, appropriate values of c are typically in the range $0.5\mathrm{ms}^{-1}$ to $3\mathrm{ms}^{-1}$, so the order of the equatorial Rossby radius is

$$a_e \sim 100\,\mathrm{km},$$

which is effectively very small compared with the range of validity of the betaplane approximation.

1.4. The Cauchy problem for the betaplane model

Before describing the equatorial waves and the asymptotic behaviour of the ocean in the fast rotation limit, we need to give the mathematical framework for our study, and therefore to specify the dissipative operators A and K occuring in (1.2.1).

From a physical point of view, it would be relevant to model the viscous effects by the following operator

$$A(h,u) = \nu\nabla\cdot(h\nabla u),$$

meaning in particular that the viscosity cancels when h vanishes. Then, in order for the Cauchy problem to be globally well-posed, it is necessary to get some control on the cavitation. Results by Bresch and Desjardins [**2**] show that capillary or friction effects can prevent the formation of singularities in the Saint-Venant system (without Coriolis force). On the other hand, in the absence of such dissipative effects, Mellet and Vasseur [**26**] have proved the weak stability of this same system under a suitable integrability assumption on the initial velocity field. All these results are based on a new entropy inequality [**2**] which controls in particular the first derivative of $\sqrt{h}$. In particular, they cannot be easily extended to (1.2.1) since the betaplane approximation of the Coriolis force prevents from deriving such an entropy inequality.

For the sake of simplicity, as we are interested in some asymptotic regime where the depth h is just a fluctuation around a mean value H, we will thus consider the following viscosity operator

$$A(h,u) = \nu\Delta u,$$

and we will neglect the capillarity

$$K(h) = 0,$$

so that the usual theory of the isentropic Navier-Stokes equations can be applied (see for instance [**23**]).

***Theorem 1* (Existence of weak solutions).** — *Let (h^0,u^0) be some measurable nonnegative function and vector-field on $\mathbf{R}\times\mathbf{T}$ such that*

$$\mathcal{E}^0 \stackrel{def}{=} \int \left(\frac{(h^0-H)^2}{2Fr^2} + \frac{h^0}{2}|u^0|^2\right) dx < +\infty. \tag{1.4.1}$$

Then there exists a global weak solution to (1.2.1) satisfying the initial condition

$$h_{|t=0} = h^0, \quad u_{|t=0} = u^0,$$

and which furthermore satisfies for almost every $t \geq 0$ the energy estimate

$$\int \left(\frac{(h-H)^2}{2Fr^2} + \frac{h}{2}|u|^2 \right)(t,x)dx + \nu \int_0^t \int |\nabla u|^2(t',x)dxdt' \leq \mathcal{E}^0. \tag{1.4.2}$$

In this paper we are interested in describing the behaviour of the ocean in the equatorial zone. We thus expect the Froude number Fr, which is the ratio of the fluid speed to a measure of the internal wave speed, to be small. More precisely we will consider depth variations

$$h = H(1+\varepsilon\eta)$$

where ε stands for the order of magnitude of the Froude number.

As seen in the introduction, in order for gravity waves to be notably modified by rotation effects, the Rossby radius of deformation has to be comparable to the typical length scales. In order to derive the quasi-geostrophic equations with free-surface term used in oceanography, we will assume that ε is also the order of magnitude of the Rossby number.

In non-dimensional variables, the viscous Saint-Venant system (1.2.1) can therefore be rewritten (normalizing H to $H = 1$ for simplicity)

$$\begin{aligned} &\partial_t \eta + \frac{1}{\varepsilon}\nabla\cdot\Big((1+\varepsilon\eta)u\Big) = 0,\\ \partial_t\Big((1+\varepsilon\eta)u\Big) + \nabla\cdot\Big((1+\varepsilon\eta)u\otimes u\Big) + \frac{\beta x_1}{\varepsilon}(1+\varepsilon\eta)u^\perp + &\frac{1}{\varepsilon}(1+\varepsilon\eta)\nabla\eta - \nu\Delta u = 0,\\ &\eta_{|t=0} = \eta^0, \quad u_{|t=0} = u^0. \end{aligned} \tag{1.4.3}$$

In such a framework, the energy inequality (1.4.2) provides uniform bounds on any family $(\eta_\varepsilon, u_\varepsilon)_{\varepsilon>0}$ of weak solutions of (1.4.3).

In all the sequel we will denote respectively $\dot H^s$ and H^s the homogeneous and inhomogeneous Sobolev spaces of order s, defined by

$$\begin{aligned} \dot H^s(\mathbf{R}\times\mathbf{T}) = \Big\{ f \in \mathcal{S}'(\mathbf{R}\times\mathbf{T}) \Big/ \mathcal{F}f \in L^1_{loc}(\mathbf{R}\times\mathbf{T}) \\ \text{and } \|f\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})} = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}} |\xi^2+k^2|^s |\mathcal{F}f(\xi,k)|^2\, d\xi < \infty \Big\}, \end{aligned}$$

and

$$H^s(\mathbf{R}\times\mathbf{T}) = \left\{ f \in \mathcal{S}'(\mathbf{R}\times\mathbf{T}) \Big/ \|f\|^2_{H^s(\mathbf{R}\times\mathbf{T})} = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}} |1+\xi^2+k^2|^s |\mathcal{F}f(\xi,k)|^2\, d\xi < \infty \right\},$$

where $\mathcal{F}$ denotes the Fourier transform

$$\forall k\in\mathbf{Z},\ \forall \xi\in\mathbf{R} \quad \mathcal{F}f(\xi,k) = \int e^{-ikx_2} e^{-i\xi x_1} f(x_1,x_2)\, dx_2 dx_1.$$

We will also denote, for all subsets Ω of $\mathbf{R}\times\mathbf{T}$ and for all $s>0$, by $H^s_0(\Omega)$, the closure of $\mathcal{D}(\Omega)$ for the H^s norm, and by $H^{-s}(\Omega)$ its dual space.

The following result is a consequence of Theorem 1.

Corollary 1.4.1. — *Let* $(\eta^0, u^0) \in L^2(\mathbf{R}\times\mathbf{T})$ *and* $(\eta^0_\varepsilon, u^0_\varepsilon)$ *such that*

$$\begin{aligned}\frac{1}{2}\int \left(|\eta^0_\varepsilon|^2 + (1+\varepsilon\eta^0_\varepsilon)|u^0_\varepsilon|^2\right) dx \le \mathcal{E}^0,\\ (\eta^0_\varepsilon, u^0_\varepsilon) \to (\eta^0, u^0) \text{ in } L^2(\mathbf{R}\times\mathbf{T}).\end{aligned} \tag{1.4.4}$$

Then, for all $\varepsilon > 0$*, System (1.4.3) has at least one weak solution* $(\eta_\varepsilon, u_\varepsilon)$ *with initial data* $(\eta^0_\varepsilon, u^0_\varepsilon)$ *satisfying the uniform bound*

$$\frac{1}{2}\int (\eta_\varepsilon^2 + (1+\varepsilon\eta_\varepsilon)|u_\varepsilon|^2)(t,x)dx + \int_0^t\int \nu|\nabla u_\varepsilon|^2(s,x)dxds \le \mathcal{E}^0. \tag{1.4.5}$$

Furthermore u_ε *is uniformly bounded in* $L^2_{loc}(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T}))$*.*

In particular, there exist $\eta \in L^\infty(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T}))$ *and* $u \in L^\infty(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T})) \cap L^2(\mathbf{R}^+; \dot{H}^1(\mathbf{R}\times\mathbf{T}))$ *such that, up to extraction of a subsequence,*

$$(\eta_\varepsilon, u_\varepsilon) \rightharpoonup (\eta, u) \text{ in w-}L^2_{loc}(\mathbf{R}^+\times\mathbf{R}\times\mathbf{T}).$$

Proof. — Replacing h by $1+\varepsilon\eta$ in the energy inequality (1.4.2), we get (1.4.5) from which we deduce that there exist $\eta \in L^\infty(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T}))$ and $u \in L^2(\mathbf{R}^+; \dot{H}^1(\mathbf{R}\times\mathbf{T}))$ such that, up to extraction of a subsequence,

$$(\eta_\varepsilon, u_\varepsilon) \rightharpoonup (\eta, u) \text{ in w-}L^2_{loc}(\mathbf{R}^+\times\mathbf{R}\times\mathbf{T}).$$

Furthermore we have the following inequality:

$$\int_\Omega |u_\varepsilon|^2(t,x)dx \le \int_\Omega (1+\varepsilon\eta_\varepsilon)|u_\varepsilon|^2(t,x)dx + \varepsilon\left(\int_\Omega \eta_\varepsilon^2(t,x)dx\right)^{1/2}\left(\int_\Omega |u_\varepsilon|^4(t,x)dx\right)^{1/2}$$

from which we deduce that

$$u_\varepsilon \in L^2_{loc}(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T})),$$

where we have used the interpolation inequality

$$\int_\Omega |u_\varepsilon|^4(t,x)dx \le C\int_\Omega |u_\varepsilon|^2(t,x)dx \int_\Omega |\nabla u_\varepsilon|^2(t,x)dx.$$

By Fatou's lemma we get that u belongs to $L^\infty(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T}))$. That concludes the proof. □

CHAPTER 2

EQUATORIAL WAVES

The aim of this chapter is to describe precisely the various waves induced by the singular perturbation

$$L : (\eta, u) \in L^2(\mathbf{R} \times \mathbf{T}) \mapsto (\nabla\cdot u, \beta x_1 u^\perp + \nabla\eta). \tag{2.0.1}$$

In the first paragraph we study the kernel of the operator, which describes the mean flow as we will see in Chapter 4. In the second paragraph we describe all the other waves, using the Hermite functions in x_1 and the Fourier transform in x_2; this enables us to recover results which are well-known from physicists (see for instance [**14**],[**28**],[**30**]-[**32**], as well as [**7**] for a mathematical study). Finally in the last paragraph we study the possible resonances between all those waves; that result will be useful in Chapter 3 to prove regularity estimates for the limit system introduced in Paragraph 2.3 below.

2.1. The geostrophic constraint

In this section we are going to study the kernel of the singular perturbation L defined in (2.0.1).

Proposition 2.1.1. — *Define the linear operator L by (2.0.1). Then $(\eta, u) \in L^2(\mathbf{R} \times \mathbf{T})$ belongs to* Ker$L$ *if and only if (η, u) belongs to $L^2(\mathbf{R}_{x_1})$ and*

$$u_1 = 0, \quad \beta x_1 u_2 + \partial_1 \eta = 0. \tag{2.1.1}$$

Proof. — If (η, u) belongs to $L^2(\mathbf{R} \times \mathbf{T}) \cap \mathrm{Ker}L$, then we have

$$\nabla\cdot u = 0, \quad \beta x_1 u^\perp + \nabla\eta = 0,$$

in the sense of distributions. Computing the vorticity in the second identity leads to

$$\nabla^\perp \cdot (\beta x_1 u^\perp + \nabla\eta) = \beta(x_1 \nabla \cdot u + u_1) = 0,$$

from which we deduce, since $\nabla\cdot u = 0$, that $u_1 = 0$. Plugging this identity respectively in the divergence-free condition and in the second component of the vectorial condition gives

$$\partial_2 u_2 = 0, \quad \partial_2 \eta = 0,$$

meaning that η and u depend only on the x_1 variable. The last condition can then be rewritten

$$\beta x_1 u_2 + \partial_1 \eta = 0.$$

Conversely, it is easy to check that any $(\eta, u) \in L^2(\mathbf{R})$ satisfying (2.1.1) belongs to Ker$L$. □

In the following we will denote by Π_0 the orthogonal projection of $L^2(\mathbf{R}\times\mathbf{T})$ onto Ker$L$. It is given by the following formula.

Proposition 2.1.2. — *Define the linear operator L by (2.0.1). Denote by Π_0 the orthogonal projection of $L^2(\mathbf{R}\times\mathbf{T})$ onto* Ker$L$. *Then, for all $(\eta,u)\in L^2(\mathbf{R}\times\mathbf{T})$*

$$\Pi_0(\eta,u)=\left(\int(DD^T+Id)^{-1}(\eta+Du_2)dx_2,0,\int D^T(DD^T+Id)^{-1}(\eta+Du_2)dx_2\right), \tag{2.1.2}$$

where D is the differential operator defined by $D\cdot=\partial_1\left(\dfrac{\cdot}{\beta x_1}\right)$.

Proof. — By Proposition 2.1.1, for all $(\eta,u)\in L^2(\mathbf{R}\times\mathbf{T})$, $(\eta^*,u^*)\stackrel{\text{def}}{=}\Pi_0(\eta,u)$ belongs to $L^2(\mathbf{R})$ and satisfies

$$u_1^*=0,\quad \beta x_1u_2^*+\partial_1\eta^*=0.$$

Averaging with respect to the x_2-variable, one is reduced to the case when $(\eta,u)\in L^2(\mathbf{R})$.

By definition $(\eta-\eta^*,u-u^*)$ is orthogonal in L^2 to any element (ρ,v) of Ker$L$: that implies that

$$\int((\eta-\eta^*)\rho+(u_2-u_2^*)v_2)\,dx_1=\int\left((\eta-\eta^*)\rho-(u_2+\frac{1}{\beta x_1}\partial_1\eta^*)\frac{1}{\beta x_1}\partial_1\rho\right)dx_1=0.$$

An integration by parts leads then to

$$\left(-\partial_1\frac{1}{\beta^2x_1^2}\partial_1+Id\right)\eta^*=\eta+\partial_1\frac{u_2}{\beta x_1}\cdot$$

Plugging this identity in the second constraint equation gives the expected formula for u_2^*.

That proves Proposition 2.1.2. □

2.2. Description of the waves

In this section we are going to describe precisely the various waves created by L. In the first paragraph of this section (Paragraph 2.2.1) we compute the eigenvalues of L and present its eigenvectors, which constitute a Hilbertian basis of $L^2(\mathbf{R}\times\mathbf{T})$ (that is proved in Paragraph 2.2.2). That basis enables us in the last paragraph to introduce the filtering operator and formally derive the limit filtered system, in the spirit of S. Schochet [**33**] (see also [**17**]).

2.2.1. Precise description of the oscillations. — In this paragraph, we are going to explain how to obtain the various eigenmodes of L. The crucial point is that the description of these eigenmodes can be achieved using the Fourier transform with respect to x_2 and the decomposition on the Hermite functions $(\psi_n)_{n\in\mathbf{N}}$ with respect to x_1. Here the Hermite functions are conveniently rescaled so that

$$\psi_n(x_1)=e^{-\frac{\beta x_1^2}{2}}P_n(\sqrt{\beta}x_1),$$

where P_n is a polynomial of degree n, and $(\psi_n)_{n\in\mathbf{N}}$ satisfy

$$-\psi_n''+\beta^2x_1^2\psi_n=\beta(2n+1)\psi_n.$$

We recall that $(\psi_n)_{n\in\mathbf{N}}$ constitutes a Hermitian basis of $L^2(\mathbf{R})$.

Moreover we have the identities

$$\psi_n'(x_1) + \beta x_1 \psi_n(x_1) = \sqrt{2\beta n}\psi_{n-1}(x_1),$$
$$\psi_n'(x_1) - \beta x_1 \psi_n(x_1) = -\sqrt{2\beta(n+1)}\psi_{n+1}(x_1). \tag{2.2.1}$$

We have used the convention that $\psi_n = 0$ if $n < 0$.

In the following we will then denote by $\widehat{f}(n,k)$, for $n \in \mathbf{N}$ and $k \in \mathbf{Z}$, the components of any function f in the Hermite-Fourier basis $(2\pi)^{-1/2}\psi_n(x_1)e^{ikx_2}$. In other words we have

$$\forall (n,k) \in \mathbf{N} \times \mathbf{Z}, \quad \widehat{f}(n,k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}\times\mathbf{T}} \psi_n(x_1) e^{-ikx_2} f(x_1,x_2)\, dx_1 dx_2,$$

along with the inversion formula

$$\forall (x_1,x_2) \in \mathbf{R} \times \mathbf{T}, \quad f(x_1,x_2) = \frac{1}{\sqrt{2\pi}} \sum_{\substack{n\in\mathbf{N}\\ k\in\mathbf{Z}}} \psi_n(x_1) e^{ikx_2} \widehat{f}(n,k).$$

In order to investigate the spectrum of L (which is an unbounded skew-symmetric operator), we are interested in the non trivial solutions to

$$L(\eta,u) = i\tau(\eta,u). \tag{2.2.2}$$

If one looks for the L^2 solutions of (2.2.2) with u_1 non identically zero, one gets as a necessary condition that the Fourier transform of u_1 with respect to x_2 (denoted by $\mathcal{F}_2 u_1$) satisfies

$$(\mathcal{F}_2 u_1)'' + \left(\tau^2 - k^2 + \frac{\beta k}{\tau} - \beta^2 x_1^2\right) \mathcal{F}_2 u_1 = 0,$$

from which we deduce that $\mathcal{F}_2 u_1$ is proportional to some ψ_n and that

$$\tau^3 - (k^2 + \beta(2n+1))\tau + \beta k = 0, \tag{2.2.3}$$

for some $n \in \mathbf{N}$.

The following lemma is proved by elementary algebraic computations.

Lemma 2.2.1. — *For any $\beta > 0$ and any $(n,k) \in \mathbf{N}^* \times \mathbf{Z}$, the polynomial*

$$P(\tau) = \tau^3 - (k^2 + \beta(2n+1))\tau + \beta k$$

has three distinct roots in $\mathbf{R}$, denoted in the following way:

$$\tau(n,k,-1) < \tau(n,k,0) < \tau(n,k,1). \tag{2.2.4}$$

Moreover if $\tau(n,k,j) = \tau(n',k,j') \neq 0$ for some $(n,n') \in \mathbf{N}^2$ with $n \neq 0$ and $(j,j') \in \{-1,0,1\}^2$, then necessarily $n = n'$ and $j = j'$.

Finally the following asymptotics hold if n or $|k|$ goes to infinity:

$$\tau(n,k,\pm 1) \sim \pm\sqrt{k^2 + \beta(2n+1)}, \quad \text{and} \quad \tau(n,k,0) \sim \frac{\beta k}{k^2 + \beta(2n+1)}\cdot$$

Proof. — To prove that the polynomial has three distinct roots we simply analyze the function $P(\tau)$. Its derivative $P'(\tau)$ vanishes in $\pm\alpha$, where

$$\alpha = \sqrt{\frac{k^2 + \beta(2n+1)}{3}}\cdot$$

It is then enough to prove that $P(\alpha) < 0$ and $P(-\alpha) > 0$. Let us write the argument for $P(\alpha)$. We have

$$P(\alpha) = -2\alpha^3 + \beta k.$$

But for $n \neq 0$, one checks easily that $2\alpha^3 > \beta|k|$. Indeed one has

$$4\alpha^6 = \frac{4}{27}(k^2 + \beta(2n+1))^3 > k^2\beta^2$$

as soon as $n \geq 1$. So the first result of the lemma is proved.

To prove the second result, we notice that if $\tau(n,k,j) = \tau(n',k,j') = \tau \neq 0$, then $2(n-n')\tau = 0$ from which we deduce that $n = n'$, and therefore that $j = j'$ since the polynomial (2.2.3) admits three separate roots for $n \neq 0$.

Finally the asymptotics of the eigenvalues is an easy computation. The lemma is proved. □

Remark 2.2.2. — *In the case when $n = 0$, the three roots of P are*

$$\tau(0,k,-1) = -\frac{k}{2} - \frac{1}{2}\sqrt{k^2+4\beta}, \ \tau(0,k,0) = k, \ \text{and} \ \tau(0,k,1) = -\frac{k}{2} + \frac{1}{2}\sqrt{k^2+4\beta}. \tag{2.2.5}$$

It follows that in the case when $\beta = 2k^2$, the roots become k (double) and $-2k$.

Now let us study more precisely the waves generated by L.

• If $k \neq 0$ and $n \neq 0$, (2.2.3) admits three solutions according to Lemma 2.2.1, and one can check (see Paragraph 2.2.2 below) that these solutions are eigenvalues of L associated to the following unitary eigenvectors

$$\Psi_{n,k,j}(x_1,x_2) = C_{n,k,j}e^{ikx_2}\begin{pmatrix} \frac{i\tau(n,k,j)}{k^2-\tau(n,k,j)^2}\psi_n'(x_1) - \frac{ik}{k^2-\tau(n,k,j)^2}\beta x_1\psi_n(x_1) \\ \psi_n(x_1) \\ \frac{ik}{k^2-\tau(n,k,j)^2}\psi_n'(x_1) - \frac{i\tau(n,k,j)}{k^2-\tau(n,k,j)^2}\beta x_1\psi_n(x_1) \end{pmatrix}$$

which can be rewritten

$$\Psi_{n,k,j}(x_1,x_2) = C_{n,k,j}e^{ikx_2}\begin{pmatrix} \frac{-i}{\tau(n,k,j)+k}\sqrt{\frac{\beta n}{2}}\psi_{n-1}(x_1) + \frac{i}{\tau(n,k,j)-k}\sqrt{\frac{\beta(n+1)}{2}}\psi_{n+1}(x_1) \\ \psi_n(x_1) \\ \frac{i}{\tau(n,k,j)+k}\sqrt{\frac{\beta n}{2}}\psi_{n-1}(x_1) + \frac{i}{\tau(n,k,j)-k}\sqrt{\frac{\beta(n+1)}{2}}\psi_{n+1}(x_1) \end{pmatrix} \tag{2.2.6}$$

because of the identities (2.2.1). The factor $C_{n,k,j}$ ensures that $\Psi_{n,k,j}$ is of norm 1 in $L^2(\mathbf{R}\times\mathbf{T})$, its precise value is given in (2.2.12) below.

The modes corresponding to $\tau(n,k,-1)$ and $\tau(n,k,1)$ are called *Poincaré modes*, and satisfy

$$\tau(n,k,\pm 1) \sim \pm\sqrt{k^2+\beta(2n+1)} \text{ as } |k| \text{ or } n \to \infty,$$

which are the frequencies of the gravity waves.

The modes corresponding to $\tau(n,k,0)$ are called *Rossby modes*, and satisfy

$$\tau(n,k,0) \sim \frac{\beta k}{k^2+\beta(2n+1)} \text{ as } |k| \text{ or } n \to \infty,$$

meaning that the oscillation frequency is very small : the planetary waves $\Psi_{n,k,0}$ satisfy indeed the quasi-geostrophic approximation.

• If $k = 0$ and $n \neq 0$, the three distinct solutions to (2.2.3) are the two Poincaré modes

$$\tau(n, 0, \pm 1) = \pm\sqrt{\beta(2n + 1)}$$

and the *non-oscillating, or geostrophic, mode* $\tau(n, 0, 0) = 0$. The corresponding eigenvectors of L are given by (2.2.6) if $j \neq 0$ and by

$$\Psi_{n,0,0}(x_1) = \frac{1}{\sqrt{2\pi(2n+1)}} \begin{pmatrix} -\sqrt{\frac{n+1}{2}}\psi_{n-1}(x_1) - \sqrt{\frac{n}{2}}\psi_{n+1}(x_1) \\ 0 \\ \sqrt{\frac{n+1}{2}}\psi_{n-1}(x_1) - \sqrt{\frac{n}{2}}\psi_{n+1}(x_1) \end{pmatrix}. \tag{2.2.7}$$

• If $n = 0$, the three solutions to (2.2.3) are the two Poincaré and *mixed Poincaré-Rossby modes*

$$\tau(0, k, \pm 1) = -\frac{k}{2} \pm \frac{1}{2}\sqrt{k^2 + 4\beta} \tag{2.2.8}$$

with asymptotic behaviours given by

$$\begin{aligned} \tau(0, k, -\operatorname{sgn}(k)) &\sim -k \text{ as } |k| \to \infty, \\ \tau(0, k, \operatorname{sgn}(k)) &\sim \frac{\beta}{k} \text{ as } |k| \to \infty, \end{aligned}$$

and the *Kelvin mode* $\tau(0, k, 0) = k$. The corresponding eigenvectors of L are given by (2.2.6) if $j \neq 0$ and by

$$\Psi_{0,k,0}(x_1, x_2) = \frac{1}{\sqrt{4\pi}} e^{ikx_2} \begin{pmatrix} \psi_0(x_1) \\ 0 \\ \psi_0(x_1) \end{pmatrix}. \tag{2.2.9}$$

Note that in the case when the fluid studied is the atmosphere rather than the ocean, the mixed Poincaré-Rossby waves are known as *Yanai waves.*

We recall that the functions ψ_n are defined by

$$\psi_n(x_1) = e^{-\frac{\beta x_1^2}{2}} P_n(\sqrt{\beta} x_1),$$

where P_n is a polynomial of degree n. We therefore have an exponential decay far from the equator.

As mentioned in the introduction, the adjustment processes are somewhat special in the vicinity of the equator (when the Coriolis acceleration vanishes). A very important property of the equatorial zone is that it acts as a *waveguide*, i.e., disturbances are trapped in the vicinity of the equator. The waveguide effect is due entirely to the variation of Coriolis parameter with latitude.

Note that another important effect of the waveguide is the separation into a discrete set of modes $n = 0, 1, 2, ...$ as occurs in a channel.

2.2.2. Diagonalization of the singular perturbation. — In this paragraph we are going to show that the previous study does provide a Hilbertian basis of eigenvectors.

Proposition 2.2.3. — *For all $(n,k,j) \in \mathbf{N} \times \mathbf{Z} \times \{-1,0,1\}$, denote by $\tau(n,k,j)$ the three roots of (2.2.3) and by $\Psi_{n,k,j}$ the unitary vector defined in Paragraph 2.2.1.*

Then $(\Psi_{n,k,j})_{(n,k,j)\in\mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}}$ is a Hilbertian basis of $L^2(\mathbf{R}\times\mathbf{T})$ constituted of eigenvectors of L :

$$\forall (n,k,j) \in \mathbf{N} \times \mathbf{Z} \times \{-1,0,1\}, \quad L\Psi_{n,k,j} = i\tau(n,k,j)\Psi_{n,k,j}. \tag{2.2.10}$$

Furthermore we have the following estimates : for all $s \geq 0$, there exists a nonnegative constant C_s such that, for all $(n,k,j) \in \mathbf{N} \times \mathbf{Z} \times \{-1,0,1\}$,

$$\begin{aligned} \|\Psi_{n,k,j}\|_{L^\infty(\mathbf{R}\times\mathbf{T})} \leq C_0, \quad \|\Psi_{n,k,j}\|_{W^{s,\infty}(\mathbf{R}\times\mathbf{T})} \leq C_s(1+|k|^2+n)^{s/2}, \\ C_s^{-1}(1+|k|^2+n)^{s/2} \leq \|\Psi_{n,k,j}\|_{H^s(\mathbf{R}\times\mathbf{T})} \leq C_s(1+|k|^2+n)^{s/2}, \end{aligned} \tag{2.2.11}$$

where $W^{s,\infty}$ denotes the usual Sobolev space. Moreover the eigenspace associated with any non zero eigenvalue is of finite dimension.

Proof. — In order to establish the diagonalization result, the three points to be checked are the identity (2.2.10), the orthonormality of the family $(\Psi_{n,k,j})$, and the fact that it generates the whole space $L^2(\mathbf{R}\times\mathbf{T})$.

- $\Psi_{n,k,j}$ is an eigenvector of L

We start by establishing the identity (2.2.10), where $\tau(n,k,j)$ is defined by (2.2.4) and (2.2.5) and $\Psi_{n,k,j}$ is defined either by (2.2.6) (for the Poincaré and Rossby modes) or by (2.2.7) (for the non-oscillating modes), or by (2.2.9) (for the Kelvin modes).

For the Poincaré, Rossby and mixed Poincaré-Rossby modes, we start from formula (2.2.6)

$$\Psi_{n,k,j} = C_{n,k,j} e^{ikx_2} \begin{pmatrix} \dfrac{-i}{\tau(n,k,j)+k}\sqrt{\dfrac{\beta n}{2}}\psi_{n-1}(x_1) + \dfrac{i}{\tau(n,k,j)-k}\sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n+1}(x_1) \\ \psi_n(x_1) \\ \dfrac{i}{\tau(n,k,j)+k}\sqrt{\dfrac{\beta n}{2}}\psi_{n-1}(x_1) + \dfrac{i}{\tau(n,k,j)-k}\sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n+1}(x_1) \end{pmatrix}.$$

We have $L\Psi_{n,k,j} = C_{n,k,j}e^{ikx_2}V_{n,k,j}$ where $V_{n,k,j}$ denotes

$$\begin{pmatrix} \psi_n'(x_1) + ik\sqrt{\dfrac{\beta}{2}}\left(\dfrac{i\sqrt{n}}{\tau(n,k,j)+k}\psi_{n-1}(x_1) + \dfrac{i\sqrt{n+1}}{\tau(n,k,j)-k}\psi_{n+1}(x_1)\right) \\ \sqrt{\dfrac{\beta}{2}}\left(\dfrac{i\sqrt{n}}{\tau(n,k,j)+k}(\beta x_1\psi_{n-1}(x_1) - \psi_{n-1}'(x_1)) + \dfrac{i\sqrt{n+1}}{\tau(n,k,j)-k}(\beta x_1\psi_{n+1}(x_1) + \psi_{n+1}'(x_1))\right) \\ -\beta x_1\psi_n(x_1) + ik\sqrt{\dfrac{\beta}{2}}\left(\dfrac{-i\sqrt{n}}{\tau(n,k,j)+k}\psi_{n-1}(x_1) + \dfrac{i\sqrt{n+1}}{\tau(n,k,j)-k}\psi_{n+1}(x_1)\right) \end{pmatrix}$$

which can be rewritten using the identities (2.2.1)

$$\begin{pmatrix} \dfrac{\tau(n,k,j)}{\tau(n,k,j)+k}\sqrt{\dfrac{\beta n}{2}}\psi_{n-1}(x_1) - \dfrac{\tau(n,k,j)}{\tau(n,k,j)-k}\sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n+1}(x_1) \\ \dfrac{i}{\tau(n,k,j)+k}\beta n\psi_n(x_1) + \dfrac{i}{\tau(n,k,j)-k}\beta(n+1)\psi_n(x_1) \\ -\dfrac{\tau(n,k,j)}{\tau(n,k,j)+k}\sqrt{\dfrac{\beta n}{2}}\psi_{n-1}(x_1) - \dfrac{\tau(n,k,j)}{\tau(n,k,j)-k}\sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n+1}(x_1) \end{pmatrix}.$$

As $\tau(n,k,j)$ satisfies (2.2.3), we have

$$\frac{i}{\tau(n,k,j)+k}\beta n + \frac{i}{\tau(n,k,j)-k}\beta(n+1) = i\frac{(2n+1)\beta\tau(n,k,j)+\beta k}{\tau(n,k,j)^2-k^2} = i\tau(n,k,j)$$

from which we deduce that

$$L\Psi_{n,k,j} = i\tau(n,k,j)\Psi_{n,k,j} \text{ for all } (n,k,j)\in \mathbf{N}\times\mathbf{Z}\times\{-1,1\}\cup\mathbf{N}^*\times\mathbf{Z}^*\times\{0\}.$$

For the Kelvin modes we start from formula (2.2.9)

$$\Psi_{0,k,0} = \frac{1}{\sqrt{4\pi}}e^{ikx_2}\begin{pmatrix}\psi_0\\0\\\psi_0\end{pmatrix}.$$

We have

$$L\Psi_{0,k,0} = \frac{1}{\sqrt{4\pi}}e^{ikx_2}\begin{pmatrix}ik\psi_0\\ \beta x_1\psi_0+\psi_0'\\ ik\psi_0\end{pmatrix} = \frac{ik}{\sqrt{4\pi}}e^{ikx_2}\begin{pmatrix}\psi_0\\0\\\psi_0\end{pmatrix},$$

or equivalently

$$L\Psi_{0,k,0} = ik\Psi_{0,k,0} \text{ for all } k\in\mathbf{Z}.$$

For the non-oscillating modes we start from formula (2.2.7)

$$\Psi_{n,0,0} = \frac{1}{\sqrt{2\pi(2n+1)}}\begin{pmatrix}-\sqrt{\frac{(n+1)}{2}}\psi_{n-1}-\sqrt{\frac{n}{2}}\psi_{n+1}\\0\\\sqrt{\frac{n+1}{2}}\psi_{n-1}-\sqrt{\frac{n}{2}}\psi_{n+1}\end{pmatrix}.$$

An easy computation shows that

$$L\Psi_{n,0,0} = \frac{1}{\sqrt{2\pi(2n+1)}}\begin{pmatrix}0\\ \sqrt{\frac{(n+1)}{2}}(x_1\psi_{n-1}-\psi_{n-1}')-\sqrt{\frac{n}{2}}(x_1\psi_{n+1}+\psi_{n+1}')\\0\end{pmatrix}$$

which is zero by (2.2.1). Thus,

$$L\Psi_{n,0,0} = 0 \text{ for all } n\in\mathbf{N}^*.$$

• $(\Psi_{n,k,j})$ is an orthonormal family

By identity (2.2.10) and the fact that $((2\pi)^{-1/2}e^{ikx_2})_{k\in\mathbf{Z}}$ and $(\psi_n(x_1))_{n\in\mathbf{N}}$ are respectively Hilbertian basis of $L^2(\mathbf{T})$ and $L^2(\mathbf{R})$ we are going to deduce that $(\Psi_{n,k,j})$ is an orthonormal family.

In formula (2.2.6), we choose

$$C_{n,k,j} = (2\pi)^{-1/2}\left(\frac{\beta n}{(\tau(n,k,j)+k)^2}+\frac{\beta(n+1)}{(\tau(n,k,j)-k)^2}+1\right)^{-1/2} \tag{2.2.12}$$

so that

$$\|\Psi_{n,k,j}\|^2_{L^2(\mathbf{R}\times\mathbf{T})} = 1,$$

for all $(n,k,j)\in\mathbf{N}\times\mathbf{Z}\times\{-1,1\}\cup\mathbf{N}^*\times\mathbf{Z}^*\times\{0\}$.

In the same way, it is immediate to check that

$$\|\Psi_{n,0,0}\|^2_{L^2(\mathbf{R}\times\mathbf{T})} = 1,$$

for all $n \in \mathbf{N}^*$, and that

$$\|\Psi_{0,k,0}\|^2_{L^2(\mathbf{R}\times\mathbf{T})} = 1,$$

for all $k \in \mathbf{Z}$.

In order to establish the orthogonality property we proceed in two steps.

If $\tau(n,k,j) \neq \tau(n',k',j')$, as L is a skew-symmetric operator, we have

$$\begin{aligned} i\tau(n,k,j)\big(\Psi_{n,k,j}|\Psi_{n',k',j'}\big) &= -\big(L\Psi_{n,k,j}|\Psi_{n',k',j'}\big) \\ &= \big(\Psi_{n,k,j}|L\Psi_{n',k',j'}\big) \\ &= i\tau(n',k',j')\big(\Psi_{n,k,j}|\Psi_{n',k',j'}\big) \end{aligned}$$

from which we deduce that

$$\big(\Psi_{n,k,j}|\Psi_{n',k',j'}\big) = 0.$$

If $\tau(n,k,j) = \tau(n',k',j')$, we first note that

$$\text{for } k \neq k', \quad \big(\Psi_{n,k,j}|\Psi_{n',k',j'}\big) = 0$$

using the orthogonality of e^{ikx_2} and $e^{ik'x_2}$. So we are left with the case when $k = k'$. First, if $\tau(n,k,j) = \tau(n',k,j') = \tau \neq 0$ and $n \neq 0$, then Lemma 2.2.1 implies that $n = n'$ and $j = j'$. Then in the case when $\tau(0,k,j) = \tau(0,k,j') = \tau \neq 0$, with $j \neq j'$, we just have to consider the explicit definition of $\Psi_{0,k,j}$ and $\Psi_{0,k,j'}$ given in Paragraph 2.2.1 to find that

$$\big(\Psi_{0,k,j}|\Psi_{0,k,j'}\big) = 0.$$

Finally, if $\tau(n,k,j) = \tau(n',k',j') = 0$, we have $k = k' = 0$ and $j = j' = 0$ and we deduce from formula (2.2.7) that

$$\text{for } n \neq n', \quad \big(\Psi_{n,0,0}|\Psi_{n',0,0}\big) = 0.$$

We thus conclude that

$$\big(\Psi_{n,k,j}|\Psi_{n',k',j'}\big) = 0,$$

as soon as $(n,k,j) \neq (n',k',j')$.

- $(\Psi_{n,k,j})$ spans $L^2(\mathbf{R}\times\mathbf{T})$

It remains therefore to see that any vector Φ of $L^2(\mathbf{R}\times\mathbf{T})$ can be decomposed on the family $(\Psi_{n,k,j})$.

We first decompose each component on the Hermite-Fourier basis

$$\Phi(x_1,x_2) = \frac{1}{\sqrt{2\pi}} \sum_{n,k} e^{ikx_2} \begin{pmatrix} \hat\Phi_0(k,n)\psi_n(x_1) \\ \hat\Phi_1(k,n)\psi_n(x_1) \\ \hat\Phi_2(k,n)\psi_n(x_1) \end{pmatrix},$$

which can be rewritten

$$\frac{1}{2\sqrt{2\pi}}\sum_{\substack{n>0\\k}} e^{ikx_2}\begin{pmatrix}(\hat\Phi_0(k,n+1)+\hat\Phi_2(k,n+1))\psi_{n+1}(x_1)+(\hat\Phi_0(k,n-1)-\hat\Phi_2(k,n-1))\psi_{n-1}(x_1)\\ 2\hat\Phi_1(k,n)\psi_n(x_1)\\ (\hat\Phi_0(k,n+1)+\hat\Phi_2(k,n+1))\psi_{n+1}(x_1)-(\hat\Phi_0(k,n-1)-\hat\Phi_2(k,n-1))\psi_{n-1}(x_1)\end{pmatrix}$$
$$+\frac{1}{2\sqrt{2\pi}}\sum_{k} e^{ikx_2}\begin{pmatrix}(\hat\Phi_0(k,0)+\hat\Phi_2(k,0))\psi_0(x_1)+(\hat\Phi_0(k,1)+\hat\Phi_2(k,1))\psi_1(x_1)\\ 2\hat\Phi_1(k,0)\psi_0(x_1)\\ (\hat\Phi_0(k,0)+\hat\Phi_2(k,0))\psi_0(x_1)+(\hat\Phi_0(k,1)+\hat\Phi_2(k,1))\psi_1(x_1)\end{pmatrix}.$$

We then introduce for all $(n,k)\in\mathbf{N}\times\mathbf{Z}$ the matrix $M_{n,k}\in M_3(\mathbf{R})$ defined by

$$M_{n,k}=\begin{pmatrix}\dfrac{-iC_{n,k,-1}\sqrt{\beta n/2}}{\tau(n,k,-1)+k} & \dfrac{-iC_{n,k,0}\sqrt{\beta n/2}}{\tau(n,k,0)+k} & \dfrac{-iC_{n,k,1}\sqrt{\beta n/2}}{\tau(n,k,1)+k}\\ C_{n,k,-1} & C_{n,k,0} & C_{n,k,1}\\ \dfrac{iC_{n,k,-1}\sqrt{\beta(n+1)/2}}{\tau(n,k,-1)-k} & \dfrac{iC_{n,k,0}\sqrt{\beta(n+1)/2}}{\tau(n,k,0)-k} & \dfrac{iC_{n,k,1}\sqrt{\beta(n+1)/2}}{\tau(n,k,1)-k}\end{pmatrix} \tag{2.2.13}$$

if $n\neq 0$ and $k\neq 0$, by

$$M_{n,0}=\begin{pmatrix}\dfrac{-iC_{n,0,-1}\sqrt{\beta n/2}}{\tau(n,0,-1)} & -C_{n,0,0}\sqrt{\beta(n+1)/2} & \dfrac{-iC_{n,0,1}\sqrt{\beta n/2}}{\tau(n,0,1)}\\ C_{n,0,-1} & 0 & C_{n,0,1}\\ \dfrac{iC_{n,0,-1}\sqrt{\beta(n+1)/2}}{\tau(n,0,-1)} & -C_{n,0,0}\sqrt{\beta n/2} & \dfrac{iC_{n,0,1}\sqrt{\beta(n+1)/2}}{\tau(n,0,1)}\end{pmatrix} \tag{2.2.14}$$

if $n\neq 0$ and by

$$M_{0,k}=\begin{pmatrix}0 & \sqrt{1/4\pi} & 0\\ C_{0,k,-1} & 0 & C_{0,k,1}\\ \dfrac{iC_{0,k,-1}\sqrt{\beta/2}}{\tau(0,k,-1)-k} & 0 & \dfrac{iC_{0,k,1}\sqrt{\beta/2}}{\tau(0,k,1)-k}\end{pmatrix}. \tag{2.2.15}$$

As the eigenvectors $\Psi_{n,k,-1}$, $\Psi_{n,k,0}$ and $\Psi_{n,k,1}$ are orthogonal in $L^2(\mathbf{R}\times\mathbf{T})$, these matrices are necessarily invertible.

We conclude by checking that one can write

$$\Phi=\sum_{n,k,j}\varphi_{n,k,j}\Psi_{n,k,j}$$

where $\varphi_{n,k,j}$ is defined by

$$\begin{pmatrix}\varphi_{n,k,-1}\\ \varphi_{n,k,0}\\ \varphi_{n,k,1}\end{pmatrix}=\frac{1}{\sqrt{2\pi}}M_{n,k}^{-1}\begin{pmatrix}\frac{1}{2}(\hat\Phi_0(k,n-1)-\hat\Phi_2(k,n-1))\\ \hat\Phi_1(k,n)\\ \frac{1}{2}(\hat\Phi_0(k,n+1)+\hat\Phi_2(k,n+1))\end{pmatrix}$$

for $n\neq 0$, and by

$$\begin{pmatrix}\varphi_{0,k,-1}\\ \varphi_{0,k,0}\\ \varphi_{0,k,1}\end{pmatrix}=\frac{1}{\sqrt{2\pi}}M_{0,k}^{-1}\begin{pmatrix}\frac{1}{2}(\hat\Phi_0(k,0)+\hat\Phi_2(k,0))\\ \hat\Phi_1(k,0)\\ \frac{1}{2}(\hat\Phi_0(k,1)+\hat\Phi_2(k,1))\end{pmatrix}.$$

• The regularity estimates are obtained using the explicit formulas (2.2.6), (2.2.7) and (2.2.9), as well as the following bounds on the elementary Fourier and Hermite functions :

$$\|e^{ikx_2}\|_{\dot H^s(\mathbf{T})} = |k|^s \|e^{ikx_2}\|_{L^2(\mathbf{T})}, \quad \|e^{ikx_2}\|_{\dot W^{s,\infty}(\mathbf{T})} = |k|^s \|e^{ikx_2}\|_{L^\infty(\mathbf{T})}$$

and

$$\|\psi_n\|_{H^s(\mathbf{R})} \sim (1+n)^{s/2} \sup_n \|\psi_n\|_{L^2(\mathbf{R})}, \quad \|\psi_n\|_{W^{s,\infty}(\mathbf{R})} \leq C_s (1+n)^{s/2} \sup_n \|\psi_n\|_{L^\infty(\mathbf{R})}$$

coming from identities (2.2.1) by a simple recurrence. The crucial point is therefore to have a uniform L^∞-bound on the Hermite functions, which is stated for instance in [**20**]:

$$\forall n \in \mathbf{N}, \quad \|\psi_n\|_{L^\infty(\mathbf{R})} \leq C_\infty \text{ with } C_\infty \sim 1.086435. \tag{2.2.16}$$

Finally let us prove that the eigenspace associated with a nonzero eigenvalue is of finite dimension. Suppose by contradiction that there is $\lambda \neq 0$ and a sequence $(n_M, k_M, j_M)_{M\in\mathbf{N}}$ in $\mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}$ such that

$$\tau(n_M, k_M, j_M) = \lambda \quad \text{and} \quad n_M + |k_M| \to \infty, \text{ as } M \to \infty.$$

By Lemma 2.2.1, as n or $|k|$ goes to infinity, the eigenvalue $\tau(n,k,j)$ goes to zero or to $\pm\infty$, which contradicts the assumption that $\tau(n_M, k_M, j_M) = \lambda$.

This concludes the proof of Proposition 2.2.3. □

As the behaviour of the eigenmodes are expected to depend strongly of their type, i.e. of the class of the corresponding eigenvalue, we split $L^2(\mathbf{R}\times\mathbf{T})$ into five supplementary subsets, namely the Poincaré modes, the Rossby modes, the mixed Poincaré-Rossby modes, the Kelvin modes and the non-oscillating modes.

Definition 2.2.4. — *With the above notation, let us define*

$$\begin{aligned}
P &= Vect\Big\{\Psi_{n,k,j} \,/\, (n,k,j) \in \mathbf{N}^*\times\mathbf{Z}\times\{-1,1\} \cup \{0\}\times\{(k,-sign(k))_{k\in\mathbf{Z}^*}\} \cup \{0\}\times\{0\}\times\{-1,1\}\Big\},\\
R &= Vect\{\Psi_{n,k,0} \,/\, (n,k) \in \mathbf{N}^*\times\mathbf{Z}^*\},\\
M &= Vect\{\Psi_{0,k,j} \,/\, k\in\mathbf{Z}^*, j = sign(k)\},\\
K &= Vect\{\Psi_{0,k,0} \,/\, k \in \mathbf{Z}^*\},
\end{aligned}$$

so that $L^2(\mathbf{R}\times\mathbf{T}) = P\oplus R\oplus M\oplus K\oplus \mathrm{Ker}L$. Then we denote by Π_P (resp. Π_R, Π_M, Π_K and Π_0) the L^2 orthogonal projection on P (resp. on R, M, K and $\mathrm{Ker}L$).

Moreover we define $\mathfrak{S}$ the set of all eigenvalues of L, as well as the following subsets of $\mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}$:

$$\begin{aligned}
\mathfrak{S}_P &= \Big\{\tau(n,k,j) \Big/ (n,k,j) \in \mathbf{N}^*\times\mathbf{Z}\times\{-1,1\}\Big\} \cup \Big\{\tau(0,k,-sign(k)) \Big/ k\in\mathbf{Z}^*\Big\} \cup \{\pm\sqrt{\beta}\},\\
\mathfrak{S}_R &= \Big\{\tau(n,k,0) \Big/ (n,k,j) \in \mathbf{N}^*\times\mathbf{Z}^*\Big\}, \quad \textit{and} \quad \mathfrak{S}_K = \mathbf{Z}^*.
\end{aligned}$$

Finally it can be useful for the rest of the study to sum up the previous notation in the following picture.

wave	n	k	j	definition of $\Psi_{n,k,j}$	definition of $\tau(n,k,j)$
Poincaré	$\mathbf{N}^*$	$\mathbf{Z}$	{-1,1}	(2.2.6) page 10	$\tau(n,k,\pm 1) \sim \pm\sqrt{k^2+\beta(2n+1)}$
	$\{0\}$	$\mathbf{Z}^*$	$-\mathrm{sign}(k)$	(2.2.6) page 11	$\tau(0,k,-\mathrm{sign}(k)) \sim -k$
	$\{0\}$	$\{0\}$	$\{-1,1\}$	(2.2.8) page 11	$\tau(0,0,\pm 1) = \pm\sqrt{\beta}$
Mixed	$\{0\}$	$\mathbf{Z}^*$	$\mathrm{sign}(k)$	(2.2.6) page 11	$\tau(0,k,\mathrm{sign}(k)) \sim \dfrac{\beta}{k}$
Kelvin	$\{0\}$	$\mathbf{Z}^*$	$\{0\}$	(2.2.9) page 11	$\tau(0,k,0) = k$
Rossby	$\mathbf{N}^*$	$\mathbf{Z}^*$	$\{0\}$	(2.2.6) page 10	$\tau(n,k,0) \sim \dfrac{\beta k}{k^2+\beta(2n+1)}$
non oscillating	$\mathbf{N}$	$\{0\}$	$\{0\}$	(2.2.7) page 11	$\tau(n,0,0) = 0$

TABLE 1. Description of the waves

2.2.3. Orthogonality properties of the eigenvectors. — In this section we are going to give some additional properties on the $\Psi_{n,k,j}$ defined above, which will be useful in the next chapters. We will write $\Pi_{n,k,j}$ for the projection on the eigenmode $\Psi_{n,k,j}$ of L, and Π_λ for the projection on the eigenspace associated with the eigenvalue $i\lambda$ of L. The main result is the following, which states an orthogonality property for the ageostrophic modes (meaning the eigenvectors in $(KerL)^\perp$). Note that there is no analogue of that result for geostrophic modes.

