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The upper envelope of positive self-similar Markov processes.

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Abstract: We establish integral tests and laws of the iterated logarithm at 0 and at $+\infty$, for the upper envelope of positive self-similar Markov processes. Our arguments are based on the Lamperti representation, time reversal arguments and on the study of the upper envelope of their future infimum due to Pardo [19]. These results extend integral test and laws of the iterated logarithm for Bessel processes due to Dvoretsky and Erdös [10] and stable Lévy processes conditioned to stay positive with no positive jumps due to Bertoin [1].

Key words: Self-similar Markov process, Self-similar additive processes, Future infimum process, Lévy process, Lamperti representation, First and last passage time, integral test, law of the iterated logarithm.

A.M.S. Classification: 60 G 18, 60 G 17, 60 G 51, 60 F 15.

1 Introduction.

A real Markov process $X = (X_t, t \geq 0)$ with càdlàg paths is a self-similar process if for every $k > 0$ and every initial state $x \geq 0$ it satisfies the scaling property, i.e., for some $\alpha > 0$

$$\text{the law of } (kX_{k^{-\alpha}t}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{kx},$$

where $\mathbb{P}_x$ denotes the law of the process X starting from $x \geq 0$.

In this article, we focus on positive self-similar Markov processes and we will refer to them as *pssMp*. We denote by $X^{(x)}$ or $(X, \mathbb{P}_x)$ for the pssMp starting from $x \geq 0$. Well-known examples of pssMp are: Bessel processes and stable subordinators or more generally, stable processes conditioned to stay positive.

Self-similar processes were the object of a systematic study first done by Lamperti [17]. In a later work, Lamperti [18] studied the markovian case in detail. According to Lamperti [18] any pssMp starting from a strictly positive state satisfies one of the following conditions:

i) it never reaches the state 0,

ii) it hits the state 0 continuously or

iii) it hits the state 0 by a negative jump.

The main result in [18] prove that any pssMp starting from a strictly positive state is a time-change of the exponential of a Lévy process. More precisely, let $X^{(x)}$ be a self-similar Markov process started from $x > 0$ that fulfills the scaling property for some $\alpha > 0$, then

$$X_t^{(x)} = x \exp\left\{\xi_{\tau(tx^{-\alpha})}\right\}, \qquad 0 \leq t \leq x^\alpha I(\xi), \tag{1.1}$$

where

$$\tau_t = \inf\left\{s \geq 0 : I_s(\xi) > t\right\}, \qquad I_s(\xi) = \int_0^s \exp\left\{\alpha\xi_u\right\}\mathrm{d}u, \qquad I(\xi) = \lim_{t\to+\infty} I_t(\xi),$$

and ξ is either a real Lévy process which drifts towards $-\infty$, if $X^{(x)}$ satisfies condition (ii) or ξ is a Lévy process killed at an independent exponential time if $X^{(x)}$ satisfies condition (iii) or ξ is a Lévy process which does not drift towards $-\infty$, if $X^{(x)}$ satisfies condition (i). This is the well-known Lamperti representation.

Several authors have studied the problem of when an entrance law at 0 for $(X, \mathbb{P}_x)$ can be defined, see for instance Bertoin and Caballero [3], Bertoin and Yor [4], and Caballero and Chaumont [7]. Bertoin and Caballero [3] studied for the first time this problem for the increasing case. Later Bertoin and Yor [4] generalized the results obtained in [3]. The main result of [4] proves that this limiting process is also a pssMp, and that it has the same semigroup as $(X, \mathbb{P}_x)$ for $x > 0$. Bertoin and Yor [4] also gave sufficient conditions for the weak convergence of $\mathbb{P}_x$ to hold when x tends to 0, in the sense of finite dimensional distributions. The entrance law was also computed in the mentioned works, such law will be written below in (2.10).

Caballero and Chaumont [7] gave necessary and sufficient conditions for the weak convergence of $X^{(x)}$ on the Skorokhod's space. Caballero and Chaumont also gave a path construction of this weak limit, that we will denote by $X^{(0)}$.

The aim of this work is to describe the upper envelope at 0 and at $+\infty$ for a large class of pssMp satisfying condition (i) through integral test and laws of the iterated logarithm. We will give special attention to the case with no positive jumps since we may obtain general integral tests and compare the rate of growth of $X^{(x)}$ with that of its future infimum process and the pssMp $X^{(x)}$ reflected at its future infimum.

Several partial results on the upper envelope of $X^{(0)}$ have already been established before, the most important of which being due to Dvoretsky and Erdös [10] who studied the case of Bessel process. More precisely, we have the well known Kolmogorov and the Dvoretsky-Erdös integral test, see for instance Itô and McKean [13].

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Theorem 1 (Kolmogorov-Dvoretsky-Erdös) *Let h be a nondecreasing, positive and unbounded function as t goes to $+\infty$. Then the upper envelope of $X^{(0)}$, a Bessel process with index $\delta \geq 2$, at 0 is as follows,*

$$\mathbb{P}\big(X_t^{(0)} > \sqrt{t}h(t), \ i.o., \ as \ t \to 0\big) = 0 \ or \ 1,$$

according as,

$$\int_0 h^\delta(t) \exp\{-h^2(t