Proposition 2.2.5. — *Let $s \geq 0$ be a given real number. There is a constant $C > 0$ such that for any non zero eigenvalue $i\lambda$ of L and for any three component vector field Φ in* $(\mathrm{Ker}L)^\perp$*, we have*

$$C_s^{-1} \sum_{\tau(n,k,j)=\lambda} \|\Pi_{n,k,j}\Phi\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})} \leq \|\Pi_\lambda\Phi\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})} \leq C_s \sum_{\tau(n,k,j)=\lambda} \|\Pi_{n,k,j}\Phi\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})}. \tag{2.2.17}$$

Proof. — Let Φ in $(\mathrm{Ker}L)^\perp$ be given and let s be any integer (the result for all $s \geq 0$ will follow by interpolation). We have

$$\begin{aligned}\|\partial^s(\Pi_\lambda\Phi)\|^2_{L^2(\mathbf{R}\times\mathbf{T})} = &\sum_{\tau(n,k,j)=\lambda} \|\partial^s(\Pi_{n,k,j}\Phi)\|^2_{L^2(\mathbf{R}\times\mathbf{T})}\\ &+ \sum_{\substack{\tau(n,k,j)=\tau(n^*,k^*,j^*)=\lambda,\\ (n,k,j)\neq(n^*,k^*,j^*)}} \left(\partial^s(\Pi_{n,k,j}\Phi)|\partial^s(\Pi_{n^*,k^*,j^*}\Phi)\right)_{L^2(\mathbf{R}\times\mathbf{T})}.\end{aligned}$$

Of course,

$$\left(\partial^s\Psi_{n,k,j}|\partial^s\Psi_{n^*,k^*,j^*}\right)_{L^2(\mathbf{R}\times\mathbf{T})} = 0 \text{ if } k \neq k^*. \tag{2.2.18}$$

Moreover we know by Proposition 2.2.3, page 12 that if $\tau(n,k,j) = \tau(n^*,k,j^*) = \lambda \neq 0$ and $n \neq 0$, then necessarily $n = n^*$ and $j = j^*$. Therefore one has in fact

$$\begin{aligned}&\sum_{\substack{\tau(n,k,j)=\tau(n^*,k^*,j^*)=\lambda,\\ (n,k,j)\neq(n^*,k^*,j^*)}} \left(\partial^s(\Pi_{n,k,j}\Phi)|\partial^s(\Pi_{n^*,k^*,j^*}\Phi)\right)_{L^2(\mathbf{R}\times\mathbf{T})}\\ &\qquad = \sum_{\substack{\tau(0,k,j)=\tau(0,k,j^*)=\lambda,\\ j\neq j^*}} \left(\partial^s(\Pi_{0,k,j}\Phi)|\partial^s(\Pi_{0,k,j^*}\Phi)\right)_{L^2(\mathbf{R}\times\mathbf{T})}.\end{aligned}$$

But according to Remark 2.2.2 page 10, such a situation occurs only if $2k^2 = \beta$, in which case $\tau(0,k,j)$ is equal to k. So there is at most one possible value for k ($k = \lambda$) which occurs only in the case when $\lambda = \pm\sqrt{\beta/2}$. In this last case, we have obviously

$$\begin{aligned}\|\Pi_\lambda \Phi\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})} &\sim \|\Pi_\lambda \Phi\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \\ &= \sum_{\tau(n,k,j)=\lambda} \|\Pi_{n,k,j}\Phi\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \\ &\sim \sum_{\tau(n,k,j)=\lambda} \|\Pi_{n,k,j}\Phi\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})}.\end{aligned}$$

The result follows. □

Remark 2.2.6. — *Note that the same argument allows actually to prove similar estimates for the components separately :*

$$C_s^{-1} \sum_{\tau(n,k,j)=\lambda} \|(\Pi_{n,k,j}\Phi)'\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})} \leq \|(\Pi_\lambda\Phi)'\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})} \leq C_s \sum_{\tau(n,k,j)=\lambda} \|(\Pi_{n,k,j}\Phi)'\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})},$$
$$C_s^{-1} \sum_{\tau(n,k,j)=\lambda} \|(\Pi_{n,k,j}\Phi)_0\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})} \leq \|(\Pi_\lambda\Phi)_0\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})} \leq C_s \sum_{\tau(n,k,j)=\lambda} \|(\Pi_{n,k,j}\Phi)_0\|^2_{\dot H^s(\mathbf{R}\times\mathbf{T})},$$

denoting by Φ_0 the first coordinate and by Φ' the two other coordinates of Φ.

2.3. The filtering operator and the formal limit system

In the previous paragraph we have presented a Hilbertian basis of $L^2(\mathbf{R}\times\mathbf{T})$ consisting in eigenvectors of the singular penalization L. We are then able to define, in the spirit of S. Schochet [**33**], the "filtering operator" associated with the system.

Let $\mathcal{L}$ be the semi-group generated by L : we write $\mathcal{L}(t) = \exp(-tL)$. Then, for any three component vector field $\Phi \in L^2(\mathbf{R}\times\mathbf{T})$, we have

$$\mathcal{L}(t)\Phi = \sum_{i\lambda\in\mathfrak{S}} e^{-it\lambda}\Pi_\lambda\Phi, \tag{2.3.1}$$

where Π_λ denotes the L^2 orthogonal projection on the eigenspace of L corresponding to the eigenvalue $i\lambda$, and where $\mathfrak{S}$ denotes the set of all the eigenvalues of L.

Now let us consider $(\eta_\varepsilon, u_\varepsilon)$ a weak solution to (1.4.3), which is formally equivalent to

$$\begin{aligned}\partial_t\eta_\varepsilon + \frac{1}{\varepsilon}\nabla\cdot\Big((1+\varepsilon\eta_\varepsilon)u_\varepsilon\Big) = 0,\\ \partial_t u_\varepsilon + u_\varepsilon\cdot\nabla u_\varepsilon + \frac{\beta x_1}{\varepsilon}u_\varepsilon^\perp + \frac{1}{\varepsilon}\nabla\eta_\varepsilon - \frac{\nu}{1+\varepsilon\eta_\varepsilon}\Delta u_\varepsilon = 0,\\ \eta_{\varepsilon|t=0} = \eta_\varepsilon^0, \quad u_{\varepsilon|t=0} = u_\varepsilon^0,\end{aligned}$$

and let us define

$$\Phi_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon). \tag{2.3.2}$$

Conjugating formally equation (2.3.2) by the semi-group leads to

$$\partial_t\Phi_\varepsilon + \mathcal{L}\left(-\frac{t}{\varepsilon}\right)Q\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_\varepsilon, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_\varepsilon\right) - \nu\mathcal{L}\left(-\frac{t}{\varepsilon}\right)\Delta'\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_\varepsilon = R_\varepsilon, \tag{2.3.3}$$

where Δ' and Q are the linear and symmetric bilinear operator defined by

$$\Delta'\Phi = (0, \Delta\Phi') \text{ and } Q(\Phi,\Phi) = (\nabla\cdot(\Phi_0\Phi'), (\Phi'\cdot\nabla)\Phi') \tag{2.3.4}$$

and

$$R_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(0, -\nu\frac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon}\Delta u_\varepsilon).$$

We therefore expect to get a bound on the time derivative of Φ_ε in some space of distributions. A formal passage to the limit in (2.3.3) as ε goes to zero (based on formula (2.3.1) and on a nonstationary phase argument) leads then to

$$\partial_t \Phi + Q_L(\Phi, \Phi) - \nu\Delta'_L\Phi = 0, \tag{2.3.5}$$

where Δ'_L and Q_L denote the linear and symmetric bilinear operator defined by

$$\Delta'_L\Phi = \sum_{i\lambda\in\mathfrak{S}} \Pi_\lambda \Delta' \Pi_\lambda \Phi \text{ and } Q_L(\Phi,\Phi) = \sum_{\substack{i\lambda, i\mu, i\tilde{\mu}\in\mathfrak{S}\\ \lambda=\mu+\tilde{\mu}}} \Pi_\lambda Q(\Pi_\mu\Phi, \Pi_{\tilde{\mu}}\Phi). \tag{2.3.6}$$

The study of (2.3.5) is the object of Chapter 3. The proof that (2.3.5) is indeed the limit system to (2.3.3) is the object of Chapter 4.

In the next section we study the resonances associated with the operator L: more precisely we describe in what cases the equality

$$\tau(n,k,j) + \tau(n^*,k^*,j^*) = \tau(m,k+k^*,\ell)$$

can hold. That will be very important in the rest of the study, to understand the structure of the nonlinear terms in (2.3.5).

2.4. Interactions between equatorial waves

In this section we will study the nonlinear term in (2.3.5). We will first study the resonances of L, and then prove that the projection of (2.3.5) onto the kernel of L is a linear equation.

2.4.1. Study of the resonances.

— Let us prove the following result.

Proposition 2.4.1. — *Except for a countable number of β and with the notation of Section 2.2.1, the following condition of non resonance holds for all $n, n^*, m \in \mathbf{N}$, all $k, k^* \in \mathbf{Z}$ and all $j, j^*, \ell \in \{-1,0,1\}$:*

$$\tau(n,k,j) + \tau(n^*,k^*,j^*) = \tau(m,k+k^*,\ell)$$

implies

$$\textit{either } \tau(n,k,j) = 0 \textit{ or } \tau(n^*,k^*,j^*) = 0 \textit{ or } \tau(m,k+k^*,\ell) = 0,$$
$$\textit{or } \tau(n,k,j), \tau(n^*,k^*,j^*), \tau(m,k+k^*,\ell) \in \mathfrak{S}_K,$$

meaning that, among the ageostrophic modes, only three Kelvin waves may interact.

Proof. — Let us start by noticing that by definition of Kelvin waves, Kelvin resonances necessarily take place simply because they correspond to convolution in Fourier space.

Before starting with technical results, let us describe the main ideas of the proof. The crucial argument is that the eigenvalues of the penalization operator L are defined as the roots of a countable number of polynomials whose coefficients depend (linearly) on the ratio β. In particular, for fixed $n, n^*, m \in \mathbf{N}$ and $k, k^* \in \mathbf{Z}$, the occurence of a resonant triad

$$\tau(n,k,j) + \tau(n^*,k^*,j^*) = \tau(m,k+k^*,\ell)$$

is controlled by the cancellation of some polynomial $P_{n,n^*,m,k,k^*}(\beta)$. Therefore, either this polynomial has a finite number of zeros, or it is identically zero. The difficulty here is that we are not able to eliminate the second possibility using only the asymptotics $\beta \to \infty$. We therefore also study the asymptotics $\beta \to 0$, and in the case when $n = 0$ or $n^* = 0$, we have to refine the previous argument introducing an auxiliary polynomial.

• *Definition of the polynomial $P_{n,n^*,m,k,k^*}(\beta)$*

For fixed $n, n^*, m \in \mathbf{N}$ and $k, k^* \in \mathbf{Z}$, it is natural to consider the following quantity

$$P_{n,n^*,m,k,k^*}(\beta) = \prod_{j,j^*,\ell \in \{-1,0,1\}} \left(\tau(n,k,j) + \tau(n^*,k^*,j^*) - \tau(m,k+k^*,\ell)\right).$$

Considerations of symmetry show that this quantity can be rewritten as a polynomial of the symmetric functions of $(\tau(n,k,j))_{j\in\{-1,0,1\}}$, the symmetric functions of $(\tau(n^*,k^*,j^*))_{j^*\in\{-1,0,1\}}$ and the symmetric functions of $(\tau(m,k+k^*,\ell))_{\ell\in\{-1,0,1\}}$.

Therefore, as the eigenvalues $(\tau(n,k,j))_{j\in\{-1,0,1\}}$ of the linear penalization L are defined as the three roots of a polynomial (2.2.3) whose coefficients depend (linearly) on β

$$\tau^3 - (k^2 + \beta(2n+1))\tau + \beta k = 0,$$

the symmetric functions of $(\tau(n,k,j))_{j\in\{-1,0,1\}}$ satisfy

$$\begin{aligned} \sum_{j\in\{-1,0,1\}} \tau(n,k,j) &= 0, \\ \sum_{j\in\{-1,0,1\}} \prod_{j'\neq j} \tau(n,k,j') &= -(k^2 + (2n+1)\beta), \\ \prod_{j\in\{-1,0,1\}} \tau(n,k,j) &= -\beta k, \end{aligned} \tag{2.4.7}$$

from which we deduce that $P_{n,n^*,m,k,k^*}(\beta)$ is a polynomial (of degree at most 27) with respect to β.

In particular, for fixed $n, n^*, m \in \mathbf{N}$ and $k, k^* \in \mathbf{Z}$, either $P_{n,n^*,m,k,k^*}(\beta)$ is identically zero or it has a finite number of roots. In other words, that means that

(a) either, for all $\beta \in \mathbf{R}^*$, there is a resonance of the type

$$\tau(n,k,j) + \tau(n^*,k^*,j^*) = \tau(m,k+k^*,\ell)$$

for some $j, j^*, \ell \in \{-1,0,1\}$,

(b) or, except for a finite number of β, such resonances do not occur.

• *Asymptotic behaviour of $P_{n,n^*,m,k,k^*}(\beta)$ as $\beta \to \infty$*

In order to discard one of these alternatives, we are interested in the asymptotic behaviour of the polynomial $P_{n,n^*,m,k,k^*}(\beta)$ as $\beta \to \infty$.

We start by describing the asymptotic behaviour of each root $(\tau(n,k,j))_{j\in\{-1,0,1\}}$ as $\beta \to \infty$.

Lemma 2.4.2. — *With the notation of Paragraph 2.2.1, for all $k \in \mathbf{Z}$ and all $n \in \mathbf{N}$, the following expansions hold as $\beta \to \infty$:*

$$\begin{aligned} \tau(n,k,1) &= \sqrt{(2n+1)\beta} - \frac{k}{2(2n+1)} + o(1), \\ \tau(n,k,-1) &= -\sqrt{(2n+1)\beta} - \frac{k}{2(2n+1)} + o(1), \\ \tau(n,k,0) &= \frac{k}{2n+1} - \frac{4n(n+1)k^3}{(2n+1)^4\beta} + o\left(\frac{1}{\beta}\right). \end{aligned} \tag{2.4.8}$$

Proof. — We start with the most general case, namely the case when $k \neq 0$. We proceed by successive approximations. As the product of the roots $-\beta k$ tends to infinity as $\beta \to \infty$, there is at least one root which tends to infinity. Therefore, we get at leading order

$$\tau^3 - \beta(2n+1)\tau = 0,$$

which implies that the Poincaré and mixed Poincaré-Rossby modes are approximately given by

$$\tau(n,k,\pm 1) \sim \pm\sqrt{(2n+1)\beta}.$$

Plugging this Ansatz in the formula

$$\tau^2 = (2n+1)\beta + k^2 - \frac{\beta k}{\tau} = (2n+1)\beta\left(1 - \frac{k}{(2n+1)\tau} + \frac{k^2}{\beta(2n+1)}\right)$$

provides the next order approximation of the Poincaré modes, namely

$$\tau(n,k,j) \sim j\sqrt{(2n+1)\beta} - \frac{k}{2(2n+1)}.$$

Then, as the sum of the roots is zero (see (2.4.7)), we deduce that the third mode, i.e. the Kelvin or Rossby mode, satisfies

$$\tau(k,n,0) = \frac{k}{2n+1} + o(1).$$

Plugging this Ansatz in the formula

$$\tau = \frac{\beta k + \tau^3}{(2n+1)\beta + k^2}$$

leads then to

$$\tau(n,k,0) = \frac{k}{2n+1}\left(1 + \frac{1}{\beta k}\frac{k^3}{(2n+1)^3} - \frac{k^2}{(2n+1)\beta}\right) + o\left(\frac{1}{\beta}\right).$$

The other case (when $k = 0$) is dealt with in a very simple way. The Poincaré modes are exactly

$$\tau(n,0,\pm 1) = \pm\sqrt{(2n+1)\beta},$$

whereas the third mode is zero

$$\tau(n,0,0) = 0,$$

and thus they satisfy the general identities (2.4.8).

The result is proved. □

Equipped with this technical lemma, we are now able to characterize the asymptotic behaviour of most of the factors

$$\tau(n,k,j)+\tau(n^*,k^*,j^*)-\tau(m,k+k^*,\ell)$$

in $P_{n,n^*,m,k,k^*}(\beta)$ as $\beta\to\infty$.

Lemma 2.4.3. — *With the notations of Paragraph 2.2.1, any triad of non zero modes*

$$\big(\tau(n,k,j),\tau(n^*,k^*,j^*),\tau(m,k+k^*,\ell)\big)$$

with $k,k^*\in\mathbf{Z}$ and $n,n^*,m\in\mathbf{N}$, which is not constituted of three Kelvin or three Rossby modes, is asymptotically non resonant as $\beta\to\infty$.

More precisely the following expansions hold as $\beta\to\infty$:

(i) for three Poincaré or mixed Poincaré-Rossby modes ($j\neq 0$ and $j^*\neq 0$ and $\ell\neq 0$)

$$\tau(n,k,j)+\tau(n^*,k^*,j^*)-\tau(m,k+k^*,\ell)\sim\sqrt{\beta}\left(j\sqrt{2n+1}+j^*\sqrt{2n^*+1}-\ell\sqrt{2m+1}\right)\;;$$

(ii) for one Poincaré or mixed Poincaré-Rossby mode and two Rossby or Kelvin or zero modes

$$\tau(n,k,j)+\tau(n^*,k^*,j^*)-\tau(m,k+k^*,\ell)\sim\sqrt{\beta}\left(j\sqrt{2n+1}+j^*\sqrt{2n^*+1}-\ell\sqrt{2m+1}\right)\;;$$

(iii) for two Poincaré or mixed Poincaré-Rossby modes and one Rossby or Kelvin mode

$$\exists C\equiv C(n,n^*,m,k,k^*)>0,\quad \big|\tau(n,k,j)+\tau(n^*,k^*,j^*)-\tau(m,k+k^*,\ell)\big|\geq C\,;$$

(iv) for two Kelvin modes and one Rossby mode

$$\exists C\equiv C(n,n^*,m,k,k^*)>0,\quad \big|\tau(n,k,j)+\tau(n^*,k^*,j^*)-\tau(m,k+k^*,\ell)\big|\geq \frac{C}{\beta}\,;$$

(v) for two Rossby modes and one Kelvin mode

$$\exists C\equiv C(n,n^*,m,k,k^*)>0,\quad \big|\tau(n,k,j)+\tau(n^*,k^*,j^*)-\tau(m,k+k^*,\ell)\big|\geq \frac{C}{\beta}\cdot$$

Proof. — The proof of these results is based on Lemma 2.4.2.

(i) In the case of three Poincaré or mixed Poincaré-Rossby modes, Lemma 2.4.2 provides

$$\tau(n,k,j)+\tau(n^*,k^*,j^*)-\tau(m,k+k^*,\ell)=\sqrt{\beta}\left(j\sqrt{2n+1}+j^*\sqrt{2n^*+1}-\ell\sqrt{2m+1}\right)+o(\sqrt{\beta}),$$

and it is easy to check, using considerations of parity, that the constant

$$\left(j\sqrt{2n+1}+j^*\sqrt{2n^*+1}-\ell\sqrt{2m+1}\right)$$

cannot be zero.

(ii) In the case of one Poincaré or mixed Poincaré-Rossby mode, we have one term which is exactly of order $\sqrt{\beta}$ whereas the others are negligible compared with $\sqrt{\beta}$, thus the sum is equivalent to the Poincaré mode, and the same formula holds

$$\tau(n,k,j)+\tau(n^*,k^*,j^*)-\tau(m,k+k^*,\ell)=\sqrt{\beta}\left(j\sqrt{2n+1}+j^*\sqrt{2n^*+1}-\ell\sqrt{2m+1}\right)+o(\sqrt{\beta}).$$

(iii) The third case is a bit more difficult to deal with, since the leading order terms can cancel each other out. Without loss of generality, we can assume that $\ell=0$ and $j,j^*\neq 0$ (the other cases being obtained by exchanging j,j^* and $-\ell$).

If $j = j^*$, or if $j + j^* = 0$ and $n \neq n^*$, the same arguments as previously show that the same formula holds

$$\tau(n, k, j) + \tau(n^*, k^*, j^*) - \tau(m, k + k^*, \ell) \sim \sqrt{\beta}\left(j\sqrt{2n+1} + j^*\sqrt{2n^*+1} - \ell\sqrt{2m+1}\right),$$

since the factor of $\sqrt{\beta}$ is not zero.

If $j + j^* = 0$ and $n = n^*$, the factor of $\sqrt{\beta}$ cancels and we have to determine the next term in the asymptotic expansion :

$$\tau(n, k, j) + \tau(n^*, k^*, j^*) - \tau(m, k + k^*, \ell) = -\frac{k + k^*}{2(2n+1)} - \frac{k + k^*}{2m+1} + o(1).$$

Considerations of parity show therefore that the limit cannot be zero if $k + k^* \neq 0$, or equivalently if $\tau(m, k + k^*, 0) \neq 0$.

(iv) In the case of one Rossby and two Kelvin modes, we are not able in general to prove that the leading order term, i.e. the limit as $\beta \to \infty$ of $\tau(n, k, j) + \tau(n^*, k^*, j^*) - \tau(m, k + k^*, \ell)$ is not zero. But we can look directly at the second term of the expansion, i.e. the factor of β^{-1} :

$$\omega_1 = -\frac{4k^3 n(n+1)}{(2n+1)^4} - \frac{4(k^*)^3 n^*(n^*+1)}{(2n^*+1)^4} + \frac{4(k+k^*)^3 m(m+1)}{(2m+1)^4}.$$

Considering one Rossby and two Kelvin modes means that k, k^* and $k + k^*$ are not zero, and that exactly two indices among n, n^* and m are zero. Thus $\omega_1 \neq 0$ and

$$\left|\tau(n, k, j) + \tau(n^*, k^*, j^*) - \tau(m, k + k^*, \ell)\right| \geq \frac{|\omega_1|}{2\beta}$$

for β large enough.

(v) The last situation is the most difficult to deal with, since the only thing we will be able to prove is that the two first terms of the asymptotic expansion of $\tau(n, k, j) + \tau(n^*, k^*, j^*) - \tau(m, k + k^*, \ell)$ with respect to β cannot cancel together. By Lemma 2.4.2, we deduce that for one Kelvin and two Rossby modes

$$\tau(n, k, j) + \tau(n^*, k^*, j^*) - \tau(m, k + k^*, \ell) = \omega_0 + \frac{\omega_1}{\beta} + o\left(\frac{1}{\beta}\right)$$

with

$$\omega_0 = \frac{k}{2n+1} + \frac{k^*}{2n^*+1} - \frac{k + k^*}{2m+1},$$

and

$$\omega_1 = -\frac{4k^3 n(n+1)}{(2n+1)^4} - \frac{4(k^*)^3 n^*(n^*+1)}{(2n^*+1)^4} + \frac{4(k+k^*)^3 m(m+1)}{(2m+1)^4}.$$

Recall moreover that k, k^* and $k + k^*$ are not zero, and that exactly one index among n, n^* and m is zero. Without loss of generality, we can assume that $m = 0$ and $n, n^* \neq 0$ (the other cases being obtained by exchanging n, n^* and m).

Then, if $\omega_0 = \omega_1 = 0$,

$$\frac{kn}{2n+1} + \frac{k^* n^*}{2n^*+1} = 0,$$
$$\frac{k^3 n(n+1)}{(2n+1)^4} + \frac{(k^*)^3 n^*(n^*+1)}{(2n^*+1)^4} = 0,$$

from which we deduce that

$$\frac{(n+1)}{n^2(2n+1)} - \frac{(n^*+1)}{(n^*)^2(2n^*+1)} = 0.$$

Therefore, as the function

$$x \mapsto \frac{x+1}{x^2(2x+1)}$$

decreases strictly on $\mathbf{R}^+$, we get $n = n^*$ and thus $k = -k^*$, which contradicts the fact that $k + k^* \neq 0$.

We conclude that either $\omega_0 \neq 0$ or $\omega_1 \neq 0$, so that

$$\left|\tau(n,k,j) + \tau(n^*,k^*,j^*) - \tau(m,k+k^*,\ell)\right| \geq \frac{|\omega_1|}{2\beta}$$

for β large enough. Lemma 2.4.3 is proved. □

Let us go back to the proof of Proposition 2.4.1, and first consider the case when $k \neq 0$, $k^* \neq 0$ and $k + k^* \neq 0$. In view of Lemma 2.4.3, the asymptotic behaviour of $P_{n,n^*,m,k,k^*}(\beta)$ as $\beta \to \infty$ is completely determined by the behaviour of the factor

$$\sigma_{n,n^*,m,k,k^*}(\beta) = \tau(n,k,0) + \tau(n^*,k^*,0) - \tau(m,k+k^*,0).$$

Indeed, $P_{n,n^*,m,k,k^*}(\beta)$ is defined as a product, eight factors of which involve triads of type (i), six of which involve triads of type (ii), twelve of which involve triads of type (iii) and the last factor of which is $\sigma_{k,k^*,n,n^*,m}(\beta)$. By Lemma 2.4.3 we then deduce that there exists a nonnegative constant C (depending on k, k^*, n, n^*, m) such that

$$\left|P_{n,n^*,m,k,k^*}(\beta)\right| \geq C\beta^7 \left|\sigma_{n,n^*,m,k,k^*}(\beta)\right|.$$

If one or two among n, n^* and m are zero, properties (iv) and (v) in Lemma 2.4.3 allow to conclude that for β large enough

$$\left|P_{n,n^*,m,k,k^*}(\beta)\right| \geq C\beta^6,$$

and thus $P_{,n,n^*,m,k,k^*}$ has a finite number of roots.

If n, n^*, m are all equal to zero or n, n^*, m are all non zero, we cannot conclude as no estimate on $\sigma_{n,n^*,m,k,k^*}(\beta)$ at infinity is available. Therefore, either $\sigma_{n,n^*,m,k,k^*}(\beta)$ is identically zero for β large enough, or $P_{n,n^*,m,k,k^*}(\beta)$ has a finite number of roots.

Thus at this stage, in order to prove Proposition 2.4.1, it remains

(1) to consider the case when $k, k^*, k + k^* \neq 0$ and $\sigma_{n,n^*,m,k,k^*}(\beta)$ is identically zero for β large enough, with n, n^*, m all zero or all non zero;

(2) to study the case when k or k^* or $k + k^*$ is zero (in order to establish that only the triads involving a zero mode may be resonant).

• *Conclusion in the case of (1)*

In the case when n, n^*, m are all zero, then the resonances corresponding to $\sigma_{0,0,0,k,k^*}(\beta) = 0$ are precisely Kelvin resonances, which cannot be removed.

In the case when n, n^*, m are all non zero, then $\sigma_{n,n^*,m,k,k^*}(\beta)$ is an analytic function of β (the roots of (2.2.3) – defined explicitely with Cardan's formula – do not cross each other according to Lemma 2.2.1, and thus depend analytically on β) : in particular, if $\sigma_{n,n^*,m,k,k^*}(\beta)$ cancels for β large enough, then it is identically zero. Let us describe the asymptotics of the roots as β goes to zero.

Lemma 2.4.4. — *With the notation of Paragraph 2.2.1, for all $k \in \mathbf{Z}$ and all $n \in \mathbf{N}^*$, the following expansions hold as $\beta \to 0$:*

$$\tau(n,k,0) = \frac{\beta}{k} + o(1).$$

Proof. — Since the product of the roots goes to zero as β goes to zero, we infer that at least one root goes to zero with β. Let us consider that root. Since $\beta(2n+1)$ is negligible with respect to k^2 and τ^3 is negligible with respect to $k^2\tau$, we find that

$$k^2\tau - k\beta \sim 0,$$

so that one root is equivalent to $\frac{\beta}{k}$ as β goes to zero. It is easy to see that the two other roots are then equivalent to $\pm k$, so that we do have $\tau(n,k,0) \sim \frac{\beta}{k}$ (we recall that for $n \neq 0$, the roots are numbered in increasing order). The lemma is proved. □

Now going back to the study of case (1), in view of Lemma 2.4.4 it is obvious that $\sigma_{n,n^*,m,k,k^*}(\beta)$ cannot vanish indentically.

• <u>*Conclusion in the case of (2)*</u>

In this situation, we need to refine the previous analysis by introducing an auxiliary polynomial. We thus define

$$I_{k,k^*} = \big\{(j,j^*,\ell) \in \{-1,0,1\}^3 \,/\, \ell \neq 0 \text{ if } k+k^* = 0, \quad j \neq 0 \text{ if } k = 0 \text{ and } j^* \neq 0 \text{ if } k^* = 0\big\}$$

and

$$Q_{n,n^*,m,k,k^*}(\beta) = \prod_{(j,j^*,\ell)\in I_{k,k^*}} \big(\tau(n,k,j) + \tau(n^*,k^*,j^*) - \tau(m,k+k^*,\ell')\big).$$

That corresponds to the remaining possible resonances, where we have omitted the trivial case when one wave is geostrophic ($\tau = 0$).

As previously, considerations of symmetry show that this quantity can be rewritten in terms of the symmetric functions of $(\tau(n,k,j))_{j\in\{-1,0,1\}}$ (or $(\tau(n,k,j))_{j\in\{-1,1\}}$ if $k=0$), the symmetric functions of $(\tau(n^*,k^*,j^*))_{j^*\in\{-1,0,1\}}$ (or $(\tau(n^*,k^*,j^*))_{j^*\in\{-1,1\}}$ if $k^*=0$) and the symmetric functions of $(\tau(m,k+k^*,\ell))_{\ell\in\{-1,1\}}$ (or $(\tau(m,k+k^*,\ell))_{\ell\in\{-1,0,1\}}$ if $k+k^*=0$). Noticing that the symmetric functions of $(\tau(n,0,j))_{j\in\{-1,1\}}$ are affine in β, we conclude that $Q_{n,n^*,m,k,k^*}(\beta)$ is a polynomial in β.

The asymptotic analysis of the various factors as $\beta \to \infty$ shows that

$$|Q_{n,n^*,m,k,k^*}(\beta)| \geq C\beta^3$$

for β large enough. Therefore, $Q_{n,n^*,m,k,k^*}(\beta)$ has a finite number of roots, meaning that there exist a finite number of β such that resonant triads with $k=0$ or $k^*=0$ or $k+k^*=0$ (other than the triads involving a non-oscillating mode) can occur.

We have therefore proved that

(1) in the case when $k, k^*, k+k^* \neq 0$ and $\sigma_{n,n^*,m,k,k^*}(\beta)$ is identically zero, only the triads involving three Kelvin modes are resonant for an infinite number of β;

(2) when k or k^* or $k+k^*$ is zero, only triads involving zero modes are resonant for an infinite number of β.

Combining this result with the conclusion of the previous paragraph achieves the proof of Proposition 2.4.1. □

2.4.2. The special case of Ker$L$. — In this short section we are going to write an algebraic computation which in particular allows to derive the following proposition.

Proposition 2.4.5. — *Let Φ and Φ^* be two smooth vector fields. Then for every $n \in \mathbf{N}$, we have*

$$(\Psi_{n,0,0}|Q_L(\Phi,\Phi^*))_{L^2(\mathbf{R}\times\mathbf{T})} = 0.$$

Remark 2.4.6. — *That proposition implies that the projection of the limit system (2.3.5) onto* Ker$L$ *can be formally written*

$$\partial_t \Pi_0 \Phi - \nu \Delta'_L \Pi_0 \Phi = 0.$$

Proof. — We are going to prove a more general result, computing the quantity

$$\Big(\Phi_\lambda | Q_L(\Phi_\mu, \Phi_{\tilde\mu})\Big)_{L^2(\mathbf{R}\times\mathbf{T})}$$

where Φ_λ, Φ_μ and $\Phi_{\tilde\mu}$ are three eigenmodes of L associated respectively with the eigenvalues $i\lambda$, $i\mu$ and $i\tilde\mu$ where $\lambda = \mu + \tilde\mu$. The proposition corresponds of course to the case when $\lambda = 0$.

We have

$$\Big(\Phi_\lambda | Q_L(\Phi_\mu, \Phi_{\tilde\mu})\Big)_{L^2(\mathbf{R}\times\mathbf{T})} = \Big(\Phi_\lambda | Q(\Phi_\mu, \Phi_{\tilde\mu})\Big)_{L^2(\mathbf{R}\times\mathbf{T})}$$

hence denoting by $\bar\Phi_\lambda$ the complex conjugate of Φ_λ, we get

$$\begin{aligned}
&\Big(\Phi_\lambda | Q_L(\Phi_\mu, \Phi_{\tilde\mu})\Big)_{L^2(\mathbf{R}\times\mathbf{T})}\\
&\quad = \frac{1}{2}\int \big(\bar\Phi_{\lambda,0}\nabla\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu} + \Phi_{\tilde\mu,0}\Phi'_\mu) + \bar\Phi'_\lambda\cdot(\Phi'_\mu\cdot\nabla\Phi'_{\tilde\mu} + \Phi'_{\tilde\mu}\cdot\nabla\Phi'_\mu)\big)\,dx\\
&\quad = \frac{1}{2}\int \big(-\nabla\bar\Phi_{\lambda,0}\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu} + \Phi_{\tilde\mu,0}\Phi'_\mu) + \bar\Phi'_\lambda\cdot(\nabla(\Phi'_\mu\cdot\Phi'_{\tilde\mu}) + \Phi'^{\perp}_\mu\nabla^\perp\cdot\Phi'_{\tilde\mu} + \Phi'^{\perp}_{\tilde\mu}\nabla^\perp\cdot\Phi'_\mu)\big)\,dx\\
&\quad = -\frac{1}{2}\int \big((\beta x_1\bar\Phi'^{\perp}_\lambda + \nabla\bar\Phi_{\lambda,0})\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu} + \Phi_{\tilde\mu,0}\Phi'_\mu) + (\nabla\cdot\bar\Phi'_\lambda)\Phi'_\mu\cdot\Phi'_{\tilde\mu}\big)\,dx\\
&\qquad\qquad + \frac{1}{2}\int \bar\Phi'_\lambda\cdot(\Phi'^{\perp}_\mu(\nabla^\perp\cdot\Phi'_{\tilde\mu} - \beta x_1\Phi_{\tilde\mu,0}) + \Phi'^{\perp}_{\tilde\mu}(\nabla^\perp\cdot\Phi'_\mu - \beta x_1\Phi_{\mu,0}))dx.
\end{aligned}$$

Using the identities

$$\nabla\cdot\Phi'_\lambda = i\lambda\Phi_{\lambda,0},$$
$$\beta x_1\Phi'^{\perp}_\lambda + \nabla\Phi_{\lambda,0} = i\lambda\Phi'_\lambda,$$

as well as their combination

$$\beta\Phi_{\lambda,1} = i\lambda(\nabla^\perp\cdot\Phi'_\lambda - \beta x_1\Phi_{\lambda,0})$$

and similar formulas for Φ_μ and $\Phi_{\tilde\mu}$, we get

$$\begin{aligned}
&\Big(\Phi_\lambda | Q_L(\Phi_\mu, \Phi_{\tilde\mu})\Big)_{L^2(\mathbf{R}\times\mathbf{T})}\\
&= \frac{1}{2}\int \big(i\lambda\bar\Phi'_\lambda\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu} + \Phi_{\tilde\mu,0}\Phi'_\mu) + i\lambda\bar\Phi_{\lambda,0}\Phi'_\mu\cdot\Phi'_\lambda\big)\,dx\\
&- \frac{1}{2\beta}\int \bar\Phi_{\lambda,2}(i\mu + i\tilde\mu)(\nabla^\perp\cdot\Phi'_\mu - \beta x_1\Phi_{\mu,0})(\nabla^\perp\cdot\Phi'_{\tilde\mu} - \beta x_1\Phi_{\tilde\mu,0})dx\\
&- \frac{1}{2\beta}\int i\lambda(\nabla^\perp\cdot\bar\Phi'_\lambda - \beta x_1\bar\Phi_{\lambda,0})(\Phi_{\mu,2}(\nabla^\perp\cdot\Phi'_{\tilde\mu} - \beta x_1\Phi_{\tilde\mu,0}) + \Phi_{\tilde\mu,2}(\nabla^\perp\cdot\Phi'_\mu - \beta x_1\Phi_{\mu,0}))dx
\end{aligned}$$

from which we deduce

$$
\begin{aligned}
&\Big(\Phi_\lambda | Q_L(\Phi_\mu,\Phi_{\tilde\mu})\Big)_{L^2(\mathbf{R}\times\mathbf{T})} \\
&\quad = \frac{i\lambda}{2}\int \left(\bar\Phi'_\lambda\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu}+\Phi_{\tilde\mu,0}\Phi'_\mu)+\bar\Phi_{\lambda,0}\Phi'_\mu\cdot\Phi'_\lambda\right)dx \\
&\quad - \frac{i\lambda}{2\beta}\int \bar\Phi_{\lambda,2}(\nabla^\perp\cdot\Phi'_\mu-\beta x_1\Phi_{\mu,0})(\nabla^\perp\cdot\Phi'_{\tilde\mu}-\beta x_1\Phi_{\tilde\mu,0})dx \\
&\quad - \frac{i\lambda}{2\beta}\int (\nabla^\perp\cdot\bar\Phi'_\lambda-\beta x_1\bar\Phi_{\lambda,0})(\Phi_{\mu,2}(\nabla^\perp\cdot\Phi'_{\tilde\mu}-\beta x_1\Phi_{\tilde\mu,0})+\Phi_{\tilde\mu,2}(\nabla^\perp\cdot\Phi'_\mu-\beta x_1\Phi_{\mu,0}))dx.
\end{aligned}
\tag{2.4.9}
$$

In particular for $\lambda=0$ this quantity is always zero, which proves Proposition 2.4.5. □

CHAPTER 3

THE ENVELOPE EQUATIONS

The aim of this chapter is to study the system (SW_0) obtained formally page 19 as the limit of the filtered system (2.3.3) as $\varepsilon \to 0$. Let us recall the system:

$$(SW_0) \qquad \begin{cases} \partial_t \Phi + Q_L(\Phi, \Phi) - \nu \Delta'_L \Phi = 0 \\ \Phi_{|t=0} = (\eta^0, u^0), \end{cases}$$

where Δ'_L and Q_L denote the linear and symmetric bilinear operator defined by (2.3.6) page 19.

Two different types of wellposedness results will be proved on (SW_0): first we will prove the existence of weak solutions in L^2 and of a unique, strong solution if the data is smooth enough (on a short time interval, which becomes infinite for small data). Then we will show that except for a countable number of β, the strong solutions exists globally in time as soon as the initial data is only in L^2, of arbitrary norm.

The statements of both theorems can be found in Paragraph 3.2, and their proofs are respectively the object of Paragraphs 3.4 and 3.5. In order to establish those results we will need to define, in Paragraph 3.1, suitable function spaces, compatible with the penalization operator L as well as the diffusion operator. Some technical preliminaries devoted to those spaces are proved in Paragraph 3.3: in particular in Paragraph 3.3.2 we prove the continuity of the bilinear operator Q_L in those function spaces. Finally the last part of this chapter is devoted to an additional smoothing property on the divergence.

3.1. Definition of suitable functional spaces

By construction the operators Δ'_L and Q_L appearing in the limiting filtered system (SW_0) are defined in terms of the projections $(\Pi_\lambda)_{i\lambda \in \mathfrak{S}}$ on the eigenspaces of L. In particular, they are not expected to satisfy "good" commutation properties with the usual derivation ∇. Therefore in order to establish a priori estimates on the solutions to (SW_0) we have to introduce some weighted Sobolev spaces associated with some derivation-like operator which acts separately on each eigenmode of L.

Let us therefore introduce the following norms. We will write as previously $\Pi_{n,k,j}$ for the projection on the eigenmode $\Psi_{n,k,j}$ of L and Π_λ for the projection on the eigenspace associated with the eigenvalue $i\lambda$

of L. Finally we define

$$S = \mathbf{N} \times \mathbf{Z} \times \{-1, 0, 1\}.$$

Definition 3.1.1. — *Let $s \geq 0$ be a given real number. We define the space H^s_L as the subspace of $(L^2(\mathbf{R} \times \mathbf{T}))^3$ given by the following norm:*

$$\|\Phi\|_{H^s_L} \stackrel{def}{=} \left(\sum_{(n,k,j) \in S} (1 + n + k^2)^s \|\Pi_{n,k,j}\Phi\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \right)^{\frac{1}{2}}.$$

Due to the definition of the eigenvectors of L seen in the previous chapter, one can prove the following proposition.

Proposition 3.1.2. — *Let $s \geq 0$ be given. Then one has the following property:*

$$\forall \Phi \in H^s_L, \quad \|\Phi\|_{H^s_L} \sim \|(Id - \Delta + \beta^2 x_1^2)^{s/2}\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}.$$

In particular, H^s_L is continuously embedded in $H^s(\mathbf{R} \times \mathbf{T})$, and for all compact subsets Ω of $\mathbf{R} \times \mathbf{T}$, $H^s_0(\Omega)$ is continuously embedded in H^s_L.

Moreover for all $\Phi \in H^s_L \cap (\mathrm{Ker}L)^\perp$, we have

$$\|\Phi\|_{H^s_L} \sim \left(\sum_{i\lambda \in \mathfrak{S} \setminus \{0\}} \|\Pi_\lambda \Phi\|^2_{H^s(\mathbf{R}\times\mathbf{T})} \right)^{\frac{1}{2}}.$$

Finally if Φ belongs to $K \cup P$, as defined in Definition 2.2.4 page 16, then

$$\|\Phi\|_{H^s_L} \sim \left(\sum_{i\lambda \in \mathfrak{S} \setminus \{0\}} (1 + \lambda^2)^s \|\Pi_\lambda \Phi\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \right)^{\frac{1}{2}}.$$

Proof. — Let us first prove the first equivalence: let $\Phi \in H^s_L$ be given. Then we have

$$\|(\mathrm{Id} - \Delta + \beta^2 x_1^2)^{s/2}\Phi\|^2_{L^2} = \Big\| \sum_{(n,k,j)\in S} (\mathrm{Id} - \Delta + \beta^2 x_1^2)^{s/2}\Pi_{n,k,j}\Phi \Big\|^2_{L^2}.$$

By the identity

$$-\psi_n'' + \beta x_1^2 \psi_n = \beta(2n+1)\psi_n$$

the orthogonality of the family $(\psi_n)_{n\in\mathbf{N}}$ and the explicit formulas (2.2.6), (2.2.7) and (2.2.9) for $\Psi_{n,k,j}$, we infer that, for all integers σ,

$$\|(\mathrm{Id} - \Delta + \beta^2 x_1^2)^\sigma \Psi_{n,k,j} - (1 + n + k^2)^\sigma \Psi_{n,k,j}\|_{L^2} \leq C(1 + n + k^2)^{\sigma - 1},$$

which implies in particular that

$$\Big| \Big((\mathrm{Id} - \Delta + \beta x_1^2)^\sigma \Psi_{n,k,j} | (\mathrm{Id} - \Delta + \beta x_1^2)^\sigma \Psi_{n,k,j^*} \Big)_{L^2(\mathbf{R}\times\mathbf{T})} \Big| \leq C(1 + n + k^2)^{\sigma-1}.$$

On the other hand,

$$\Big((\mathrm{Id} - \Delta + \beta x_1^2)^\sigma \Psi_{n,k,j} | (\mathrm{Id} - \Delta + \beta x_1^2)^\sigma \Psi_{n^*,k^*,j^*} \Big)_{L^2(\mathbf{R}\times\mathbf{T})} = 0 \text{ if } n \neq n^* \text{ or } k \neq k^*,$$

so we find that

$$\begin{aligned} \|(\mathrm{Id} - \Delta + \beta x_1^2)^\sigma \Phi\|^2_{L^2(\mathbf{R}\times\mathbf{T})} &= \sum_{n,k,j} \sum_{j^*} \Big((\mathrm{Id} - \Delta + \beta x_1^2)^\sigma \Pi_{n,k,j}\Phi | (\mathrm{Id} - \Delta + \beta x_1^2)^\sigma \Pi_{n,k,j^*}\Phi \Big)_{L^2} \\ &\sim \sum_{n,k,j} (1 + n + k^2)^{2\sigma} \|\Pi_{n,k,j}\Phi\|^2_{L^2(\mathbf{R}\times\mathbf{T})}. \end{aligned}$$

We then obtain the first equivalence for all $s \geq 0$ by interpolation.

Then, from the inequality

$$\forall \Phi \in H^s_L, \quad \|\Phi\|_{H^s(\mathbf{R}\times\mathbf{T})} \leq C\|(\mathrm{Id}\ - \Delta + \beta x_1^2)^{s/2}\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}$$

along with the fact that and for all $\Phi \in C^\infty(\mathbf{R}\times\mathbf{T})$ supported in $[-R, R]\times\mathbf{T}$,

$$\|(\mathrm{Id}\ - \Delta + \beta x_1^2)^{s/2}\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})} \leq C(1+R^2)^{s/2}\|\Phi\|_{H^s(\mathbf{R}\times\mathbf{T})}$$

we get the embeddings $H^s_0(\Omega) \subset H^s_L \subset H^s(\mathbf{R}\times\mathbf{T})$ for all $\Omega \subset\subset \mathbf{R}\times\mathbf{T}$.

The second result of the proposition is easy, using Proposition 2.2.5 page 17:

$$\forall i\lambda \in \mathfrak{S}\setminus\{0\}, \quad \frac{1}{C_s}\sum_{\tau(n,k,j)=\lambda}\|\Pi_{n,k,j}\Phi\|^2_{H^s(\mathbf{R}\times\mathbf{T})} \leq \|\Pi_\lambda\Phi\|^2_{H^s(\mathbf{R}\times\mathbf{T})} \leq C_s\sum_{\tau(n,k,j)=\lambda}\|\Pi_{n,k,j}\Phi\|^2_{H^s(\mathbf{R}\times\mathbf{T})},$$

and recalling that by Proposition 2.2.3 page 12, we have

$$\frac{1}{C_s}(1+n+k^2)^{s/2} \leq \|\Psi_{n,k,j}\|_{H^s(\mathbf{R}\times\mathbf{T})} \leq C_s(1+n+k^2)^{s/2}.$$

Finally the last result, concerning Kelvin and Poincaré modes is simply due to Lemma 2.2.1 and Proposition 2.2.3 .

The proposition is proved. □

Remark 3.1.3. — *The H^s_L estimates are both regularity and decay estimates. In particular, the embedding $H^s_L \subset L^2(\mathbf{R}\times\mathbf{T})$ is compact, and we have the following equality*

$$\bigcap_{s\geq 0} H^s_L = \mathcal{S}(\mathbf{R}\times\mathbf{T}).$$

Note that these spaces are also used by Dutrifoy and Majda [**7**] *to study the uniform wellposedness of a non viscous version of* (SW_ε).

3.2. Statement of the wellposedness result

The main results of this chapter are the following two theorems. We have written $\Pi_\perp\Phi$ for the projection of Φ onto $(\mathrm{Ker}L)^\perp$. In the next theorem, we state the global existence of weak solutions and the local in time existence (and uniqueness) of strong solutions.

Theorem 2 **(Wellposedness results for all** β**)**. — *There is a constant C such that the following results hold. Let $\Phi^0 \in L^2(\mathbf{R}\times\mathbf{T};\mathbf{R}^3)$ be given. Then*

• *there exists a global weak solution $\Phi \in L^\infty(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T}))$ to (SW_0), such that $\Pi_\perp\Phi$ belongs to the space $L^2(\mathbf{R}^+; H^1_L)$, and which satisfies for every $t\geq 0$ the energy estimate*

$$\frac{1}{2}\|\Phi(t)\|^2_{L^2(\mathbf{R}\times\mathbf{T})} + \nu\int_0^t\|\nabla(\Pi_0\Phi)'(t')\|^2_{L^2(\mathbf{R}\times\mathbf{T})}dt' + \frac{\nu}{C}\int_0^t\|\nabla(\Pi_\perp\Phi)(t')\|^2_{L^2(\mathbf{R}\times\mathbf{T})}dt' \leq \frac{1}{2}\|\Phi^0\|^2_{L^2(\mathbf{R}\times\mathbf{T})}.$$

• *if we further assume that $\Pi_0\Phi^0$ belongs to H^s_L for $s \geq 0$, then $\Pi_0\Phi$ (which is unique) belongs to $L^\infty_{loc}(\mathbf{R}^+; H^s_L)$.*

• *if* $\Pi_0\Phi^0$ *belongs to* $L^2(\mathbf{R}\times\mathbf{T})$ *and* $\Pi_\perp\Phi^0$ *belongs to* $H_L^{1/2}$*, then there exists a maximal time interval* $[0,T^*[$*, with* $T^* = +\infty$ *under the smallness assumption*

$$\|\Pi_0\Phi^0\|_{L^2(\mathbf{R}\times\mathbf{T})} + \|\Pi_\perp\Phi^0\|_{H_L^{1/2}} \leq C^{-1}\nu,$$

such that Φ *is the unique solution to* (SW_0)*, and* $\Pi_\perp\Phi$ *belongs to* $L^\infty_{loc}([0,T^*[,H_L^{1/2})\cap L^2_{loc}([0,T^*[,H_L^{3/2})$.

• *if* $\Pi_\perp\Phi^0$ *belongs to* H_L^s *for some* $1/2 \leq s \leq 1$*, then* $\Pi_\perp\Phi$ *belongs to* $L^\infty_{loc}([0,T^*[,H_L^s)\cap L^2_{loc}([0,T^*[,H_L^{s+1})$.

The previous theorem is much improved if a countable set of values for β is removed.

Theorem 3 **(Wellposedness results for generic** β**).** — *There is a constant* C *and a countable subset* $\mathcal{N}$ *of* $\mathbf{R}^+$ *such that for any* $\beta\in\mathbf{R}^+\setminus\mathcal{N}$*, the following result holds. Let* $\Phi^0\in L^2(\mathbf{R}\times\mathbf{T};\mathbf{R}^3)$ *be given. Then* (SW_0) *is globally wellposed, in the sense that there is a unique, global solution* Φ *in* $L^\infty(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T}))$ *such that* $\Pi_\perp\Phi$ *belongs to the space* $L^2(\mathbf{R}^+;H_L^1)$*, and which satisfies the energy inequality of Theorem 2.*

• *if we further assume that* $\Pi_\perp\Phi^0$ *belongs to* H_L^s*, for* $0\leq s\leq 1$*, then* $\Pi_\perp\Phi$ *belongs to* $L^\infty_{loc}(\mathbf{R}^+,H_L^s)\cap L^2_{loc}(\mathbf{R}^+,H_L^{s+1})$.

Remark 3.2.1. — *These results are based on a precise study of the structure of* (SW_0)*, and in particular of the ageostrophic part of that equation, meaning its projection onto* $(\mathrm{Ker}L)^\perp$*. One can prove in particular that the ageostrophic part of* (SW_0) *is in fact fully parabolic. That should be compared to the case of the incompressible limit of the compressible Navier-Stokes equations, where again the limit system is parabolic, contrary to the original compressible system (see* [**5**], [**8**], [**25**]*). Note however that* (SW_0) *actually satisfies the same type of trilinear estimates as the three-dimensional incompressible Navier-Stokes system, which accounts for the fact that in Theorem 2 unique solutions are only obtained for a short life span (despite the fact that the space variable runs in the two dimensional domain* $\mathbf{R}\times\mathbf{T}$*). In the case of Theorem 3, we use the study of resonances of the previous chapter which shows that the limit system is linear, except for its projection onto Kelvin modes; but Kelvin modes are essentially one-dimensional so energy estimates are much improved compared to the case of Theorem 2, and that is why global wellposedness is true in* L^2*, for arbitrarily large initial data.*

The rest of this chapter is devoted to the proof of those theorems. Some preliminary results are proved in Section 3.3 below, namely the fact that the ageostrophic part of the limit system is parabolic, along with trilinear estimates. In Section 3.4 we prove Theorem 2, whereas the proof of Theorem 3 can be found in Section 3.5. The last section will be devoted to an additional regularity result, giving an estimate of the divergence of Φ' in both cases, which will be useful in the next chapter.

3.3. Preliminary results

Let us prove some results that will be used throughout this chapter: in Section 3.3.1 below, we prove that the limit system, projected onto $(\mathrm{Ker}L)^\perp$ is parabolic. In Section 3.3.2 we prove crucial trilinear estimates.

3.3.1. Parabolicity of the ageostrophic limit equation. — In this section we are going to prove that the projection of the limit system onto $(\mathrm{Ker}L)^\perp$ is parabolic. To obtain that result, the important remark is that, for each eigenmode of L, the first and third components of the eigenvectors (corresponding to η and u_2) have very similar behaviours, and thus controlling the regularity of the last two components is sufficient to have an estimate on $\Pi_\perp\Phi$ in H^1_L. A result of quasi-orthogonality in $H^s(\mathbf{R}\times\mathbf{T})$ of the nonzero eigenmodes of L leads indeed to the following result.

Lemma 3.3.1. — *Let $s \geq 0$ be given. There is a constant C_s such that for any $\Phi \in (\mathrm{Ker}L)^\perp$, we have*

$$\|\Phi\|^2_{H^{s+1}_L} \leq C_s(\Phi| - \Delta'_L\Phi)_{H^s_L},$$

meaning in particular that the projection of the system (SW_0) onto $(\mathrm{Ker}L)^\perp$ is fully parabolic.

Proof. — The proof of that result consists in using the structure of the eigenmodes to prove that the diffusion – acting a priori only on the velocity field – has also a smoothing effect on the pressure, and more precisely that

$$\forall (n,k,j) \in S, \quad \|\Pi_{n,k,j}\Phi\|_{H^s(\mathbf{R}\times\mathbf{T})} \leq C'\|(\Pi_{n,k,j}\Phi)'\|_{H^s(\mathbf{R}\times\mathbf{T})}, \tag{3.3.1}$$

where $\Pi_{n,k,j}$ denotes the projection on the eigenmode $\Psi_{n,k,j}$ of L (with the notation of the previous chapter) and C' is a nonnegative constant (independent of n, k and j). By formulas (2.2.6), (2.2.7) and (2.2.9) we deduce that for any integer s, we have

$$\|\partial_2^s(\Psi_{n,k,j})_0\|_{L^2(\mathbf{R}\times\mathbf{T})} = \|\partial_2^s(\Psi_{n,k,j})_2\|_{L^2(\mathbf{R}\times\mathbf{T})}$$

using the orthogonality of ψ_{n-1} and ψ_{n+1}, and that

$$\frac{1}{C}\|\partial_1^s(\Psi_{n,k,j})_2\|_{L^2(\mathbf{R}\times\mathbf{T})} \leq \|\partial_1^s(\Psi_{n,k,j})_0\|_{L^2(\mathbf{R}\times\mathbf{T})} \leq C\|\partial_1^s(\Psi_{n,k,j})_2\|_{L^2(\mathbf{R}\times\mathbf{T})}.$$

This implies in particular (3.3.1).

By Remark 2.2.6, page 18 we then deduce that for all $\Phi \in (\mathrm{Ker}L)^\perp$

$$\begin{aligned}
(\Phi| - \Delta'_L\Phi)_{L^2(\mathbf{R}\times\mathbf{T})} &= \sum_{i\lambda\in\mathfrak{S}\setminus\{0\}} (\Pi_\lambda\Phi| - \Delta'_L(\Pi_\lambda\Phi))_{L^2(\mathbf{R}\times\mathbf{T})} = \sum_{i\lambda\in\mathfrak{S}\setminus\{0\}} \|(\Pi_\lambda\Phi)'\|^2_{\dot H^1(\mathbf{R}\times\mathbf{T})} \\
&\geq \frac{1}{C}\sum_{(n,k,j)\in S^*} \|(\Pi_{n,k,j}\Phi)'\|^2_{\dot H^1(\mathbf{R}\times\mathbf{T})} \\
&\geq \frac{1}{CC'}\sum_{(n,k,j)\in S^*} \|\Pi_{n,k,j}\Phi\|^2_{\dot H^1(\mathbf{R}\times\mathbf{T})} \\
&\geq \frac{1}{CC'C'_1}\|\Phi\|^2_{H^1_L}
\end{aligned}$$

recalling that

$$\frac{1}{C'_s}(1+n+k^2)^{s/2} \leq \|\Psi_{n,k,j}\|_{\dot H^s(\mathbf{R}\times\mathbf{T})} \leq C'_s(1+n+k^2)^{s/2}.$$

We therefore obtain the first inequality using Proposition 3.1.2.

In a similar way, by Proposition 2.2.3, page 12, we have for all $\Phi \in (\mathrm{Ker}L)^\perp$

$$\begin{aligned}(\Phi| - \Delta'_L \Phi)_{H^s_L} &= \sum_{(n,k,j)\in S^*} (1+n+k^2)^s (\Pi_{n,k,j}\Phi| - \Delta'_L \Phi)_{L^2(\mathbf{R}\times\mathbf{T})}\\ &= \sum_{(n,k,j)\in S^*} (1+n+k^2)^s \sum_{\tau(n^*,k^*,j^*)=\tau(n,k,j)} (\Pi_{n,k,j}\Phi| - \Delta'(\Pi_{n^*,k^*,j^*}\Phi))_{L^2(\mathbf{R}\times\mathbf{T})}\\ &= \sum_{\substack{(n,k,j)\in S^*\\ \tau(n,k,j)\neq\pm\sqrt{\beta/2}}} (1+n+k^2)^s \|(\Pi_{n,k,j}\Phi)'\|^2_{\dot H^1(\mathbf{R}\times\mathbf{T})} + R_\beta\end{aligned}$$

where R_β is the contribution of the modes $\pm\sqrt{\beta/2}$, defined by

$$\begin{aligned}R_\beta &= \sum_{k=\pm\sqrt{\beta/2}} (1+\frac{\beta}{2})^s \sum_{(j,j^*)\in(0,\mathrm{sign}k)^2} (\Pi_{0,k,j}\Phi| - \Delta'\Pi_{0,k,j^*}\Phi)\\ &= \sum_{k=\pm\sqrt{\beta/2}} (1+\frac{\beta}{2})^s \left\| \sum_{(j,j^*)\in(0,\mathrm{sign}k)^2} (\Pi_{0,k,j}\Phi)' \right\|^2_{\dot H^1}.\end{aligned}$$

Using (3.3.1) and Proposition 2.2.5 page 17 leads then to the expected estimate. □

Remark 3.3.2. — *Note that these inequalities indicate in particular that the notion of homogeneous or inhomogeneous spaces does not make sense for these weighted Sobolev spaces.*

Moreover we recall that there is no analogue of Proposition 2.2.5 in the case of geostrophic modes, so that the geostrophic equation does not have that ellipticity property.

3.3.2. Derivation of the trilinear estimates. — An important step in the proof of Theorems 2 and 3 consists in establishing some control on the nonlinear term arising in (SW_0) in terms of the Sobolev norms introduced in Section 3.1. Such estimates are obtained using classical para-differential methods. Obviously a more general statement could be written, at the price of more technicalities. In order to keep the proof as simple as possible we choose to state only those estimates that will be used in the following.

Proposition 3.3.3. — *Denote by Q_L the limit nonlinear operator defined by (2.3.6), and let α be any real number greater than $3/2$. Then the following trilinear estimates hold :*

$$\begin{aligned}\left|(\Phi_*|Q_L(\Phi,\Phi^*))_{L^2(\mathbf{R}\times\mathbf{T})}\right| &\le C\|\Pi_\perp\Phi_*\|^{3/4}_{H^1_L}\|\Pi_\perp\Phi_*\|^{1/4}_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi\|^{3/4}_{H^1_L}\|\Pi_\perp\Phi^*\|^{3/4}_{H^1_L}\\ &\quad\times\left(\|\Pi_\perp\Phi^*\|^{1/4}_{H^1_L}\|\Pi_\perp\Phi\|^{1/4}_{L^2(\mathbf{R}\times\mathbf{T})} + \|\Pi_\perp\Phi\|^{1/4}_{H^1_L}\|\Pi_\perp\Phi^*\|^{1/4}_{L^2(\mathbf{R}\times\mathbf{T})}\right)\\ &\quad + C\|\Pi_\perp\Phi_*\|_{L^2(\mathbf{R}\times\mathbf{T})}\left(\|\Pi_0\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi^*\|_{H^1_L} + \|\Pi_0\Phi^*\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi\|_{H^1_L}\right)\\ \left|(\Phi_*|Q_L(\Phi,\Phi^*))_{L^2(\mathbf{R}\times\mathbf{T})}\right| &\le C\|\Pi_\perp\Phi\|_{H^{3/2}_L}\|\Pi_\perp\Phi_*\|^{1/2}_{H^1_L}\|\Pi_\perp\Phi_*\|^{1/2}_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi^*\|^{1/2}_{H^1_L}\|\Pi_\perp\Phi^*\|^{1/2}_{L^2(\mathbf{R}\times\mathbf{T})}\\ &\quad + C\|\Pi_\perp\Phi\|_{H^1_L}\|\Pi_\perp\Phi^*\|_{H^1_L}\|\Pi_\perp\Phi_*\|^{1/2}_{H^1_L}\|\Pi_\perp\Phi_*\|^{1/2}_{L^2(\mathbf{R}\times\mathbf{T})}\\ &\quad + C\|\Phi_*\|_{L^2(\mathbf{R}\times\mathbf{T})}\left(\|\Pi_0\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi^*\|_{H^1_L} + \|\Pi_0\Phi^*\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi\|_{H^1_L}\right)\\ \left|(\Phi_*|Q_L(\Phi,\Phi^*))_{L^2(\mathbf{R}\times\mathbf{T})}\right| &\le C_\alpha\|\Pi_\perp\Phi_*\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi\|_{H^\alpha_L}\|\Pi_\perp\Phi^*\|_{H^\alpha_L}\\ &\quad + C_\alpha\|\Pi_\perp\Phi_*\|_{L^2(\mathbf{R}\times\mathbf{T})}\left(\|\Pi_0\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi^*\|_{H^1_L} + \|\Pi_0\Phi^*\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi\|_{H^1_L}\right),\end{aligned}$$

and for all $s \leq 1$

$$\begin{aligned}\left|(\Phi_*|Q_L(\Phi,\Phi^*))_{H^s_L}\right| &\leq C\left(\|\Pi_\perp\Phi\|_{H^{3/2}_L}\|\Pi_\perp\Phi^*\|_{H^1_L}+\|\Pi_\perp\Phi^*\|_{H^{3/2}_L}\|\Pi_\perp\Phi\|_{H^1_L}\right)\|\Pi_\perp\Phi_*\|_{H^{2s}_L}\\ &\quad +C\|\Pi_\perp\Phi_*\|_{H^{s+1}_L}\left(\|\Pi_0\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi^*\|_{H^s_L}+\|\Pi_0\Phi^*\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi\|_{H^s_L}\right)\\ &\leq C\left(\|\Pi_\perp\Phi\|_{H^{3/2}_L}\|\Pi_\perp\Phi^*\|^s_{H^s_L}\|\Pi_\perp\Phi^*\|^{1-s}_{H^{s+1}_L}+\|\Pi_\perp\Phi^*\|_{H^{3/2}_L}\|\Pi_\perp\Phi\|^s_{H^s_L}\|\Pi_\perp\Phi\|^{1-s}_{H^{s+1}_L}\right)\\ &\qquad\times\|\Pi_\perp\Phi_*\|^{1-s}_{H^s_L}\|\Pi_\perp\Phi_*\|^s_{H^{s+1}_L}\\ &\quad +C\|\Pi_\perp\Phi_*\|_{H^{s+1}_L}\left(\|\Pi_0\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi^*\|_{H^s_L}+\|\Pi_0\Phi^*\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi\|_{H^s_L}\right).\end{aligned}$$

Remark 3.3.4. — *1.The estimates presented in that proposition are exactly the analogue of the usual trilinear estimate for the three-dimensional Navier-Stokes equations. For instance in three space dimensions one has*

$$\left|(\Phi_*|div\,(\Phi\otimes\Phi^*))_{L^2(\mathbf{R}^3)}\right|\leq C\|\Phi_*\|_{\dot H^{\frac34}(\mathbf{R}^3)}\left(\|\Phi\|_{\dot H^{\frac34}(\mathbf{R}^3)}\|\nabla\Phi^*\|_{L^2(\mathbf{R}^3)}+\|\Phi^*\|_{\dot H^{\frac34}(\mathbf{R}^3)}\|\nabla\Phi\|_{L^2(\mathbf{R}^3)}\right)$$

whereas in two space dimensions one would expect

$$\left|(\Phi_*|div\,(\Phi\otimes\Phi^*))_{L^2(\mathbf{R}^2)}\right|\leq C\|\Phi_*\|_{\dot H^{\frac12}(\mathbf{R}^2)}\left(\|\Phi\|_{\dot H^{\frac12}(\mathbf{R}^2)}\|\nabla\Phi^*\|_{L^2(\mathbf{R}^2)}+\|\Phi^*\|_{\dot H^{\frac12}(\mathbf{R}^2)}\|\nabla\Phi\|_{L^2(\mathbf{R}^2)}\right).$$

The reason for the loss of one half derivative compared to the usual two dimensional case is linked to the fact that differentiation with respect to x_1 corresponds to a multiplication by $\sqrt{n}$ instead of n.

2. The restriction $s \leq 1$ is due do the particular structure of the nonlinear term, in particular to the coupling between Rossby modes. For the Rossby mode associated to the eigenvalue $i\lambda$ the regularity is indeed measured by $1/\lambda$. Therefore the condition of resonance $\lambda = \mu + \tilde\mu$ (which is of course not equivalent to $\lambda^{-1} = \mu^{-1} + \tilde\mu^{-1}$) does not allow to distribute the derivatives as in the usual paradifferential calculus. Note nevertheless that the computation (2.4.9) page 27 allows actually to distribute one derivative and to obtain a trilinear estimate of the form

$$\left|(\Phi|Q_L(\Phi,\Phi))_{H^s_L}\right|\leq C\|\Pi_\perp\Phi\|_{H^{s+1}_L}\|\Pi_\perp\Phi\|_{H^s_L}\left(\|\Pi_\perp\Phi\|_{H^{3/2}_L}+\|\Pi_0\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}\right),$$

for all $s \leq 2$.

Proof. — The method used to establish these estimates is rather standard : we decompose each vector on the eigenmodes of L, then compute each elementary trilinear term, and finally determine summability conditions. The fundamental result we will use to estimate the sums is the following (see [**21**])

$$\begin{aligned}&\forall v\in\ell^p(\mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}),\ \forall w\in\ell^{q,\infty}(\mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}),\ v*w\in\ell^r(\mathbf{N}\times\mathbf{Z}\times\{-1,0,1\})\\ &\qquad\text{with } p,q,r\in]1,+\infty[\text{ and }\frac1r=\frac1p+\frac1q-1,\end{aligned}\tag{3.3.2}$$

where the convolution is to be understood in k and n, coupled with the classical result

$$((1+n+k^2)^{-1})\in\ell^{3/2,\infty}(\mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}).\tag{3.3.3}$$

In the sequel we will use the following notation

$$S=\mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}\ \text{ and }\ S^*=\mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}\setminus\mathbf{N}\times\{0\}\times\{0\}.$$

We have by definition of the space H^s_L,

$$(\Phi_*|Q_L(\Phi,\Phi^*))_{H^s_L}=\sum_{(n_*,k_*,j_*)\in S}(1+n_*+k_*^2)^s\left(\Pi_{n_*,k_*,j_*}\Phi_*\Big|\Pi_{n_*,k_*,j_*}Q_L(\Phi,\Phi^*)\right)_{L^2}.$$

We can then write

$$(\Phi_*|Q_L(\Phi,\Phi^*))_{H^s_L} = \sum_{(n_*,k_*,j_*)\in S} \sum_{\substack{i\lambda, i\lambda^*\in\mathfrak{S}\\ \lambda+\lambda^*=\tau(n_*,k_*,j_*)}} (1+n_*+k_*^2)^s \left(\Pi_{n_*,k_*,j_*}\Phi_* \middle| \Pi_{n_*,k_*,j_*} Q(\Pi_\lambda\Phi, \Pi_{\lambda^*}\Phi^*)\right)_{L^2}.$$

• Let us start by estimating the purely ageostrophic part of Q_L, denoted $\tilde{Q}_L$ and defined by

$$\tilde{Q}_L(\Phi,\Phi^*) = \sum_{\substack{i\lambda, i\mu, i\mu^*\in\mathfrak{S}\setminus\{0\}\\ \lambda=\mu+\mu^*}} \Pi_\lambda Q(\Pi_\mu\Phi, \Pi_{\mu^*}\Phi^*). \tag{3.3.4}$$

We have

$$\left(\Phi_*|\tilde{Q}_L(\Phi,\Phi^*)\right)_{H^s_L} = \sum_{\substack{k+k^*=k_*\\ \tau(n,k,j)+\tau(n^*,k^*,j^*)=\tau(n_*,k_*,j_*)}} (1+n_*+k_*^2)^s \left(\Pi_{n_*,k_*,j_*}\Phi_* \middle| \Pi_{n_*,k_*,j_*} Q(\Pi_{n,k,j}\Phi, \Pi_{n^*,k^*,j^*}\Phi^*)\right)_{L^2},$$

where the eigenvalues $i\tau(n,k,j), i\tau(n_*,k_*,j_*)$ and $i\tau(n^*,k^*,j^*)$ run over $\mathfrak{S}\setminus\{0\}$.

Thus using the regularity estimates on the eigenvectors $(\Psi_{n,k,j})$ of L stated in Proposition 2.2.3, page 12, we get (writing to simplify τ for $\tau(n,k,j)$, and similarly $\tau_* = \tau(n_*,k_*,j_*)$ and $\tau^* = \tau(n^*,k^*,j^*)$)

$$\left| \sum_{\substack{\tau_*=\tau+\tau^*\\ k_*=k+k^*\\ i\tau, i\tau_*, i\tau^*\in\mathfrak{S}\setminus\{0\}}} (1+n_*+k_*^2)^s \left(\Pi_{n_*,k_*,j_*}\Phi_* \middle| \Pi_{n_*,k_*,j_*} Q(\Pi_{n,k,j}\Phi, \Pi_{n^*,k^*,j^*}\Phi^*)\right)_{L^2} \right|$$
$$\leq C \sum_{\substack{\tau_*=\tau+\tau^*\\ k_*=k+k^*\\ i\tau, i\tau_*, i\tau^*\in\mathfrak{S}\setminus\{0\}}} (1+n_*+k_*^2)^s |(\varphi_*)_{n_*,k_*,j_*}||\varphi_{n,k,j}||\varphi^*_{n^*,k^*,j^*}|((n+k^2)^{1/2}+(n^*+(k^*)^2)^{1/2}),$$

where $\varphi_{n,k,j}$ is defined as in the proof of Proposition 2.2.3 by

$$\varphi_{n,k,j} = (\Psi_{n,k,j}|\Phi)_{L^2(\mathbf{R}\times\mathbf{T})}.$$

For the sake of clearness, we will simplify (abusively) the notations as follows : we will denote respectively by φ, φ_* and φ^* the coefficients $\varphi_{n,k,j}$, $(\varphi_*)_{n_*,k_*,j_*}$ and $\varphi^*_{n^*,k^*,j^*}$, and by $\sum$ the sum over $(n,k,j)(n_*,k_*,j_*)(n^*,k^*,j^*)\in(S^*)^3$ satisfying the following constraints

$$\tau(n_*,k_*,j_*) = \tau(n,k,j)+\tau(n^*,k^*,j^*) \text{ and } k_* = k+k^*.$$

It is fundamental for the following estimates to notice that those constraints in fact imply that when k and k^* as well as j, j^* and j_* are fixed, then the condition $\tau(n_*,k+k^*,j_*) = \tau(n,k,j)+\tau(n^*,k^*,j^*)$ implies that a given n and n^* constrain the value of n_*. Indeed we recall that according to Lemma 2.2.1, page 9, there is only one value of n_* associated with one value of $\tau(n_*,k_*,j_*)\neq 0$. Note however that contrary to the usual case when there is an actual convolution (as is the case for the Fourier variable k here), we have no ovbious estimate on n_*, as a function of n and n^*. So the usual methods of distribution of derivatives cannot be fully used here, as derivatives in the x_1 direction, acting on Φ_* cannot be traded for derivatives on Φ or Φ^*.

(i) If $s = 0$, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\sum |\varphi_*||\varphi||\varphi^*|(n+k^2)^{1/2}\\ &\le \Big(\sum (n+k^2)^{1/2}|\varphi|(1+n_*+k_*^2)^{3/4}|\varphi_*|^{3/2}(1+n^*+(k^*)^2)^{-3/4}|\varphi^*|^{1/2}\Big)^{1/2}\\ &\times \Big(\sum (n+k^2)^{1/2}|\varphi|(1+n_*+k_*^2)^{-3/4}|\varphi_*|^{1/2}(1+n^*+(k^*)^2)^{3/4}|\varphi^*|^{3/2}\Big)^{1/2}.\end{aligned}$$

By (3.3.2) and (3.3.3), we therefore get

$$\begin{aligned}&\sum |\varphi_*||\varphi||\varphi^*|(n+k^2)^{1/2}\\ &\le \Big(\|(n+k^2)^{1/2}\varphi\|_{\ell^2(S^*)}\|(1+n_*+k_*^2)^{1/2}\varphi_*\|^{3/2}_{\ell^2(S^*)}\|(1+n^*+(k^*)^2)^{-3/4}\|_{\ell^{2,\infty}(S)}\|\varphi^*\|^{1/2}_{\ell^2(S^*)}\Big)^{1/2}\\ &\times \Big(\|(n+k^2)^{1/2}\varphi\|_{\ell^2(S^*)}\|(1+n_*+k_*^2)^{-3/4}\|_{\ell^{2,\infty}(S)}\|\varphi_*\|^{1/2}_{\ell^2(S^*)}\|(1+n^*+(k^*)^2)^{1/2}\varphi^*\|^{3/2}_{\ell^2(S^*)}\Big)^{1/2}\end{aligned}$$

and a similar estimate for the symmetric term.

By definition,

$$\|(1+n+k^2)^{1/2}\varphi\|_{\ell^2(S^*)} \le C\|\Pi_\perp\Phi\|_{H^1_L}. \tag{3.3.5}$$

Plugging this estimate in the previous inequality leads to

$$\begin{aligned}\left|\Big(\Phi_*|\tilde{Q}_L(\Phi,\Phi^*)\Big)_{L^2(\mathbf{R}\times\mathbf{T})}\right| &\le C\|\Pi_\perp\Phi_*\|^{3/4}_{H^1_L}\|\Pi_\perp\Phi_*\|^{1/4}_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi\|^{3/4}_{H^1_L}\|\Pi_\perp\Phi^*\|^{3/4}_{H^1_L}\\ &\times\Big(\|\Pi_\perp\Phi^*\|^{1/4}_{H^1_L}\|\Pi_\perp\Phi\|^{1/4}_{L^2(\mathbf{R}\times\mathbf{T})}+\|\Pi_\perp\Phi\|^{1/4}_{H^1_L}\|\Pi_\perp\Phi^*\|^{1/4}_{L^2(\mathbf{R}\times\mathbf{T})}\Big).\end{aligned}\tag{3.3.6}$$

(ii) If $s = 0$, another way to estimate the L^2 scalar product is as follows

$$\begin{aligned}&\sum |\varphi_*||\varphi||\varphi^*|(n+k^2)^{1/2}\\ &\le \left(\sum \Big((1+n+k^2)^{3/4}|\varphi|\Big)^{3/2}\Big((1+n_*+k_*^2)^{1/4}|\varphi_*|\Big)^{3/2}(1+n^*+(k^*)^2)^{-3/4}\right)^{1/3}\\ &\times \left(\sum \Big((1+n+k^2)^{3/4}|\varphi|\Big)^{3/2}\Big((1+n^*+(k^*)^2)^{1/4}|\varphi^*|\Big)^{3/2}(1+n_*+k_*^2)^{-3/4}\right)^{1/3}\\ &\times \left(\sum (1+n+k^2)^{-3/4}\Big((1+n_*+k_*^2)^{1/4}|\varphi_*|\Big)^{3/2}\Big((1+n^*+(k^*)^2)^{1/4}|\varphi^*|\Big)^{3/2}\right)^{1/3}\end{aligned}$$

and

$$\begin{aligned}&\sum |\varphi_*||\varphi||\varphi^*|(n^*+(k^*)^2)^{1/2}\\ &\le \left(\sum (n^*+(k^*)^2)^{1/2}|\varphi^*|\Big((1+n+k^2)^{1/2}|\varphi|\Big)^{7/6}\Big((1+n_*+k_*^2)^{1/4}|\varphi_*|\Big)^{7/6}(1+n_*+k_*^2)^{-1/2}\right)^{1/2}\\ &\times \left(\sum (n^*+(k^*)^2)^{1/2}|\varphi^*|\Big((1+n+k^2)^{1/2}|\varphi|\Big)^{5/6}\Big((1+n_*+k_*^2)^{1/4}|\varphi_*|\Big)^{5/6}(1+n+k^2)^{-1/4}\right)^{1/2}.\end{aligned}$$

By (3.3.2) and (3.3.3), we therefore get

$$\begin{aligned}\sum |\varphi_*||\varphi||\varphi^*|(n+k^2)^{1/2} &\le \|(1+n+k^2)^{3/4}\varphi\|_{\ell^2(S^*)}\|(1+n_*+k_*^2)^{1/4}\varphi_*\|_{\ell^2(S^*)}\\ &\times\|(1+n^*+(k^*)^2)^{1/4}\varphi^*\|_{\ell^2(S^*)}\|(1+n+k^2)^{-3/4}\|_{\ell^{2,\infty}(S^*)}\end{aligned}$$

and

$$\sum |\varphi_*||\varphi||\varphi^*|(n^*+(k^*)^2)^{1/2} \leq \|(n^*+(k^*)^2)^{1/2}\varphi^*\|_{\ell^2(S^*)}\|(1+n+k^2)^{1/2}\varphi\|_{\ell^2(S^*)} \times\|(1+n_*+k_*^2)^{1/4}\varphi_*\|_{\ell^2(S^*)}\|(1+n+k^2)^{-3/4}\|_{\ell^{2,\infty}(S^*)}.$$

So we find

(3.3.7)
$$\left|\left(\Phi_*|\tilde Q_L(\Phi,\Phi^*)\right)_{L^2(\mathbf{R}\times\mathbf{T})}\right| \leq C\|\Pi_\perp\Phi\|_{H_L^{3/2}}\|\Pi_\perp\Phi_*\|_{H_L^1}^{1/2}\|\Pi_\perp\Phi_*\|_{L^2(\mathbf{R}\times\mathbf{T})}^{1/2}\|\Pi_\perp\Phi^*\|_{H_L^1}^{1/2}\|\Pi_\perp\Phi^*\|_{L^2(\mathbf{R}\times\mathbf{T})}^{1/2} + C\|\Pi_\perp\Phi\|_{H_L^1}\|\Pi_\perp\Phi^*\|_{H_L^1}\|\Pi_\perp\Phi_*\|_{H_L^1}^{1/2}\|\Pi_\perp\Phi_*\|_{L^2(\mathbf{R}\times\mathbf{T})}^{1/2}.$$

(iii) If $s=0$ and we have additional regularity on Φ and Φ^*, then again one can write a different estimate. We can write indeed

(3.3.8)
$$\sum |\varphi_*||\varphi||\varphi^*|(n^*+(k^*)^2)^{1/2} \leq \|\varphi_*\|_{\ell^2(S^*)}\|(1+n^*+(k^*)^2)^{\alpha/2}\varphi^*\|_{\ell^2(S^*)} \times \|(1+n^*+(k^*)^2)^{1/2-\alpha/2}\|_{\ell^\infty(S^*)}\|(1+n+k^2)^{s/2}\varphi\|_{\ell^2(S^*)}\|(1+n+k^2)^{-\alpha/2}\|_{\ell^2(S^*)},$$

which, coupled with the similar estimate for the symmetric term, gives the expected result.

(iv) If $s\leq 1$, we have by Hölder's inequality

$$\sum(1+n_*+k_*^2)^s|\varphi_*||\varphi||\varphi^*|(n+k^2)^{1/2}$$
$$\leq \left(\sum(1+n_*+k_*^2)^s|\varphi_*|\left((1+n+k^2)^{3/4}|\varphi|\right)^{7/6}\left((1+n^*+(k^*)^2)^{1/2}|\varphi^*|\right)^{7/6}(1+n+k^2)^{-1/2}\right)^{1/2}$$
$$\times\left(\sum(1+n_*+k_*^2)^s|\varphi_*|\left((1+n+k^2)^{3/4}|\varphi|\right)^{5/6}\left((1+n^*+(k^*)^2)^{1/2}|\varphi^*|\right)^{5/6}(1+n^*+(k^*)^2)^{-1}\right)^{1/2}$$

from which we deduce

$$\sum(1+n_*+k_*^2)^s|\varphi_*||\varphi||\varphi^*|(n+k^2)^{1/2} \leq C\|(1+n_*+k_*^2)^s\varphi_*\|_{\ell^2(S^*)}\|(1+n+k^2)^{3/4}\varphi\|_{\ell^2(S^*)} \times\|(1+n^*+(k^*)^2)^{1/2}\varphi^*\|_{\ell^2(S^*)}\|(1+n+k^2)^{-3/4}\|_{\ell^{2,\infty}(S^*)}$$

and a similar estimate for the symmetric term.

Therefore, we get

$$\left|\left(\Phi_*|\tilde Q_L(\Phi,\Phi^*)\right)_{H_L^s}\right| \leq C\|\Pi_\perp\Phi_*\|_{H_L^{2s}}(\|\Pi_\perp\Phi\|_{H_L^{3/2}}\|\Pi_\perp\Phi^*\|_{H_L^1}+\|\Pi_\perp\Phi^*\|_{H_L^{3/2}}\|\Pi_\perp\Phi\|_{H_L^1}),$$

and we conclude by interpolation

(3.3.9)
$$\left|\left(\Phi_*|\tilde Q_L(\Phi,\Phi^*)\right)_{H_L^s}\right| \leq C\left(\|\Pi_\perp\Phi\|_{H_L^{3/2}}\|\Pi_\perp\Phi^*\|_{H_L^s}^s\|\Pi_\perp\Phi^*\|_{H_L^{s+1}}^{1-s}+\|\Pi_\perp\Phi^*\|_{H_L^{3/2}}\|\Pi_\perp\Phi\|_{H_L^s}^s\|\Pi_\perp\Phi\|_{H_L^{s+1}}^{1-s}\right) \times\|\Pi_\perp\Phi_*\|_{H_L^s}^{1-s}\|\Pi_\perp\Phi_*\|_{H_L^{s+1}}^s.$$

• Proposition 2.4.5 shows that

$$Q_L(\Phi,\Phi^*) = Q_L(\Pi_0\Phi,\Pi_\perp\Phi^*)+Q_L(\Pi_\perp\Phi,\Pi_0\Phi^*)+\tilde Q_L(\Phi,\Phi^*).$$

In order to end the proof of the proposition, it remains therefore to estimate the terms coupling the geostrophic and ageostrophic parts. We start by noticing that the constraint $\tau(n,k,j)=\tau(n',k,j')$

implies by Proposition 2.2.1 page 9 that necessarily $n = n'$ and $j = j'$, except if $n = 0$. But according to Remark 2.2.2 that case corresponds to two different values of j for the same eigenvalue only if $2k^2 = \beta$, in which case the multiple root is $k = \pm\sqrt{\beta/2}$. That means that one can write

$$\begin{aligned}|\,(\Phi_*|Q_L(\Pi_0\Phi,\Pi_\perp\Phi^*))_{H^s_L}\,| \leq & \left|\sum_{(n,k,j)\in S^*}(1+n+k^2)^s\,(\Pi_{n,k,j}\Phi_*|Q(\Pi_0\Phi,\Pi_{n,k,j}\Phi^*))_{L^2(\mathbf{R}\times\mathbf{T})}\right| \\ & + \sum_{j=\pm1}\left|(1+\frac{\beta}{2})^s\left(\Pi_{0,j\sqrt{\beta/2},0}\Phi_*|Q(\Pi_0\Phi,\Pi_{0,\sqrt{\beta/2},j}\Phi^*)\right)_{L^2(\mathbf{R}\times\mathbf{T})}\right| \\ & + \sum_{j=\pm1}\left|(1+\frac{\beta}{2})^s\left(\Pi_{0,j\sqrt{\beta/2},j}\Phi_*|Q(\Pi_0\Phi,\Pi_{0,\sqrt{\beta/2},0}\Phi^*)\right)_{L^2(\mathbf{R}\times\mathbf{T})}\right|.\end{aligned}$$

Integrating by parts when the derivative acts on $\Pi_0\Phi$, we get

$$\left|(\Pi_{n,k,j}\Phi_*|Q(\Pi_0\Phi,\Pi_{n,k,j^*}\Phi^*))_{L^2(\mathbf{R}\times\mathbf{T})}\right| \leq C|(\varphi_*)_{n,k,j}||(\varphi^*)_{n,k,j^*}|(n+k^2)^{1/2}\|\Pi_0\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}.$$

By the Cauchy-Schwarz inequality, we then get

$$\begin{aligned}&|\,(\Phi_*|Q_L(\Pi_0\Phi,\Pi_\perp\Phi^*))_{H^s_L}\,| \\ &\leq C\|\Pi_0\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}\|(1+n_*+k_*^2)^{s/2}\varphi_*\|_{\ell^2(S^*)}\|(1+n^*+(k^*)^2)^{(s+1)/2}\varphi^*\|_{\ell^2(S^*)}\end{aligned}$$

Remark that the derivatives can be distributed either on Φ_* or on Φ^*.

We finally deduce that

$$\begin{aligned}&\left|\left(\Phi_*|\tilde{Q}_L(\Pi_0\Phi,\Phi^*)\right)_{H^s_L}+\left(\Phi_*|\tilde{Q}_L(\Phi,\Pi_0\Phi^*)\right)_{H^s_L}\right| \\ &\leq C\|\Pi_\perp\Phi_*\|_{H^s_L}\left(\|\Pi_0\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi^*\|_{H^{s+1}_L}+\|\Pi_0\Phi^*\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi\|_{H^{s+1}_L}\right)\end{aligned} \tag{3.3.10}$$

Note that this term is not zero as in the case of the usual Sobolev spaces, because the spectrum of L is not symmetric with respect to 0.

One should also remark that in (3.3.10), no derivative acts on any vector field in Ker$L$. This can seem somewhat surprising, but is due to the very strong constraint induced by the resonance: instead of a summation over three types of indexes (namely (n,k), (n^*,k^*) and (n_*,k_*)), one only sums over n.

Combining (3.3.6), (3.3.7) and (3.3.8) with (3.3.10) (with $s = 0$) provides the first estimate of the proposition, while (3.3.9) and (3.3.10) (with $s \leq 1$) give the second one.

The proposition is proved. □

3.4. Proof of Theorem 2

The proof of Theorem 2 is divided into four steps. In Paragraph 3.4.1 is proved the existence of weak solutions, and the propagation of regularity of the geostrophic part is proved in Paragraph 3.4.2. The construction of strong solutions is performed in Paragraph 3.4.3, while the propagation of regularity of the ageostrophic part is proved in Paragraph 3.4.4.

3.4.1. Weak solutions. — In this section we are going to prove the existence of weak solutions to the limit filtered system (SW_0). We follow the lines of the classical proof of the Leray theorem, stating the existence of weak solutions to the Navier-Stokes equations.

• *Definition of the approximation scheme*

Denote by K_N the truncation operator defined by

$$K_N = \sum_{\substack{(n,k,j)\in S \\ (n+k^2)^{1/2}\leq N}} \Pi_{n,k,j}\,. \tag{3.4.1}$$

Clearly the operator $K_N Q_L(K_N\Phi, K_N\Phi) - \nu K_N \Delta'_L K_N \Phi$ is continuous on $L^2(\mathbf{R}\times\mathbf{T})$ (with a norm depending on N). Therefore, we deduce from the Cauchy-Lipschitz theorem that there exists a unique maximal solution $\Phi_{(N)} \in C([0,T_N[, L^2(\mathbf{R}\times\mathbf{T}))$ to

$$\begin{aligned} \partial_t \Phi_{(N)} + K_N Q_L(K_N\Phi_{(N)}, K_N\Phi_{(N)}) - \nu K_N \Delta'_L K_N \Phi_{(N)} = 0, \\ \Phi_{(N)}(t=0) = K_N \Phi^0\,. \end{aligned} \tag{3.4.2}$$

Note that the uniqueness implies in particular that $\Phi_{(N)} = K_N \Phi_{(N)}$.

Now let us write an energy estimate on (3.4.2). The quadratic form being skew-symmetric in L^2 (this can easily be seen by its definition as the limit of the filtered quadratic form in (2.3.3) page 18) we find that

$$\frac{1}{2}\frac{d}{dt}\|\Phi_{(N)}(t)\|^2_{L^2(\mathbf{R}\times\mathbf{T})} - \nu(\Delta'_L\Phi_{(N)}|\Phi_{(N)})_{L^2(\mathbf{R}\times\mathbf{T})} = 0.$$

Applying Lemma 3.3.1 implies that

$$-(\Delta'_L \Pi_\perp \Phi_{(N)}|\Pi_\perp\Phi_{(N)})_{L^2(\mathbf{R}\times\mathbf{T})} \geq C_0^{-1}\|\Pi_\perp\Phi_{(N)}\|^2_{H^1(\mathbf{R}\times\mathbf{T})}$$

so we infer by Gronwall's lemma that

$$\begin{aligned} \|\Phi_{(N)}(t)\|^2_{L^2(\mathbf{R}\times\mathbf{T})} &+ 2\nu\int_0^t \|\nabla(\Pi_0\Phi_{(N)})'(t')\|^2_{L^2(\mathbf{R}\times\mathbf{T})}dt' + 2\frac{\nu}{C_0}\int_0^t \|\Pi_\perp\Phi_{(N)}(t')\|^2_{H^1(\mathbf{R}\times\mathbf{T})}dt' \\ &\leq \|K_N\Phi^0\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \leq \|\Phi^0\|^2_{L^2(\mathbf{R}\times\mathbf{T})}, \end{aligned} \tag{3.4.3}$$

so that the approximate solution is defined globally, i.e., $T_N = +\infty$. Moreover the proof of Lemma 3.3.1 also implies that

$$\Pi_\perp\Phi_{(N)} \text{ is bounded in } L^2(\mathbf{R}^+; H^1_L).$$

• *Existence of a weak solution*

We will only sketch the proof of the existence of a weak solution, as it is very similar to the case of the 3D incompressible Navier-Stokes equations. By (3.4.3) we deduce that

$$\begin{aligned} ((\Phi_{(N)})_0) &\text{ is uniformly bounded in } L^\infty(\mathbf{R}^+, L^2(\mathbf{R}\times\mathbf{T})) \\ (\Phi'_{(N)}) &\text{ is uniformly bounded in } L^\infty(\mathbf{R}^+, L^2(\mathbf{R}\times\mathbf{T}))\cap L^2_{loc}(\mathbf{R}^+, H^1(\mathbf{R}\times\mathbf{T})) \text{ and} \\ \Pi_\perp\Phi_{(N)} &\text{ is uniformly bounded in } L^\infty(\mathbf{R}^+, L^2(\mathbf{R}\times\mathbf{T}))\cap L^2(\mathbf{R}^+, H^1_L). \end{aligned}$$

For any $h > 0$, denote by $\delta_h \Phi_{(N)}(t,x) = \Phi_{(N)}(t+h,x) - \Phi_{(N)}(t,x)$. Then

$$\begin{aligned}
&\|\delta_h \Phi_{(N)}(t)\|^2_{L^2([0,T]\times\mathbf{R}\times\mathbf{T})} \\
&= \int_0^T \left(\delta_h \Phi_{(N)}(t) \Big| \int_t^{t+h} \partial_t \Phi_{(N)}(s) ds \right)_{L^2(\mathbf{R}\times\mathbf{T})} dt \\
&= - \int_0^T \int_t^{t+h} \left(\delta_h \Phi_{(N)}(t) | Q_L(\Phi_{(N)}(s), \Phi_{(N)}(s)) \right)_{L^2(\mathbf{R}\times\mathbf{T})} dsdt \\
&+ \nu \int_0^T \int_t^{t+h} \left(\delta_h \Phi_{(N)}(t) | \Delta'_L \Phi_{(N)}(s) \right)_{L^2(\mathbf{R}\times\mathbf{T})} dsdt
\end{aligned}$$

By Proposition 3.3.3 and the positivity of $-\Delta'_L$, we deduce that

$$\begin{aligned}
&\|\delta_h \Phi_{(N)}(t)\|^2_{L^2([0,T]\times\mathbf{R}\times\mathbf{T})} \\
&\leq C \int_0^T \int_t^{t+h} \|\Pi_\perp \delta_h \Phi_{(N)}(t)\|^{3/4}_{H^1_L} \|\Pi_\perp \delta_h \Phi_{(N)}(t)\|^{1/4}_{L^2(\mathbf{R}\times\mathbf{T})} \|\Pi_\perp \Phi_{(N)}(s)\|^{7/4}_{H^1_L} \\
&\qquad\qquad \times \|\Pi_\perp \Phi_{(N)}(s)\|^{1/4}_{L^2(\mathbf{R}\times\mathbf{T})} dsdt \\
&+ C \int_0^T \int_t^{t+h} \|\delta_h \Pi_\perp \Phi_{(N)}(t)\|_{L^2} \|\Pi_0 \Phi_{(N)}(s)\|_{L^2(\mathbf{R}\times\mathbf{T})} \|\Pi_\perp \Phi_{(N)}(s)\|_{H^1_L} dsdt \\
&+ \nu \int_0^T \int_t^{t+h} \left(\Phi_{(N)}(s) | \Delta'_L \Phi_{(N)}(s) \right)^{1/2}_{L^2(\mathbf{R}\times\mathbf{T})} \left(\delta_h \Phi_{(N)}(t) | \Delta'_L \delta_h \Phi_{(N)}(t) \right)^{1/2}_{L^2(\mathbf{R}\times\mathbf{T})} dsdt
\end{aligned}$$

Therefore, using the uniform $L^\infty(\mathbf{R}^+, L^2(\mathbf{R}\times\mathbf{T}))$ bounds on $\Phi_{(N)}$ and $\delta_h \Phi_{(N)}$, and the uniform $L^2(\mathbf{R}^+, H^1_L)$ bounds on $\Pi_\perp \Phi_{(N)}$ and $\Pi_\perp \delta_h \Phi_{(N)}$ coming from the energy estimate, we get by Hölder's inequality

$$\begin{aligned}
&\|\delta_h \Phi_{(N)}(t)\|^2_{L^2([0,T]\times\mathbf{R}\times\mathbf{T})} \\
&\leq C T^{5/8} \|\Pi_\perp \delta_h \Phi_{(N)}\|^{3/4}_{L^2([0,T],H^1_L)} \|\Pi_\perp \Phi_{(N)}\|^{7/4}_{L^2([0,T],H^1_L)} h^{1/8} + CT \|\Pi_\perp \delta_h \Phi_{(N)}\|_{L^2([0,T],H^1_L)} h^{1/2} \\
&+ C\nu T^{1/2} \left(\Phi_{(N)} | \Delta'_L \Phi_{(N)} \right)^2_{L^2(\mathbf{R}\times\mathbf{R}\times\mathbf{T})} h^{1/2},
\end{aligned}$$

and thus

$$\|\delta_h \Phi_{(N)}(t)\|^2_{L^2([0,T]\times\mathbf{R}\times\mathbf{T})} \leq C_T h^{1/8}.$$

By interpolation, on gets therefore that (up to extraction)

$$\begin{aligned}
\Pi_0 \Phi_{(N)} &\rightharpoonup \Pi_0 \Phi \text{ weakly in } L^2(\mathbf{R}^+, L^2_{loc}(\mathbf{R}\times\mathbf{T})) \\
\Phi'_{(N)} &\to \Phi' \text{ strongly in } L^2_{loc}(\mathbf{R}^+, L^2_{loc}(\mathbf{R}\times\mathbf{T})) \\
\Pi_\perp \Phi_{(N)} &\to \Pi_\perp \Phi \text{ strongly in } L^2_{loc}(\mathbf{R}^+, L^2(\mathbf{R}\times\mathbf{T})).
\end{aligned}$$

Note that, because of Remark 3.1.3, the last convergence is actually global in space. We are then able, as in the usual case of the 3D Navier-Stokes equations, to take limits in the weak formulation of (3.4.2), which proves that Φ is a weak solution to (SW_0).

• *Strong-weak uniqueness*

In general such a weak solution is not unique and the Cauchy problem is not well-posed in $L^2(\mathbf{R}\times\mathbf{T})$. Nevertheless we have the following strong-weak uniqueness principle.

Proposition 3.4.1. — *There is a positive constant C and a nondecreasing, positive function $C(t)$ such that the following holds. Let Φ and Φ_* be two weak solutions to (SW_0) with respective initial data*

Φ^0 and Φ^0_*, satisfying the energy estimate. Assume that there exists some $T>0$ such that $\Pi_\perp\Phi$ belongs to $L^\infty([0,T],H^{1/2}_L)\cap L^2([0,T],H^{3/2}_L)$. Then for all $t\in[0,T]$, the function $\delta\Phi=\Phi_*-\Phi$ satisfies

$$\|\delta\Phi(t)\|^2_{L^2}\leq\|\delta\Phi(0)\|^2_{L^2}\exp\Big(C(t)(1+\|\Phi^0\|^2_{L^2})+C_\nu(1+\|\Pi_\perp\Phi\|^2_{L^\infty([0,t];H^{1/2}_L)})\int_0^t\|\Pi_\perp\Phi(t')\|^2_{H^{3/2}_L}dt'\Big).$$

In particular, $\Phi_*=\Phi$ on $[0,T]\times\mathbf{R}\times\mathbf{T}$ if $\Phi^0_*=\Phi^0$.

Proof. — In order to establish the stability inequality we start by writing (formally) the equation on $\delta\Phi=\Phi_*-\Phi$

$$\partial_t\delta\Phi+Q_L(\delta\Phi,\delta\Phi)+2Q_L(\delta\Phi,\Phi)-\nu\Delta'_L\delta\Phi=0. \tag{3.4.4}$$

Proposition 3.3.3 implies that

$$\begin{aligned}|\,(\delta\Phi|Q_L(\delta\Phi,\Phi))_{L^2}\,|&\leq C\|\Pi_\perp\Phi\|_{H^{3/2}_L}\|\Pi_\perp\delta\Phi\|_{H^1_L}\|\Pi_\perp\delta\Phi\|_{L^2}\\&\quad+C\|\Pi_\perp\Phi\|_{H^1_L}\|\Pi_\perp\delta\Phi\|^{3/2}_{H^1_L}\|\Pi_\perp\delta\Phi\|^{1/2}_{L^2}\\&\quad+C\|\Pi_\perp\delta\Phi\|_{L^2}(\|\Pi_0\Phi\|_{L^2}\|\Pi_\perp\delta\Phi\|_{H^1_L}+\|\Pi_0\delta\Phi\|_{L^2}\|\Pi_\perp\Phi\|_{H^1_L}).\end{aligned}$$

We then deduce (using the same argument as in the construction of a weak solution page 40 for the Δ'_L term), that

$$\begin{aligned}\|\delta\Phi(t)\|^2_{L^2(\mathbf{R}\times\mathbf{T})}-\|\delta\Phi^0\|^2_{L^2(\mathbf{R}\times\mathbf{T})}+2\nu\int_0^t\|\Pi_0(\delta\Phi)'(t')\|_{L^2(\mathbf{R}\times\mathbf{T})}dt'+\frac{\nu}{C}\int_0^t\|\Pi_\perp\delta\Phi(t')\|^2_{H^1_L}dt'\\\leq C_\nu\int_0^t\Big(\|\Pi_\perp\Phi(t')\|^2_{H^{3/2}_L}+\|\Pi_\perp\Phi(t')\|^4_{H^1_L}\Big)\|\Pi_\perp\delta\Phi(t')\|^2_{L^2}dt'\\+C_\nu\int_0^t\Big(\|\Pi_0\Phi(t')\|^2_{L^2}\|\Pi_\perp\delta\Phi(t')\|^2_{L^2}+\|\Pi_\perp\Phi(t')\|^2_{H^1_L}\|\Pi_0\delta\Phi\|^2_{L^2}\Big)dt',\end{aligned}$$

using the embedding $H^1_L\subset L^2$. Gronwall's lemma yields

$$\|\delta\Phi(t)\|^2_{L^2}\leq\|\delta\Phi^0\|^2_{L^2}\exp\Big(C_\nu\int_0^t\big(1+\|\Pi_0\Phi(\tau)\|^2_{L^2(\mathbf{R}\times\mathbf{T})}+\|\Pi_\perp\Phi(\tau)\|^4_{H^1_L}+\|\Pi_\perp\Phi(\tau)\|^2_{H^{3/2}_L}\big)d\tau\Big),$$

and the conclusion comes from the fact that $\Pi_\perp\Phi$ belongs to $L^4([0,T],H^1_L)$ by interpolation between $L^\infty([0,T],H^{1/2}_L)$ and $L^2([0,T],H^{3/2}_L)$, along with the energy estimate on Φ. The proposition is proved. □

3.4.2. Propagation of the geostrophic regularity. — The following regularity result for the geostrophic equation is inspired by the Weyl-Hörmander symbolic calculus, even if that theory does not seem to be appliable directly due to the possible singularity at $x_1=0$.

Using the formula giving Π_0 in Proposition 2.1.2, we first see that the geostrophic equation

$$\partial_t\Phi-\nu\Pi_0\Delta'\Pi_0\Phi=0$$

can be brought back (at least formally) to the scalar equation

$$\partial_t u_2-\nu D(DD^T+Id)^{-1}D^T\Delta u_2=0,$$

where we recall that D is the differential operator defined by $D\cdot=\partial_1\left(\dfrac{\cdot}{\beta x_1}\right)$. Then by a simple change of variables this scalar equation becomes

$$\partial_t\varphi-\nu A\varphi=0,$$

where A is some self-adjoint scalar pseudo-differential operator (possibly singular at $x_1 = 0$), the principal symbol of which is given by

$$a(x_1, \xi_1) = \frac{\xi_1^4}{\beta^2 x_1^2 + \xi_1^2}$$

neglecting the possible singularity at $x_1 = 0$. Then, in order to propagate Sobolev regularity on φ, we should have to control some commutator of the type $[A, \nabla^s]$, which is not so easy because A cannot be written simply in terms of the usual derivatives ∇.

In order to find a convenient way to measure the regularity, we therefore use formally the results of symbolic calculus. Note that the Weyl-Hörmander theory is used here just to guide intuition, the result of propagation being actually proved by explicit computations. In order to determine the class of operators A should belong to, we have first to characterize the metric. Computing the partial derivatives of a with respect to x_1 and ξ_1

$$\frac{\partial_{x_1} a}{a}(x_1, \xi_1) = -\frac{2\beta^2 x_1}{\beta^2 x_1^2 + \xi_1^2}, \quad \frac{\partial_{\xi_1} a}{a}(x_1, \xi_1) = \frac{4}{\xi_1} - \frac{2\xi_1}{\beta^2 x_1^2 + \xi_1^2}$$

shows that the Hörmander metric to be considered is the one associated to the harmonic oscillator

$$g(dx_1, d\xi_1) = \beta^2 \frac{dx_1^2}{1 + x_1^2} + \frac{d\xi_1^2}{1 + \xi_1^2}.$$

Then it is natural to measure the regularity by powers of the harmonic oscillator, and therefore to study the propagation equation

$$\partial_t (-\partial^2_{x_1 x_1} + \beta^2 x_1^2)^s \varphi - \nu A (-\partial^2_{x_1 x_1} + \beta^2 x_1^2)^s \varphi = \nu [(-\partial^2_{x_1 x_1} + \beta^2 x_1^2)^s, A] \varphi$$

The fundamental result of the Weyl-Hörmander theory states the following: if A is a pseudo-differential operator (meaning in particular that there is no singularity at $x_1 = 0$), the commutator occuring in the right-hand side of the previous equation is a pseudo-differential operator of lower order (for the metric g), meaning that we expect $\|[(-\partial^2_{x_1 x_1} + \beta^2 x_1^2)^s, A]\varphi\|_{L^2(\mathbf{R})}$ to be controlled by $\|(-\partial^2_{x_1 x_1} + \beta^2 x_1^2)^s \varphi\|_{L^2(\mathbf{R})}$ and $-\big((-\partial^2_{x_1 x_1} + \beta^2 x_1^2)^s \varphi | A(-\partial^2_{x_1 x_1} + \beta^2 x_1^2)^s \varphi\big)$.

Nevertheless, as we are not able to prove in a simple way that there is no singularity at $x_1 = 0$, we shall not use the general theory of pseudo-differential operators and will proceed instead using explicit computations. We have seen in the previous chapter that the family $(\Psi_{n,0,0})_{n \in \mathbf{N}}$ defined by

$$\Psi_{n,0,0}(x_1) = \begin{pmatrix} -\sqrt{\frac{(n+1)}{2(2n+1)}}\psi_{n-1}(x_1) - \sqrt{\frac{n}{2(2n+1)}}\psi_{n+1}(x_1) \\ 0 \\ \sqrt{\frac{(n+1)}{2(2n+1)}}\psi_{n-1}(x_1) - \sqrt{\frac{n}{2(2n+1)}}\psi_{n+1}(x_1) \end{pmatrix}$$

constitutes an Hermitian basis of Ker$L$, and that, due to the properties of the Hermite functions,

$$\forall n \in \mathbf{N}, \quad \left\|(-\partial^2_{x_1 x_1} + \beta^2 x_1^2)^s \Psi_{n,0,0}\right\|_{L^2(\mathbf{R})} \sim (1+n)^s \left\|\Psi_{n,0,0}\right\|_{L^2(\mathbf{R})}.$$

Therefore it is natural to study the propagation of the H^s_L norm of $\Pi_0 \Phi$, recalling that

$$\|\Pi_0 \Phi\|^2_{H^s_L} = \sum_{n \in \mathbf{N}} (1+n)^s |\underline{\varphi}_n|^2,$$

where we have defined

$$\underline{\varphi}_n = (\Psi_{n,0,0} | \Pi_0 \Phi).$$

In the following we will denote by N_s the operator defined by

$$\forall n \in \mathbf{N}, \quad N_s \Pi_0 \Phi = \sum (1+n)^s \underline{\varphi}_n \Psi_{n,0,0},$$

so that

$$\|N_s \Pi_0 \Phi\|_{L^2(\mathbf{R})} = \|\Pi_0 \Phi\|_{H_L^{2s}}.$$

We have

$$\partial_t N_s \Pi_0 \Phi - \nu \Pi_0 \Delta' N_s \Pi_0 \Phi = \nu [N_s, \Pi_0 \Delta'] \bar{\Phi}.$$

• We start by computing $\Pi_0 \Delta' \Psi_{n,0,0}$. From (2.2.1) page 9 we deduce that

$$\partial_{x_1} \Psi'_{n,0,0}(x_1) = \begin{pmatrix} 0 \\ \\ \sqrt{\frac{\beta(n+1)(n-1)}{4(2n+1)}} \psi_{n-2}(x_1) - \sqrt{\frac{\beta n(n+1)}{2n+1}} \psi_n(x_1) + \sqrt{\frac{\beta n(n+2)}{4(2n+1)}} \psi_{n+2}(x_1) \end{pmatrix}$$

and

$$\partial^2_{x_1 x_1} \Psi'_{n,0,0}(x_1) = \frac{\beta}{2\sqrt{2(2n+1)}} \begin{pmatrix} 0 \\ \\ \sqrt{(n+1)(n-1)(n-2)} \psi_{n-3}(x_1) - (3n-1)\sqrt{n+1} \psi_{n-1}(x_1) \\ +(3n+4)\sqrt{n} \psi_{n+1}(x_1) - \sqrt{n(n+2)(n+3)} \psi_{n+3}(x_1) \end{pmatrix}$$

with the usual convention that $\psi_n \equiv 0$ for $n < 0$. Therefore, using the orthogonality of the Hermite functions in $L^2(\mathbf{R})$, we get

$$\Pi_0 \Delta' \Psi_{n,0,0} = \alpha_n^{(-4)} \Psi_{n-4,0,0} + \alpha_n^{(-2)} \Psi_{n-2,0,0} + \alpha_n^{(0)} \Psi_{n,0,0} + \alpha_n^{(2)} \Psi_{n+2,0,0} + \alpha_n^{(4)} \Psi_{n+4,0,0}$$

with

$$\begin{aligned} \alpha_n^{(-4)} &= -\frac{\beta}{4} \sqrt{\frac{(n-4)(n-2)(n-1)(n+1)}{(2n+1)(2n-7)}}, \\ \alpha_n^{(-2)} &= \frac{\beta}{4}(4n-2) \sqrt{\frac{(n-2)(n+1)}{(2n+1)(2n-3)}}, \\ \alpha_n^{(0)} &= -\frac{\beta}{4} \frac{6n^2+6n-1}{(2n+1)}, \\ \alpha_n^{(2)} &= \frac{\beta}{4}(4n+6) \sqrt{\frac{n(n+3)}{(2n+1)(2n+5)}}, \\ \alpha_n^{(4)} &= -\frac{\beta}{4} \sqrt{\frac{n(n+2)(n+3)(n+5)}{(2n+1)(2n+9)}}. \end{aligned} \tag{3.4.5}$$

• From the previous computation we deduce that

$$\begin{aligned} [N_s, \Pi_0 \Delta'] \Psi_{n,0,0} &= ((n-3)^s - (n+1)^s) \alpha_n^{(-4)} \Psi_{n-4,0,0} + ((n-1)^s - (n+1)^s) \alpha_n^{(-2)} \Psi_{n-2,0,0} \\ &+ ((n+2)^s - (n+1)^s) \alpha_n^{(2)} \Psi_{n+2,0,0} + ((n+5)^s - (n+1)^s) \alpha_n^{(4)} \Psi_{n+4,0,0} \end{aligned}$$

Thus, using the definition (3.4.5) of the coefficients α, we get

$$|\alpha_n| \leq C(n+1)$$

and

$$\|[N_s,\Pi_0\Delta']\Psi_{n,0,0}\|_{L^2(\mathbf{R})} \leq C_s(n+1)^s \leq C_s\|N_s\Psi_{n,0,0}\|_{L^2(\mathbf{R})}.$$

Because of the quasi-orthogonality of $([N_s,\Pi_0\Delta']\Psi_{n,0,0})_{n\in\mathbf{N}}$, we have actually the more general commutator estimate

$$\begin{aligned}
&\|[N_s,\Pi_0\Delta']\Pi_0\Phi\|^2_{L^2(\mathbf{R})}\\
&= \sum_n \Big|((n+1)^s-(n-3)^s)\alpha^{(-4)}_{n-4}\underline{\varphi}_{n+4} + ((n+1)^s-(n-1)^s)\alpha^{(-2)}_{n-2}\underline{\varphi}_{n+2}\\
&\qquad + ((n+1)^s-(n+3)^s)\alpha^{(2)}_{n+2}\underline{\varphi}_{n-2} + ((n+1)^s-(n+5)^s)\alpha^{(4)}_{n+4}\underline{\varphi}_{n-4}\Big|^2\\
&\leq C_s\sum_n (n+1)^{2s}\left(|\underline{\varphi}_{n+4}|^2+|\underline{\varphi}_{n+2}|^2+|\underline{\varphi}_{n-2}|^2+|\underline{\varphi}_{n-4}|^2\right)\\
&\leq C_s\sum_n (n+1)^{2s}|\varphi_n|^2
\end{aligned}$$

which can be rewritten

$$\|[N_s,\Pi_0\Delta']\Pi_0\Phi\|_{L^2(\mathbf{R})} \leq C_s\|N_s\Pi_0\Phi\|_{L^2(\mathbf{R})}. \tag{3.4.6}$$

Note that, due to the particular choice of the operator N_s, there is some additional cancellation, meaning that the commutator $[N_s,\Pi_0\Delta']$ which is expected to be a pseudodifferential operator of order $(2s+1)$ is actually of order $2s$.

• It is now very easy to propagate regularity using Gronwall's lemma. We recall that

$$\partial_t N_s\Pi_0\Phi - \nu\Pi_0\Delta' N_s\Pi_0\Phi = \nu[N_s,\Pi_0\Delta']\Pi_0\Phi,$$

from which we deduce that

$$\|N_s\Pi_0\Phi(t)\|^2_{L^2(\mathbf{R})} + \nu\int_0^t \|\nabla(N_s\Pi_0\Phi)'(\tau)\|^2_{L^2(\mathbf{R})}d\tau \leq \|N_s\Pi_0\Phi^0\|^2_{L^2(\mathbf{R})} + C_s\int_0^t \|N_s\Pi_0\Phi(\tau)\|^2_{L^2(\mathbf{R})}d\tau$$

and finally

$$\|N_s\Pi_0\Phi(t)\|^2_{L^2(\mathbf{R})} + \nu\int_0^t \|\nabla(N_s\Pi_0\Phi)'(\tau)\|^2_{L^2(\mathbf{R})}d\tau \leq \|N_s\Pi_0\Phi^0\|^2_{L^2(\mathbf{R})}\exp(C_st). \tag{3.4.7}$$

This concludes the proof.

3.4.3. Local strong solutions. — In this section we are going to prove the existence of unique, strong solutions for smooth enough initial data. As in the case of weak solutions discussed in Section 3.4.1 above, we will not write the full proof, but detail the estimates enabling one to use the usual Fujita-Kato theory of strong solutions to the 3D Navier-Stokes equations (see [**4**] for instance).

• *Global existence of strong solutions for small data*

We prove here that under a suitable smallness assumption there exists a (unique) global strong solution to (SW_0) such that $\Pi_\perp\Phi$ belongs to $L^\infty(\mathbf{R}^+,H^{1/2}_L)\cap L^2(\mathbf{R}^+,H^{3/2}_L)$.

As previously we start from the solutions $\Phi_{(N)}$ of the approximation scheme (3.4.2). We have of course

$$\frac{d}{dt}\|\Pi_0\Phi_{(N)}(t)\|^2_{L^2(\mathbf{R}\times\mathbf{T})} + 2\nu\|\nabla(\Pi_0\Phi_{(N)})'(t)\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \leq 0$$

and, by Proposition 3.3.3,

$$\begin{aligned}
&\frac{d}{dt}\|\Pi_\perp \Phi_{(N)}(t)\|^2_{H^{1/2}_L} - 2\nu\left(\Pi_\perp\Phi_{(N)}|\Delta'_L\Pi_\perp\Phi_{(N)}\right)_{H^{1/2}_L}(t)\\
&\quad\le -2\left(\Pi_\perp\Phi_{(N)}|\Pi_\perp Q_L(\Phi_{(N)},\Phi_{(N)})\right)_{H^{1/2}_L}(t)\\
&\quad\le C\Big(\|\Pi_\perp\Phi_{(N)}\|^2_{H^{3/2}_L}\|\Pi_\perp\Phi_{(N)}\|_{H^{1/2}_L} + \|\Pi_0\Phi_{(N)}\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi_{(N)}\|_{H^{3/2}_L}\|\Pi_\perp\Phi_{(N)}\|_{H^{1/2}_L(\mathbf{R}\times\mathbf{T})}\Big)(t).
\end{aligned}$$

By Lemma 3.3.1 and the obvious embedding $H^{3/2}_L \subset H^{1/2}_L$, that inequality can be written

$$\begin{aligned}
&\frac{d}{dt}\|\Pi_\perp\Phi_{(N)}(t)\|^2_{H^{1/2}_L} + 2\frac{\nu}{C_{1/2}}\|\Pi_\perp\Phi_{(N)}(t)\|^2_{H^{3/2}_L}\\
&\le C\|\Pi_\perp\Phi_{(N)}\|^2_{H^{3/2}_L}\Big(\|\Pi_\perp\Phi_{(N)}\|_{H^{1/2}_L} + \|\Pi_0\Phi_{(N)}\|_{L^2(\mathbf{R}\times\mathbf{T})}\Big)(t).
\end{aligned} \tag{3.4.8}$$

As usual we notice that this inequality is useful only if $\|\Pi_\perp\Phi_{(N)}(t)\|_{H^{1/2}_L}$ and $\|\Pi_0\Phi_{(N)}\|_{L^2(\mathbf{R}\times\mathbf{T})}$ are small, which is a typical phenomena of global results under a smallness condition.

Define

$$D_N = \left\{t\in\mathbf{R}^+ \,/\, \forall t'\le t,\ \|\Pi_0\Phi_{(N)}(t')\|^2_{L^2(\mathbf{R}\times\mathbf{T})} + \|\Pi_\perp\Phi_{(N)}(t')\|^2_{H^{1/2}_L} \le \left(\frac{\nu}{2CC_{1/2}}\right)^2\right\},$$

where C is the constant appearing in (3.4.8), and let us impose the following smallness assumption on the initial data:

$$\|\Pi_0\Phi^0_{(N)}\|^2_{L^2(\mathbf{R}\times\mathbf{T})} + \|\Pi_\perp\Phi^0_{(N)}\|^2_{H^{1/2}_L} \le \left(\frac{\nu}{4CC_{1/2}}\right)^2.$$

Then clearly D_N is not empty. By construction, $\Pi_0\Phi_{(N)}$ belongs to $C(\mathbf{R}^+, L^2(\mathbf{R}\times\mathbf{T}))$ and $\Pi_\perp\Phi_{(N)}$ belongs to $C(\mathbf{R}^+, H^{1/2}_L)$ thus D_N is a closed set.

Denote by $T_N = \max D_N$. If $T_N < +\infty$, then

$$\|\Pi_0\Phi_{(N)}(T_N)\|^2_{L^2(\mathbf{R}\times\mathbf{T})} + \|\Pi_\perp\Phi_{(N)}(T_N)\|^2_{H^{1/2}_L} \le \left(\frac{\nu}{2CC_{1/2}}\right)^2$$

and we deduce from (3.4.8) that

$$\frac{d}{dt}\left(\|\Pi_0\Phi_{(N)}\|^2_{L^2(\mathbf{R}\times\mathbf{T})} + \|\Pi_\perp\Phi_{(N)}\|^2_{H^{1/2}_L(\mathbf{R}\times\mathbf{T})}\right)(T_N) < 0,$$

which is in contradiction with the definition of $T_N = \max D_N$. Therefore $T_N = +\infty$.

Then we deduce immediately that for all $t\in\mathbf{R}^+$

$$\|\Pi_\perp\Phi_{(N)}(t)\|^2_{H^{1/2}_L} + \frac{\nu}{C_{1/2}}\int_0^t \|\Pi_\perp\Phi_{(N)}(t')\|^2_{H^{3/2}_L}dt' \le \|\Pi_\perp\Phi^0_{(N)}\|^2_{H^{1/2}_L}.$$

Up to the extraction of a subsequence that converges to a Leray solution Φ of (SW_0), the previous estimate implies that

$$\Pi_\perp\Phi \in L^\infty(\mathbf{R}^+, H^{1/2}_L)\cap L^2(\mathbf{R}^+, H^{3/2}_L).$$

The strong-weak stability principle established in the previous section provides then the uniqueness of such a solution.

• *Local existence of strong solutions*

Let us now consider the case of large data. The idea (see for instance [**4**]) is to split $\Phi_{(N)}$ in two parts as follows

$$\Phi_{(N)} = \Phi^{<}_{(N)} + \Phi^{>}_{(N)}$$

where $\Phi^{<}_{(N)}$ is the unique solution to

$$\begin{aligned} &\partial_t \Phi^{<}_{(N)} - \nu \Delta'_L \Phi^{<}_{(N)} = 0 \\ &\Phi^{<}_{(N)}(0) = \sum_{\substack{(n,k,j)\in S \\ (n+k^2)^{1/2}\le A}} \Pi_{n,k,j}\Pi_\perp \Phi^0_{(N)} + \Pi_0 \Phi_{(N)}, \end{aligned} \tag{3.4.9}$$

and $A > 0$ is a truncation parameter to be determined (independent of N). Using Proposition 3.3.1, it is easy to check that

$$\begin{aligned} \|\Pi_\perp \Phi^{<}_{(N)}\|_{L^\infty(\mathbf{R}^+, L^2(\mathbf{R}\times\mathbf{T}))} &\le C\|\Phi^0\|_{L^2(\mathbf{R}\times\mathbf{T})}, \\ \|\Pi_\perp \Phi^{<}_{(N)}\|_{L^\infty(\mathbf{R}^+, H^s_L)} &\le CA^s \|\Phi^0\|_{L^2(\mathbf{R}\times\mathbf{T})}. \end{aligned}$$

By (3.4.2) and (3.4.9) we deduce the equation satisfied by $\Phi^{>}_{(N)}$:

$$\begin{aligned} &\partial_t \Phi^{>}_{(N)} - \nu\Delta'_L \Phi^{>}_{(N)} + J_N Q_L(\Phi^{>}_{(N)}, \Phi^{>}_{(N)}) + 2J_N Q_L(\Phi^{>}_{(N)}, \Phi^{<}_{(N)}) = -J_N Q_L(\Phi^{<}_{(N)}, \Phi^{<}_{(N)}) \\ &\Phi^{>}_{(N)}(0) = \Phi^0_{(N)} - \left(\sum_{\substack{(n,k,j)\in S \\ (n+|k|)^{1/2}\le A}} \Pi_{n,k,j}\Pi_\perp \Phi^0_{(N)} + \Pi_0\Phi_{(N)} \right). \end{aligned} \tag{3.4.10}$$

We are going to show that $\Phi^{>}_{(N)}$ remains small in $H^{1/2}_L$ on a time interval which does not depend on N.

Let us write an energy inequality in $H^{1/2}_L$ on (3.4.10): we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Phi^{>}_{(N)}(t)\|^2_{H^{1/2}_L} + \frac{\nu}{C_{1/2}}\|\Phi^{>}_{(N)}(t)\|^2_{H^{3/2}_L} &= -(J_N Q_L(\Phi^{>}_{(N)}, \Phi^{>}_{(N)})|\Phi^{>}_{(N)})_{H^{1/2}_L}(t) \\ &+2(J_N Q_L(\Phi^{>}_{(N)}, \Phi^{<}_{(N)})|\Phi^{>}_{(N)})_{H^{1/2}_L}(t) - (J_N Q_L(\Phi^{<}_{(N)}, \Phi^{<}_{(N)})|\Phi^{>}_{(N)})_{H^{1/2}_L}(t), \end{aligned}$$

and Proposition 3.3.3 yields

$$\begin{aligned} &\frac{d}{dt}\|\Phi^{>}_{(N)}(t)\|^2_{H^{1/2}_L} + 2\frac{\nu}{C_{1/2}}\|\Phi^{>}_{(N)}(t)\|^2_{H^{3/2}_L} \le C\|\Phi^{>}_{(N)}(t)\|_{H^{1/2}_L}\|\Phi^{>}_{(N)}(t)\|^2_{H^{3/2}_L} \\ &+ C\|\Phi^{>}_{(N)}(t)\|_{H^1_L}\big(\|\Phi^{>}_{(N)}(t)\|_{H^{3/2}_L}\|\Pi_\perp\Phi^{<}_{(N)}(t)\|_{H^1_L} + \|\Phi^{>}_{(N)}(t)\|_{H^1_L}\|\Pi_\perp\Phi^{<}_{(N)}(t)\|_{H^{3/2}_L}\big) \\ &+ C\|\Phi^{>}_{(N)}(t)\|_{H^1_L}\big(\|\Phi^{>}_{(N)}(t)\|_{H^1_L}\|\Pi_0\Phi_{(N)}(t)\|_{L^2(\mathbf{R}\times\mathbf{T})} + \|\Pi_\perp\Phi^{<}_{(N)}(t)\|_{H^1_L}\|\Pi_\perp\Phi^{<}_{(N)}(t)\|_{H^{3/2}_L}\big) \\ &+ C\|\Phi^{>}_{(N)}(t)\|_{H^{1/2}_L}\|\Pi_0\Phi_{(N)}(t)\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi^{<}_{(N)}(t)\|_{H^{3/2}_L}. \end{aligned}$$

Now consider the set

$$D_N = \left\{ t\in\mathbf{R}^+ \,/\, \forall t'\le t,\ \|\Phi^{>}_{(N)}(t')\|_{H^{1/2}_L} \le \frac{\nu}{CC_{1/2}} \right\},$$

(where C is the constant appearing in the right-hand side of the previous inequality) and $T_N = \sup D_N$. We are going to prove that there exists $T > 0$ such that

$$\forall N\in\mathbf{N}^*,\quad T_N \ge T.$$

We notice that if A is chosen large enough (independently of N), then D_N is not empty: we can indeed choose A so that

$$\|\Phi^{>}_{(N)}(0)\|_{H^{1/2}_L} \le \frac{\nu}{4CC_{1/2}},\quad \forall N\in\mathbf{N}. \tag{3.4.11}$$

As long as $t \leq T_N$, we can write the previous inequality in the following way:

$$\begin{aligned}\frac{d}{dt}\|\Phi^{>}_{(N)}(t)\|^2_{H^{1/2}_L} + \frac{\nu}{2C_{1/2}}\|\Phi^{>}_{(N)}(t)\|^2_{H^{3/2}_L} &\leq \frac{C}{\nu^3}\|\Phi^{>}_{(N)}(t)\|^2_{H^{1/2}_L}\|\Pi_\perp\Phi^{<}_{(N)}(t)\|^4_{H^1_L}\\
&+ \frac{C}{\nu}\|\Phi^{>}_{(N)}(t)\|^2_{H^{1/2}_L}\left(\|\Pi_\perp\Phi^{<}_{(N)}(t)\|^2_{H^{3/2}_L}\| + \|\Pi_0\Phi_{(N)}(t)\|^2_{L^2(\mathbf{R}\times\mathbf{T})}\right)\\
&+ \frac{C}{\nu^{1/3}}\|\Phi^{>}_{(N)}(t)\|^{2/3}_{H^{1/2}_L}\|\Pi_\perp\Phi^{<}_{(N)}(t)\|^{4/3}_{H^1_L}\|\Pi_\perp\Phi^{<}_{(N)}(t)\|^{4/3}_{H^{3/2}_L}\\
&+ C\|\Phi^{>}_{(N)}(t)\|_{H^{1/2}_L}\|\Pi_0\Phi_{(N)}(t)\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp\Phi^{<}_{(N)}(t)\|_{H^{3/2}_L}\end{aligned}$$

So we get

$$\frac{d}{dt}\|\Phi^{>}_{(N)}(t)\|^2_{H^{1/2}_L} + \frac{\nu}{2C_{1/2}}\|\Phi^{>}_{(N)}(t)\|^2_{H^{3/2}_L} \leq (1+\|\Phi^{>}_{(N)}(t)\|^2_{H^{1/2}_L})F(A,\Phi^0,\nu),$$

where

$$F(A,\Phi^0,\nu) = C\left(1+\frac{1}{\nu^3}\right)(1+A^4)(1+\|\Phi^0\|^4_{L^2(\mathbf{R}\times\mathbf{T})}).$$

Gronwall's lemma enables us to infer that for all $t \leq T_N$,

$$\|\Phi^{>}_{(N)}(t)\|^2_{H^{1/2}_L} \leq (1+\|\Phi^{>}_{(N)}(0)\|^2_{H^{1/2}_L})\exp\left(tF(A,\Phi^0,\nu)\right) - 1$$

Since A is chosen so that (3.4.11) is satisfied, it suffices now to choose T in such a way that

$$\exp\left(TF(A,\Phi^0,\nu)\right) \leq 2, \quad \exp\left(TF(A,\Phi^0,\nu)\right) - 1 \leq \frac{1}{2}\left(\frac{\nu}{CC_{1/2}}\right)^2$$

so that for any $t \leq T$,

$$\|\Phi^{>}_{(N)}(t)\|^2_{H^{1/2}_L} \leq \left(\frac{\nu}{CC_{1/2}}\right)^2$$

hence necessarily

$$\forall N \in \mathbf{N}^*, \quad T_N \geq T.$$

• Gathering those results, we infer that any limiting point of $\Phi^{<}_{(N)} + \Phi^{>}_{(N)}$ (in particular any Leray solution to (SW_0)) satisfies

$$\begin{aligned}\Pi_0\Phi \in L^\infty(\mathbf{R}^+, L^2(\mathbf{R}\times\mathbf{T})), \quad (\Pi_0\Phi)' \in L^2(\mathbf{R}^+, H^1(\mathbf{R}\times\mathbf{T}))\\
\Pi_\perp\Phi \in L^\infty([0,T], H^{1/2}_L) \cap L^2([0,T], H^{3/2}_L).\end{aligned}$$

The weak-strong stability principle gives then the uniqueness of such a solution on $[0,T]$.

3.4.4. Propagation of the ageostrophic regularity. — In order to construct a strong approximation of the filtered solution to the Saint-Venant system in the next chapter, we will actually need further regularity on the solution of the limit filtered system, which is obtained in a very standard way from the trilinear estimates stated in Proposition 3.3.3. So suppose that $\Pi_\perp\Phi^0$ belongs to H^s_L, with $1/2 \leq s \leq 1$, and consider as previously the sequence $(\Phi_{(N)})$ of approximate solutions to (SW_0) defined by (3.4.2). From Lemma 3.3.1 and the last estimate in Proposition 3.3.3, we deduce that for all $s \leq 1$,

$$\begin{aligned}&\frac{d}{dt}\|\Pi_\perp\Phi_{(N)}(t)\|^2_{H^s_L} + \frac{2\nu}{C_s}\|\Pi_\perp\Phi_{(N)}(t)\|^2_{H^{s+1}_L}\\
&\leq C\|\Pi_\perp\Phi_{(N)}(t)\|_{H^s_L}\|\Pi_\perp\Phi_{(N)}(t)\|_{H^{s+1}_L}\left(\|\Pi_\perp\Phi_{(N)}(t)\|_{H^{3/2}_L} + \|\Pi_0\Phi_{(N)}(t)\|_{L^2(\mathbf{R}\times\mathbf{T})}\right)\\
&\leq \frac{\nu}{C_s}\|\Pi_\perp\Phi_{(N)}(t)\|^2_{H^{s+1}_L} + \frac{C}{\nu}\|\Pi_\perp\Phi_{(N)}(t)\|^2_{H^s_L}\left(\|\Pi_\perp\Phi_{(N)}(t)\|_{H^{3/2}_L} + \|\Pi_0\Phi_{(N)}(t)\|_{L^2(\mathbf{R}\times\mathbf{T})}\right)^2,\end{aligned}$$

and thus by Gronwall's lemma

$$\|\Pi_\perp \Phi_{(N)}(t)\|^2_{H^s_L} + \frac{\nu}{C_s}\int_0^t \|\Pi_\perp \Phi_{(N)}(t')\|^2_{H^{s+1}_L} dt' \\ \leq \|\Pi_\perp \Phi^0_{(N)}\|^2_{H^s_L} \exp\left(\frac{C}{\nu}\int_0^t \left(\|\Pi_\perp \Phi_{(N)}(t')\|_{H^{3/2}_L} + \|\Pi_0 \Phi_{(N)}(t')\|_{L^2(\mathbf{R}\times\mathbf{T})}\right)^2 dt'\right).$$

Taking limits as $N \to \infty$ in the previous inequality shows that

$$\Pi_\perp \Phi \in L^\infty_{loc}([0,T^*[,H^s_L) \cap L^2_{loc}([0,T^*[,H^{s+1}_L),$$

and that

$$\|\Pi_\perp \Phi(t)\|^2_{H^s_L} + \frac{\nu}{C_s}\int_0^t \|\Pi_\perp \Phi(t')\|^2_{H^{s+1}_L} dt' \\ \leq \|\Pi_\perp \Phi^0\|^2_{H^s_L} \exp\left(\frac{C}{\nu}\int_0^t \left(\|\Pi_\perp \Phi(t')\|_{H^{3/2}_L} + \|\Pi_0 \Phi(t')\|_{L^2(\mathbf{R}\times\mathbf{T})}\right)^2 dt'\right),$$

which proves the propagation of regularity result, and completes the proof of Theorem 2. □

3.5. Proof of Theorem 3

In this section we shall prove Theorem 3: Paragraph 3.5.1 is devoted to the global wellposedness result in L^2, while the propagation of regularity results are given in Paragraphs 3.5.2 and 3.6.

3.5.1. Global wellposedness. — In this section we shall prove the first part of Theorem 3, namely the fact that except for a countable number of β, the limit system is globally wellposed in L^2. This turns out to be an easy matter in view of the resonance results obtained in Section 2.4 in the previous chapter.

Indeed Proposition 2.4.1, page 19 indicates that except for a countable set of values for β, the limit system reduces to the following:

$$\begin{aligned} \partial_t \Pi_0 \Phi - \nu \Pi_0 \Delta'_L \Phi &= 0, \\ \partial_t \Pi_R \Phi + 2Q_L(\Pi_0\Phi, \Pi_R\Phi) - \nu \Pi_R \Delta'_L \Phi &= 0, \\ \partial_t \Pi_M \Phi + 2Q_L(\Pi_0\Phi, \Pi_M\Phi) - \nu \Pi_M \Delta'_L \Phi &= 0, \\ \partial_t \Pi_P \Phi + 2Q_L(\Pi_0\Phi, \Pi_P\Phi) - \nu \Pi_P \Delta'_L \Phi &= 0, \\ \partial_t \Pi_K \Phi + 2Q_L(\Pi_0\Phi, \Pi_K\Phi) + Q_L(\Pi_K\Phi, \Pi_K\Phi) - \nu \Pi_K \Delta'_L \Phi &= 0. \end{aligned}$$

So the limit system is a linear equation on all modes but Kelvin modes; Kelvin modes being essentially one-dimensional, it will be easy to prove the wellposedness of the system. In fact the only point to be proved is the uniqueness of the solution, since existence was proved in the previous section. Uniqueness will be immediate in the case of all non-Kelvin modes so let us concentrate on the equation on $\Pi_K\Phi$. Let Φ and Φ_* be two solutions, and define $\delta\Phi = \Phi_* - \Phi$. Then

$$\partial_t \Pi_K \delta\Phi + 2Q_L(\Pi_0\delta\Phi, \Pi_K\delta\Phi) + 2Q_L(\Pi_0\Phi, \Pi_K\delta\Phi) + 2Q_L(\Pi_0\delta\Phi, \Pi_K\Phi) + Q_L(\Pi_K\delta\Phi, \Pi_K\delta\Phi) \\ +2Q_L(\Pi_K\Phi, \Pi_K\delta\Phi) - \nu \Pi_K \Delta'_L \delta\Phi = 0.$$

Now let us write an energy estimate in L^2, in the spirit of the computations of Section 3.4.1 above. We note that in the case of Π_K, the decomposition on eigenmodes of L simply corresponds to the Fourier decomposition, so that

$$(\Pi_K\delta\Phi|Q_L(\Pi_K\Phi,\Pi_K\delta\Phi))_{L^2(\mathbf{R}\times\mathbf{T})} = (\Pi_K\delta\Phi|Q(\Pi_K\Phi,\Pi_K\delta\Phi))_{L^2(\mathbf{R}\times\mathbf{T})}\,.$$

Moreover in that case, the usual Sobolev spaces H^s coincide with the H^s_L spaces, since $n=0$. Using the fact that Π_0 projects onto x_2-independent functions and that the dependence in x_1 of $\Pi_K\delta\Phi$ is that of the Gaussian ψ_0, it is easy to see that

$$\left|(\Pi_K\delta\Phi|Q_L(\Pi_0\Phi,\Pi_K\delta\Phi))_{L^2(\mathbf{R}\times\mathbf{T})}\right| \leq \|\Pi_K\delta\Phi\|_{H^1(\mathbf{R}\times\mathbf{T})}\|\Pi_0\Phi\|_{L^2(\mathbf{R})}\|\Pi_K\delta\Phi\|_{L^\infty(\mathbf{R};L^2(\mathbf{T}))},$$

and that

$$\begin{aligned}\left|(\Pi_K\delta\Phi|Q(\Pi_K\Phi,\Pi_K\delta\Phi))_{L^2(\mathbf{R}\times\mathbf{T})}\right| &\leq \|\Pi_K\delta\Phi\|_{H^1(\mathbf{R}\times\mathbf{T})}\|\Pi_K\Phi\|_{L^\infty(\mathbf{R}\times\mathbf{T})}\|\Pi_K\delta\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}\\ &\leq \frac{\nu}{2C_1}\|\Pi_K\delta\Phi\|^2_{H^1(\mathbf{R}\times\mathbf{T})}+\frac{C}{\nu}\|\Pi_K\Phi\|^2_{L^\infty(\mathbf{R}\times\mathbf{T})}\|\Pi_K\delta\Phi\|^2_{L^2(\mathbf{R}\times\mathbf{T})}.\end{aligned}$$

We recall indeed that

$$\Pi_K\cdot = \sum_{k\in\mathbf{Z}^*}(\Psi_{0,k,0}\,|\,\cdot\,)_{L^2(\mathbf{R}\times\mathbf{T})}\Psi_{0,k,0} \text{ where } \Psi_{0,k,0}(x_1,x_2)=\frac{1}{\sqrt{4\pi}}e^{ikx_2}\begin{pmatrix}\psi_0(x_1)\\ 0\\ \psi_0(x_1)\end{pmatrix}.$$

One sees easily that $\|\Pi_K\Phi\|_{L^\infty(\mathbf{R}\times\mathbf{T})}\leq C\|\Pi_K\Phi\|_{H^1(\mathbf{R}\times\mathbf{T})}$, so finally an energy estimate (coupled with a Gronwall lemma) gives

$$\begin{aligned}\|\Pi_K\delta\Phi(t)\|^2_{L^2(\mathbf{R}\times\mathbf{T})}+\frac{\nu}{C}\int_0^t\|\Pi_K\delta\Phi(t')\|^2_{H^1(\mathbf{R}\times\mathbf{T})}\,dt' &\leq \|\Pi_K\delta\Phi(0)\|^2_{L^2(\mathbf{R}\times\mathbf{T})}\\ &\quad\times\exp\left(\frac{C}{\nu}\int_0^t\|\Pi_K\Phi(t')\|^2_{H^1}\,dt'\right).\end{aligned}$$

Since $\Pi_K\delta\Phi(0)=0$, uniqueness follows from the energy bound on $\Pi_K\Phi$. Note that we have recovered here the usual, two-dimensional Navier-Stokes type estimates since in the case of purely Kelvin modes, the quadratic form and the spaces involved are the same as in the Navier-Stokes case. In fact Kelvin modes are even one-dimensional (up to a multiplication by $\psi_0(x_1)$) so it is possible to improve those estimates (that will be done in the last section of this chapter).

3.5.2. Propagation of regularity. — Let us now prove that if the initial data $\Pi_\perp\Phi^0$ is in H^s_L with $s\in[0,1]$, then that regularity is propagated to $\Pi_\perp\Phi$. Recalling the special form of the limit system for almost all β, the only equation we need to study is the one on $\Pi_K\Phi$. Indeed denoting

$$\Psi=(\Pi_R+\Pi_P+\Pi_M)\Phi,$$

we have

$$\partial_t\Psi+2Q_L(\Pi_0\Phi,\Psi)-\nu\Delta'_L\Psi=0,$$

and Proposition 3.3.3 gives directly

$$\|\Psi(t)\|^2_{H^s_L}+\frac{\nu}{C}\int_0^t\|\Psi(t')\|^2_{H^{s+1}_L}\leq\|\Psi(0)\|^2_{H^s_L}\exp\left(\frac{C}{\nu}\|\Pi_0\Phi\|^2_{L^2([0,T];L^2)}\right).$$

Now let us turn to the Kelvin modes. In that case we simply use again the fact that H^s and H^s_L spaces coincide in the case of Π_K. We have, using Proposition 3.3.3 again,

$$\|\Pi_K\Phi(t)\|^2_{H^s(\mathbf{R}\times\mathbf{T})} + \frac{\nu}{C_1}\int_0^t \|\Pi_K\Phi(t')\|^2_{H^{s+1}(\mathbf{R}\times\mathbf{T})} \leq \left|\int_0^t (\Pi_K\Phi|Q(\Pi_K\Phi,\Pi_K\Phi))_{H^s(\mathbf{R}\times\mathbf{T})}(t')\,dt'\right|$$
$$+ C\int_0^t \|\Pi_0\Phi(t')\|_{L^2}\|\Pi_K\Phi(t')\|_{H^s(\mathbf{R}\times\mathbf{T})}\|\Pi_K\Phi(t')\|_{H^{s+1}(\mathbf{R}\times\mathbf{T})}\,dt'.$$

Two-dimensional product rules give

$$\begin{aligned}\left|(\Pi_K\Phi|Q(\Pi_K\Phi,\Pi_K\Phi))_{H^s(\mathbf{R}\times\mathbf{T})}\right| &\leq \|\Pi_K\Phi\|_{H^{s+1}(\mathbf{R}\times\mathbf{T})}\|\Pi_K\Phi\otimes\Pi_K\Phi\|_{H^s(\mathbf{R}\times\mathbf{T})}\\ &\leq \|\Pi_K\Phi\|_{H^{s+1}(\mathbf{R}\times\mathbf{T})}\|\Pi_K\Phi\|_{H^{s+\frac12}(\mathbf{R}\times\mathbf{T})}\|\Pi_K\Phi\|_{H^{\frac12}(\mathbf{R}\times\mathbf{T})}\\ &\leq \|\Pi_K\Phi\|^{\frac32}_{H^{s+1}(\mathbf{R}\times\mathbf{T})}\|\Pi_K\Phi\|^{\frac12}_{H^s(\mathbf{R}\times\mathbf{T})}\|\Pi_K\Phi\|_{H^{\frac12}(\mathbf{R}\times\mathbf{T})}\\ &\leq \frac{\nu}{2C_1}\|\Pi_K\Phi\|^2_{H^{s+1}(\mathbf{R}\times\mathbf{T})} + \frac{C}{\nu}\|\Pi_K\Phi\|^2_{H^s(\mathbf{R}\times\mathbf{T})}\|\Pi_K\Phi\|^4_{H^{\frac12}(\mathbf{R}\times\mathbf{T})}\end{aligned}$$

so that finally

$$\|\Pi_K\Phi(t)\|^2_{H^s(\mathbf{R}\times\mathbf{T})} + \frac{\nu}{4C_1}\int_{0^t} \|\Pi_K\Phi(t')\|^2_{H^{s+1}(\mathbf{R}\times\mathbf{T})} \leq \frac{C}{\nu}\int_0^t \|\Pi_0\Phi(t')\|^2_{L^2}\|\Pi_K\Phi(t')\|^2_{H^s(\mathbf{R}\times\mathbf{T})}\,dt'$$
$$+\frac{C}{\nu}\int_0^t \|\Pi_K\Phi(t')\|^2_{H^s(\mathbf{R}\times\mathbf{T})}\|\Pi_K\Phi(t')\|^4_{H^{\frac12}(\mathbf{R}\times\mathbf{T})}\,dt',$$

and the result follows from Gronwall's lemma and the energy estimate on $\Pi_K\Phi$.

That concludes the proof of Theorem 3. □

3.6. A regularity result for the divergence

In this section we are going to prove an additional regularity result for the system (SW_0), which will be useful to study the strong asymptotics of the rotating shallow-water system in the next chapter.

Proposition 3.6.1. — *Let Φ^0 belong to $L^2(\mathbf{R}\times\mathbf{T})$.*

For all $\beta \in \mathbf{R}^+_*$, if $\Pi_\perp\Phi^0$ belongs to $H^{1/2}_L$ and $(\Pi_P+\Pi_K)\Phi^0$ belongs to H^α_L for $\alpha > 3/2$, then the solution Φ of (SW_0) with initial data Φ^0 defined on $[0, T*[$ satisfies for all $t\in[0,T^*[$

$$\int_0^t \|\nabla\cdot\Phi'(t')\|_{L^\infty(\mathbf{R}\times\mathbf{T})}dt' < +\infty,$$

where we recall that Φ' denotes the two last components of Φ.

Furthermore the regularity assumption on the initial data can be relaxed for all but a countable number of β. Indeed, for all $\beta\in\mathbf{R}^+\setminus\mathcal{N}$ where $\mathcal{N}$ is the countable subset of $\mathbf{R}^+$ defined in Theorem 3, if $(\Pi_P+\Pi_K)\Phi^0$ belongs to H^α_L for $\alpha > 1/2$, then the solution Φ of (SW_0) with initial data Φ^0 satisfies for all $t\in\mathbf{R}^+$

$$\int_0^t \|\nabla\cdot\Phi'(t')\|_{L^\infty(\mathbf{R}\times\mathbf{T})}dt' < +\infty.$$

Proof. — Such a result is established by decoupling the equations on the various parts of Φ, and proving that the regularity is propagated for the Poincaré and Kelvin modes, while a smoothing property on the divergence holds for the nonoscillating, Rossby and mixed Rossby-Poincaré modes.

Let us decompose Φ on the supplementary subsets Ker$L$, R, M, P and K

$$\Phi = \Pi_0\Phi + \Pi_R\Phi + \Pi_M\Phi + \Pi_P\Phi + \Pi_K\Phi,$$

and estimate each projection separately.

• For nonoscillating, Rossby and mixed modes the smoothness of the divergence is not due to a propagation result but to a stationnary property of the eigenvectors.

Clearly, by definition of Ker$L$, $\nabla\cdot(\Pi_0\Phi)' = 0$.

Let us consider the Rossby and mixed Rossby-Poincaré modes. By definition of the Rossby modes we deduce the following relation

$$\forall\Phi\in R,\quad \nabla\cdot\Phi' = \sum_{i\lambda\in\mathfrak{S}_R}\nabla\cdot\Phi'_\lambda = \sum_{i\lambda\in\mathfrak{S}_R} i\lambda(\Phi_\lambda)_0$$

with the notation $\Phi_\lambda = \Pi_\lambda\Phi$. It follows that

$$\begin{aligned}\|\nabla\cdot\Phi'\|^2_{H^2} &= \Big\|\sum_{i\lambda\in\mathfrak{S}_R} i\lambda(\Phi_\lambda)_0\Big\|^2_{H^2}\\ &\leq C\sum_{i\lambda\in\mathfrak{S}_R}\|\lambda(\Phi_\lambda)_0\|^2_{H^2(\mathbf{R}\times\mathbf{T})},\end{aligned}$$

by Proposition 3.1.2. But, as Rossby waves correspond to $j=0$, we have

$$\|\lambda(\Phi_\lambda)_0\|^2_{H^2(\mathbf{R}\times\mathbf{T})} \leq C|\lambda|^2\sum_{\tau(n,k,0)=\lambda}\|(\Pi_{n,k,0}\Phi)_0\|^2_{H^2(\mathbf{R}\times\mathbf{T})},$$

using Remark 2.2.6. Recalling the explicit form of $(\Psi_{n,k,j})_0$, we see that

$$\|(\Pi_{n,k,0}\Phi)_0\|^2_{H^2(\mathbf{R}\times\mathbf{T})} \leq (1+n+k^2)\|(\Pi_{n,k,0}\Phi)_0\|^2_{H^1(\mathbf{R}\times\mathbf{T})}.$$

But for Rossby modes, the following asymptotics hold as $|k|$ or n goes to infinity:

$$\lambda = \tau(n,k,0) \sim \frac{\beta k}{k^2+\beta(2n+1)}.$$

So we infer that as $|k|$ or n goes to infinity,

$$|\lambda|^2\|(\Pi_{n,k,0}\Phi)_0\|^2_{H^2(\mathbf{R}\times\mathbf{T})} \leq C\|(\Pi_{n,k,0}\Phi)_0\|^2_{H^1(\mathbf{R}\times\mathbf{T})}.$$

Finally we infer that

$$\begin{aligned}\|\nabla\cdot\Phi'\|^2_{H^2} &\leq C\sum_{i\lambda\in\mathfrak{S}_R}\|\lambda(\Phi_\lambda)_0\|^2_{H^2(\mathbf{R}\times\mathbf{T})}\\ &\leq C\sum_{(n,k,0)\in\mathfrak{S}_R}\|(\Pi_{n,k,0}\Phi)_0\|^2_{H^1(\mathbf{R}\times\mathbf{T})}\\ &\leq C\|\Phi\|^2_{H^1_L}.\end{aligned}$$

By the embedding of $H^2(\mathbf{R}\times\mathbf{T})$ into $L^\infty(\mathbf{R}\times\mathbf{T})$ we conclude that $\nabla\cdot(\Pi_R\Phi)'$ belongs to the space $L^2([0,T];L^\infty(\mathbf{T}\times\mathbf{R}))$. The same result can easily be extended to the mixed Poincaré-Rossby modes (it is in fact easier since $n=0$ in that case) and we obtain

$$\|\Pi_M\Phi\|_{L^2([0,T],H^1_L)} \leq C_T,\quad \|\nabla\cdot(\Pi_M\Phi)'\|_{L^2([0,T],L^\infty(\mathbf{R}\times\mathbf{T}))} \leq C_T\,.$$

Finally we deduce that

$$\|\nabla\cdot((\Pi_0+\Pi_R+\Pi_M)\Phi)'\|_{L^2([0,T],L^\infty(\mathbf{R}\times\mathbf{T}))} \leq C_T\,.$$

• In order to establish a similar estimate for the Poincaré and Kelvin modes, we prove that the equation governing these modes propagates the H^s_L regularity without restriction on s. Indeed we have seen in Remark 3.3.4 that the restriction to $s \leq 1$ for the propagation of regularity stated in Theorem 2 is due to the coupling between Rossby modes.

The idea here is to study the propagation of the following norm on $(\mathrm{Ker} L)^\perp$

$$\|\Phi\|_s^2 = \sum_{\lambda \in \mathfrak{S}} (1+\lambda^2)^s \|\Pi_\lambda \Phi\|^2_{L^2(\mathbf{R}\times\mathbf{T})},$$

which controls the H^s_L regularity of the Poincaré and Kelvin modes only, due to the following easy estimate (see Proposition 3.1.2):

$$C^{-1}\|(\Pi_K+\Pi_P)\Phi\|_{H^s_L} \leq \|\Pi_\perp\Phi\|_s \leq C(\|(\Pi_K+\Pi_P)\Phi\|_{H^s_L} + \|\Pi_\perp\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}). \tag{3.6.1}$$

This norm is convenient to deal with the condition of resonance occuring in the nonlinear term of (SW_0).

Similar arguments as in Proposition 3.3.3 allow to write the following trilinear estimate:

$$|(\Phi|\tilde{Q}_L(\Phi,\Phi))_s| \leq C_s\|\Pi_\perp\Phi\|_s(\|\Pi_\perp\Phi\|_{s+1} + \|\Pi_\perp\Phi\|_{H^1_L})(\|\Pi_\perp\Phi\|_{H^{3/2}_L} + \|\Pi_0\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}). \tag{3.6.2}$$

With the same notation as in the proof of Proposition 3.3.3, we have indeed

$$(\Phi_*|\tilde{Q}_L(\Phi,\Phi^*))_s \stackrel{\text{def}}{=} \left| \sum_{\substack{\tau_*=\tau+\tau^* \\ k_*=k+k^* \\ i\tau, i\tau_*, i\tau^* \in \mathfrak{S}\setminus\{0\}}} (1+\tau_*^2)^s \left(\Pi_{n_*,k_*,j_*}\Phi_* \Big| \Pi_{n_*,k_*,j_*} Q(\Pi_{n,k,j}\Phi, \Pi_{n^*,k^*,j^*}\Phi^*)\right)_{L^2} \right|$$

$$\leq C_s \sum (1+\tau_*^2)^{s/2}\Big((1+\tau^2)^{s/2} + (1+(\tau^*)^2)^{s/2}\Big)|\varphi_*||\varphi||\varphi^*|((n+k^2)^{1/2} + (n^*+(k^*)^2)^{1/2}),$$

from which we deduce that

$$|(\Phi|\tilde{Q}_L(\Phi,\Phi))_s \leq C_s\|\Pi_\perp\Phi\|_s(\|\Pi_\perp\Phi\|_{s+1} + \|\Pi_\perp\Phi\|_{H^1_L})\|\Pi_\perp\Phi\|_{H^{3/2}_L}.$$

In the same way, for the geostrophic part, we have

$$(\Phi|\tilde{Q}_L(\Pi_0\Phi,\Pi_\perp\Phi))_s \leq C_s\|\Pi_\perp\Phi\|_s(\|\Pi_\perp\Phi\|_{s+1} + \|\Pi_\perp\Phi\|_{H^1_L})\|\Pi_0\Phi\|_{L^2(\mathbf{R}\times\mathbf{T})}.$$

Note that the H^1 norm appearing above is used to control the gradient of the Rossby and mixed modes.

Using (3.6.2) and Gronwall's lemma, we get the following propagation result

$$\|\Pi_\perp\Phi(t)\|_s^2 + \frac{\nu}{C_s}\int_0^t \Big(\|\Pi_\perp\Phi(t')\|^2_{s+1} + \|\Pi_\perp\Phi(t')\|^2_{H^1_L}dt'\Big)$$
$$\leq \|\Pi_\perp\Phi^0\|_s^2 \exp\left(\frac{C_s}{\nu}\int_0^t \Big(\|\Pi_\perp\Phi(t')\|_{H^{3/2}_L} + \|\Pi_0\Phi(t')\|_{L^2(\mathbf{R}\times\mathbf{T})}\Big)^2 dt'\right).$$

(see Paragraph 3.4.4 for the detailed proof), and therefore by (3.6.1) we get

$$\|(\Pi_K+\Pi_P)\Phi(t)\|^2_{H^s_L} + \frac{\nu}{C_s}\int_0^t \|(\Pi_K+\Pi_P)\Phi(t')\|^2_{H^{s+1}_L}dt'$$
$$\leq (\|(\Pi_K+\Pi_P)\Phi^0\|^2_{H^s_L} + \|\Phi^0\|^2_{L^2(\mathbf{R}\times\mathbf{T})}) \exp\left(\frac{C_s}{\nu}\int_0^t \Big(\|\Pi_\perp\Phi(t')\|_{H^{3/2}_L} + \|\Pi_0\Phi(t')\|_{L^2(\mathbf{R}\times\mathbf{T})}\Big)^2 dt'\right).$$

By Sobolev embeddings we then deduce the expected control on the divergence

$$\|\nabla \cdot ((\Pi_K + \Pi_P)\Phi)'\|_{L^2([0,T],L^\infty(\mathbf{R}\times\mathbf{T}))} \le C_T$$

for all $T \le T^*$ where T^* is the lifespan of the strong solution Φ.

That concludes the proof of the proposition in the case of general β.

• By Proposition 2.4.1 page 19, we recall that, for all $\beta \in \mathbf{R}^+ \setminus \mathcal{N}$, the only possible resonances are Kelvin resonances.

Let us now consider the equation governing the Poincaré modes which can be seen as a linear parabolic equation whose coefficients depend on $\Pi_0\Phi$. Therefore it is very easy to propagate regularity once one has noticed that the multiplication by $|\lambda|$ for the Poincaré mode $\Pi_\lambda\Phi$ is "equivalent" to a derivation.

Introduce as previously the notation

$$\Phi = \sum_{(n,k,j)\in S} \varphi_{n,k,j}\Psi_{n,k,j},$$

so that

$$\Pi_P\Phi = \sum_{(n,k,j)\in S_P} \varphi_{n,k,j}\Psi_{n,k,j},$$

where

$$S_P = \mathbf{N}^* \times \mathbf{Z} \times \{-1,1\} \cup \{0\} \times \mathbf{Z}_*^+ \times \{1\} \cup \{0\} \times \mathbf{Z}_*^- \times \{-1\} \cup 0 \times 0 \times \{-1,1\}.$$

We can use Proposition 2.2.3 page 12 to deduce that for each (n,k,j) in S_P the equation governing $\varphi_{n,k,j}$ can be decoupled (recall that $\Pi_0\Phi$ only depends on x_1):

$$\partial_t\varphi_{n,k,j} - \nu\varphi_{n,k,j}(\Psi_{n,k,j}|\Delta'\Psi_{n,k,j})_{L^2(\mathbf{R}\times\mathbf{T})} = -2\varphi_{n,k,j}(\Psi_{n,k,j}|Q(\Psi_{n,k,j},\Pi_0\Phi))_{L^2(\mathbf{R}\times\mathbf{T})}$$

which can be rewritten

$$\begin{aligned}\partial_t &\left(\varphi_{n,k,j}\exp\left(-\nu t(\Psi_{n,k,j}|\Delta'\Psi_{n,k,j})_{L^2(\mathbf{R}\times\mathbf{T})}\right)\right)\\ &= -2\varphi_{n,k,j}(\Psi_{n,k,j}|Q(\Psi_{n,k,j},\Pi_0\Phi))_{L^2(\mathbf{R}\times\mathbf{T})}\exp\left(-\nu t(\Psi_{n,k,j}|\Delta'\Psi_{n,k,j})_{L^2(\mathbf{R}\times\mathbf{T})}\right).\end{aligned}$$

By Gronwall's lemma and the estimates

$$\begin{aligned}\left|(\Psi_{n,k,j}|Q(\Psi_{n,k,j},\Pi_0\Phi))_{L^2(\mathbf{R}\times\mathbf{T})}\right| &\le C_1(n+k^2)^{1/2},\\ -(\Psi_{n,k,j}|\Delta'\Psi_{n,k,j})_{L^2(\mathbf{R}\times\mathbf{T})} &\ge C_2(n+k^2),\end{aligned}$$

we then deduce that there exists a nonnegative constant C_ν (depending only on ν) such that,

$$\forall (n,k,j)\in S_P, \quad |\varphi_{n,k,j}(t)| \le |\varphi_{n,k,j}(0)|\exp(-C_\nu(n+k^2)t). \tag{3.6.3}$$

We have

$$\begin{aligned}\|\nabla\cdot(\Pi_P\Phi)'(t)\|_{L^\infty(\mathbf{R}\times\mathbf{T})} &\le \sum_{(n,k,j)\in S_P} |\varphi_{n,k,j}(t)|\,\|\nabla\cdot(\Psi_{n,k,j})'(t)\|_{L^\infty(\mathbf{R}\times\mathbf{T})}\\ &\le C\sum_{(n,k,j)\in S_P}|\varphi_{n,k,j}(t)|(n+k^2)^{1/2}\end{aligned}$$

since $(\Psi_{n,k,j})$ is uniformly bounded in $L^\infty(\mathbf{R}\times\mathbf{T})$. Thus, by (3.6.3),

$$\|\nabla\cdot(\Pi_P\Phi)'(t)\|_{L^\infty(\mathbf{R}\times\mathbf{T})} \le C\sum_{(n,k,j)\in S_P}|\varphi_{n,k,j}(0)|\exp(-C_\nu(n+k^2)t)(n+k^2)^{1/2}.$$

Integrating with respect to time leads then to

$$\|\nabla\cdot(\Pi_P\Phi)'\|_{L^1([0,T];L^\infty(\mathbf{R}\times\mathbf{T}))}\leq C'_\nu\sum_{(n,k,j)\in S_P}|\varphi_{n,k,j}(0)|(n+k^2)^{-1/2}$$

$$\leq C'_\nu\left(\sum_{(n,k,j)\in S_P}|\varphi_{n,k,j}(0)|^2(n+k^2)^\alpha\right)^{1/2}\left(\sum_{(n,k,j)\in S_P}(n+k^2)^{-1-\alpha}\right)^{1/2},$$

from which we deduce that for $\alpha>1/2$,

$$\|\nabla\cdot(\Pi_P\Phi)'\|_{L^1([0,T],L^\infty(\mathbf{R}\times\mathbf{T}))}\leq C\|\Pi_P\Phi^0\|_{H^\alpha_L}$$

where C depends only on ν and α.

It remains then to establish the propagation of regularity for the Kelvin part of the equation, which is nonlinear and has no smoothing effect for the divergence as the Rossby part.

The crucial point here is to recall as above that this equation is actually one-dimensional (modulo a smooth function with respect to x_1). The propagation of regularity result proved in Paragraph 3.5.2 implies that as soon as the initial data is in H^α_L with $0\leq\alpha\leq1$, then the solution lies in $L^2(\mathbf{R}^+;H^{\alpha+1}_L)$. In particular $\nabla\cdot(\Pi_K\Phi)'$ lies in $L^2(\mathbf{R}^+;H^\alpha_L)$. So if $\alpha>1/2$, using the fact that $H^\alpha(\mathbf{T})$ is embedded in $L^\infty(\mathbf{T})$, the result follows directly.

Proposition 3.6.1 is proved. □

CHAPTER 4

CONVERGENCE RESULTS

The aim of this chapter is to study the asymptotics of the rotating shallow-water system (1.4.3) presented page 5. In particular we will see that the system (2.3.5) obtained formally in Section 2.3, page 19, is indeed the limit system, after application of the filtering operator $\exp(-tL/\varepsilon)$. In order to simplify the presentation let us recall here the two main systems we will be considering in this chapter, namely the shallow-water system

$$(SW_\varepsilon) \qquad \begin{cases} \partial_t\eta + \dfrac{1}{\varepsilon}\nabla\cdot \Big((1+\varepsilon\eta)u\Big) = 0, \\ \partial_t\Big((1+\varepsilon\eta)u\Big) + \nabla\cdot\Big((1+\varepsilon\eta)u\otimes u\Big) + \dfrac{\beta x_1}{\varepsilon}(1+\varepsilon\eta)u^\perp + \dfrac{1}{\varepsilon}(1+\varepsilon\eta)\nabla\eta - \nu\Delta u = 0, \\ \eta_{|t=0} = \eta^0, \quad u_{|t=0} = u^0, \end{cases}$$

and the limit system

$$(SW_0) \qquad \begin{cases} \partial_t\Phi + Q_L(\Phi,\Phi) - \nu\Delta'_L\Phi = 0 \\ \Phi_{|t=0} = (\eta^0, u^0), \end{cases}$$

where Δ'_L and Q_L denote the linear and symmetric bilinear operator defined by (2.3.6) page 19.

We also recall the formal equivalent form of (SW_ε),

$$\begin{cases} \partial_t(\eta,u) + \dfrac{1}{\varepsilon}L(\eta,u) + Q\left((\eta,u),(\eta,u)\right) - \nu\Delta'(\eta,u) = R \\ (\eta,u)_{|t=0} = (\eta^0,u^0), \end{cases}$$

where Q and Δ' are defined by (2.3.4) page 18 and

$$R = (0, -\nu\frac{\varepsilon\eta}{1+\varepsilon\eta}\Delta u).$$

The study of the asymptotics of (SW_ε) will be achieved through three different methods, which provide three different types of results. In Section 4.1 we describe the weak limit of the weak solutions to (SW_ε) as ε goes to zero, which is proved to satisfy the geostrophic equation studied in the previous chapter, i.e., the projection of (SW_0) onto Ker$L$. The statement is given in Theorem 4 below. Then for smooth enough initial data, we prove in Section 4.2 the strong convergence of the filtered sequence of solutions towards the unique solution of (SW_0). The precise statement depends on the setting, as in Chapter 3: for all $\beta > 0$ we are only able to prove results locally in time (globally for small data) whereas if a countable set of values of β is removed, then the convergence is strong for all times, and the smoothness assumptions on the initial data are less restrictive (see Theorems 5 and 6). Finally in Section 4.3 we propose an intermediate study between those two asymptotic results, by considering the asymptotic behaviour of the filtered sequence $e^{-tL/\varepsilon}(\eta_\varepsilon, u_\varepsilon)$, where $(\eta_\varepsilon, u_\varepsilon)$ is a weak solution to (SW_ε). We prove a strong convergence result towards a weak solution to (2.3.5), where unfortunately due to the lack of compactness of η_ε in space, a defect measure remains (see Theorem 7). In order to circumvent that difficulty we propose an alternate system to the Saint-Venant equations (SW_ε), where capillarity effects are included. Technically the effect of capillarity is to have a uniform control on $\varepsilon\eta_\varepsilon$ in strong enough norms so as to obtain an evolution equation for u_ε. A strong convergence result for $e^{-tL/\varepsilon}(\eta_\varepsilon, u_\varepsilon)$ is established in that new setting, see Theorem 9.

In this chapter, many results and notation of the previous chapters will be used. However precise references will be made each time, so that this chapter can be read independently of the others (assuming the results of course).

4.1. Weak convergence of weak solutions

The first aim of the chapter is to describe the weak limit (η, u) of $(\eta_\varepsilon, u_\varepsilon)$ as ε goes to zero.

***Theorem 4* (Weak convergence).** — *Let $(\eta^0, u^0) \in L^2(\mathbf{R} \times \mathbf{T})$ and $(\eta^0_\varepsilon, u^0_\varepsilon)$ be such that*

$$\begin{aligned} &\frac{1}{2}\int \left(|\eta^0_\varepsilon|^2 + (1+\varepsilon\eta^0_\varepsilon)|u^0_\varepsilon|^2\right) dx \le \mathcal{E}^0, \\ &(\eta^0_\varepsilon, u^0_\varepsilon) \to (\eta^0, u^0) \text{ in } L^2(\mathbf{R} \times \mathbf{T}). \end{aligned} \tag{4.1.1}$$

For all $\varepsilon > 0$, denote by $(\eta_\varepsilon, u_\varepsilon)$ a solution of (SW_ε) with initial data $(\eta^0_\varepsilon, u^0_\varepsilon)$, as constructed in Corollary 1.4.1 page 6. Then $(\eta_\varepsilon, u_\varepsilon)$ converges weakly in $L^2_{loc}(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{T})$ to the solution $(\eta, u) \in L^\infty(\mathbf{R}^+, L^2(\mathbf{R}))$, with u also belonging to $L^2(\mathbf{R}^+, \dot H^1(\mathbf{R}))$, of the following linear equation (given in weak formulation)

$$u_1 = 0, \quad \beta x_1 u_2 + \partial_1 \eta = 0, \tag{4.1.2}$$

and for all $(\eta^*, u^*) \in L^2 \times H^1(\mathbf{R})$ satisfying (4.1.2)

$$\int (\eta\eta^* + u_2 u_2^*)(t,x)\, dx + \nu \int_0^t \int \nabla u_2 \cdot \nabla u_2^*(t',x)\, dx\, dt' = \int (\eta^0\eta^* + u^0_2 u^*_2)(x)\, dx. \tag{4.1.3}$$

Remark 4.1.1. — • *Theorem 4 shows that the system satisfied by the weak limits of η_ε and u_ε is linear. There is therefore no convective term in the mean flow: system (4.1.2, 4.1.3) actually corresponds to the projection of (SW_0) onto* Ker$L$*: as seen in Section 2.4.2, that projection can indeed be formally written*

$$\begin{aligned} \partial_t(\eta, 0, u_2) - \nu\Pi_0(0,0,\Delta u_2) &= 0, \\ (\eta, u)(t) &= \Pi_0(\eta, u)(t)\ \forall t \ge 0, \\ (\eta, u)_{|t=0} &= \Pi_0(\eta^0, u^0). \end{aligned}$$

• *Note that (η^0, u^0) do not necessarily satisfy the constraints (4.1.2), so in general $(\eta, u)_{|t=0}$ is not equal to (η^0, u^0).*

• *The study of the waves induced by L, in Chapter 3, revealed the presence of trapped equatorial waves, which however do not appear in the mean flow described by Equation (4.1.3): no constructive interferences take place in the limiting process, in other words the fast oscillating modes decouple from the mean flow, without creating any additional term in the limit system (that feature was already observed in* [**10**] *in the case of inhomogeneous rotating fluid equations, modelling the ocean or the atmosphere at midlatitudes). This will be obtained by a compensated compactness argument in Section 4.1.4.*

4.1.1. Constraints on the weak limit. — We recall (see Chapter 1) that the uniform energy bound on $(\eta_\varepsilon, u_\varepsilon)$ implies the existence of a weak limit (η, u). In this paragraph we are going to prove that the weak limit belongs to Ker$L$.

Proposition 4.1.2. — *Let $(\eta^0, u^0) \in L^2(\mathbf{R} \times \mathbf{T})$. Denote by $(\eta_\varepsilon, u_\varepsilon)_{\varepsilon>0}$ a family of solutions of (SW_ε), and by (η, u) any of its limit points. Then, $(\eta, u) \in L^\infty(\mathbf{R}^+, L^2(\mathbf{R}))$ belongs to* Ker$L$*, and in particular satisfies the constraints*

$$u_1 = 0, \quad \beta x_1 u_2 + \partial_1 \eta = 0. \tag{4.1.4}$$

Proof. — Let $\chi, \psi \in \mathcal{D}(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{T})$ be any test functions. Multiplying the conservation of mass in (SW_ε) by $\varepsilon\chi$ and integrating with respect to all variables leads to

$$\iint (\varepsilon \eta_\varepsilon \partial_t \chi + (1 + \varepsilon \eta_\varepsilon) u_\varepsilon \cdot \nabla \chi)\, dxdt = 0.$$

Because of the bounds coming from the energy estimate (1.4.5), we can take limits in the previous identity as ε goes to 0 to get

$$\iint u \cdot \nabla \chi dxdt = 0.$$

Similarly, multiplying the conservation of momentum by $\varepsilon\psi$ and integrating with respect to all variables leads to

$$\iint \Big(\varepsilon(1+\varepsilon\eta_\varepsilon)u_\varepsilon \partial_t \psi + \varepsilon(1+\varepsilon\eta_\varepsilon)u_\varepsilon \cdot (u_\varepsilon \cdot \nabla)\psi + \beta x_1 (1+\varepsilon\eta_\varepsilon) u_\varepsilon \cdot \psi^\perp + (1+\frac{\varepsilon}{2}\eta_\varepsilon)\eta_\varepsilon \nabla \cdot \psi + \nu u_\varepsilon \cdot \Delta \psi\Big) dxdt = 0.$$

Once again the bounds coming from the energy estimate (1.4.5) enable us to take the limit as ε goes to 0, and find that

$$\iint (\eta \nabla \cdot \psi + \beta x_1 u \cdot \psi^\perp) dxdt = 0.$$

It follows that $(\eta(t), u(t))$ belongs to Ker$L$ for almost all $t \in \mathbf{R}^+$, and we conclude by Proposition 2.1.1 page 7 that (η, u) does not depend on x_2 and satisfies the constraints (4.1.4). □

To go further in the description of the weak limit (η, u), we have to isolate the fast oscillations generated by the singular perturbation L, which produce "big" terms in (SW_ε), but converge weakly to 0.

Therefore, a natural idea consists in introducing the following decomposition

$$(\eta_\varepsilon, u_\varepsilon) = \Pi_0(\eta_\varepsilon, u_\varepsilon) + \Pi_\perp(\eta_\varepsilon, u_\varepsilon),$$

where Π_0 is the L^2 orthogonal projection onto Ker$L$ and $\Pi_\perp$ the L^2 orthogonal projection onto $(\text{Ker}L)^\perp$.

The idea to get the mean motion is then to apply Π_0 to (SW_ε) : since L is a skew-symmetric operator, we have $\Pi_0 L = 0$ and we expect $\partial_t \Pi_0(\eta_\varepsilon, u_\varepsilon)$ to be uniformly bounded in some distribution space. The difficulty comes from the fact that one has no uniform spatial regularity on η_ε. That is why we will actually consider the weak form of the evolution equations; the point is then to take limits in the nonlinear terms.

4.1.2. Rough description of the oscillations. — The analysis of the nonlinear terms lies essentially on the structure of the oscillations. A rough description of those fast oscillations will be enough to prove that they do not produce any constructive interference, and therefore do not appear in the equation governing the mean (geostrophic) motion. The much more precise description given in Chapter 2 will not be used in this section, but will be necessary to discuss the strong asymptotic behaviour of the solutions in the next sections.

In the following statement we have considered a regularization kernel defined as follows: let κ be a function of $C^\infty_c(\mathbf{R}^2,\mathbf{R}^+)$ such that $\kappa(x)=0$ if $|x|\geq 1$ and $\int \kappa dx = 1$. Then for any $\delta>0$ we define κ_δ by

$$\kappa_\delta(x)=\delta^{-2}\kappa(\delta^{-1}x).$$

Proposition 4.1.3. — *Let $(\eta^0,u^0)\in L^2(\mathbf{R}\times\mathbf{T})$ and $(\eta^0_\varepsilon,u^0_\varepsilon)$ satisfy assumptions (4.1.1), and denote by $((\eta_\varepsilon,u_\varepsilon))_{\varepsilon>0}$ a family of solutions of (SW_ε) with respective initial data $(\eta^0_\varepsilon,u^0_\varepsilon)$.*

Then $\eta^\delta_\varepsilon=\kappa_\delta\star\eta_\varepsilon$ and $m^\delta_\varepsilon=\kappa_\delta\star((1+\varepsilon\eta_\varepsilon)u_\varepsilon)=u^\delta_\varepsilon+\varepsilon(\eta_\varepsilon u_\varepsilon)^\delta$ satisfy, for all $T>0$, the uniform convergences for all $\Omega\subset\subset\mathbf{R}\times\mathbf{T}$

$$\begin{aligned}&\|\eta_\varepsilon-\eta^\delta_\varepsilon\|_{L^\infty(\mathbf{R}^+,H^s(\Omega))}\to 0 \text{ as } \delta\to 0 \text{ uniformly in } \varepsilon>0, \text{ for all } s<0,\\ &\|u_\varepsilon-u^\delta_\varepsilon\|_{L^2([0,T];H^s(\Omega))}\to 0 \text{ as } \delta\to 0 \text{ uniformly in } \varepsilon>0, \text{ for all } s<1,\\ &\|\eta_\varepsilon u_\varepsilon-(\eta_\varepsilon u_\varepsilon)^\delta\|_{L^2([0,T];H^s(\Omega))}\to 0 \text{ as } \delta\to 0 \text{ uniformly in } \varepsilon>0, \text{ for all } s<0,\end{aligned}\tag{4.1.5}$$

as well as the approximate wave equations

$$\begin{aligned}\varepsilon\partial_t\eta^\delta_\varepsilon+\nabla\cdot m^\delta_\varepsilon&=0,\\ \varepsilon\partial_t m^\delta_\varepsilon+\beta x_1(m^\delta_\varepsilon)^\perp+\nabla\eta^\delta_\varepsilon&=\varepsilon s^\delta_\varepsilon+\delta\sigma^\delta_\varepsilon,\end{aligned}\tag{4.1.6}$$

denoting by s^δ_ε and $\sigma^\delta_\varepsilon$ some quantities satisfying, for all $T>0$,

$$\begin{aligned}&\sup_{\delta>0}\sup_{\varepsilon>0}\|\sigma^\delta_\varepsilon\|_{L^2([0,T];H^1(\mathbf{R}\times\mathbf{T}))}<+\infty,\\ &\forall\delta>0,\quad \sup_{\varepsilon>0}\|s^\delta_\varepsilon\|_{L^1([0,T];H^1(\mathbf{R}\times\mathbf{T}))}<\infty.\end{aligned}\tag{4.1.7}$$

In particular the approximate vorticity $\omega^\delta_\varepsilon=\nabla^\perp\cdot m^\delta_\varepsilon$ satisfies

$$\varepsilon\partial_t(\omega^\delta_\varepsilon-\beta x_1\eta^\delta_\varepsilon)+\beta m^\delta_{\varepsilon,1}=\varepsilon q^\delta_\varepsilon+\delta p^\delta_\varepsilon,\tag{4.1.8}$$

with, for all $T>0$,

$$\sup_{\substack{\varepsilon>0\\ \delta>0}}\|p^\delta_\varepsilon\|_{L^2([0,T];L^2(\mathbf{R}\times\mathbf{T}))}<+\infty \text{ and } \forall\delta>0,\quad \sup_{\varepsilon>0}\|q^\delta_\varepsilon\|_{L^1([0,T];L^2(\mathbf{R}\times\mathbf{T}))}<+\infty.\tag{4.1.9}$$

Proof. — We proceed in two steps, first stating the wave equations for $(\eta_\varepsilon,m_\varepsilon)$, then introducing the regularization $(\eta^\delta_\varepsilon,m^\delta_\varepsilon)$.

• The first step consists in establishing some bounds for

$$\varepsilon\partial_t(\eta_\varepsilon,m_\varepsilon)+L(\eta_\varepsilon,m_\varepsilon).$$

We have

$$\varepsilon\partial_t\eta_\varepsilon+\nabla\cdot m_\varepsilon=0,$$

and

$$\varepsilon\partial_t m_\varepsilon + \beta x_1 m_\varepsilon^\perp + \nabla\eta_\varepsilon = \varepsilon r_\varepsilon,$$

with

$$r_\varepsilon = -\nabla\cdot(m_\varepsilon\otimes u_\varepsilon) - \eta_\varepsilon\nabla\eta_\varepsilon + \nu\Delta u_\varepsilon.$$

Let us now find a bound for r_ε. It is made of three contributions. The easiest to handle is Δu_ε. Indeed u_ε is bounded in $L^2(\mathbf{R}^+,\dot H^1(\mathbf{R}\times\mathbf{T}))$, so Δu_ε is bounded in $L^2(\mathbf{R}^+,H^{-1}(\mathbf{R}\times\mathbf{T}))$.

Next let us consider the nonlinear terms

$$-\nabla\cdot(m_\varepsilon\otimes u_\varepsilon) - \eta_\varepsilon\nabla\eta_\varepsilon = -\nabla\cdot(m_\varepsilon\otimes u_\varepsilon) - \frac{1}{2}\nabla\eta_\varepsilon^2.$$

By the energy bound (1.4.5) we infer that they are bounded in $L^\infty(\mathbf{R}^+,W^{-1,1}(\mathbf{R}\times\mathbf{T}))$.

Therefore in particular

$$\|r_\varepsilon\|_{L^2([0,T];H^{-5/2}(\mathbf{R}\times\mathbf{T}))} \leq C_T. \tag{4.1.10}$$

• Now let us proceed to the regularization. We recall that the energy inequality (1.4.5) provides the following uniform bounds

$$\begin{aligned}
&\|\sqrt{1+\varepsilon\eta_\varepsilon}u_\varepsilon\|_{L^\infty(\mathbf{R}^+,L^2(\mathbf{R}\times\mathbf{T}))} \leq C,\\
&\|\eta_\varepsilon\|_{L^\infty(\mathbf{R}^+,L^2(\mathbf{R}\times\mathbf{T}))} \leq C,\\
&\|u_\varepsilon\|_{L^2(\mathbf{R}^+,\dot H^1(\mathbf{R}\times\mathbf{T}))} \leq C.
\end{aligned}$$

In particular we have

$$\|u_\varepsilon\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \leq \|\sqrt{1+\varepsilon\eta_\varepsilon}u_\varepsilon\|^2_{L^2(\mathbf{R}\times\mathbf{T})} + C\varepsilon\|\eta_\varepsilon\|_{L^2(\mathbf{R}\times\mathbf{T})}\|u_\varepsilon\|_{L^2(\mathbf{R}\times\mathbf{T})}\|u_\varepsilon\|_{\dot H^1(\mathbf{R}\times\mathbf{T})},$$

thus by the Cauchy-Schwarz inequality

$$\frac{1}{2}\|u_\varepsilon\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \leq \|\sqrt{1+\varepsilon\eta_\varepsilon}u_\varepsilon\|^2_{L^2(\mathbf{R}\times\mathbf{T})} + 8C^2\varepsilon^2\|\eta_\varepsilon\|^2_{L^2(\mathbf{R}\times\mathbf{T})}\|\nabla u_\varepsilon\|^2_{L^2(\mathbf{R}\times\mathbf{T})}.$$

and finally

$$\|u_\varepsilon\|_{L^2([0,T];H^1(\mathbf{R}\times\mathbf{T}))} \leq C_T. \tag{4.1.11}$$

We also deduce from the usual product laws that

$$\|\eta_\varepsilon u_\varepsilon\|_{L^2([0,T];H^s(\mathbf{R}\times\mathbf{T}))} \leq C_s \text{ for all } s<0. \tag{4.1.12}$$

The convergences (4.1.5) are then obtained by the Rellich Kondrachov theorem.

• By convolution we get (with obvious notation)

$$\varepsilon\partial_t\eta_\varepsilon^\delta + \nabla\cdot u_\varepsilon^\delta = 0,$$

and

$$\varepsilon\partial_t m_\varepsilon^\delta + \beta x_1(m_\varepsilon^\delta)^\perp + \nabla\eta_\varepsilon^\delta = \varepsilon r_\varepsilon^\delta + \beta x_1(m_\varepsilon^\delta)^\perp - (\beta x_1 m_\varepsilon^\delta)^\perp.$$

Notice that

$$\begin{aligned}
x_1 m_\varepsilon^\delta(x) - (x_1 m_\varepsilon)^\delta(x) &= \int \kappa_\delta(y)m_\varepsilon(x-y)(x_1-(x_1-y_1))\,dy\\
&= \int y_1\kappa_\delta(y)m_\varepsilon(x-y)\,dy.
\end{aligned}$$

That implies that

$$x_1 m_\varepsilon^\delta(x) - (x_1 m_\varepsilon)^\delta(x) = \delta\kappa_\delta^{(1)}(y)\star m_\varepsilon, \tag{4.1.13}$$

where $\kappa_\delta^{(1)}(y) = \delta^{-2}\kappa^{(1)}(\delta^{-1}x)$ and $\kappa^{(1)}(x) = x_1\kappa(x)$. We therefore infer that

$$\beta x_1 (m_\varepsilon^\delta)^\perp(x) - \beta(x_1 m_\varepsilon^\perp)^\delta(x) = \delta\sigma_\varepsilon^\delta(x) + \varepsilon\beta\delta\kappa_\delta^{(1)} * (\eta_\varepsilon u_\varepsilon),$$

where for all $T > 0$,

$$\sup_{\delta>0}\sup_{\varepsilon>0} \|\sigma_\varepsilon^\delta\|_{L^2([0,T];H^1(\mathbf{R}\times\mathbf{T}))} < +\infty.$$

It remains then to control

$$s_\varepsilon^\delta = r_\varepsilon^\delta + \beta\delta\kappa_\delta^{(1)} * (\eta_\varepsilon u_\varepsilon)\,.$$

By (4.1.10) and (4.1.12) we get

$$\forall \delta > 0, \quad \sup_{\varepsilon>0} \|s_\varepsilon^\delta\|_{L^1([0,T];H^1(\mathbf{R}\times\mathbf{T}))} < \infty.$$

• Taking the vorticity in the second equation of (4.1.6) leads then to

$$\partial_t\omega_\varepsilon^\delta + \nabla^\perp \cdot (\beta x_1 (m_\varepsilon^\delta)^\perp + \nabla\eta_\varepsilon^\delta) = \varepsilon\nabla^\perp \cdot s_\varepsilon^\delta + \delta\nabla^\perp \cdot \sigma_\varepsilon^\delta,$$

from which we deduce that

$$\partial_t\omega_\varepsilon^\delta + \beta x_1 \nabla \cdot m_\varepsilon^\delta + \beta m_{\varepsilon,1}^\delta = \varepsilon\nabla^\perp \cdot s_\varepsilon^\delta + \delta\nabla^\perp \cdot \sigma_\varepsilon^\delta.$$

Combining this last equation with the first one in (4.1.6) gives finally

$$\partial_t(\omega_\varepsilon^\delta - \beta x_1\eta_\varepsilon^\delta) + \beta m_{\varepsilon,1}^\delta = \varepsilon\nabla^\perp \cdot s_\varepsilon^\delta + \delta\nabla^\perp \cdot \sigma_\varepsilon^\delta,$$

from which we deduce the estimate (4.1.9) on the remainder. The proposition is proved. □

4.1.3. Proof of Theorem 4. — We consider an initial data $(\eta^0, u^0) \in L^2(\mathbf{R}\times\mathbf{T})$ and a family $(\eta_\varepsilon^0, u_\varepsilon^0)$ such that

$$\frac{1}{2}\int \left(|\eta_\varepsilon^0|^2 + (1+\varepsilon\eta_\varepsilon^0)|u_\varepsilon^0|^2\right) dx \le \mathcal{E}^0,$$
$$(\eta_\varepsilon^0, u_\varepsilon^0) \to (\eta^0, u^0) \text{ in } L^2(\mathbf{R}\times\mathbf{T}).$$

We consider a family $((\eta_\varepsilon, u_\varepsilon))_{\varepsilon>0}$ of solutions of (SW_ε) with respective initial data $(\eta_\varepsilon^0, u_\varepsilon^0)$ (given by Corollary 1.4.1 page 6), and (η, u) any of its limit points. Finally we consider (η^*, u^*) in $(L^2\times H^1)(\mathbf{R})$ such that (η^*, u^*) belongs to the kernel of L. In particular by Proposition 2.1.1 we know that

$$u_1^* = 0 \quad \text{and} \quad \beta x_1 u_2^* + \partial_1\eta^* = 0.$$

Our aim is to prove that

$$\int (\eta\eta^* + u_2 u_2^*)(t,x)\,dx + \nu\int_0^t\int \nabla u_2 \cdot \nabla u_2^*(t',x)\,dx\,dt' = \int (\eta^0\eta^* + u_2^0 u_2^*)(x)\,dx. \tag{4.1.14}$$

• In order to establish such an identity, the idea is to take limits in the weak form of System (SW_ε), which will require some further regularity on (η^*, u^*), and then to extend the limiting equality to all vector fields $(\eta^*, u^*) \in L^2\times H^1(\mathbf{R}) \cap \mathrm{Ker}L$ by a density argument.

Note that the classical regularization method cannot be applied here, since the kernel of L is not stable by convolution. In view of the explicit formula (2.1.2) page 8 giving the projector Π_0 (which is written in terms of the singular pseudo-differential operator $\partial_1((\beta x_1)^{-1}\cdot)$), it is actually natural to consider the Hermite functions introduced in the previous chapter, and we recall (see (2.2.7) page 11) that any element of Ker$L$ is a linear combination of the following

$$(\eta_n, u_n) = \frac{1}{\sqrt{2\pi(2n+1)}} \begin{pmatrix} -\sqrt{\frac{n+1}{2}}\psi_{n-1}(x_1) - \sqrt{\frac{n}{2}}\psi_{n+1}(x_1) \\ 0 \\ \sqrt{\frac{n+1}{2}}\psi_{n-1}(x_1) - \sqrt{\frac{n}{2}}\psi_{n+1}(x_1) \end{pmatrix} \text{ for } n \geq 1,$$

$$\text{and } (\eta_0, u_0) = \begin{pmatrix} \psi_0(x_1) \\ 0 \\ \psi_0(x_1) \end{pmatrix}.$$

We will therefore restrict our attention to these particular vector fields which are smooth and integrable against any polynomial in x_1 (recall that

$$\psi_n(x_1) = \exp\left(-\frac{\beta x_1^2}{2}\right) P_n(x_1\sqrt{\beta})$$

where P_n is the n-th Hermite polynomial), and then conclude by a density argument.

• Using the conservations of mass and momentum (SW_ε) it is easy to see that

$$\begin{aligned} &\int (\eta_\varepsilon \eta_n + m_{\varepsilon,2} u_{n,2})(t,x)\, dx + \nu \int_0^t \int \nabla u_{\varepsilon,2} \cdot \nabla u_{n,2}(t',x)\, dx\, dt' \\ = \ &\int (\eta_\varepsilon^0 \eta_n + m_{\varepsilon,2}^0 u_{n,2})(x)\, dx + \int_0^t \int (m_\varepsilon \cdot (u_\varepsilon \cdot \nabla u_n))(t',x)\, dx dt'. \end{aligned}$$

Now we need to take limits as ε goes to zero in all four ε-dependent integrals appearing in that expression.

Clearly the three first terms converge to their expected limits, as

$$\int (\eta_\varepsilon \eta_n + u_{\varepsilon,2} u_{n,2})(t,x)\, dx \to \int (\eta \eta_n + u_2 u_{n,2})(t,x)\, dx$$

$$\int \left(\eta_\varepsilon^0 \eta_n + u_{\varepsilon,2}^0 u_{n,2}\right)(x)\, dx \to \int \left(\eta^0 \eta_n + u_2^0 u_{n,2}\right)(x)\, dx$$

and

$$\nu \int_0^t \int \nabla u_{\varepsilon,2} \cdot \nabla u_{n,2}(t',x)\, dx\, dt' \to \nu \int_0^t \int \nabla u_2 \cdot \nabla u_{n,2}(t',x)\, dx\, dt'$$

for all $t \geq 0$, as ε goes to zero.

So the only term we need to worry about is the coupling term

$$\int_0^t \int m_{\varepsilon,2} u_{\varepsilon,1} \partial_1 u_{n,2}(t',x)\, dx dt'.$$

We will prove in the following lemma that it actually converges to 0 :

$$\lim_{\varepsilon \to 0} \int_0^t \int m_{\varepsilon,2} u_{\varepsilon,1} \partial_1 u_{n,2}(t',x)\, dx dt' = 0,$$

which is due to the special structure of the oscillations pointed out in Proposition 4.1.3. This result clearly ends the proof of Theorem 4, and is proved in the next paragraph.

4.1.4. The compensated compactness argument. — Let us prove the following lemma.

Lemma 4.1.4. — *With the previous notation, we have locally uniformly in t*

$$\lim_{\varepsilon\to 0}\int_0^t\int m_{\varepsilon,2}u_{\varepsilon,1}\partial_1 u_{n,2}(t',x)\,dxdt' = 0.$$

Proof. — Let us introduce the same regularization as in Proposition 4.1.3, defining

$$\eta_\varepsilon^\delta = \eta_\varepsilon \star \kappa_\delta, \quad u_\varepsilon^\delta = u_\varepsilon \star \kappa_\delta \quad \text{and} \quad m_\varepsilon^\delta = m_\varepsilon \star \kappa_\delta.$$

Then

$$\begin{aligned}\int_0^t\int m_{\varepsilon,2}u_{\varepsilon,1}\partial_1 u_{n,2}(t',x)\,dxdt' &= \int_0^t\int m^\delta_{\varepsilon,2}m^\delta_{\varepsilon,1}\partial_1 u_{n,2}(t',x)\,dxdt'\\ &+\int_0^t\int m^\delta_{\varepsilon,2}(u^\delta_{\varepsilon,1}-m^\delta_{\varepsilon,1})\partial_1 u_{n,2}(t',x)\,dxdt'\\ &+\int_0^t\int m^\delta_{\varepsilon,2}(u_{\varepsilon,1}-u^\delta_{\varepsilon,1})\partial_1 u_{n,2}(t',x)\,dxdt'\\ &+\int_0^t\int (m_{\varepsilon,2}-m^\delta_{\varepsilon,2})u_{\varepsilon,1}\partial_1 u_{n,2}(t',x)\,dxdt'.\end{aligned}\tag{4.1.15}$$

• By the energy estimates and the bounds on the Hermite functions given in Proposition 2.2.3 page 12, we can prove that the two last integrals converge towards zero as δ goes to zero uniformly in ε. Indeed for all $\alpha > 0$ there exists some bounded subset $\Omega_\alpha \times \mathbf{T}$ of $\mathbf{R}\times\mathbf{T}$ such that (recalling that n is fixed)

$$\|\partial_1 u_{n,2}\|_{W^{1,\infty}(\mathbf{R}\setminus\Omega_\alpha)} \le \alpha.$$

Then, for $0 < s < 1$ and for any $s' > 0$,

$$\begin{aligned}&\left|\int_0^t\int (m_{\varepsilon,2}-m^\delta_{\varepsilon,2})u_{\varepsilon,1}\partial_1 u_{n,2}(t',x)\,dxdt'\right|\\ &\le \|m_{\varepsilon,2}-m^\delta_{\varepsilon,2}\|_{L^2([0,T];H^{-s-s'}(\Omega_\alpha\times\mathbf{T}))}\|u_{\varepsilon,1}\|_{L^2([0,T];H^1(\mathbf{R}\times\mathbf{T}))}\|\partial_1 u_{n,2}\|_{W^{1,\infty}(\mathbf{R})}\\ &+2\alpha\|m_{\varepsilon,2}\|_{L^2([0,T];H^{-s-s'}(\mathbf{R}\times\mathbf{T}))}\|u_{\varepsilon,1}\|_{L^2([0,T];H^1(\mathbf{R}\times\mathbf{T}))}\end{aligned}$$

which goes to zero as α then δ go to zero, uniformly in ε by (1.4.5), (4.1.11) and (4.1.5).

Similarly, we get, for $0 < s < 1$,

$$\begin{aligned}&\left|\int_0^t\int m^\delta_{\varepsilon,2}(u_{\varepsilon,1}-u^\delta_{\varepsilon,1})\partial_1 u_{n,2}(t',x)\,dxdt'\right|\\ &\le \|m^\delta_{\varepsilon,2}\|_{L^2([0,T];H^{-s}(\mathbf{R}\times\mathbf{T}))}\|u_{\varepsilon,1}-u^\delta_{\varepsilon,1}\|_{L^2([0,T];H^{s}(\Omega_\alpha\times\mathbf{T}))}\|\partial_1 u_{n,2}\|_{W^{1,\infty}(\mathbf{R}\times\mathbf{T})}\\ &+2\alpha\|m_{\varepsilon,2}\|_{L^2([0,T];H^{-s}(\mathbf{R}\times\mathbf{T}))}\|u_{\varepsilon,1}\|_{L^2([0,T];H^1(\mathbf{R}\times\mathbf{T}))}\end{aligned}$$

which goes to zero as α then δ go to zero, uniformly in ε by (1.4.5), (4.1.11) and (4.1.5).

Next we prove that for all $\delta > 0$, the second integral in the right-hand side of (4.1.15) goes to zero as ε goes to 0. We have seen in (4.1.12) that $\eta_\varepsilon u_\varepsilon$ and consequently m_ε are uniformly bounded in the space $L^2([0,T];H^s(\mathbf{R}\times\mathbf{T}))$ for $s < 0$. Therefore, for fixed $\delta > 0$, $(\eta_\varepsilon u_\varepsilon)^\delta$ and m^δ_ε are uniformly bounded in $L^2([0,T]\times\mathbf{R}\times\mathbf{T})$. Then,

$$\begin{aligned}&\left|\int_0^t\int m^\delta_{\varepsilon,2}(u^\delta_{\varepsilon,1}-m^\delta_{\varepsilon,1})\partial_1 u_{n,2}(t',x)\,dxdt'\right|\\ &\le \varepsilon\|m^\delta_{\varepsilon,2}\|_{L^2([0,T]\times\mathbf{R}\times\mathbf{T})}\|(\eta_\varepsilon u_{\varepsilon,1})^\delta\|_{L^2([0,T]\times\mathbf{R}\times\mathbf{T})}\|\partial_1 u_{n,2}\|_{L^\infty(\mathbf{R}\times\mathbf{T})}\end{aligned}$$

which goes to zero as $\varepsilon \to 0$ for all fixed $\delta > 0$.

• So finally we need to consider the first term in the right-hand side of (4.1.15). We are going to prove that the limit of that term is zero using Proposition 4.1.4. Integrating by parts, we have, recalling that $\omega_\varepsilon^\delta = \nabla^\perp m_\varepsilon^\delta$,

$$
\begin{aligned}
&\int_0^t \int m_{\varepsilon,2}^\delta m_{\varepsilon,1}^\delta \partial_1 u_{n,2}(t',x)\, dxdt' \\
&\quad = -\int_0^t \int \left((\partial_1 m_{\varepsilon,2}^\delta) m_{\varepsilon,1}^\delta + m_{\varepsilon,2}^\delta (\partial_1 m_{\varepsilon,1}^\delta) \right) u_{n,2}(t',x)\, dxdt' \\
&\quad = -\int_0^t \int \left((-\omega_\varepsilon^\delta + \partial_2 m_{\varepsilon,1}^\delta) m_{\varepsilon,1}^\delta + m_{\varepsilon,2}^\delta (\nabla \cdot m_\varepsilon^\delta - \partial_2 m_{\varepsilon,2}^\delta) \right) u_{n,2}(t',x)\, dxdt' \\
&\quad = -\int_0^t \int \left((-\omega_\varepsilon^\delta + \beta x_1 \eta_\varepsilon^\delta) m_{\varepsilon,1}^\delta + \eta_\varepsilon^\delta(-\beta x_1 m_{\varepsilon,1}^\delta + \partial_2 \eta_\varepsilon^\delta) + m_{\varepsilon,2}^\delta (\nabla \cdot m_\varepsilon^\delta) \right) u_{n,2}(t',x)\, dxdt' \\
&\qquad - \frac{1}{2} \int_0^t \int \partial_2 \left((m_{\varepsilon,1}^\delta)^2 - (m_{\varepsilon,2}^\delta)^2 - (\eta_\varepsilon^\delta)^2 \right) u_{n,2}(t',x) dxdt'
\end{aligned}
$$

and the last term is zero because $\partial_2 u_{n,2} = 0$.

Proposition 4.1.4 now implies that

$$
\begin{aligned}
\varepsilon \partial_t (\omega_\varepsilon^\delta - \beta x_1 \eta_\varepsilon^\delta) + \beta m_{\varepsilon,1}^\delta &= \varepsilon q_\varepsilon^\delta + \delta p_\varepsilon^\delta, \\
\varepsilon \partial_t m_{\varepsilon,2}^\delta - \beta x_1 m_{\varepsilon,1}^\delta + \partial_2 \eta_\varepsilon^\delta &= \varepsilon s_{\varepsilon,2}^\delta + \delta \sigma_{\varepsilon,2}^\delta, \\
\varepsilon \partial_t \eta_\varepsilon^\delta + \nabla \cdot m_\varepsilon^\delta &= 0,
\end{aligned}
$$

where q_ε^δ and s_ε^δ are bounded respectively in $L^1([0,T]; L^2(\mathbf{R} \times \mathbf{T}))$ and $L^1([0,T]; H^1(\mathbf{R} \times \mathbf{T}))$ for any $T > 0$ uniformly in ε (by a constant depending on δ), and where p_ε^δ and $\sigma_\varepsilon^\delta$ are uniformly bounded in ε and δ, respectively in the spaces $L^2([0,T]; L^2(\mathbf{R} \times \mathbf{T}))$ and $L^2([0,T]; H^1(\mathbf{R} \times \mathbf{T}))$ for any $T > 0$. It follows that

$$
\begin{aligned}
&\int_0^t \int m_{\varepsilon,2}^\delta m_{\varepsilon,1}^\delta \partial_1 u_{n,2}(t',x)\, dxdt' \\
&\quad = -\int_0^t \int \left(\frac{\varepsilon}{2\beta} \partial_t (\beta x_1 \eta_\varepsilon^\delta - \omega_\varepsilon^\delta)^2 + \frac{\varepsilon}{\beta} (\beta x_1 \eta_\varepsilon^\delta - \omega_\varepsilon^\delta) q_\varepsilon^\delta + \frac{\delta}{\beta} (\beta x_1 \eta_\varepsilon^\delta - \omega_\varepsilon^\delta) p_\varepsilon^\delta \right) u_{n,2}(t',x)\, dxdt' \\
&\quad - \int_0^t \int \left(-\varepsilon \partial_t (\eta_\varepsilon^\delta m_{\varepsilon,2}^\delta) + \varepsilon \eta_\varepsilon^\delta s_{\varepsilon,2}^\delta + \delta \eta_\varepsilon^\delta \sigma_{\varepsilon,2}^\delta \right) u_{n,2}(t',x) dxdt'
\end{aligned}
$$

Now we notice that

$$
\begin{aligned}
\left| \int_0^t \int (\beta x_1 \eta_\varepsilon^\delta - \omega_\varepsilon^\delta) p_\varepsilon^\delta u_{n,2}(t',x)\, dxdt' \right| \leq & C \left(T^{1/2} \|\eta_\varepsilon^\delta\|_{L^\infty(\mathbf{R}^+; L^2(\mathbf{R} \times \mathbf{T}))} + \|\omega_\varepsilon^\delta\|_{L^2([0,T]; L^2(\mathbf{R} \times \mathbf{T}))} \right) \\
& \times \|(1 + x_1^2)^{1/2} u_{n,2}\|_{L^\infty(\mathbf{R} \times \mathbf{T})} \|p_\varepsilon^\delta\|_{L^2([0,T]; L^2(\mathbf{R} \times \mathbf{T}))},
\end{aligned}
$$

and similarly

$$
\left| \int_0^t \int \eta_\varepsilon^\delta \sigma_{\varepsilon,2}^\delta u_{n,2}(t',x)\, dxdt' \right| \leq C T^{1/2} \|\eta_\varepsilon^\delta\|_{L^\infty(\mathbf{R}^+; L^2(\mathbf{R} \times \mathbf{T}))} \|u_{n,2}\|_{L^\infty(\mathbf{R} \times \mathbf{T})} \|\sigma_\varepsilon^\delta\|_{L^2([0,T]; L^2(\mathbf{R} \times \mathbf{T}))}.
$$

So writing

$$
\|\omega_\varepsilon^\delta\|_{L^2([0,T]; L^2(\mathbf{R} \times \mathbf{T}))} \leq \|\nabla^\perp \cdot u_\varepsilon^\delta\|_{L^2([0,T]; L^2(\mathbf{R} \times \mathbf{T}))} + \varepsilon \|\nabla^\perp \cdot (\eta_\varepsilon^\delta u_\varepsilon^\delta)\|_{L^2([0,T]; L^2(\mathbf{R} \times \mathbf{T}))},
$$

we infer that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left(\frac{\delta}{\beta}\left|\int_0^t\int(\beta x_1\eta_\varepsilon^\delta-\omega_\varepsilon^\delta)p_\varepsilon^\delta u_{n,2}(t',x)\,dxdt'\right|\right)=0,\quad\text{and}$$

$$\lim_{\delta\to 0}\left(\delta\left|\int_0^t\int\eta_\varepsilon^\delta\sigma_{\varepsilon,2}^\delta u_{n,2}(t',x)\,dxdt'\right|\right)=0,\quad\text{uniformly in }\varepsilon.$$

On the other hand,

$$\begin{aligned}\left|\int_0^t\int(\beta x_1\eta_\varepsilon^\delta-\omega_\varepsilon^\delta)q_\varepsilon^\delta u_{n,2}(t',x)\,dxdt'\right| &\leq C\left(\|\eta_\varepsilon^\delta\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T}))}+\|\omega_\varepsilon^\delta\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T}))}\right)\\ &\quad\times\|(1+x_1^2)^{1/2}u_{n,2}\|_{L^\infty(\mathbf{R}\times\mathbf{T})}\|q_\varepsilon^\delta\|_{L^1([0,T];L^2(\mathbf{R}\times\mathbf{T}))}\\ &\leq C\left(\|\eta_\varepsilon^\delta\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T}))}+\frac{1}{\delta}\|u_\varepsilon\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T}))}+\varepsilon\|\nabla^\perp\cdot(\eta_\varepsilon^\delta u_\varepsilon^\delta)\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T}))}\right)\\ &\quad\times\|(1+x_1^2)u_{n,2}\|_{L^\infty(\mathbf{R}\times\mathbf{T})}\|q_\varepsilon^\delta\|_{L^1([0,T];L^2(\mathbf{R}\times\mathbf{T}))},\end{aligned}$$

and

$$\left|\int_0^t\int\eta_\varepsilon^\delta s_{\varepsilon,2}^\delta u_{n,2}(t',x)\,dxdt'\right|\leq C\|\eta_\varepsilon^\delta\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T}))}\|u_{n,2}\|_{L^\infty(\mathbf{R}\times\mathbf{T})}\|s_\varepsilon^\delta\|_{L^1([0,T];L^2(\mathbf{R}\times\mathbf{T}))}$$

so

$$\lim_{\varepsilon\to 0}\left(\frac{\varepsilon}{\beta}\left|\int_0^t\int(\beta x_1\eta_\varepsilon^\delta-\omega_\varepsilon^\delta)q_\varepsilon^\delta u_{n,2}(t',x)\,dxdt'\right|\right)=0,\quad\text{for all }\delta>0,$$

$$\lim_{\varepsilon\to 0}\left(\varepsilon\left|\int_0^t\int\eta_\varepsilon^\delta s_{\varepsilon,2}^\delta u_{n,2}(t',x)\,dxdt'\right|\right)=0,\quad\text{for all }\delta>0.$$

So we simply need to let ε go to zero, then δ, and the result follows. □

4.2. Strong convergence of filtered weak solutions towards a strong solution

In this paragraph we will prove the following strong convergence theorems. We recall that $\Pi_\perp$ denotes the projection onto $(\mathrm{Ker}L)^\perp$, and that the spaces H^s_L were defined and studied in Chapter 3 and defined again in this chapter, page 79.

The first result we will state concerns the case of smooth enough initial data, and requires no restriction on β.

***Theorem 5* (strong convergence for all β).** — *Let $\Phi^0=(\eta^0,u^0)$ belong to $L^2(\mathbf{R}\times\mathbf{T})$, and consider a family $((\eta_\varepsilon^0,u_\varepsilon^0))_{\varepsilon>0}$ such that*

$$\frac{1}{2}\int\left(|\eta_\varepsilon^0|^2+(1+\varepsilon\eta_\varepsilon^0)|u_\varepsilon^0|^2\right)dx\leq\mathcal{E}^0\quad\textit{and}$$

$$\frac{1}{2}\int\left(|\eta_\varepsilon^0-\eta^0|^2+(1+\varepsilon\eta_\varepsilon^0)|u_\varepsilon^0-u^0|^2\right)dx\to 0\ \textit{as}\ \varepsilon\to 0.$$

For all $\varepsilon>0$ denote by $(\eta_\varepsilon,u_\varepsilon)$ a solution of (SW_ε) with initial data $(\eta_\varepsilon^0,u_\varepsilon^0)$. Finally suppose that $\Pi_\perp\Phi^0$ belongs to $H_L^{1/2}$ and that $\Pi_P\Phi^0$ and $\Pi_K\Phi^0$ belong to H_L^α for some $\alpha>3/2$. Then the sequence of filtered solutions (Φ_ε) to (SW_ε) defined by

$$\Phi_\varepsilon=\mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon,u_\varepsilon),\tag{4.2.1}$$

converges strongly towards Φ in $L^2_{loc}([0,T^*[;L^2(\mathbf{R}\times\mathbf{T}))$, where Φ is the unique solution on $[0,T^*[$ of (SW_0) constructed in Theorem 2, page 31 .

The next theorem requires less assumptions on the initial data; on the other hand one must first remove a countable set of values for β.

***Theorem 6* (strong convergence for generic β).** — *There is a countable subset $\mathcal{N}$ of $\mathbf{R}^+$ such that for any $\beta \in \mathbf{R}^+ \setminus \mathcal{N}$, the following result holds. Let $\Phi^0 \in L^2(\mathbf{R} \times \mathbf{T}; \mathbf{R}^3)$ be given, and consider a family $((\eta_\varepsilon^0, u_\varepsilon^0))_{\varepsilon>0}$ such that*

$$
\begin{aligned}
&\frac{1}{2} \int \left(|\eta_\varepsilon^0|^2 + (1 + \varepsilon \eta_\varepsilon^0) |u_\varepsilon^0|^2 \right) dx \leq \mathcal{E}^0 \quad \text{and} \\
&\frac{1}{2} \int \left(|\eta_\varepsilon^0 - \eta^0|^2 + (1 + \varepsilon \eta_\varepsilon^0) |u_\varepsilon^0 - u^0|^2 \right) dx \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}
\tag{4.2.2}
$$

For all $\varepsilon > 0$ denote by $(\eta_\varepsilon, u_\varepsilon)$ a solution of (SW_ε) with initial data $(\eta_\varepsilon^0, u_\varepsilon^0)$. Finally suppose that $\Pi_P \Phi^0$ and $\Pi_K \Phi^0$ belong to H_L^α for some $\alpha > 1/2$. Then the sequence of filtered solutions (Φ_ε) to (SW_ε) defined by (4.2.1)

$$\Phi_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon)$$

converges strongly towards Φ in $L^2_{loc}(\mathbf{R}^+; L^2(\mathbf{R} \times \mathbf{T}))$, where Φ is the unique solution of (SW_0) constructed in Theorem 3, page 32 .

Remark 4.2.1. — • *Note that definition (4.2.1) of Φ_ε does make sense since as stated in Corollary 1.4.1, one has a L^2 bound on u_ε.*

• *The strong compactness of (Φ_ε) in $L^2_{loc}(\mathbf{R}^+, L^2(\mathbf{R} \times \mathbf{T}))$ cannot be obtained directly using some a priori estimates. Indeed we have a priori no uniform regularity on η_ε with respect to the space variable x (besides we expect the limiting system to be a mixed hyperbolic-parabolic system).*

• *The proof of both convergence results is based on a weak-strong stability property of (SW_ε). It is therefore crucial to be able to construct a smooth approximate solution Φ_{app} to Φ_ε, writing an asymptotic expansion in ε whose first term is Φ. The regularity assumptions on the initial data stated in both theorems are precisely that enabling one to guarantee that the limit system has a unique, stable solution and propagates regularity. In particular it should be noted that in both cases, the assumptions on the initial data imply that $\nabla \cdot \Phi'$ belongs to $L^1([0,T]; L^\infty(\mathbf{R} \times \mathbf{T}))$ (see Proposition 3.6.1). Once the setting is posed so that the limit system does satisfy those properties, the proofs are very much the same in both cases. So in the following we will only prove Theorem 6, and leave to the reader the easy adaptations in the case of Theorem 5.*

Proof. — As noted in Remark 4.2.1 above, we will only prove Theorem 6 here.

The idea is, as usual in filtering methods, to start by approximating the solution of the limit system, and then to use a weak-strong stability method to conclude.

So let us consider the solution Φ of (SW_0) constructed in the previous chapter (see Theorem 3 page 32), which we truncate in the following way:

$$\Phi_N = J_N \Pi_\perp \Phi + \Pi_0 \Phi_N, \tag{4.2.3}$$

where J_N is the spectral truncation defined by

$$J_N = \sum_{i\lambda \in \mathfrak{S}_N} \Pi_\lambda \tag{4.2.4}$$

with

$$\mathfrak{S}_N = \left\{ i\tau(n,k,j) \in \mathfrak{S} \Big/ n \leq N,\ |k| \leq N \right\},$$

and $\Pi_\perp$ denotes as previously the projection onto $(\mathrm{Ker}L)^\perp$. Finally $\Pi_0\Phi_N$ solves

$$\partial_t \Pi_0 \Phi_N - \nu \Pi_0 \Delta' \Pi_0 \Phi_N = 0$$

$$\Pi_0 \Phi_{N|t=0} = \sum_{0 \leq n \leq N} \Pi_{n,0,0} \Phi^0.$$

We recall that $\Pi_{n,0,0}$ denotes the projection onto the eigenvector $\Psi_{n,0,0}$ of $\mathrm{Ker}L$. Then for all fixed $N \in \mathbf{N}$ we have (see Theorem 2)

$$\Pi_0 \Phi_N \text{ belongs to } L^\infty(\mathbf{R}^+; H^\sigma_L),\ \forall \sigma \geq 0. \tag{4.2.5}$$

Recall that such a result means that $\Pi_0\Phi_N$ is as smooth as needed, and decays as fast as needed when x_1 goes to infinity.

Moreover by the stability of the limit system (which is linear) we have of course, for all $T > 0$,

$$\lim_{N \to \infty} \|\Pi_0 \Phi_N - \Pi_0 \Phi\|_{L^\infty([0,T]; L^2(\mathbf{R} \times \mathbf{T}))} = 0. \tag{4.2.6}$$

Note also that for all fixed $N \in \mathbf{N}$, using the smoothness and the decay of the eigenvectors of L, we get for any polynomial $Q \in \mathbf{R}[X]$

$$Q(x_1)\Phi_N \in L^\infty([0,T]; C^\infty(\mathbf{R} \times \mathbf{T})) \tag{4.2.7}$$

We have moreover, for all $T > 0$,

$$\left(\|\Pi_\perp(\Phi - \Phi_N)\|_{L^\infty([0,T];L^2(\mathbf{R}\times\mathbf{T}))} + \|(\Pi_K + \Pi_P)(\Phi - \Phi_N)\|_{L^\infty([0,T];H^\alpha_L)} \right) \to 0 \text{ as } N \to \infty, \tag{4.2.8}$$

and

$$\left(\|\Pi_\perp(\Phi - \Phi_N)\|_{L^2([0,T];H^1_L)} + \|(\Pi_K + \Pi_P)(\Phi - \Phi_N)\|_{L^2([0,T];H^{\alpha+1}_L)} \right) \to 0 \text{ as } N \to \infty. \tag{4.2.9}$$

Finally since J_N commutes with Δ'_L, the vector field Φ_N satisfies the approximate limit filtered system

$$\begin{aligned} &\partial_t \Phi_N + J_N Q_L(\Phi, \Phi) - \nu \Delta'_L \Phi_N = 0, \\ &\Phi_{N|t=0} = J_N \Phi^0. \end{aligned} \tag{4.2.10}$$

Conjugating this equation by the semi-group $\mathcal{L}$ leads then to

$$\partial_t \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N \right) + \frac{1}{\varepsilon} L \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N \right) + J_N Q_L \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi, \mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi \right) - \nu \Delta'_L \mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N = 0,$$

using the definitions (2.3.6) of Q_L and Δ'_L. Let us now rewrite this last equation in a convenient way

$$\begin{aligned} &\partial_t \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N \right) + \frac{1}{\varepsilon} L \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N \right) + Q \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N \right) - \nu \Delta' \mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N \\ &= (Q - Q_L) \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N \right) - \nu(\Delta' - \Delta'_L) \mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N \\ &+ (Id - J_N) Q_L \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi, \mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi \right) + Q_L \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) (\Phi_N - \Phi), \mathcal{L}\left(\frac{t}{\varepsilon}\right) (\Phi_N + \Phi) \right). \end{aligned}$$

Because of (4.2.8) and (4.2.6), the last term in the right-hand side is expected to be small when N is large, uniformly in ε, and similarly for the third term, using the stability of the limit system proved in the previous chapter. So we are left with the first two terms, which as usual cannot be dealt with so easily since they do not converge strongly towards zero. However they are fast oscillating terms, and will be treated by introducing a small quantity $\varepsilon\phi_N$ (which will be small when ε goes to zero, for each fixed N), so that

$$(\partial_t + \frac{1}{\varepsilon}L)\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\varepsilon\phi_N\right) \sim -(Q-Q_L)\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) + \nu(\Delta' - \Delta'_L)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N.$$

Let us now define

$$\phi_N = -\sum_{\substack{\lambda\neq\mu+\tilde\mu\\ i\lambda\in\mathfrak{S}, i\mu, i\tilde\mu\in\mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda-\mu-\tilde\mu)}}{i(\lambda-\mu-\tilde\mu)}\Pi_\lambda Q(\Pi_\mu\Phi_N, \Pi_{\tilde\mu}\Phi_N) + \nu \sum_{\substack{\lambda\neq\mu,\\ i\lambda\in\mathfrak{S}, i\mu\in\mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda-\mu)}}{i(\lambda-\mu)}\Pi_\lambda\Delta'\Pi_\mu\Phi_N, \tag{4.2.11}$$

and consider

$$\Phi_{\varepsilon,N} = \Phi_N + \varepsilon\phi_N.$$

Let us prove the following result.

Proposition 4.2.2. — *For all but a countable number of β, the following result holds. Consider a vector field $\Phi^0 = (\eta_0, u_0) \in L^2(\mathbf{R}\times\mathbf{T})$, with $(\Pi_P + \Pi_K)\Phi^0$ in H^α_L for some $\alpha > 1/2$. Denote by Φ the associate solution of (SW_0). Then there exists a family $(\eta_{\varepsilon,N}, u_{\varepsilon,N}) = \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_{\varepsilon,N}$, bounded in the space $L^\infty_{loc}(\mathbf{R}^+, L^2) \cap L^2_{loc}(\mathbf{R}^+, H^1)$, such that $(\Pi_P + \Pi_K)(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ is uniformly bounded in the space $L^\infty_{loc}(\mathbf{R}^+, H^\alpha_L) \cap L^2_{loc}(\mathbf{R}^+, H^{\alpha+1}_L)$, and satisfying the following properties:*

- *$\Phi_{\varepsilon,N}$ behaves asymptotically as Φ as $\varepsilon \to 0$ and $N \to \infty$:*

$$\forall T > 0, \quad \lim_{N\to\infty}\lim_{\varepsilon\to 0} \|\Phi_{\varepsilon,N} - \Phi\|_{L^\infty([0,T];L^2(\mathbf{R}\times\mathbf{T}))} = 0; \tag{4.2.12}$$

- *for all $N \in \mathbf{N}$, $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ is smooth: for all $T > 0$ and all $Q \in \mathbf{R}[X]$,*

$$Q(x_1)(\eta_{\varepsilon,N}, u_{\varepsilon,N}) \text{ is bounded in } L^\infty([0,T]; C^\infty(\mathbf{R}\times\mathbf{T})), \text{ uniformly in } \varepsilon; \tag{4.2.13}$$

- *$(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ satisfies the uniform regularity estimate*

$$\forall T > 0, \quad \sup_{N\in\mathbf{N}}\lim_{\varepsilon\to 0} \|\nabla\cdot u_{\varepsilon,N}\|_{L^1([0,T];L^\infty(\mathbf{R}\times\mathbf{T}))} \le C_T; \tag{4.2.14}$$

- *$(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ satisfies approximatively the viscous Saint-Venant system (SW_ε) :*

$$\partial_t(\eta_{\varepsilon,N}, u_{\varepsilon,N}) + \frac{1}{\varepsilon}L(\eta_{\varepsilon,N}, u_{\varepsilon,N}) + Q\left((\eta_{\varepsilon,N}, u_{\varepsilon,N}), (\eta_{\varepsilon,N}, u_{\varepsilon,N})\right) - \nu\Delta'(\eta_{\varepsilon,N}, u_{\varepsilon,N}) = R_{\varepsilon,N} \tag{4.2.15}$$

where $R_{\varepsilon,N}$ goes to 0 as $\varepsilon \to 0$ then $N \to \infty$:

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\left(\|R_{\varepsilon,N}\|_{L^1([0,T];L^2(\mathbf{R}\times\mathbf{T}))} + \varepsilon\|R_{\varepsilon,N}\|_{L^\infty([0,T]\times\mathbf{R}\times\mathbf{T})}\right) = 0. \tag{4.2.16}$$

Proof. — Let us define Φ_N as in (4.2.3) and ϕ_N as in (4.2.11). We can write

$$\phi_N = \phi_N^{(1)} + \phi_N^{(2)}, \quad \text{with}$$

$$\phi_N^{(1)} = -\sum_{\substack{\lambda\neq\mu+\tilde\mu\\ i\lambda\in\mathfrak{S}, i\mu, i\tilde\mu\in\mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda-\mu-\tilde\mu)}}{i(\lambda-\mu-\tilde\mu)}\Pi_\lambda Q(\Pi_\mu\Phi_N, \Pi_{\tilde\mu}\Phi_N)$$

$$\phi_N^{(2)} = \nu\sum_{\substack{\lambda\neq\mu,\\ i\lambda\in\mathfrak{S}, i\mu\in\mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda-\mu)}}{i(\lambda-\mu)}\Pi_\lambda\Delta'\Pi_\mu\Phi_N.$$

We will check that the approximate solution $\Phi_{\varepsilon,N}$ defined by

$$\Phi_{\varepsilon,N} = \Phi_N + \varepsilon\phi_N \tag{4.2.17}$$

satisfies the required properties.

It will be useful to notice that there are two positive functions $\Lambda_\pm(N)$ such that if μ belongs to $\mathfrak{S}_N$, then either $\mu = 0$ or

$$0 < \Lambda_-(N) \le \mu \le \Lambda_+(N) < +\infty.$$

It will also be useful to recall that, considering the asymptotics of $\lambda = \tau(n,k,j)$ as k or n go to infinity, the operator $\displaystyle\sum_{\lambda\neq 0} \frac{\Pi_\lambda}{\lambda}$ is continuous from H^σ_L to $H^{\sigma-1}_L$ for any given $\sigma \in \mathbf{R}$.

Finally we recall that the spectrum of L admits only 0 and ∞ as accumulation points.

• The correction ϕ_N is defined as the sum of two terms.

Let us consider the first one, $\phi_N^{(1)}$. It can in turn be written

$$\phi_N^{(1)} = \Pi_0\phi_N^{(1)} + \Pi_\perp\phi_N^{(1)}.$$

The first part, $\Pi_0\phi_N^{(1)}$, is easy to handle since Π_0 is of course continuous from H^σ_L to H^σ_L for any σ.

Let us now study $\Pi_\perp\phi_N^{(1)}$. Clearly $Q(\Pi_\mu\Phi_N, \Pi_{\tilde\mu}\Phi_N)$ is in H^σ_L for any $\sigma \ge 0$. We infer that $\Pi_\perp\phi_N^{(1)}$ belongs to H^σ_L for any $\sigma \ge 0$. Indeed if $\mu+\tilde\mu \neq 0$ then $|\lambda-\mu-\tilde\mu|$ is bounded from below since μ and $\tilde\mu$ are in $\mathfrak{S}_N$ and the accumulation points of λ are 0 and ∞. On the other hand if $\mu+\tilde\mu = 0$ then we use the continuity property of $\displaystyle\sum_{\lambda\neq 0} \frac{\Pi_\lambda}{\lambda}$ recalled above.

So we find that for all polynomials $Q \in \mathbf{R}[X]$,

$$Q(x_1)\phi_N^{(1)} \in L^\infty([0,T]; C^\infty(\mathbf{R}\times\mathbf{T})),$$

as well as

$$Q(x_1)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\phi_N^{(1)} \in L^\infty([0,T]; C^\infty(\mathbf{R}\times\mathbf{T})).$$

This obviously implies that for all $k \in \mathbf{N}$

$$\forall N \in \mathbf{N}, \quad \lim_{\varepsilon\to 0} \|\varepsilon Q(x_1)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\phi_N^{(1)}\|_{L^\infty([0,T];C^k(\mathbf{R}\times\mathbf{T}))} = 0.$$

The second term $\phi_N^{(2)}$ is dealt with similarly, splitting it into two terms:

$$\phi_N^{(2)} = \nu \sum_{\substack{\lambda\neq\mu,\\ i\lambda\in\mathfrak{S}, i\mu\in\mathfrak{S}_N\setminus\{0\}}} \frac{e^{i\frac{t}{\varepsilon}(\lambda-\mu)}}{i(\lambda-\mu)}\Pi_\lambda\Delta'\Pi_\mu\Phi_N + \nu \sum_{\substack{\lambda\neq 0,\\ i\lambda\in\mathfrak{S}}} \frac{e^{i\frac{t}{\varepsilon}(\lambda)}}{i\lambda}\Pi_\lambda\Delta'\Pi_0\Phi_N.$$

Because of the relations (2.2.1) satisfied by the Hermite functions, it is easy to see that the first contribution can be rewritten as a finite combination of some eigenvectors of L (which are smooth functions rapidly decaying in x_1), and the second contribution is dealt with again by using the fact that the operator $\displaystyle\sum_{\lambda\neq 0} \frac{\Pi_\lambda}{\lambda}$ is continuous from H^σ_L to $H^{\sigma-1}_L$.

We conclude that for all $Q \in \mathbf{R}[X]$

$$Q(x_1)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\phi_N^{(2)} \in L^\infty([0,T]; C^\infty(\mathbf{R}\times\mathbf{T})).$$

and thus

$$\forall N \in \mathbf{N}, \quad \lim_{\varepsilon\to 0} \|\varepsilon Q(x_1)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\phi_N^{(2)}\|_{L^\infty([0,T];C^k(\mathbf{R}\times\mathbf{T}))} = 0, \ \forall k \in \mathbf{N}.$$

Combining these results with (4.2.5), (4.2.6), (4.2.7) and (4.2.8) leads to (4.2.12) and (4.2.13).

• The uniform regularity estimate (4.2.14) is obtained in a very similar way. Of course the regularity of the correction established previously shows that its contribution to $\nabla \cdot u_{\varepsilon,N}$ converges to zero as ε goes to 0 in the sense of smooth functions rapidly decaying with respect to x_1. Therefore the only point to be checked is that

$$\nabla \cdot \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N \right)' \text{ is uniformly bounded in } L^1([0,T]; L^\infty(\mathbf{R} \times \mathbf{T})),$$

which is obtained as the regularity property stated in Proposition 3.6.1, remarking that the L^∞ bound comes from estimates which are stable by the truncation J_N and by conjugation by the semi-group $\mathcal{L}$. Indeed for almost all β, provided that

$$(\Pi_K + \Pi_P)\Phi^0 \in H^\alpha_L \text{ for some } \alpha > \frac{1}{2},$$

any weak solution Φ to (SW_0) satisfies the following estimates

$$\|\nabla \cdot \Phi'\|_{L^1([0,T];L^\infty(\mathbf{R}\times\mathbf{T}))} \leq C_T$$

as well as

$$\left\| \nabla \cdot \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right) \Phi_N \right)' \right\|_{L^1([0,T];L^\infty(\mathbf{R}\times\mathbf{T}))} \leq C_T,$$

where C_T depends only on $T \in \mathbf{R}^+$, $\|\Phi^0\|_{L^2(\mathbf{R}\times\mathbf{T})}$ and $\|(\Pi_K + \Pi_P)\Phi^0\|_{H^\alpha_L}$ (neither on N nor on ε.)

• It remains then to establish the equation satisfied by $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$. A direct computation provides

$$\begin{aligned}
\varepsilon \partial_t \phi_N = &- \sum_{\substack{\lambda \neq \mu + \tilde\mu \\ i\lambda \in \mathfrak{S}, i\mu, i\tilde\mu \in \mathfrak{S}_N}} e^{i\frac{t}{\varepsilon}(\lambda - \mu - \tilde\mu)} \Pi_\lambda Q(\Pi_\mu \Phi_N, \Pi_{\tilde\mu} \Phi_N) \\
&+ \nu \sum_{\substack{\lambda \neq \mu, \\ i\lambda \in \mathfrak{S}, i\mu \in \mathfrak{S}_N}} e^{i\frac{t}{\varepsilon}(\lambda - \mu)} \Pi_\lambda \Delta' \Pi_\mu \Phi_N \\
&- 2\varepsilon \sum_{\substack{\lambda \neq \mu + \tilde\mu \\ i\lambda \in \mathfrak{S}, i\mu, i\tilde\mu \in \mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda - \mu - \tilde\mu)}}{i(\lambda - \mu - \tilde\mu)} \Pi_\lambda Q(\Pi_\mu \partial_t \Phi_N, \Pi_{\tilde\mu} \Phi_N) \\
&+ \varepsilon\nu \sum_{\substack{\lambda \neq \mu, \\ i\lambda \in \mathfrak{S}, i\mu \in \mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda - \mu)}}{i(\lambda - \mu)} \Pi_\lambda \Delta' \Pi_\mu \partial_t \Phi_N
\end{aligned}$$

By (4.2.10) we infer that $\partial_t \Phi_N$ is smooth and rapidly decaying (recalling in particular that $\Pi_0 Q_L = 0$ due to Proposition 2.4.5 page 26), and thus the last two terms go to zero as $\varepsilon \to 0$ (for all fixed N). The previous identity can therefore be rewritten

(4.2.18)
$$\varepsilon \partial_t \phi_N = -\mathcal{L}\left(-\frac{t}{\varepsilon}\right)(Q - Q_L)\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) + \nu \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\Delta' - \Delta'_L)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N + r_{\varepsilon,N}$$

where

$$\forall k \in \mathbf{N}, \forall N \in \mathbf{N}, \forall Q \in \mathbf{R}[X], \quad \lim_{\varepsilon \to 0} \|Q(x_1) r_{\varepsilon,N}\|_{L^\infty([0,T];C^k(\mathbf{R}\times\mathbf{T}))} = 0. \tag{4.2.19}$$

Now let us recall that

$$\begin{aligned}
&\partial_t \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) + \frac{1}{\varepsilon} L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) + Q\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) - \nu\Delta'\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N \\
&= (Q - Q_L)\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) - \nu(\Delta' - \Delta'_L)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N \\
&+ (Id - J_N)Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right) + Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N - \Phi), \mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N + \Phi)\right).
\end{aligned} \tag{4.2.20}$$

We thus have, recalling that

$$(\eta_{\varepsilon,N}, u_{\varepsilon,N}) = \mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N + \varepsilon\phi_N),$$

(4.2.21)

$$\begin{aligned}
&\partial_t(\eta_{\varepsilon,N}, u_{\varepsilon,N}) + \frac{1}{\varepsilon}L(\eta_{\varepsilon,N}, u_{\varepsilon,N}) + Q\left((\eta_{\varepsilon,N}, u_{\varepsilon,N}), (\eta_{\varepsilon,N}, u_{\varepsilon,N})\right) - \nu\Delta'(\eta_{\varepsilon,N}, u_{\varepsilon,N}) \\
&\quad = (Id - J_N)Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right) + Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N - \Phi), \mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N + \Phi)\right) \\
&\quad + \varepsilon Q\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)(2\Phi_N + \varepsilon\phi_N)\right) - \varepsilon\nu\Delta'\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\phi_N\right) + r_{\varepsilon,N}
\end{aligned}$$

Note that the regularity estimates on Φ_N and ϕ_N allow to prove that the two last explicit terms in the right-hand side go to zero as $\varepsilon \to 0$ (for all fixed N), and therefore to incorporate them into the remainder $r_{\varepsilon,N}$.

The stability of the limiting filtered system (SW_0) allows to prove that the second term in the right-hand side of (4.2.21) goes to zero as $N \to \infty$ uniformly in ε. We have indeed

$$\left\| Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N - \Phi), \mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N + \Phi)\right)\right\|_{L^2(\mathbf{R}\times\mathbf{T})} = \|Q_L(\Phi_N - \Phi, \Phi_N + \Phi)\|_{L^2(\mathbf{R}\times\mathbf{T})}$$

and recalling that only the Kelvin waves can have resonances,

$$\begin{aligned}
&\|Q_L(\Phi_N - \Phi, \Phi_N + \Phi)\|_{L^2(\mathbf{R}\times\mathbf{T})} \leq \|Q_L(\Pi_K(\Phi_N - \Phi), \Pi_K(\Phi_N + \Phi))\|_{L^2(\mathbf{R}\times\mathbf{T})} \\
&\quad + \|Q_L(\Pi_0(\Phi_N - \Phi), \Pi_\perp(\Phi_N + \Phi))\|_{L^2(\mathbf{R}\times\mathbf{T})} + \|Q_L(\Pi_\perp(\Phi_N - \Phi), \Pi_0(\Phi_N + \Phi))\|_{L^2(\mathbf{R}\times\mathbf{T})},
\end{aligned}$$

so by Proposition 3.3.3 page 34 and two-dimensional product rules on the Kelvin part (recall as in the previous chapter that H^s and H^s_L spaces coincide in the case of Kelvin modes) we infer that

$$\begin{aligned}
\|Q_L(\Phi_N - \Phi, \Phi_N + \Phi)\|_{L^2(\mathbf{R}\times\mathbf{T})} &\leq C_\alpha\|\Pi_K(\Phi_N - \Phi)\|_{H_L^{\alpha+1}}\|\Pi_K(\Phi_N + \Phi)\|_{H_L^\alpha} \\
&+ C\|\Pi_0(\Phi_N - \Phi)\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp(\Phi_N + \Phi)\|_{H_L^1} \\
&+ C\|\Pi_0(\Phi_N + \Phi)\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\Pi_\perp(\Phi_N - \Phi)\|_{H_L^1}.
\end{aligned} \tag{4.2.22}$$

So by (4.2.8) and (4.2.9) we conclude that

$$\lim_{N\to\infty}\|Q_L(\Phi_N - \Phi, \Phi_N + \Phi)\|_{L^2([0,T];L^2(\mathbf{R}\times\mathbf{T}))} = 0.$$

Let us estimate the first term in the right side of (4.2.21). We can write as above (recalling that $Q(\Pi_0\,\cdot\,, \Pi_0\,\cdot\,) = 0$)

$$\left\|Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right)\right\|_{L^2} \leq \|Q_L(\Pi_K\Phi, \Pi_K\Phi)\|_{L^2} + 2\|Q_L(\Pi_0\Phi, \Pi_\perp\Phi)\|_{L^2}$$

so we find that

$$\left\|Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right)\right\|_{L^2} \leq C\left(\|\Pi_K\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\|^2_{H^1_L} + \|\Pi_0\Phi_N\|^2_{L^2}\right) \tag{4.2.23}$$

and thus

$$\lim_{N\to\infty} \sup_{\varepsilon} \|(Id - J_N)Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right)\|_{L^2([0,T];L^2(\mathbf{R}\times\mathbf{T}))} = 0.$$

Note that in the case when β belongs to $\mathcal{N}$ (Theorem 5), equations (4.2.22) and (4.2.23) must be adapted using the third estimate of Proposition 3.3.3.

Finally to prove that for all $N \in \mathbf{N}$, for all $T > 0$, the quantity

$$\varepsilon Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N - \Phi), \mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N + \Phi)\right) + \varepsilon(Id - J_N)Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right)$$

goes to zero as ε goes to zero, in the space $L^\infty([0,T]\times\mathbf{R}\times\mathbf{T})$, we simply notice that

$$\begin{aligned} &Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N - \Phi), \mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N + \Phi)\right) + (Id - J_N)Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right) \\ &\quad = J_N Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N - \Phi), \mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N + \Phi)\right) + (Id - J_N)Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) \end{aligned}$$

and the convergence result is obvious if one considers the right-hand side, simply because those terms are smooth for each fixed N.

Combining all the previous estimates shows that $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ satisfies the expected approximate equation (4.2.15), where $R_{\varepsilon,N}$ satisfies the expected estimate (4.2.16) as well as

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0} \varepsilon\|R_{\varepsilon,N}\|_{L^\infty([0,T]\times\mathbf{R}\times\mathbf{T})} = 0. \tag{4.2.24}$$

Proposition 4.2.2 is proved. □

Equipped with that result, we are now ready to prove the strong convergence theorem. The method relies on a weak-strong stability method which we shall now detail. We are going to prove that

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0} \|(\eta_\varepsilon, u_\varepsilon) - (\eta_{\varepsilon,N}, u_{\varepsilon,N})\|_{L^2([0,T]\times\mathbf{R}\times\mathbf{T})} = 0, \tag{4.2.25}$$

where $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ is the approximate solution to (SW_ε) defined in Proposition 4.2.2. Note that combining this estimate with the fact that $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ is close to $\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi$ provides the expected convergence, namely the fact that

$$\forall T > 0, \quad \lim_{\varepsilon\to 0}\left\|(\eta_\varepsilon, u_\varepsilon) - \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right\|_{L^2([0,T]\times\mathbf{R}\times\mathbf{T})} = 0.$$

The key to the proof of (4.2.25) lies in the following proposition.

Proposition 4.2.3. — *There is a constant C such that the following property holds. Let (η^0, u^0) and $(\eta^0_\varepsilon, u^0_\varepsilon)$ satisfy assumption (4.2.2), and let $T > 0$ be given. For all $\varepsilon > 0$, denote by $(\eta_\varepsilon, u_\varepsilon)$ a solution of (SW_ε) with initial data (η^0, u^0). For any couple of vector fields $(\underline{\eta}, \underline{u})$ belonging to $L^\infty([0,T]; C^\infty(\mathbf{R}\times\mathbf{T}))$ and rapidly decaying with respect to x_1, define*

$$\mathcal{E}_\varepsilon(t) = \frac{1}{2}\int\left((\eta_\varepsilon - \underline{\eta})^2 + (1 + \varepsilon\eta_\varepsilon)|u_\varepsilon - \underline{u}|^2\right)(t,x)dx + \nu\int_0^t\int |\nabla(u_\varepsilon - \underline{u})|^2(t',x)dxdt'.$$

Then the following stability inequality holds for all $t \in [0,T]$:

$$\begin{aligned}
&\mathcal{E}_\varepsilon(t) \le C\mathcal{E}_\varepsilon(0)\exp(\chi(t)) + \omega_\varepsilon(t) \\
&+ C\int_0^t e^{\chi(t)-\chi(t')}\int \left(\partial_t \underline{\eta} + \frac{1}{\varepsilon}\nabla\cdot\underline{u} + \nabla\cdot(\underline{\eta}\underline{u})\right)(\underline{\eta}-\eta_\varepsilon)(t',x)dxdt' \\
&+ C\int_0^t e^{\chi(t)-\chi(t')}\int (1+\varepsilon\eta_\varepsilon)\left(\partial_t\underline{u} + \frac{1}{\varepsilon}(\beta x_1\underline{u}^\perp + \nabla\underline{\eta}) + (\underline{u}\cdot\nabla)\underline{u} - \nu\Delta\underline{u}\right)\cdot(\underline{u}-u_\varepsilon)(t',x)dxdt',
\end{aligned}$$

where $\omega_\varepsilon(t)$ depends on $\underline{u}$ and goes to zero with ε, uniformly in time, and where

$$\chi(t) = C\int_0^t \left(\|\nabla\cdot\underline{u}\|_{L^\infty(\mathbf{R}\times\mathbf{T})} + \|\nabla\underline{u}\|^2_{L^2(\mathbf{R}\times\mathbf{T})}\right)(t')dt'.$$

Let us postpone the proof of that result, and end the proof of the strong convergence. We apply that proposition to $(\underline{\eta},\underline{u}) = (\eta_{\varepsilon,N}, u_{\varepsilon,N})$, where $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ is the approximate solution given by Proposition 4.2.2. We will denote by $\chi_{\varepsilon,N}$ and $\mathcal{E}_{\varepsilon,N}$ the quantities defined in Proposition 4.2.3, where $(\underline{\eta},\underline{u})$ has been replaced by $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$.

Because of the uniform regularity estimates on $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$, we have

$$\forall T>0, \quad \sup_N \lim_{\varepsilon\to 0}\left(\|\nabla u_{\varepsilon,N}\|^2_{L^2([0,T];L^2(\mathbf{R}\times\mathbf{T}))} + \|\nabla\cdot u_{\varepsilon,N}\|_{L^1([0,T];L^\infty(\mathbf{R}\times\mathbf{T}))}\right) \le C_T,$$

so we get a uniform bound on $\chi_{\varepsilon,N}$:

$$\sup_N \lim_{\varepsilon\to 0}\|\chi_{\varepsilon,N}\|_{L^\infty([0,T])} \le C_T.$$

Then, from the initial convergence (4.2.2) we obtain that

$$\forall N\in\mathbf{N}, \quad \mathcal{E}_{\varepsilon,N}(0)\exp(\chi_{\varepsilon,N}(t)) \to 0 \text{ as } \varepsilon\to 0 \text{ in } L^\infty([0,T]).$$

Moreover by Proposition 4.2.3 we have

$$\partial_t(\eta_{\varepsilon,N}, u_{\varepsilon,N}) + \frac{1}{\varepsilon}L(\eta_{\varepsilon,N}, u_{\varepsilon,N}) + Q((\eta_{\varepsilon,N}, u_{\varepsilon,N}),(\eta_{\varepsilon,N}, u_{\varepsilon,N})) - \nu\Delta'(\eta_{\varepsilon,N}, u_{\varepsilon,N}) = R_{\varepsilon,N}. \tag{4.2.26}$$

Let us estimate the contribution of the remainder term. We can write

$$\int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R_{\varepsilon,N}\cdot((\eta_{\varepsilon,N}-\eta_\varepsilon),(1+\varepsilon\eta_\varepsilon)(u_{\varepsilon,N}-u_\varepsilon))(t',x)dxdt' = I^{(1)}_{\varepsilon,N}(t) + I^{(2)}_{\varepsilon,N}(t),$$

with

$$\begin{aligned}
I^{(1)}_{\varepsilon,N}(t) &\stackrel{\text{def}}{=} \int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R_{\varepsilon,N,0}(\eta_{\varepsilon,N}-\eta_\varepsilon)(t',x)dxdt', \quad \text{and} \\
I^{(2)}_{\varepsilon,N}(t) &\stackrel{\text{def}}{=} \int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R'_{\varepsilon,N}(1+\varepsilon\eta_\varepsilon)(u_{\varepsilon,N}-u_\varepsilon)(t',x)dxdt'.
\end{aligned}$$

The first term can be estimated in the following way:

$$|I^{(1)}_{\varepsilon,N}(t)| \le C_T\|R_{\varepsilon,N}\|_{L^1([0,T];L^2(\mathbf{R}\times\mathbf{T}))}\|\eta_{\varepsilon,N}-\eta_\varepsilon\|_{L^\infty([0,T];L^2(\mathbf{R}\times\mathbf{T}))}.$$

For the second term we can write

$$|I^{(2)}_{\varepsilon,N}(t)| \le C_T\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_{\varepsilon,N}-u_\varepsilon)\|_{L^\infty([0,T];L^2(\mathbf{R}\times\mathbf{T}))}\|\sqrt{1+\varepsilon\eta_\varepsilon}R_{\varepsilon,N}\|_{L^1([0,T];L^2(\mathbf{R}\times\mathbf{T}))}.$$

Now we can write

$$\|\sqrt{1+\varepsilon\eta_\varepsilon}R_{\varepsilon,N}\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \le C(\|R_{\varepsilon,N}\|^2_{L^2(\mathbf{R}\times\mathbf{T})} + \varepsilon\|\eta_\varepsilon\|_{L^2(\mathbf{R}\times\mathbf{T})}\|R_{\varepsilon,N}\|^2_{L^4(\mathbf{R}\times\mathbf{T})}).$$

Since

$$\varepsilon\|R_{\varepsilon,N}\|^2_{L^4(\mathbf{R}\times\mathbf{T})} \le \varepsilon\|R_{\varepsilon,N}\|_{L^\infty(\mathbf{R}\times\mathbf{T})}\|R_{\varepsilon,N}\|_{L^2(\mathbf{R}\times\mathbf{T})},$$

we infer that the quantity $\varepsilon^{\frac{1}{2}} R_{\varepsilon,N}$ goes to zero as ε goes to zero and N goes to infinity, in the space $L^2([0,T];L^4(\mathbf{R}\times\mathbf{T}))$, so in particular

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\varepsilon^{\frac{1}{2}}\|R_{\varepsilon,N}\|_{L^1([0,T];L^4(\mathbf{R}\times\mathbf{T}))}=0.$$

Finally by the uniform bound on η_ε in $L^\infty([0,T];L^2(\mathbf{R}\times\mathbf{T}))$ and by the smallness assumptions on $R_{\varepsilon,N}$, we deduce that

$$\begin{aligned}\int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R_{\varepsilon,N}\cdot\left((\eta_{\varepsilon,N}-\eta_\varepsilon),(1+\varepsilon\eta_\varepsilon)(u_{\varepsilon,N}-u_\varepsilon)\right)(t',x)dxdt'&\\ \leq\frac{1}{2}(\|\eta_{\varepsilon,N}-\eta_\varepsilon\|^2_{L^\infty([0,T];L^2)}+\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_{\varepsilon,N}-u_\varepsilon)\|^2_{L^\infty([0,T];L^2)})+\omega_{\varepsilon,N}(t),&\end{aligned}$$

where

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\|\omega_{\varepsilon,N}(t)\|_{L^\infty([0,T])}=0.$$

We now recall that by Proposition 4.2.3, using (4.2.26), we have

$$\begin{aligned}\mathcal{E}_{\varepsilon,N}(t)\leq C\mathcal{E}_{\varepsilon,N}(0)\exp\left(\chi_{\varepsilon,N}(t)\right)+\omega_\varepsilon(t)&\\ +C\int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R_{\varepsilon,N}\cdot\left((\eta_{\varepsilon,N}-\eta_\varepsilon),(1+\varepsilon\eta_\varepsilon)(u_{\varepsilon,N}-u_\varepsilon)\right)(t',x)dxdt'&\end{aligned}$$

where

$$\mathcal{E}_{\varepsilon,N}(t)=\frac{1}{2}\left(\|(\eta_\varepsilon-\eta_{\varepsilon,N})(t)\|^2_{L^2}+\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-u_{\varepsilon,N})(t)\|^2_{L^2}\right)+\nu\int_0^t\|\nabla(u_\varepsilon-u_{\varepsilon,N})(t')\|^2_{L^2}(t')dt'.$$

Putting together the previous results we get that $\lim_{N\to\infty}\lim_{\varepsilon\to 0}\mathcal{E}_{\varepsilon,N}(t)=0$ uniformly on $[0,T]$, hence that

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\|\eta_{\varepsilon,N}-\eta_\varepsilon\|_{L^\infty([0,T];L^2(\mathbf{R}\times\mathbf{T}))}=0,$$
$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_{\varepsilon,N}-u_\varepsilon)\|_{L^\infty([0,T];L^2(\mathbf{R}\times\mathbf{T}))}=0,$$
$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\|u_{\varepsilon,N}-u_\varepsilon\|_{L^2([0,T];\dot H^1(\mathbf{R}\times\mathbf{T}))}=0.$$

By interpolation we therefore find that

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\left(\|\eta_{\varepsilon,N}-\eta_\varepsilon\|_{L^\infty([0,T];L^2(\mathbf{R}\times\mathbf{T}))}+\|u_{\varepsilon,N}-u_\varepsilon\|_{L^2([0,T];H^1(\mathbf{R}\times\mathbf{T}))}\right)=0,$$

hence (4.2.25) is proved.

To conclude the proof of Theorem 6 it remains to prove Proposition 4.2.3. As the energy is a Lyapunov functional for (SW_ε), we have

$$\begin{aligned}\mathcal{E}_\varepsilon(t)-\mathcal{E}_\varepsilon(0)\leq&\int_0^t\frac{d}{dt}\int\left((\frac{1}{2}\underline{\eta}^2-\underline{\eta}\eta_\varepsilon)+(1+\varepsilon\eta_\varepsilon)(\frac{1}{2}|\underline{u}|^2-\underline{u}\cdot u_\varepsilon)\right)(t',x)dxdt'\\ &+\int_0^t\int\nu(\nabla\underline{u}-2\nabla u_\varepsilon)\cdot\nabla\underline{u}(t',x)dxdt'\\ \leq&\int_0^t\int\left(\partial_t\underline{\eta}(\underline{\eta}-\eta_\varepsilon)+(1+\varepsilon\eta_\varepsilon)\partial_t\underline{u}\cdot(\underline{u}-u_\varepsilon)\right)(t',x)dxdt'\\ &-\int_0^t\int\left(\partial_t\eta_\varepsilon\underline{\eta}+\partial_t((1+\varepsilon\eta_\varepsilon)u_\varepsilon)\cdot\underline{u}-\frac{\varepsilon}{2}\partial_t\eta_\varepsilon|\underline{u}|^2)\right)(t',x)dxdt'\\ &-\int_0^t\int\nu\left(\Delta\underline{u}\cdot(\underline{u}-u_\varepsilon)-\Delta u_\varepsilon\cdot\underline{u}\right)(t',x)dxdt'.\end{aligned}$$

Using the conservation of mass and of momentum we get

$$\begin{aligned}\mathcal{E}_\varepsilon(t)-\mathcal{E}_\varepsilon(0)\le&\int_0^t\int\left(\partial_t\underline{\eta}(\underline{\eta}-\eta_\varepsilon)+(1+\varepsilon\eta_\varepsilon)(\partial_t\underline{u}-\nu\Delta\underline{u})\cdot(\underline{u}-u_\varepsilon)\right)(t',x)dxdt'\\&+\int_0^t\int\frac{1}{\varepsilon}\nabla\cdot((1+\varepsilon\eta_\varepsilon)u_\varepsilon)\left(\underline{\eta}-\frac{\varepsilon}{2}|\underline{u}|^2\right)(t',x)dxdt'\\&+\int_0^t\int\left(\frac{(1+\varepsilon\eta_\varepsilon)}{\varepsilon}(\beta x_1u_\varepsilon^\perp+\nabla\eta_\varepsilon)+\nabla\cdot((1+\varepsilon\eta_\varepsilon)u_\varepsilon\otimes u_\varepsilon)\right)\cdot\underline{u}(t',x)dxdt'\\&+\int_0^t\int\varepsilon\nu\eta_\varepsilon\Delta\underline{u}\cdot(\underline{u}-u_\varepsilon)(t',x)dxdt'.\end{aligned}$$

Integrating by parts leads then to

$$\begin{aligned}\mathcal{E}_\varepsilon(t)-\mathcal{E}_\varepsilon(0)\le&\int_0^t\int\left(\partial_t\underline{\eta}+\frac{1}{\varepsilon}\nabla\cdot\underline{u}+\nabla\cdot(\underline{\eta}\underline{u})\right)(\underline{\eta}-\eta_\varepsilon)(t',x)dxdt'\\&+\int_0^t\int(1+\varepsilon\eta_\varepsilon)\left(\partial_t\underline{u}+\frac{1}{\varepsilon}(\beta x_1\underline{u}^\perp+\nabla\underline{\eta})+(\underline{u}\cdot\nabla)\underline{u}-\nu\Delta\underline{u}\right)\cdot(\underline{u}-u_\varepsilon)(t',x)dxdt'\\&-\int_0^t\int(1+\varepsilon\eta_\varepsilon)D\underline{u}:(\underline{u}-u_\varepsilon)^{\otimes2}(t',x)dxdt'\\&-\int_0^t\int\left(\frac{1}{2}\eta_\varepsilon^2\nabla\cdot\underline{u}+(\underline{\eta}-\eta_\varepsilon)\nabla\cdot(\underline{\eta}\underline{u})+\eta_\varepsilon\underline{u}\cdot\nabla\underline{\eta}\right)(t',x)dxdt'+R_\varepsilon,\end{aligned}\tag{4.2.27}$$

where

$$R_\varepsilon(t)=\int_0^t\int\varepsilon\nu\eta_\varepsilon\Delta\underline{u}\cdot(\underline{u}-u_\varepsilon)(t',x)dxdt'.$$

The last term is rewritten in a convenient form by integrating by parts

$$\begin{aligned}-\int_0^t\int&\left(\frac{1}{2}\eta_\varepsilon^2\nabla\cdot\underline{u}+(\underline{\eta}-\eta_\varepsilon)\nabla\cdot(\underline{\eta}\underline{u})+\eta_\varepsilon\underline{u}\cdot\nabla\underline{\eta}\right)(t',x)dxdt'\\&=-\int_0^t\int\left(\frac{1}{2}\eta_\varepsilon^2\nabla\cdot\underline{u}+(\underline{\eta}-\eta_\varepsilon)(\underline{u}\cdot\nabla\underline{\eta}+\underline{\eta}\nabla\cdot\underline{u})+\eta_\varepsilon\underline{u}\cdot\nabla\underline{\eta}\right)(t',x)dxdt'\\&=-\int_0^t\int\left(\frac{1}{2}\eta_\varepsilon^2\nabla\cdot\underline{u}+(\underline{\eta}-\eta_\varepsilon)\underline{\eta}\nabla\cdot\underline{u}+\frac{1}{2}\underline{u}\cdot\nabla\underline{\eta}^2\right)(t',x)dxdt'\\&=-\int_0^t\int\frac{1}{2}(\eta_\varepsilon-\underline{\eta})^2\nabla\cdot\underline{u}(t',x)dxdt'.\end{aligned}\tag{4.2.28}$$

Plugging (4.2.28) into (4.2.27) leads to

$$\begin{aligned}\mathcal{E}_\varepsilon(t)-\mathcal{E}_\varepsilon(0)\le&\int_0^t\int\left(\partial_t\underline{\eta}+\frac{1}{\varepsilon}\nabla\cdot\underline{u}+\nabla\cdot(\underline{\eta}\underline{u})\right)(\underline{\eta}-\eta_\varepsilon)(t',x)dxdt'\\&+\int_0^t\int(1+\varepsilon\eta_\varepsilon)\left(\partial_t\underline{u}+\frac{1}{\varepsilon}(\beta x_1\underline{u}^\perp+\nabla\underline{\eta})+(\underline{u}\cdot\nabla)\underline{u}-\nu\Delta\underline{u}\right)\cdot(\underline{u}-u_\varepsilon)(t',x)dxdt'\\&-\int_0^t\int(1+\varepsilon\eta_\varepsilon)D\underline{u}:(\underline{u}-u_\varepsilon)^{\otimes2}(t',x)dxdt'-\int_0^t\int\frac{1}{2}(\eta_\varepsilon-\underline{\eta})^2\nabla\cdot\underline{u}(t',x)dxdt'+R_\varepsilon(t).\end{aligned}\tag{4.2.29}$$

In order to get an inequality of Gronwall type, one has to control the right hand side in terms of $\mathcal{E}_\varepsilon$. We start by estimating the flux term. We have

$$\begin{aligned}
-\int_0^t \int (1+\varepsilon\eta_\varepsilon) D\underline{u} : (\underline{u}-u_\varepsilon)^{\otimes 2}(t',x)\,dx\,dt' \\
\leq \int_0^t \left(\|\nabla \underline{u}\|_{L^2(\mathbf{R}\times\mathbf{T})} + \varepsilon\|\eta_\varepsilon\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\nabla\underline{u}\|_{L^\infty(\mathbf{R}\times\mathbf{T})}\right)\|\underline{u}-u_\varepsilon\|^2_{L^4(\mathbf{R}\times\mathbf{T})}(t')\,dt' \\
\leq C\int_0^t \left(\|\nabla \underline{u}\|_{L^2(\mathbf{R}\times\mathbf{T})} + \varepsilon\|\eta_\varepsilon\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\nabla\underline{u}\|_{L^\infty(\mathbf{R}\times\mathbf{T})}\right)\|\underline{u}-u_\varepsilon\|_{L^2(\mathbf{R}\times\mathbf{T})} \\
\times \|\underline{u}-u_\varepsilon\|_{\dot H^1(\mathbf{R}\times\mathbf{T})}(t')\,dt'
\end{aligned}$$

and

$$\|\underline{u}-u_\varepsilon\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \leq \|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-\underline{u})\|^2_{L^2(\mathbf{R}\times\mathbf{T})} + \varepsilon\|\eta_\varepsilon\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\underline{u}-u_\varepsilon\|_{L^2(\mathbf{R}\times\mathbf{T})}\|\underline{u}-u_\varepsilon\|_{\dot H^1(\mathbf{R}\times\mathbf{T})}$$

which implies

$$\|\underline{u}-u_\varepsilon\|^2_{L^2(\mathbf{R}\times\mathbf{T})} \leq 2\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-\underline{u})\|^2_{L^2(\mathbf{R}\times\mathbf{T})} + 16\varepsilon^2\|\eta_\varepsilon\|^2_{L^2(\mathbf{R}\times\mathbf{T})}\|\underline{u}-u_\varepsilon\|^2_{\dot H^1(\mathbf{R}\times\mathbf{T})}.$$

Therefore, using the uniform bounds on η_ε, $\sqrt{1+\varepsilon\eta_\varepsilon}u_\varepsilon$ and on u_ε given by the energy estimate, we gather that

$$\begin{aligned}
-\int_0^t \int (1+\varepsilon\eta_\varepsilon) D\underline{u} : (\underline{u}-u_\varepsilon)^{\otimes 2}(t',x)\,dx\,dt' \\
\leq C\int_0^t (\|\nabla\underline{u}\|_{L^2} + \varepsilon\|\nabla\underline{u}\|_{L^\infty})\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-\underline{u})\|_{L^2}\|\underline{u}-u_\varepsilon\|_{\dot H^1}(t')\,dt' \\
+ C\varepsilon\int_0^t (\|\nabla\underline{u}\|_{L^2} + \varepsilon\|\nabla\underline{u}\|_{L^\infty})\|\underline{u}-u_\varepsilon\|^2_{\dot H^1}(t')\,dt' \\
\leq \frac{\nu}{4}\int \|\underline{u}-u_\varepsilon\|^2_{\dot H^1}(t')\,dt' + \frac{C}{\nu}\int \|\nabla\underline{u}\|^2_{L^2}\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-\underline{u})\|^2_{L^2}(t')\,dt' + \omega_\varepsilon(t).
\end{aligned} \tag{4.2.30}$$

We also have

$$-\int_0^t \int \frac{1}{2}(\eta_\varepsilon-\underline{\eta})^2\nabla\cdot\underline{u}(t',x)\,dx\,dt' \leq \frac{1}{2}\int_0^t \|\nabla\cdot\underline{u}\|_{L^\infty(\mathbf{R}\times\mathbf{T})}\|\underline{\eta}-\eta_\varepsilon\|^2_{L^2(\mathbf{R}\times\mathbf{T})}(t')\,dt', \tag{4.2.31}$$

so we are left with the study of the remainder R_ε. We have

$$R_\varepsilon(t) \leq \varepsilon\nu\|\eta_\varepsilon\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T}))}\int_0^t \|\Delta\underline{u}\|_{L^4(\mathbf{R}\times\mathbf{T})}\|\underline{u}-u_\varepsilon\|_{L^4(\mathbf{R}\times\mathbf{T})}(t')\,dt'.$$

The above estimate on $\|\underline{u}-u_\varepsilon\|_{L^2(\mathbf{R}\times\mathbf{T})}$ implies in particular that $\|\underline{u}-u_\varepsilon\|_{L^2(\mathbf{R}\times\mathbf{T})}$ is bounded in $L^2([0,T])$, hence we get that $\|\underline{u}-u_\varepsilon\|_{L^4(\mathbf{R}\times\mathbf{T})}$ is also bounded in $L^2([0,T])$. So we infer directly that $R_\varepsilon(t)$ goes to zero in $L^\infty([0,T])$ as ε goes to zero. That result, joint with (4.2.30) and (4.2.31) allows to deduce from (4.2.29) the following estimate:

$$\begin{aligned}
\frac{1}{2}\mathcal{E}_\varepsilon(t) - \mathcal{E}_\varepsilon(0) \leq \int_0^t \int \left(\partial_t\underline{\eta} + \frac{1}{\varepsilon}\nabla\cdot\underline{u} + \nabla\cdot(\underline{\eta}\underline{u})\right)(\underline{\eta}-\eta_\varepsilon)(t',x)\,dx\,dt' \\
+ \int_0^t \int (1+\varepsilon\eta_\varepsilon)\left(\partial_t\underline{u} + \frac{1}{\varepsilon}(\beta x_1\underline{u}^\perp + \nabla\underline{\eta}) + (\underline{u}\cdot\nabla)\underline{u} - \nu\Delta\underline{u}\right)\cdot(\underline{u}-u_\varepsilon)(t',x)\,dx\,dt' \\
+ \frac{C}{\nu}\int \|\nabla\underline{u}\|^2_{L^2}\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-\underline{u})\|^2_{L^2(\mathbf{R}\times\mathbf{T})}(t')\,dt' + \frac{1}{2}\int_0^t \|\nabla\cdot\underline{u}\|_{L^\infty}\|\underline{\eta}-\eta_\varepsilon\|^2_{L^2}(t')\,dt' + \omega_\varepsilon(t)
\end{aligned}$$

thus applying Gronwall's lemma provides the expected stability inequality. Theorem 6 is proved. □

4.3. Strong convergence of filtered weak solutions towards a weak solution

The aim of this section is to prove an intermediate convergence result, in the sense that we will seek a strong convergence result of the filtered weak solutions, towards a weak solution of the limit system; thus no additional smoothness will be required on the initial data other than $L^2(\mathbf{R}\times\mathbf{T})$. As explained in the introduction of this chapter, the lack of compactness in the spatial variables of η_ε will prevent us from obtaining at the limit the expected system (SW_0): a defect measure remains at the limit, which we are unable to remove. In order to gain some space compactness and to get rid of that defect measure, we propose in the final paragraph of this section (Paragraph 4.3.4 below) an alternate model which takes into account capillarity effects, and for which one can prove the strong convergence of filtered solutions towards a weak solution of (SW_0).

The first result of this paragraph is the following.

***Theorem 7* (strong convergence towards weak solutions).** — *Let $(\eta^0,u^0)\in L^2(\mathbf{R}\times\mathbf{T})$ and $(\eta^0_\varepsilon,u^0_\varepsilon)$ satisfy* (4.1.1)*. For all $\varepsilon>0$, denote by $(\eta_\varepsilon,u_\varepsilon)$ a solution of (SW_ε) with initial data $(\eta^0_\varepsilon,u^0_\varepsilon)$, and by*

$$\Phi_\varepsilon=\mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon,u_\varepsilon).$$

Up to the extraction of a subsequence, Φ_ε converges strongly in $L^2_{loc}(\mathbf{R}^+;H^s_{loc}(\mathbf{R}\times\mathbf{T}))$ (for all $s<0$) towards some solution Φ of the following limiting filtered system: for all $i\lambda\in\mathfrak{S}$, there is a bounded measure $\upsilon_\lambda\in\mathcal{M}(\mathbf{R}^+\times\mathbf{R}\times\mathbf{T})$ (which vanishes if $\lambda=0$), such that for all smooth $\Phi^*_\lambda\in\operatorname{Ker}(L-i\lambda Id)$,

$$\int\Phi\cdot\bar\Phi^*_\lambda(x)\,dx-\nu\int_0^t\int\Delta'_L\Phi\cdot\bar\Phi^*_\lambda(t',x)\,dxdt' \\ +\int_0^t\int Q_L(\Phi,\Phi)\cdot\bar\Phi^*_\lambda(t',x)\,dxdt'+\int_0^t\int\nabla\cdot(\bar\Phi^*_\lambda)'\upsilon_\lambda(dt'dx)=\int\Phi^0\cdot\bar\Phi^*_\lambda(x)\,dx,$$

where Q_L and Δ'_L are defined by (2.3.6) page 19, and where $\Phi^0=(\eta^0,u^0)$.

Remark 4.3.1. — • *Note that, by interpolation with the uniform $L^2_{loc}(\mathbf{R}^+;H^1(\mathbf{R}\times\mathbf{T}))$ bound on u_ε, we get the strong convergence of u_ε in $L^2_{loc}(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T}))$: up to extraction of a subsequence,*

$$\left\|u_\varepsilon-\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right)'\right\|_{L^2(\mathbf{R}\times\mathbf{T})}\to 0 \text{ in } L^2_{loc}(\mathbf{R}^+).$$

• *As explained above, the presence of the defect measure υ_λ at the limit is due to a possible defect of compactness in space of the sequence $(\eta_\varepsilon)_{\varepsilon>0}$. As the proof of the theorem will show, that measure is zero if one is able to prove some equicontinuity in space on η_ε, or even on $\varepsilon\eta_\varepsilon$. Since we have been unable to prove such a result, we study in the final paragraph of this section a slightly different model, where capillarity effects are added in order to gain that compactness. Note that the model introduced in Paragraph 4.3.4 is unfortunately not very physical due to the particular form of the capillarity operator (see its definition in (4.3.9) below).*

Theorem 7 is proved in Sections 4.3.1 to 4.3.3, and the result in the presence of capillarity is stated and proved in Section 4.3.4 .

4.3.1. Strong compactness of $\Pi_\lambda \Phi_\varepsilon$. — Let us prove the following lemma.

Lemma 4.3.2. — *With the notation of Theorem 7, the following results hold.*

- *For all $i\lambda \in \mathfrak{S} \setminus \{0\}$, $\Pi_\lambda \Phi_\varepsilon$ is strongly compact in $L^2([0,T]; H^s(\mathbf{R}\times\mathbf{T}))$ for all $T > 0$ and all $s \in \mathbf{R}$;*
- *$\Pi_0 \Phi_\varepsilon$ is strongly compact in $L^2([0,T]; H^s_{loc}(\mathbf{R}\times\mathbf{T}))$ for all $T > 0$ and all $s < 0$.*

Proof. — • For all $\lambda \neq 0$, we recall that by Proposition 2.2.3 page 12, the eigenspace of L associated with the eigenvalue $i\lambda$ is a finite dimensional subspace of $H^\infty(\mathbf{R}\times\mathbf{T})$. Therefore the only point to be checked is the compactness with respect to time, which is obtained as follows.

Let $(n,k,j) \in \mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}$ be given, such that $\lambda = \tau(n,k,j) \neq 0$, and let $\Psi_{n,k,j}$ be the corresponding eigenvector. Multiplying the system (SW_ε) by $\Psi_{n,k,j}$ (which is smooth and rapidly decaying as $|x_1|$ goes to infinity) and integrating with respect to x leads to

$$\partial_t \int \left(\eta_\varepsilon (\bar\Psi_{n,k,j})_0 + m_\varepsilon \cdot \bar\Psi'_{n,k,j}\right)(t,x)\,dx + \frac{i\tau(n,k,j)}{\varepsilon}\int (\eta_\varepsilon(\bar\Psi_{n,k,j})_0 + m_\varepsilon\cdot\bar\Psi'_{n,k,j})(t,x)\,dx$$
$$+\nu\int \nabla u_\varepsilon : \nabla\bar\Psi'_{n,k,j}(t,x)\,dx - \int m_\varepsilon\cdot(u_\varepsilon\cdot\nabla)\bar\Psi'_{n,k,j}(t,x)\,dx - \frac{1}{2}\int \eta_\varepsilon^2 \nabla\cdot\bar\Psi'_{n,k,j}\,dx = 0$$

where $\bar\Psi_{n,k,j}$ denotes the complex conjugate of $\Psi_{n,k,j}$, or equivalently

$$\begin{aligned}
&\partial_t\left(\exp\left(\frac{it\tau(n,k,j)}{\varepsilon}\right)\int(\eta_\varepsilon(\bar\Psi_{n,k,j})_0 + m_\varepsilon\cdot\bar\Psi'_{n,k,j})(t,x)\,dx\right)\\
&+\nu\int\nabla\left(\exp\left(\frac{it\tau(n,k,j)}{\varepsilon}\right)u_\varepsilon\right) : \nabla\bar\Psi'_{n,k,j}(t,x)\,dx\\
&-\int\exp\left(\frac{it\tau(n,k,j)}{\varepsilon}\right)\left(m_\varepsilon\cdot(u_\varepsilon\cdot\nabla)\bar\Psi'_{n,k,j} + \frac{1}{2}\eta_\varepsilon^2\nabla\cdot\bar\Psi'_{n,k,j}\right)(t,x)\,dx = 0.
\end{aligned}\tag{4.3.1}$$

By the uniform estimates coming from the energy inequality we then deduce that

$$\partial_t\left(\exp\left(\frac{it\tau(n,k,j)}{\varepsilon}\right)\int(\eta_\varepsilon(\bar\Psi_{n,k,j})_0 + m_\varepsilon\cdot\bar\Psi'_{n,k,j})(t,x)dx\right) \text{ is uniformly bounded in } \varepsilon.$$

Therefore the family

$$\left(\exp\left(\frac{it\lambda}{\varepsilon}\right)\Pi_\lambda(\eta_\varepsilon, m_\varepsilon)\right)_{\varepsilon>0} \text{ is compact in } L^2([0,T];H^s(\mathbf{R}\times\mathbf{T})) \text{ for any } s\in\mathbf{R},$$

and since $\varepsilon\eta_\varepsilon u_\varepsilon$ converges to 0 in $L^2(\mathbf{R}^+; H^s(\mathbf{R}\times\mathbf{T}))$ for all $s < 0$, we deduce that

$$\exp\left(\frac{it\lambda}{\varepsilon}\right)\Pi_\lambda(\eta_\varepsilon, u_\varepsilon) = \Pi_\lambda\Phi_\varepsilon \text{ is compact in } L^2([0,T];H^s(\mathbf{R}\times\mathbf{T})) \text{ for any } s\in\mathbf{R}.$$

• For $\Pi_0\Phi_\varepsilon = \Pi_0(\eta_\varepsilon, u_\varepsilon)$ the study is a little more difficult since the compactness with respect to spatial variables has to be taken into account. By the energy estimate we have the uniform bound

$$\Phi_\varepsilon \text{ is uniformly bounded in } L^2_{loc}(\mathbf{R}^+, L^2(\mathbf{R}\times\mathbf{T})).$$

We recall that we have defined in Section 3.1 (Definition 3.1.1 page 30) the space

$$H^s_L = \left\{\psi\in L^2(\mathbf{R}\times\mathbf{T})\,/\,\sum_{n,k,j\in S}(1+n+k^2)^s\,(\psi|\Psi_{n,k,j})^2_{L^2(\mathbf{R}\times\mathbf{T})} < +\infty\right\},$$

where $S = \mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}$ or equivalently (see Proposition 3.1.2 page 30)

$$H^s_L = \left\{\psi \in L^2(\mathbf{R}\times\mathbf{T}) \,/\, (\mathrm{Id}\ - \Delta + \beta^2 x_1^2)^{s/2}\psi \in L^2(\mathbf{R}\times\mathbf{T})\right\}.$$

As $(\Psi_{n,0,0})_{n\in\mathbf{N}}$ is a Hilbertian basis of Ker$L$, we have for all $T>0$ and all $s<0$

$$\left\| \sum_{n\le N} (\Phi_\varepsilon|\Psi_{n,0,0})_{L^2(\mathbf{R}\times\mathbf{T})}\,\Psi_{n,0,0} - \Pi_0\Phi_\varepsilon \right\|_{L^2([0,T];H^s_L)} \to 0 \text{ as } N\to\infty \text{ uniformly in } \varepsilon.$$

Let Ω be any relatively compact open subset of $\mathbf{R}\times\mathbf{T}$. Proposition 3.1.2 page 30 implies that, for all $s\ge 0$

$$H^s_0(\Omega)\subset H^s_L \subset H^s(\mathbf{R}\times\mathbf{T}),$$

and conversely for $s\le 0$,

$$H^s(\mathbf{R}\times\mathbf{T})\subset H^s_L\subset H^s(\Omega). \tag{4.3.2}$$

Thus for all $s<0$ and all $T>0$, we have

$$\left\| \sum_{n\le N} (\Phi_\varepsilon|\Psi_{n,0,0})_{L^2(\mathbf{R}\times\mathbf{T})}\,\Psi_{n,0,0} - \Pi_0\Phi_\varepsilon \right\|_{L^2([0,T];H^s(\Omega))} \to 0 \text{ as } N\to\infty \text{ uniformly in } \varepsilon.$$

Moreover the same computation as previously shows that for any $n\in\mathbf{N}$,

$$\begin{aligned}\partial_t\left(\int (\eta_\varepsilon\bar\eta_{n,0,0} + m_\varepsilon\cdot \bar u_{n,0,0})(t,x)dx\right) + \nu\int \nabla u_\varepsilon : \nabla \bar u_{n,0,0}(t,x)dx \\ - \int m_\varepsilon\cdot(u_\varepsilon\cdot\nabla)\bar u_{n,0,0}(t,x)dx = 0,\end{aligned} \tag{4.3.3}$$

and, since $\varepsilon\eta_\varepsilon u_\varepsilon$ converges to 0 in $L^2(\mathbf{R}^+;H^s(\mathbf{R}\times\mathbf{T}))$ for any $s<0$ we get

$$\sum_{n\le N}\Pi_{n,0,0}(\eta_\varepsilon,u_\varepsilon) \text{ is compact in } L^2([0,T]\times\mathbf{R}\times\mathbf{T}).$$

Combining both results shows finally that

$$\Pi_0\Phi_\varepsilon \text{ is compact in } L^2([0,T];H^s_{loc}(\mathbf{R}\times\mathbf{T}))$$

for all $T>0$ and all $s<0$. Lemma 4.3.2 is proved. □

As the spectrum of L, $\mathfrak{S}$ is countable (see Chapter 2), we are therefore able to construct (by diagonal extraction) a subsequence of Φ_ε, and some $\Phi_\lambda\in\mathrm{Ker}(L-i\lambda Id)$ such that for all $s<0$ and all $T>0$

$$\forall i\lambda\in\mathfrak{S},\quad \Pi_\lambda\Phi_\varepsilon\to\Phi_\lambda \text{ in } L^2([0,T];H^s_{loc}(\mathbf{R}\times\mathbf{T})). \tag{4.3.4}$$

Note that the Φ_λ defined as the strong limit of $\Pi_\lambda\Phi_\varepsilon$ can also be obtained as the weak limit of $\exp\left(\frac{it\lambda}{\varepsilon}\right)(\eta_\varepsilon,u_\varepsilon)$. We have indeed the following lemma.

Lemma 4.3.3. — *With the notation of Theorem 7, consider a subsequence of $(\Phi_\varepsilon)_{\varepsilon>0}$, and some Φ_λ in $\mathrm{Ker}(L-i\lambda Id)$ such that for all $s<0$ and all $T>0$*

$$\forall i\lambda\in\mathfrak{S},\quad \Pi_\lambda\Phi_\varepsilon\to\Phi_\lambda \text{ in } L^2([0,T];H^s_{loc}(\mathbf{R}\times\mathbf{T})).$$

Then, for all $i\lambda\in\mathfrak{S}$, $e^{\frac{it\lambda}{\varepsilon}}(\eta_\varepsilon,u_\varepsilon)$ converges to Φ_λ weakly in $L^2([0,T]\times\mathbf{R}\times\mathbf{T})$. In particular, for all $i\lambda\in\mathfrak{S}$, the vector field Φ'_λ belongs to $L^2([0,T];H^1(\mathbf{R}\times\mathbf{T}))$.

Proof. — Denote by $(\eta_\lambda, u_\lambda)$ any weak limit point of the sequence $\exp\left(\frac{it\lambda}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon)$ (recall that the sequence is bounded in $L^2_{loc}(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{T})$). Let χ and ψ be any test function and vector field in $\mathcal{D}(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{T})$. Multiplying the conservation of mass in (SW_ε) by $\varepsilon\chi \exp\left(\frac{-it\lambda}{\varepsilon}\right)$ and integrating with respect to all variables leads to

$$\iint \left(\varepsilon\eta_\varepsilon \exp\left(\frac{it\lambda}{\varepsilon}\right)\left(\partial_t\chi + \frac{i\lambda}{\varepsilon}\chi\right) + (1+\varepsilon\eta_\varepsilon)u_\varepsilon \exp\left(\frac{it\lambda}{\varepsilon}\right)\cdot\nabla\chi\right) dxdt = 0.$$

Because of the bounds coming from the energy estimate (1.4.5) page 6, we can take limits in the previous identity as ε goes to 0 to get

$$\iint (u_\lambda \cdot \nabla\chi + i\lambda\eta_\lambda\chi) dxdt = 0.$$

Similarly, multiplying the conservation of momentum by $\varepsilon\psi \exp\left(\frac{it\lambda}{\varepsilon}\right)$ and integrating with respect to all variables leads to

$$\begin{aligned}\iint \Big(\varepsilon(1+\varepsilon\eta_\varepsilon)u_\varepsilon \exp\left(\frac{it\lambda}{\varepsilon}\right)\left(\partial_t\psi + \frac{it\lambda}{\varepsilon}\psi\right) + \varepsilon\exp\left(\frac{it\lambda}{\varepsilon}\right)(1+\varepsilon\eta_\varepsilon)u_\varepsilon\cdot(u_\varepsilon\cdot\nabla)\psi \\ +\beta x_1 \exp\left(\frac{it\lambda}{\varepsilon}\right)(1+\varepsilon\eta_\varepsilon)u_\varepsilon\cdot\psi^\perp + (1+\frac{\varepsilon}{2}\eta_\varepsilon)\eta_\varepsilon \exp\left(\frac{it\lambda}{\varepsilon}\right)\nabla\cdot\psi \\ +\varepsilon\nu u_\varepsilon \exp\left(\frac{it\lambda}{\varepsilon}\right)\cdot\Delta\psi\Big) dxdt = 0.\end{aligned}$$

Once again the bounds coming from the energy estimate (1.4.5) will enable us to take the limit as ε goes to 0, to get

$$\iint (\eta_\lambda\nabla\cdot\psi + \beta x_1 u_\lambda\cdot\psi^\perp + i\lambda u_\lambda\psi) dxdt = 0.$$

It follows that $(\eta_\lambda(t), u_\lambda(t))$ belongs to $\mathrm{Ker}(L - i\lambda Id)$ for almost all $t \in \mathbf{R}^+$, and we conclude by uniqueness of the limit and L^2 continuity of Π_λ that $\Phi_\lambda = (\eta_\lambda, u_\lambda)$. The lemma is proved. □

4.3.2. Strong convergence of Φ_ε. — As a corollary of the previous mode by mode convergence results, we get the following convergence for Φ_ε.

Lemma 4.3.4. — *With the notation of Theorem 7, the following results hold. Consider a subsequence of (Φ_ε), and some $\Phi_\lambda \in \mathrm{Ker}(L - i\lambda Id)$ such that as constructed in (4.3.4), for all $s < 0$ and all $T > 0$*

$$\forall i\lambda \in \mathfrak{S}, \quad \Pi_\lambda\Phi_\varepsilon \to \Phi_\lambda \text{ in } L^2([0,T]; H^s_{loc}(\mathbf{R}\times\mathbf{T})).$$

Then,

$$\Phi_\varepsilon \rightharpoonup \Phi = \sum_{i\lambda\in\mathfrak{S}} \Phi_\lambda \text{ weakly in } L^2_{loc}(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T})),$$
$$\text{and} \quad \Phi_\varepsilon \to \Phi \text{ strongly in } L^2_{loc}(\mathbf{R}^+; H^s_{loc}(\mathbf{R}\times\mathbf{T})) \text{ for all } s<0.$$

Moreover, defining K_N as in (3.4.1) page 40, we have for any relatively compact subset Ω of $\mathbf{R}\times\mathbf{T}$, for all $T > 0$ and for all $s < 0$,

$$\|(Id - K_N)\Phi_\varepsilon\|_{L^2([0,T];H^s(\Omega))} + \|(Id - K_N)\mathcal{L}(\frac{t}{\varepsilon})\Phi_\varepsilon\|_{L^2([0,T];H^s(\Omega))} \to 0 \text{ as } N\to\infty, \tag{4.3.5}$$

uniformly in ε.

Proof. — The first convergence statement comes directly from the uniform bound on Φ_ε in the space $L^2_{loc}(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T}))$ and the L^2 continuity of Π_λ.

In order to establish the strong convergence result, the crucial argument is to approximate (uniformly) Φ_ε by a finite number of modes, i.e. to prove (4.3.5). The main idea is the same as for the approximation of $\Pi_0\Phi_\varepsilon$ in Lemma 4.3.2. We have for all $T>0$ and all $s<0$

$$\left\| \sum_{n\le N, |k|\le N} (\Psi_{n,k,j}|\Phi_\varepsilon)_{L^2(\mathbf{R}\times\mathbf{T})}\, \Psi_{n,k,j} - \Phi_\varepsilon \right\|_{L^2([0,T];H^s_L)} \to 0 \text{ as } N\to\infty \text{ uniformly in } \varepsilon,$$

and similarly

$$\left\| \sum_{n\le N, |k|\le N} e^{-i\tau(n,k,j)\frac{t}{\varepsilon}}\, (\Psi_{n,k,j}|\Phi_\varepsilon)_{L^2(\mathbf{R}\times\mathbf{T})}\, \Psi_{n,k,j} - \mathcal{L}(\frac{t}{\varepsilon})\Phi_\varepsilon \right\|_{L^2([0,T];H^s_L)} \to 0 \text{ as } N\to\infty,$$

uniformly in ε. Therefore for all relatively compact subsets Ω of $\mathbf{R}\times\mathbf{T}$, the embedding of H^s_L into $H^s(\Omega)$ recalled in (4.3.2) implies that both quantities

$$\sum_{n\le N, |k|\le N} (\Psi_{n,k,j}|\Phi_\varepsilon)_{L^2(\mathbf{R}\times\mathbf{T})}\, \Psi_{n,k,j} - \Phi_\varepsilon$$

and

$$\sum_{n\le N, |k|\le N} e^{-i\tau(n,k,j)\frac{t}{\varepsilon}}\, (\Psi_{n,k,j}|\Phi_\varepsilon)_{L^2(\mathbf{R}\times\mathbf{T})}\, \Psi_{n,k,j} - \mathcal{L}(\frac{t}{\varepsilon})\Phi_\varepsilon$$

converge strongly towards zero in $L^2([0,T];H^s(\Omega))$ as N goes to infinity, uniformly in ε. Finally (4.3.5) is proved.

The strong convergence is therefore obtained from the following decomposition:

$$\Phi_\varepsilon - \Phi = (Id-K_N)\Phi_\varepsilon + K_N(\Phi_\varepsilon-\Phi) - (Id-K_N)\Phi$$

The first term converges to 0 as $N\to\infty$ uniformly in ε in $L^2_{loc}(\mathbf{R}^+, H^s_{loc}(\mathbf{R}\times\mathbf{T}))$ for all $s<0$ by (4.3.5). By Lemma 4.3.2, the second term (which is a finite sum of modes) converges to 0 as $\varepsilon\to 0$ for all fixed N in $L^2_{loc}(\mathbf{R}^+; H^s(\mathbf{R}\times\mathbf{T}))$ for all $s<0$. The last term converges to 0 as $N\to\infty$ in $L^2_{loc}(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T}))$. Thus taking limits as $\varepsilon\to 0$, then as $N\to\infty$ leads to the expected strong convergence. ☐

4.3.3. Taking limits in the equation on $\Pi_\lambda\Phi_\varepsilon$. — The next step is then to obtain the evolution equation for each mode Φ_λ, taking limits in (4.3.1) and (4.3.3). In the following proposition, we recall that the first result (concerning the geostrophic motion) relies on a compensated compactness argument, i.e. on both the algebraic structure of the coupling term and the particular form of the oscillating modes, which implies that there is no contribution of the equatorial waves to the geostrophic flow. That result was proved in Section 4.1 (see also Proposition 2.4.5 page 26). Here we will prove the second part of the statement, concerning the limit ageostrophic motion.

Proposition 4.3.5. — *With the notation of Theorem 7, there is a subsequence of (Φ_ε) such that the following result holds. Consider a family $(\Phi_\lambda)_{i\lambda\in\mathfrak{S}}$ such that $\Phi_\lambda\in \operatorname{Ker}(L-i\lambda Id)$ and such that for all $s<0$ and all $T>0$*

$$\forall i\lambda\in\mathfrak{S},\quad \Pi_\lambda\Phi_\varepsilon\to\Phi_\lambda \text{ in } L^2([0,T];H^s_{loc}(\mathbf{R}\times\mathbf{T})),$$

as constructed in (4.3.4).

Then, $\Phi_0 = (\eta_0, u_0)$ satisfies the geostrophic equation : for all (η^*, u^*) belonging to Ker$L$ *and satisfying $u^* \in H^1(\mathbf{R} \times \mathbf{T})$,*

$$\int (\eta_0 \eta^* + u_{0,2} u_2^*)(t,x)\, dx + \nu \int_0^t \int \nabla u_{0,2} \cdot \nabla u_2^*(t',x)\, dx\, dt' = \int (\eta^0 \eta^* + u_2^0 u_2^*)(x)\, dx.$$

Moreover for $\lambda \neq 0$, $\Phi_\lambda = (\Phi_\lambda^0, \Phi_\lambda')$ satisfies the following envelope equation : there is a measure υ_λ in $\mathcal{M}(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{T})$, such that for all smooth $\Phi_\lambda^* = (\Phi_{\lambda,0}^*, (\Phi_\lambda^*)') \in$ Ker$(L - i\lambda Id)$,

$$\int \Phi_\lambda \cdot \bar{\Phi}_\lambda^*(t,x)\, dx + \nu \int_0^t \int \nabla \Phi_\lambda' : \nabla (\bar{\Phi}_\lambda^*)'(t',x)\, dx\, dt' + \int_0^t \int \nabla \cdot (\bar{\Phi}_\lambda^*)' \upsilon_\lambda(dt', dx)$$
$$+ \sum_{\substack{i\mu, i\tilde{\mu} \in \mathfrak{S} \\ \lambda = \mu + \tilde{\mu}}} \int_0^t Q(\Phi_\mu, \Phi_{\tilde{\mu}}) \cdot \bar{\Phi}_\lambda^*(t',x)\, dx\, dt' = \int \Phi^0 \cdot \bar{\Phi}_\lambda^*(x)\, dx,$$

where Q is defined by (2.3.4) page 18.

Proof. — Let us first recall that for $\lambda \neq 0$, Ker$(L - i\lambda Id)$ is constituted of smooth, rapidly decaying vector fields, so that it makes sense to apply Π_λ to any distribution.

Starting from (4.3.1) we get that for all smooth $\Phi_\lambda^* = (\Phi_{\lambda,0}^*, (\Phi_\lambda^*)') \in$ Ker$(L - i\lambda Id)$

$$\int \exp\left(\frac{it\lambda}{\varepsilon}\right) (\eta_\varepsilon \bar{\Phi}_{\lambda,0}^* + m_\varepsilon \cdot (\bar{\Phi}_\lambda^*)')(t,x) dx - \int (\eta_\varepsilon^0 \bar{\Phi}_{\lambda,0}^* + m_\varepsilon^0 \cdot (\bar{\Phi}_\lambda^*)')(x) dx$$
$$+\nu \int_0^t \int \nabla \left(\exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \right) : \nabla (\bar{\Phi}_\lambda^*)'(t',x) dx dt' \tag{4.3.6}$$
$$- \int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) \Big(m_\varepsilon \cdot (u_\varepsilon \cdot \nabla)(\bar{\Phi}_\lambda^*)' + \frac{1}{2} \eta_\varepsilon^2 \nabla \cdot (\bar{\Phi}_\lambda^*)' \Big)(t',x) dx dt' = 0.$$

Taking limits as $\varepsilon \to 0$ in the three first terms is immediate using Lemma 4.3.3 and the assumption on the initial data. The limit as $\varepsilon \to 0$ in the two nonlinear terms is given in the following proposition.

Proposition 4.3.6. — *With the notation of Proposition 4.3.5, we have*

$$\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) m_\varepsilon \cdot (u_\varepsilon \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x) dx dt' \to \int_0^t \int \sum_{\substack{\mu + \tilde{\mu} = \lambda \\ i\mu, i\tilde{\mu} \in \mathfrak{S}}} \Phi_\mu' \cdot (\Phi_{\tilde{\mu}}' \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x) dx dt',$$

and

$$\frac{1}{2} \int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_\varepsilon^2 \nabla \cdot (\bar{\Phi}_\lambda^*)'(t',x) dx dt' \to \frac{1}{2} \int_0^t \int \sum_{\substack{\mu + \tilde{\mu} = \lambda \\ i\mu, i\tilde{\mu} \in \mathfrak{S}}} \Phi_{\mu,0} \Phi_{\tilde{\mu},0} \nabla \cdot (\bar{\Phi}_\lambda^*)'(t',x) dx dt'$$
$$- \int_0^t \nabla \cdot (\bar{\Phi}_\lambda^*)' \upsilon_\lambda(dt' dx).$$

Before proving that result, let us conclude the proof of Proposition 4.3.5. It remains to check that

$$- \int \sum_{\substack{\mu + \tilde{\mu} = \lambda \\ i\lambda, i\mu, i\tilde{\mu} \in \mathfrak{S}}} \Big(\Phi_\mu' \cdot (\Phi_{\tilde{\mu}}' \cdot \nabla)(\bar{\Phi}_\lambda^*)' + \Phi_{\mu,0} \Phi_{\tilde{\mu},0} \nabla \cdot (\bar{\Phi}_\lambda^*)' \Big)\, dx$$
$$= \int \sum_{\substack{\mu + \tilde{\mu} = \lambda \\ i\mu, i\tilde{\mu} \in \mathfrak{S}}} \Big((\Phi_\mu' \cdot \nabla) \Phi_{\tilde{\mu}}' \cdot (\bar{\Phi}_\lambda^*)' + \nabla \cdot (\Phi_{\mu,0} \Phi_{\tilde{\mu}}') \bar{\Phi}_{\lambda,0}^* \Big)\, dx,$$

since

$$\int \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \Big((\Phi'_\mu\cdot\nabla)\Phi'_{\tilde\mu}\cdot(\bar\Phi^*_\lambda)' + \nabla\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu})\bar\Phi^*_{\lambda,0}\Big)\,dx = \int \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} Q(\Phi_\mu,\Phi_{\tilde\mu})\cdot\bar\Phi^*_\lambda\,dx.$$

Clearly one has

$$-\int \Phi'_\mu\cdot(\Phi'_{\tilde\mu}\cdot\nabla)(\bar\Phi^*_\lambda)'\,dx = \int (\Phi'_{\tilde\mu}\cdot\nabla)\Phi'_\mu\cdot(\bar\Phi^*_\lambda)'\,dx + \int \Phi'_\mu\cdot(\bar\Phi^*_\lambda)'\nabla\cdot\Phi'_{\tilde\mu}\,dx,$$

so since μ and $\tilde\mu$ play symmetric roles, we just need to check that

$$\sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \Big(\int \Phi'_\mu\cdot(\bar\Phi^*_\lambda)'\nabla\cdot\Phi'_{\tilde\mu}\,dx - \frac12\int \Phi_{\mu,0}\Phi_{\tilde\mu,0}\nabla\cdot(\bar\Phi^*_\lambda)'\,dx\Big) = \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\lambda,i\mu,i\tilde\mu\in\mathfrak{S}}} \int \nabla\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu})\bar\Phi^*_{\lambda,0}\,dx.$$

Recalling that

$$\nabla\bar\Phi^*_{\lambda,0} = -i\lambda(\bar\Phi^*_\lambda)' - \beta x_1(\bar\Phi^*_\lambda)'^{\perp} = -i(\mu+\tilde\mu)(\bar\Phi^*_\lambda)' - \beta x_1(\bar\Phi^*_\lambda)'^{\perp},$$

we have

$$\int \nabla\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu})\bar\Phi^*_{\lambda,0}\,dx = i(\mu+\tilde\mu)\int \Phi_{\mu,0}\Phi'_{\tilde\mu}\cdot(\bar\Phi^*_\lambda)'\,dx + \int \Phi_{\mu,0}\beta x_1\Phi'_{\tilde\mu}\cdot(\bar\Phi^*_\lambda)'^{\perp}\,dx.$$

Then we write

$$i\mu\Phi_{\mu,0} = \nabla\cdot\Phi'_\mu$$

so that

$$\begin{aligned}\int \nabla\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu})\bar\Phi^*_{\lambda,0}\,dx &= \int \nabla\cdot\Phi'_\mu\Phi'_{\tilde\mu}\cdot(\bar\Phi^*_\lambda)'\,dx + i\tilde\mu\int \Phi_{\mu,0}\Phi'_{\tilde\mu}\cdot(\bar\Phi^*_\lambda)'\,dx\\ &\qquad - \int \Phi_{\mu,0}\beta x_1{\Phi'_{\tilde\mu}}^{\perp}\cdot(\bar\Phi^*_\lambda)'\,dx.\end{aligned}$$

Exchanging the roles of μ and $\tilde\mu$ in the first integral we get

$$\begin{aligned}\sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \int \nabla\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu})\bar\Phi^*_{\lambda,0}\,dx &= \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \Big(\int \nabla\cdot\Phi'_{\tilde\mu}\Phi'_\mu\cdot(\bar\Phi^*_\lambda)'\,dx\\ &\qquad + i\tilde\mu\int \Phi_{\mu,0}\Phi'_{\tilde\mu}\cdot(\bar\Phi^*_\lambda)'\,dx - \int \Phi_{\mu,0}(i\tilde\mu\Phi'_{\tilde\mu} - \nabla\Phi_{\tilde\mu,0})\cdot(\bar\Phi^*_\lambda)'\,dx\Big),\end{aligned}$$

so

$$\sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \int \nabla\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu})\bar\Phi^*_{\lambda,0}\,dx = \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \Big(\int \nabla\cdot\Phi'_{\tilde\mu}\Phi'_\mu\cdot(\bar\Phi^*_\lambda)'\,dx + \int \Phi_{\mu,0}\nabla\Phi_{\tilde\mu,0}\cdot(\bar\Phi^*_\lambda)'\,dx\Big).$$

The result finally follows from the fact that, by symmetry,

$$\sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \int \Phi_{\mu,0}\nabla\Phi_{\tilde\mu,0}\cdot(\bar\Phi^*_\lambda)'\,dx = \frac12 \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \int \nabla(\Phi_{\tilde\mu,0}\Phi_{\mu,0})\cdot(\bar\Phi^*_\lambda)'\,dx$$

which finally implies that

$$\sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \Big(\int \Phi'_\mu\cdot(\bar\Phi^*_\lambda)'\nabla\cdot\Phi'_{\tilde\mu}\,dx - \frac12\int \Phi_{\mu,0}\Phi_{\tilde\mu,0}\nabla\cdot(\bar\Phi^*_\lambda)'\,dx\Big) = \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \int \nabla\cdot(\Phi_{\mu,0}\Phi'_{\tilde\mu})\bar\Phi^*_{\lambda,0}\,dx.$$

Now let us prove Proposition 4.3.6. The idea is to decompose Φ_ε on the eigenmodes of L, by writing

$$(\eta_\varepsilon,u_\varepsilon)(t,x) = \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_\varepsilon(t,x) = \sum_{i\lambda\in\mathfrak{S}} e^{-\frac{it\lambda}{\varepsilon}}\Pi_\lambda\Phi_\varepsilon(t,x).$$

Note in particular that by (4.3.5), for any $s < 0$,

$$(\eta_\varepsilon, u_\varepsilon)(t) - \mathcal{L}\left(\frac{t}{\varepsilon}\right) K_N \Phi_\varepsilon(t) \to 0 \text{ in } L^2_{loc}(\mathbf{R}^+; H^s_{loc}(\mathbf{R}\times\mathbf{T}))$$

as N goes to infinity, uniformly in ε. Let us also introduce the notation

$$\begin{aligned}\Phi_{\varepsilon,N} &= \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_{\varepsilon,N}, u_{\varepsilon,N}) = K_N\Phi_\varepsilon, \quad \text{and}\\ \Phi_{\varepsilon,\lambda,N} &= \Pi_\lambda \Phi_{\varepsilon,N}.\end{aligned}$$

We will start by considering the first nonlinear term in Proposition 4.3.6, namely

$$\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) m_\varepsilon \cdot (u_\varepsilon\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'.$$

We can notice that

$$\begin{aligned}\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) m_\varepsilon \cdot (u_\varepsilon\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt' &= \int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) \varepsilon\eta_\varepsilon u_\varepsilon \cdot (u_\varepsilon\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'\\ &+ \int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \cdot (u_\varepsilon\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'.\end{aligned}$$

The uniform bounds coming from the energy estimate imply clearly that the first term converges to 0 as $\varepsilon \to 0$. Then we can decompose the second contribution in the following way:

$$\begin{aligned}&\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \cdot (u_\varepsilon\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'\\ &= \int_0^t \int_{(\mathbf{R}\setminus[-R,R])\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \cdot (u_\varepsilon\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'\\ &+ \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) (u_\varepsilon - u_{\varepsilon,N}) \cdot (u_\varepsilon\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'\\ &+ \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_{\varepsilon,N} \cdot ((u_\varepsilon - u_{\varepsilon,M})\cdot\nabla)\,(\bar\Phi^*_\lambda)'(t',x)dxdt'\\ &+ \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_{\varepsilon,N} \cdot (u_{\varepsilon,M}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'.\end{aligned} \tag{4.3.7}$$

Let us consider now all the terms in the right-hand side of (4.3.7). The uniform bound on u_ε and the decay of Φ^*_λ imply that the first term on the right-hand side converges to 0 as $R \to \infty$ uniformly in ε.

By the inequality

$$\begin{aligned}&\left|\int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) (u_\varepsilon - u_{\varepsilon,N}) \cdot (u_\varepsilon\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'\right|\\ &\le C\|u_\varepsilon - u_{\varepsilon,N}\|_{L^2([0,T];H^s([-R,R]\times\mathbf{T}))}\|u_\varepsilon\|_{L^2([0,T];H^1(\mathbf{R}\times\mathbf{T}))}\|\Phi^*_\lambda\|_{W^{2,\infty}(\mathbf{R}\times\mathbf{T})}, \ \text{with } -1 < s < 0,\end{aligned}$$

we deduce that the third term converges to 0 as $N \to \infty$ uniformly in ε.

Now let us consider the third term on the right-hand side. Since $u_{\varepsilon,N}$ corresponds to the projection of Φ_ε onto a finite number of eigenvectors of L, we deduce that

$$\forall N \in \mathbf{N}, \exists C_N, \forall\varepsilon > 0, \quad \|u_{\varepsilon,N}\|_{L^\infty(\mathbf{R}^+;H^1(\mathbf{R}\times\mathbf{T}))} \le C_N.$$

Thus

$$\left|\int_0^t\int_{[-R,R]\times\mathbf{T}}\exp\left(\frac{it'\lambda}{\varepsilon}\right)u_{\varepsilon,N}\cdot((u_\varepsilon-u_{\varepsilon,M})\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)dxdt'\right|$$
$$\le C_N\|u_\varepsilon-u_{\varepsilon,M}\|_{L^2([0,T];H^s([-R,R]\times\mathbf{T}))}\|\Phi^*_\lambda\|_{W^{2,\infty}(\mathbf{R}\times\mathbf{T})}$$

and, for all fixed N and R, the fourth term converges to 0 as $M\to\infty$ uniformly in ε.

It remains then to take limits in the last term of (4.3.7). It can be rewritten

$$\int_0^t\int_{[-R,R]\times\mathbf{T}}\exp\left(\frac{it'\lambda}{\varepsilon}\right)u_{\varepsilon,N}\cdot(u_{\varepsilon,M}\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)dxdt'$$
$$=\int_0^t\int_{[-R,R]\times\mathbf{T}}\sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}}\exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)(\Phi_{\varepsilon,\mu,N})'\cdot(\Phi'_{\varepsilon,\tilde{\mu},M}\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)dxdt'.$$

This in turn can be written in the following way:

$$\int_0^t\int_{[-R,R]\times\mathbf{T}}\sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}}\exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)\Phi'_{\varepsilon,\mu,N}\cdot(\Phi'_{\varepsilon,\tilde{\mu},M}\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)dxdt'$$
$$=\int_0^t\int_{[-R,R]\times\mathbf{T}}\sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}}\exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)(\Phi'_{\varepsilon,\mu,N}-\Phi'_{\mu,N})\cdot(\Phi'_{\varepsilon,\tilde{\mu},M}\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)dxdt'$$
$$+\int_0^t\int_{[-R,R]\times\mathbf{T}}\sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}}\exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)\Phi'_{\mu,N}\cdot((\Phi'_{\varepsilon,\tilde{\mu},M}-\Phi'_{\tilde{\mu},M})\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)dxdt'$$
$$+\int_0^t\int_{[-R,R]\times\mathbf{T}}\sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}}\exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)\Phi'_{\mu,N}\cdot(\Phi'_{\tilde{\mu},M}\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)dxdt'.$$

We have denoted

$$\Phi_{\mu,N}=\Pi_\mu\Phi_N,\quad\text{where}\quad\Phi_N=K_N\Phi.$$

The first two terms on the right-hand side go to zero as ε goes to zero, for all given N,M and R, due to the following estimates: for $-1<s<0$,

$$\int_0^t\int_{[-R,R]\times\mathbf{T}}\sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}}\left|(\Phi'_{\varepsilon,\mu,N}-\Phi'_{\mu,N})\cdot(\Phi'_{\varepsilon,\tilde{\mu},M}\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)\right|dxdt'$$
$$\le C_{N,M}\|\Phi'_{\varepsilon,N}-\Phi'_N\|_{L^2([0,T];H^s([-R,R]\times\mathbf{T}))}\|\Phi_{\varepsilon,M}\|_{L^\infty([0,T];H^1(\mathbf{R}\times\mathbf{T}))}\|\Phi^*_\lambda\|_{W^{2,\infty}(\mathbf{R}\times\mathbf{T})},$$

and similarly

$$\int_0^t\int_{[-R,R]\times\mathbf{T}}\sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}}\left|\Phi'_{\mu,N}\cdot((\Phi'_{\varepsilon,\tilde{\mu},M}-\Phi'_{\tilde{\mu},M})\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)\right|dxdt'$$
$$\le C_{N,M}\|\Phi'_{\varepsilon,M}-\Phi'_M\|_{L^2([0,T];H^s([-R,R]\times\mathbf{T}))}\|\Phi_{\varepsilon,N}\|_{L^\infty([0,T];H^1(\mathbf{R}\times\mathbf{T}))}\|\Phi^*_\lambda\|_{W^{2,\infty}(\mathbf{R}\times\mathbf{T})}.$$

Finally let us consider the last term, which can be decomposed in the following way:

$$\int_0^t\int_{[-R,R]\times\mathbf{T}}\sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}}\exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)\Phi'_{\mu,N}\cdot(\Phi'_{\tilde{\mu},N}\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)dxdt'$$
$$=\int_0^t\int_{[-R,R]\times\mathbf{T}}\sum_{\substack{i\mu,i\tilde{\mu}\in\mathfrak{S}\\ \lambda=\mu+\tilde{\mu}}}\Phi'_{\mu,N}\cdot(\Phi'_{\tilde{\mu},M}\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)dxdt'$$
$$+\int_0^t\int_{[-R,R]\times\mathbf{T}}\sum_{\substack{i\mu,i\tilde{\mu}\in\mathfrak{S}\\ \lambda\neq\mu+\tilde{\mu}}}\exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)\Phi'_{\mu,N}\cdot(\Phi'_{\tilde{\mu},M}\cdot\nabla)(\bar{\Phi}^*_\lambda)'(t',x)dxdt'.$$

For fixed N and M, the nonstationary phase theorem (which corresponds here to a simple integration by parts in the t' variable) shows that the second term is a finite sum of terms converging to 0 as $\varepsilon \to 0$. And the first term (which does not depend on ε) converges to

$$\int_0^t \int \sum_{\substack{\mu+\tilde\mu=\lambda \\ i\lambda, i\mu, i\tilde\mu \in \mathfrak{S}}} \Phi'_\mu \cdot (\Phi'_{\tilde\mu} \cdot \nabla)(\bar\Phi^*_\lambda)'(t', x) dx dt'$$

as $N, M, R \to \infty$, because Φ'_N converges towards Φ' strongly in $L^2([0,T]; L^2(\mathbf{R} \times \mathbf{T}))$ when N goes to infinity, and then by Lebesgue's theorem when R goes to infinity.

Therefore, taking limits as $\varepsilon \to 0$, then $M \to \infty$, then $N \to \infty$, then $R \to \infty$ in (4.3.7) leads to

$$\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) m_\varepsilon \cdot (u_\varepsilon \cdot \nabla)(\bar\Phi^*_\lambda)'(t', x) dx dt' \to \int_0^t \int \sum_{\substack{\mu+\tilde\mu=\lambda \\ i\lambda, i\mu, i\tilde\mu \in \mathfrak{S}}} \Phi'_\mu \cdot (\Phi'_{\tilde\mu} \cdot \nabla)(\bar\Phi^*_\lambda)'(t', x) dx dt'.$$

Finally let us consider the second term of the proposition, namely

$$\frac{1}{2} \int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_\varepsilon^2 \nabla \cdot (\bar\Phi^*_\lambda)'(t', x)\, dx dt'.$$

The first step of the above study remains valid, in the sense that one can write

$$\begin{aligned}\frac{1}{2} \int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_\varepsilon^2 \nabla \cdot (\bar\Phi^*_\lambda)'(t', x)\, dx dt' &= \frac{1}{2} \int_0^t \int_{\mathbf{R}\setminus[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_\varepsilon^2 \nabla \cdot (\bar\Phi^*_\lambda)'(t', x)\, dx dt' \\ &+ \frac{1}{2} \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_\varepsilon^2 \nabla \cdot (\bar\Phi^*_\lambda)'(t', x)\, dx dt',\end{aligned}$$

and the first term converges to zero uniformly in ε as R goes to infinity, due to the spatial decay of the eigenvectors of L. For such a result, a uniform bound of η_ε in $L^\infty(\mathbf{R}^+; L^2(\mathbf{R} \times \mathbf{T}))$ is sufficient. However the next steps of the above study do not work here, as we have no smoothness on η_ε other than that energy bound. In order to conclude, let us nevertheless decompose the remaining term as above, for any integers N and M to be chosen large enough below:

$$\begin{aligned}&\frac{1}{2} \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_\varepsilon^2 \nabla \cdot (\bar\Phi^*_\lambda)'(t', x)\, dx dt' \\ &= \frac{1}{2} \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) (\eta_\varepsilon - \eta_{\varepsilon,N})\eta_\varepsilon \nabla \cdot (\bar\Phi^*_\lambda)'(t', x)\, dx dt' \\ &+ \frac{1}{2} \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_{\varepsilon,N}(\eta_\varepsilon - \eta_{\varepsilon,M}) \nabla \cdot (\bar\Phi^*_\lambda)'(t', x)\, dx dt' \\ &+ \frac{1}{2} \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_{\varepsilon,N}\eta_{\varepsilon,M} \nabla \cdot (\bar\Phi^*_\lambda)'(t', x)\, dx dt'.\end{aligned} \tag{4.3.8}$$

The sequence $-\frac{1}{2}\exp\left(\frac{it'\lambda}{\varepsilon}\right)(\eta_\varepsilon - \eta_{\varepsilon,N})\eta_\varepsilon$ is uniformly bounded in $N \in \mathbf{N}$ and $\varepsilon > 0$ in the space $L^1_{loc}(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{T})$, so up to the extraction of a subsequence it converges weakly, as ε goes to zero, towards a measure $\upsilon_{\lambda,N}$, which in turn is uniformly bounded in $\mathcal{M}(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{T})$. Denoting by υ_λ its limit in $\mathcal{M}(\mathbf{R}^+ \times \mathbf{R} \times \mathbf{T})$ as N goes to infinity, we find that

$$\frac{1}{2} \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) (\eta_\varepsilon - \eta_{\varepsilon,N})\eta_\varepsilon \nabla \cdot (\bar\Phi^*_\lambda)'(t', x)\, dx dt' \to -\int_0^t \int_{[-R,R]\times\mathbf{T}} \nabla \cdot (\bar\Phi^*_\lambda)' \upsilon_\lambda(dt'dx)$$

as ε goes to zero and N goes to infinity, which in turn converges to

$$-\int_0^t \int \nabla \cdot (\bar{\Phi}_\lambda^*)' \upsilon_\lambda(dt'dx)$$

as R goes to infinity, due to the smoothness of $\nabla \cdot (\bar{\Phi}_\lambda^*)'$.

Note that as $\mathfrak{S}$ is countable, one can choose a subsequence such that for all $i\lambda \in \mathfrak{S}$, the sequence $-\frac{1}{2}\exp\left(\frac{it'\lambda}{\varepsilon}\right)(\eta_\varepsilon - \eta_{\varepsilon,N})\eta_\varepsilon$ converges towards υ_λ as ε goes to zero and N goes to infinity.

Now let us consider the two last terms in (4.3.8). We recall that $\eta_{\varepsilon,N}$ corresponds to the projection of Φ_ε onto a finite number of eigenvectors of L, so it is smooth for each fixed N. In particular we can write, for any $s < 0$,

$$\frac{1}{2}\left|\int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_{\varepsilon,N}(\eta_\varepsilon - \eta_{\varepsilon,M})\nabla \cdot (\bar{\Phi}_\lambda^*)'(t',x)\,dxdt'\right| \\ \leq C_N \|\eta_\varepsilon - \eta_{\varepsilon,M}\|_{L^\infty([0,T];H^s([-R,R]\times\mathbf{T}))}\|\Phi_\lambda^*\|_{W^{2,\infty}(\mathbf{R}\times\mathbf{T})}.$$

So letting M go to infinity we find that this term converges to zero uniformly in ε for each fixed N and R.

Finally for the last term of (4.3.8) we write similar computations as for the first nonlinear term in Proposition 4.3.6. We have indeed

$$\begin{aligned}
&\int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_{\varepsilon,N}\eta_{\varepsilon,M}\nabla \cdot (\bar{\Phi}_\lambda^*)'(t',x)\,dxdt' \\
&= \int_0^t \int_{[-R,R]\times\mathbf{T}} \sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)(\Phi_{\mu,\varepsilon,N})_0(\Phi_{\tilde{\mu},\varepsilon,M})_0\nabla \cdot (\bar{\Phi}_\lambda^*)'(t',x)\,dxdt' \\
&= \int_0^t \int_{[-R,R]\times\mathbf{T}} \sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)(\Phi_{\mu,\varepsilon,N}-\Phi_{\mu,N})_0(\Phi_{\tilde{\mu},\varepsilon,M})_0\nabla \cdot (\bar{\Phi}_\lambda^*)'(t',x)\,dxdt' \\
&+ \int_0^t \int_{[-R,R]\times\mathbf{T}} \sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)(\Phi_{\mu,N})_0(\Phi_{\tilde{\mu},\varepsilon,M}-\Phi_{\tilde{\mu},M})_0\nabla \cdot (\bar{\Phi}_\lambda^*)'(t',x)\,dxdt' \\
&+ \int_0^t \int_{[-R,R]\times\mathbf{T}} \sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)(\Phi_{\mu,N})_0(\Phi_{\tilde{\mu},M})_0\nabla \cdot (\bar{\Phi}_\lambda^*)'(t',x)\,dxdt'.
\end{aligned}$$

The two first terms in the right-hand side are easily shown to converge to zero as ε goes to zero, for each fixed N and M. We have indeed

$$\left|\int_0^t \int_{[-R,R]\times\mathbf{T}} \sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)(\Phi_{\mu,\varepsilon,N}-\Phi_{\mu,N})_0(\Phi_{\tilde{\mu},\varepsilon,M})_0\nabla \cdot (\bar{\Phi}_\lambda^*)'(t',x)\,dxdt'\right| \\ \leq C_M \|\Phi_{\mu,\varepsilon,N}-\Phi_{\mu,N}\|_{L^\infty([0,T];H^s([-R,R]\times\mathbf{T}))}\|\Phi_\lambda^*\|_{W^{2,\infty}},$$

and similarly

$$\left|\int_0^t \int_{[-R,R]\times\mathbf{T}} \sum_{i\mu,i\tilde{\mu}\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde{\mu})}{\varepsilon}\right)(\Phi_{\mu,N})_0(\Phi_{\tilde{\mu},\varepsilon,M}-\Phi_{\tilde{\mu},M})_0\nabla \cdot (\bar{\Phi}_\lambda^*)'(t',x)\,dxdt'\right| \\ \leq C_N \|\Phi_{\tilde{\mu},\varepsilon,M}-\Phi_{\tilde{\mu},M}\|_{L^\infty([0,T];H^s([-R,R]\times\mathbf{T}))}\|\Phi_\lambda^*\|_{W^{2,\infty}}.$$

Finally the last term on the right-hand side is dealt with by a nonstationary phase argument, and we have as above, as ε goes to zero and then M, N and R go successively to infinity,

$$\frac{1}{2}\int_0^t \int_{[-R,R]\times\mathbf{T}} \sum_{i\mu,i\tilde\mu\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde\mu)}{\varepsilon}\right)(\Phi_{\mu,N})_0(\Phi_{\tilde\mu,M})_0\nabla\cdot(\bar\Phi^*_\lambda)'(t',x)\,dxdt'$$
$$\to \frac{1}{2}\int_0^t\int \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \Phi_{\mu,0}\Phi_{\tilde\mu,0}\nabla\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt'.$$

Proposition 4.3.6 is proved, and therefore also Theorem 7. □

4.3.4. The case when capillarity is added. — In this final paragraph we propose an adaptation to the Saint-Venant model which provides some additional smoothness on η_ε, and which enables one to get rid of the defect measure present in the above study. The model is presented in the next part, and the convergence result stated and proved below.

4.3.4.1. The model. — Let us present an alternative to the Saint-Venant model studied up to now, which presents the advantage of providing the additional smoothness of $\varepsilon\eta_\varepsilon$ which is lacking in the original system. Its disadvantage however is that there is no real physical meaning to the capillarity operator we use in that model. With the notation of Chapter 1, we choose indeed the capillarity operator

$$K(h) = \kappa(-\Delta)^{2\alpha}h, \tag{4.3.9}$$

where $\kappa > 0$ and $\alpha > 1/2$ are given constants. After rescaling as in Chapter 1, we find the following system:

$$\begin{aligned} \partial_t\eta + \frac{1}{\varepsilon}\nabla\cdot\Big((1+\varepsilon\eta)u\Big) &= 0,\\ \partial_t u + u\cdot\nabla u + \frac{\beta x_1}{\varepsilon}u^\perp + \frac{1}{\varepsilon}\nabla\eta - \frac{\nu}{1+\varepsilon\eta}\Delta u + \varepsilon\kappa\nabla(-\Delta)^{2\alpha}\eta &= 0,\\ \eta_{|t=0} = \eta^0, \quad u_{|t=0} &= u^0. \end{aligned} \tag{4.3.10}$$

In the next part we discuss the existence of bounded energy solutions to that system of equations (under a smallness assumption), and the following part consists in the proof of the analogue of Theorem 7 in that setting. One should emphasize here that the additional capillarity term that is added in the system will not appear in the limit, since it comes as a $O(\varepsilon)$ term. Moreover it is a linear term, so it should not change the other asymptotics proved in this chapter. However its unphysical character (as well as the smallness condition on the initial data) made us prefer to study the original Saint-Venant system for all the convergence results of this chapter.

4.3.4.2. Existence of solutions. — The following theorem is an easy adaptation of the result by D. Bresch and B. Desjardins in [**2**] (see also [**23**] for the compressible Navier-Stokes system), we give a sketch of the proof below.

Theorem 8 **(existence of solutions in the case with capillarity)**

There is a constant $C > 0$ such that the following result holds. Let $(\eta^0_\varepsilon, u^0_\varepsilon)$ be a family of $H^{2\alpha}\times L^2(\mathbf{R}\times\mathbf{T})$ such that for all $\varepsilon > 0$,

$$\frac{1}{2}\int\Big((\eta^0_\varepsilon)^2 + \kappa\varepsilon^2|(-\Delta)^\alpha\eta^0_\varepsilon|^2 + (1+\varepsilon\eta^0_\varepsilon)|u^0_\varepsilon|^2\Big)(x)\,dx \le \mathcal{E}^0.$$

If $\mathcal{E}^0 \leq C$, then there is a family $(\eta_\varepsilon, u_\varepsilon)$ of weak solutions to (4.3.10), satisfying the energy estimate

$$\frac{1}{2}\int \Big(\eta_\varepsilon^2 + \kappa\varepsilon^2|(-\Delta)^\alpha \eta_\varepsilon|^2 + (1+\varepsilon\eta_\varepsilon)|u_\varepsilon|^2\Big)(t,x)\,dx + \nu\int_0^t\int |\nabla u_\varepsilon|^2(t',x)\,dxdt' \leq \mathcal{E}^0.$$

Proof. — Weak solutions can be constructed by a standard approximation scheme obtained by regularization : compactness on the approximate solutions comes from the a priori bounds derived from the energy inequality, which is obtained formally in a classical way by multiplying the momentum equation by u, using the mass conservation and integrating by parts. It allows to derive immediately the following a priori bounds (denoting by η and u approximate solutions) :

$$\begin{aligned}
\eta &\in L^\infty(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T}))\\
\varepsilon\eta &\in L^\infty(\mathbf{R}^+; H^{2\alpha}(\mathbf{R}\times\mathbf{T}))\\
(1+\varepsilon\eta)|u|^2 &\in L^\infty(\mathbf{R}^+; L^1(\mathbf{R}\times\mathbf{T}))\\
|\nabla u|^2 &\in L^1(\mathbf{R}^+; L^1(\mathbf{R}\times\mathbf{T})).
\end{aligned}$$

Since $\alpha > 1/2$, the first bound implies in particular that

$$\varepsilon\eta \in L^\infty(\mathbf{R}^+\times\mathbf{R}\times\mathbf{T}),$$

and in particular if $\mathcal{E}_0$ is small enough (compared to the reference height which is 1 here), then $1+\varepsilon\eta$ is bounded from below. We infer that u is bounded in $L^\infty(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T}))\times L^2(\mathbf{R}^+, \dot{H}^1(\mathbf{R}\times\mathbf{T}))$.

It is standard (see [**2**], [**23**]) to deduce that $1+\varepsilon\eta$ is compact in $L^2_{loc}(\mathbf{R}^+; H^1(\mathbf{R}\times\mathbf{T}))$, and that u is compact in $L^2_{loc}(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T}))$.

Taking the limit in the non linear terms is now possible : we need indeed to deal with the following nonlinear terms :

$$u\cdot\nabla u, \quad \nabla\cdot((1+\varepsilon\eta)u) \quad \text{and} \quad \frac{1}{1+\varepsilon\eta}\Delta u.$$

The compactness of $1+\varepsilon\eta$ and u derived above allows to deal with the two first terms in a standard fashion. For the last one we just have to recall that $1/(1+\varepsilon\eta)$ is bounded in $L^\infty(\mathbf{R}^+; H^{2\alpha}(\mathbf{R}\times\mathbf{T}))$. This completes the proof of Theorem 8. □

4.3.4.3. Convergence. — In this section our aim is to show that the capillarity term enables us to get rid of the defect measure present in the conclusion of Theorem 7 page 78. As the proof is very similar to that theorem, up to the compactness of η_ε, we will not give the full details. The result is the following.

***Theorem 9* (strong convergence in the case with capillarity).** — *Under the assumptions of Theorem 8, denote by $(\eta_\varepsilon, u_\varepsilon)$ a solution of (4.3.10) with initial data $(\eta^0_\varepsilon, u^0_\varepsilon)$, and define*

$$\Phi_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon).$$

Up to the extraction of a subsequence, Φ_ε converges weakly in $L^2_{loc}(\mathbf{R}^+; H^s_{loc}(\mathbf{R}\times\mathbf{T}))$ (for all $s<0$) toward some solution Φ of the following limiting filtered system: for all $i\lambda$ in $\mathfrak{S}$ and for all smooth Φ^*_λ in $\mathrm{Ker}(L - i\lambda Id)$*,*

$$\int \Phi\cdot\bar{\Phi}^*_\lambda(x)\,dx - \nu\int_0^t\int \Delta'_L\Phi\cdot\bar{\Phi}^*_\lambda(t',x)\,dxdt' + \int_0^t\int Q_L(\Phi,\Phi)\cdot\bar{\Phi}^*_\lambda(t',x)\,dxdt' = \int \Phi^0\cdot\bar{\Phi}^*_\lambda(x)\,dx,$$

where $\Phi^0 = (\eta^0, u^0)$.

Proof. — We will follow the lines of the proof of Theorem 7. In particular all the results of Sections 4.3.1 and 4.3.2 are true in this situation and we will not detail the proofs. So the point, as in Section 4.3.3, consists in taking the limit as ε goes to zero, of the equation on $\Pi_\lambda \Phi_\varepsilon$.

Equation (4.3.6) page 83 can be written here as follows: for all smooth $\Phi^*_\lambda = (\Phi^*_{\lambda,0}, (\Phi^*_\lambda)')$ belonging to $\mathrm{Ker}(L - i\lambda Id)$,

(4.3.11)

$$\begin{aligned}\int \exp\left(\frac{it\lambda}{\varepsilon}\right)(\eta_\varepsilon \bar\Phi^*_{\lambda,0} + u_\varepsilon \cdot (\bar\Phi^*_\lambda)')(t,x)dx - \int (\eta^0_\varepsilon \bar\Phi^*_{\lambda,0} + u^0_\varepsilon \cdot (\bar\Phi^*_\lambda)')(x)dx \\ -\varepsilon\kappa \int_0^t \int (-\Delta)^\alpha \exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon \nabla\cdot(-\Delta)^\alpha (\bar\Phi^*_\lambda)'(t',x)dxdt' \\ -\int_0^t \int \frac{\nu}{1+\varepsilon\eta_\varepsilon}\Delta\left(\exp\left(\frac{it'\lambda}{\varepsilon}\right)u_\varepsilon\right)\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt' \\ + \int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right)(u_\varepsilon\cdot\nabla)u_\varepsilon \cdot (\bar\Phi^*_\lambda)'(t',x)dxdt' - \int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon u_\varepsilon \cdot \nabla \bar\Phi^*_{\lambda,0}(t',x)dxdt' = 0.\end{aligned}$$

Remark 4.3.7. — *We have chosen to keep the unknowns $(\eta_\varepsilon, u_\varepsilon)$ and not write the analysis in terms of $(\eta_\varepsilon, m_\varepsilon)$ as previously (recall that $m_\varepsilon = (1+\varepsilon\eta_\varepsilon)u_\varepsilon$): the study of m_ε rather than u_ε is indeed unnecessary here as the factor $\dfrac{1}{1+\varepsilon\eta_\varepsilon}$ which appears in the diffusion term in the equation on u_ε can be controled in this situation, contrary to the previous case. The advantage of writing the equations on $(\eta_\varepsilon, u_\varepsilon)$ is that there is no nonlinear term in η_ε, contrary to the previous study, but of course the difficulty is transfered to the study of the diffusion operator; the gain of regularity in η_ε will appear here.*

Taking limits as $\varepsilon \to 0$ in the two first terms is immediate. For the third term, we simply recall that η_ε is bounded in $L^\infty(\mathbf{R}^+; L^2(\mathbf{R}\times\mathbf{T}))$ and $\varepsilon\eta_\varepsilon$ is bounded in $L^\infty(\mathbf{R}^+; H^{2\alpha}(\mathbf{R}\times\mathbf{T}))$, so $\varepsilon\eta_\varepsilon$ goes strongly to zero in $L^\infty(\mathbf{R}^+; H^s(\mathbf{R}\times\mathbf{T}))$ for every $s < 2\alpha$. Since Φ^*_λ is smooth, we infer that

$$\varepsilon\kappa \int_0^t \int (-\Delta)^\alpha \exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon \nabla\cdot(-\Delta)^\alpha(\bar\Phi^*_\lambda)'(t',x)dxdt' \to 0, \quad \text{as } \varepsilon \to 0.$$

Let us now consider the fourth term,

$$-\int_0^t \int \frac{\nu}{1+\varepsilon\eta_\varepsilon}\Delta\left(\exp\left(\frac{it'\lambda}{\varepsilon}\right)u_\varepsilon\right)\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt'.$$

It is here that the presence of capillarity enables us to get a better control. Let us write

$$\begin{aligned}&-\int_0^t \int \frac{\nu}{1+\varepsilon\eta_\varepsilon}\Delta\left(\exp\left(\frac{it'\lambda}{\varepsilon}\right)u_\varepsilon\right)\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt' \\ &= \nu\int_0^t \int \nabla\left(\exp\left(\frac{it'\lambda}{\varepsilon}\right)u_\varepsilon\right) : \nabla(\bar\Phi^*_\lambda)'(t',x)dxdt' \\ &- \nu\int_0^t \int \nabla\left(\exp\left(\frac{it'\lambda}{\varepsilon}\right)u_\varepsilon\right) : \nabla\left(\frac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon}(\bar\Phi^*_\lambda)'\right)(t',x)dxdt'.\end{aligned}$$

Clearly the first term on the right-hand side converges towards the expected limit: we have

$$\nu\int_0^t \int \nabla\left(\exp\left(\frac{it'\lambda}{\varepsilon}\right)u_\varepsilon\right) : \nabla(\bar\Phi^*_\lambda)'(t',x)dxdt' \to \nu\int_0^t \int \nabla\Phi'_\lambda : \nabla(\bar\Phi^*_\lambda)'(t',x)dxdt', \quad \text{as } \varepsilon \to 0.$$

To study the second one, we can notice that

$$\nabla\left(\frac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon}(\bar\Phi^*_\lambda)'\right)=\nabla\left(\frac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon}\right)(\bar\Phi^*_\lambda)'+\frac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon}\nabla(\bar\Phi^*_\lambda)',$$

and since the second term on the right-hand side is obviously easier to study than the first one, let us concentrate on the first term. We have

$$\nabla\frac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon}=\frac{\varepsilon\nabla\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon}-\frac{\varepsilon^2\eta_\varepsilon\nabla\eta_\varepsilon}{(1+\varepsilon\eta_\varepsilon)^2}.$$

Since $\varepsilon\eta_\varepsilon$ is bounded in $L^\infty(\mathbf{R}^+;H^{2\alpha}(\mathbf{R}\times\mathbf{T}))$, we infer easily, by product laws in Sobolev spaces, that

$$\varepsilon^2\eta_\varepsilon\nabla\eta_\varepsilon\quad\text{is bounded in}\quad L^\infty(\mathbf{R}^+;H^\sigma(\mathbf{R}\times\mathbf{T})),\ \text{for some}\ \sigma>0.$$

But on the other hand η_ε is bounded in $L^\infty(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T}))$, so we have also

$$\varepsilon^2\eta_\varepsilon\nabla\eta_\varepsilon\to 0\quad\text{in}\quad L^\infty(\mathbf{R}^+;H^{2\alpha-2}(\mathbf{R}\times\mathbf{T})).$$

By interpolation we gather that

$$\varepsilon^2\eta_\varepsilon\nabla\eta_\varepsilon\to 0\quad\text{in}\quad L^\infty(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T})),$$

and the lower bound on $1+\varepsilon\eta_\varepsilon$ ensures that

$$\frac{\varepsilon^2\eta_\varepsilon\nabla\eta_\varepsilon}{(1+\varepsilon\eta_\varepsilon)^2}\to 0\quad\text{in}\quad L^\infty(\mathbf{R}^+;L^2(\mathbf{R}\times\mathbf{T})).$$

The argument is similar (and easier) for the term $\dfrac{\varepsilon\nabla\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon}$, so we can conclude that

$$-\int_0^t\int\frac{\nu}{1+\varepsilon\eta_\varepsilon}\Delta\left(\exp\left(\frac{it'\lambda}{\varepsilon}\right)u_\varepsilon\right)\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt'\to\nu\int_0^t\int\nabla\Phi_\lambda:\nabla(\bar\Phi^*_\lambda)'(t',x)dxdt'.$$

Finally we are left with the nonlinear terms: let us study the limit of

$$\int_0^t\int\exp\left(\frac{it'\lambda}{\varepsilon}\right)(u_\varepsilon\cdot\nabla)u_\varepsilon\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt'-\int_0^t\int\exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon u_\varepsilon\nabla\cdot\bar\Phi^*_{\lambda,0}(t',x)dxdt'.$$

The study is very similar to the case studied in Section 4.3.3 (see Proposition 4.3.6), so we will not give all the details but merely point out the differences. First, one can truncate the integral in $x_1\in\mathbf{R}$ to $x_1\in[-R,R]$, where R is a parameter to be chosen large enough in the end. As previously that is simply due to the decay of the eigenvectors of L at infinity. So we are reduced to the study of

$$\int_0^t\int_{[-R,R]\times\mathbf{T}}\exp\left(\frac{it'\lambda}{\varepsilon}\right)(u_\varepsilon\cdot\nabla)u_\varepsilon\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt'\quad\text{and}$$

$$-\int_0^t\int_{[-R,R]\times\mathbf{T}}\exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon u_\varepsilon\cdot\nabla\bar\Phi^*_{\lambda,0}(t',x)dxdt'.$$

The limit of the first term is obtained in an identical way to Section 4.3.3, since u_ε satisfies the same bounds, so we have

$$\int_0^t\int_{[-R,R]\times\mathbf{T}}\exp\left(\frac{it'\lambda}{\varepsilon}\right)(u_\varepsilon\cdot\nabla)u_\varepsilon\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt'\to\int_0^t\int\sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}}(\Phi'_\mu\cdot\nabla)\Phi'_{\tilde\mu}\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt',$$

as ε goes to 0 and R goes to infinity.

Now let us concentrate on the last nonlinear term. With the notation of Section 4.3.3, we can write

$$\begin{aligned} &\int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_\varepsilon u_\varepsilon \cdot \nabla\bar{\Phi}^*_{\lambda,0}(t',x)dxdt' \\ &= \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) (\eta_\varepsilon - \eta_{\varepsilon,N}) u_\varepsilon \cdot \nabla\bar{\Phi}^*_{\lambda,0}(t',x)dxdt' \\ &+ \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_{\varepsilon,N} (u_\varepsilon - u_{\varepsilon,M}) \cdot \nabla\bar{\Phi}^*_{\lambda,0}(t',x)dxdt' \\ &+ \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_{\varepsilon,N} u_{\varepsilon,M} \cdot \nabla\bar{\Phi}^*_{\lambda,0}(t',x)dxdt'. \end{aligned}$$

The first two terms on the right-hand side converge to zero, due to the following estimates: for some $-1 < s < 0$ and for all $t \in [0,T]$,

$$\begin{aligned} &\left| \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) (\eta_\varepsilon - \eta_{\varepsilon,N}) u_\varepsilon \cdot \nabla\bar{\Phi}^*_{\lambda,0}(t',x)dxdt' \right| \\ &\qquad \le C_T \|\eta_\varepsilon - \eta_{\varepsilon,N}\|_{L^\infty([0,T];H^s([-R,R]\times\mathbf{T}))} \|u_\varepsilon\|_{L^2([0,T];H^1([-R,R]\times\mathbf{T}))} \|\bar{\Phi}^*_\lambda\|_{W^{2,\infty}(\mathbf{R}\times\mathbf{T})}, \end{aligned}$$

and similarly

$$\begin{aligned} &\left| \int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_{\varepsilon,N} (u_\varepsilon - u_{\varepsilon,M}) \cdot \nabla\bar{\Phi}^*_{\lambda,0}(t',x)dxdt' \right| \\ &\qquad \le C_{T,N} \|u_\varepsilon - u_{\varepsilon,M}\|_{L^2([0,T];H^s([-R,R]\times\mathbf{T}))} \|\bar{\Phi}^*_\lambda\|_{W^{2,\infty}(\mathbf{R}\times\mathbf{T})}. \end{aligned}$$

Finally the limit of the third term is obtained by the (by now) classical nonstationary phase theorem, namely we find, exactly as in the proof of Proposition 4.3.6, that

$$\int_0^t \int_{[-R,R]\times\mathbf{T}} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_{\varepsilon,N} u_{\varepsilon,M} \cdot \nabla\bar{\Phi}^*_{\lambda,0}(t',x)dxdt' \to \int_0^t \int \sum_{\substack{\mu+\tilde{\mu}=\lambda \\ i\mu, i\tilde{\mu}\in\mathfrak{S}}} \Phi_{\mu,0}\Phi'_{\tilde{\mu}} \cdot \nabla\bar{\Phi}^*_{\lambda,0}(t',x)dxdt',$$

as ε goes to 0 and M, N and R go to infinity.

That concludes the proof of the theorem. □

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NOTATION INDEX

Δ', diffusion operator in the Saint-Venant system, p. 18

Δ'_L, diffusion operator in the limit filtered system, p. 19

$\mathcal{E}_\varepsilon(t)$, modulated energy, p. 73

$\mathcal{E}_{\varepsilon,N}(t)$, modulated energy applied to a sequence of approximate solutions, p. 75

$\mathcal{F}$, Fourier transform, p. 5

$\mathcal{F}_2$, Fourier transform with respect to x_2, p. 9

$\widehat{f}(n,k)$, coefficients of f in the Hermite-Fourier basis, p. 9

H^s, inhomogeneous Sobolev space, p. 5

$\dot{H}^s$, homogeneous Sobolev space, p. 5

$H^s_0(\Omega)$, for $s>0$, Sobolev space on a bounded set Ω with Dirichlet boundary conditions, p. 5

$H^{-s}(\Omega)$, for $s>0$, dual space of $H^s_0(\Omega)$, p. 5

H^s_L, weighted Sobolev space, p. 30

J_N, spectral truncation, p. 67

K_N, truncation operator in the $(\Psi_{n,k,j})$ basis, p. 40

κ, a regularizing kernel, p. 60

$\ell^{p,\infty}$, weakly convergent series, p. 35

L, singular perturbation, p. 7

$\mathcal{L}$, semi-group generated by L, p. 18

N_s, pseudo-differential operator on Ker$L$, p. 44

$\Pi_{n,k,j}$, projection on $\Psi_{n,k,j}$, p. 17

Π_λ, projection on $\mathrm{Ker}(L - i\lambda Id)$, p. 17

Π_0, projection on $\mathrm{Ker}L$, p. 8

$\Pi_\perp$, projection on $(\mathrm{Ker}L)^\perp$, p. 31

Π_K, projection on Kelvin modes, p. 16

Π_M, projection on mixed Rossby-Poincaré modes, p. 16

Π_P, projection on Poincaré modes, p. 16

Π_R, projection on Rossby modes, p. 16

Q, quadratic operator in the Saint-Venant system, p. 18

Q_L, quadratic operator in the limit filtered system, p. 19

$\tilde{Q}_L$, purely ageostrophic quadratic operator in the limit filtered system, p. 36

S, set of indices (n, k, j), p. 30

S^*, set of indices (n, k, j) corresponding to $(\mathrm{Ker}L)^\perp$, p. 35

(SW_ε), the shallow water system, p. 57

(SW_0), the limit system after filtering, p. 57

$\mathfrak{S}$, spectrum of L, p. 16

$\mathfrak{S}_K$, subset of $\mathfrak{S}$ corresponding to Kelvin modes, p. 16

$\mathfrak{S}_P$, subset of $\mathfrak{S}$ corresponding to Poincaré modes, p. 16

$\mathfrak{S}_R$, subset of $\mathfrak{S}$ corresponding to Rossby modes, p. 16

$\mathfrak{S}_N$, subset of $\mathfrak{S}$ defined by a frequency truncation, p. 68

$i\tau_{n,k,j}$, eigenvalues of L, p. 9

$W^{s,\infty}$, Sobolev space, p. 12

Φ_0, first coordinate of the three component vector field Φ, p. 18

$\Phi' = (\Phi_1, \Phi_2)$, two last coordinates of the three component vector field Φ, p. 18

${\Phi'}^\perp = (\Phi_2, -\Phi_1)$, image of Φ' by a rotation of angle $\pi/2$, p. 3

$\bar{\Phi}$, complex conjugate of Φ, p. 26

$\Phi_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon)$, where $(\eta_\varepsilon, u_\varepsilon)$ solves the Saint-Venant system, p. 78

Φ, a solution to the limit system (SW_0), p. 78

Φ_λ, an element of $\mathrm{Ker}(L - i\lambda Id)$, p. 80

Φ_N, an approximation of Φ, p. 67

$\Phi_{(N)}$, an approximate Leray solution of (SW_0), p. 40

$\Phi_{\varepsilon,N}$, an approximation of Φ_ε, p. 85

$\Phi_{\varepsilon,\lambda,N}$, the projection of $\Phi_{\varepsilon,N}$ onto $\mathrm{Ker}(L - i\lambda Id)$, p. 85

$\underline{\Phi}$, a solution to the geostrophic equation, p. 43

$\underline{\varphi}_n$, coefficients of $\underline{\Phi} \in \mathrm{Ker} L$ in the $(\Psi_{n,0,0})$ basis, p. 43

ϕ_N, a corrector to Φ_N, p. 69

$\varphi_{n,k,j}$, coefficients of Φ in the $(\Psi_{n,k,j})$ basis, p. 15

ψ_n, Hermite functions, p. 8

$\Psi_{n,k,j}$, eigenvectors of L, p. 10–11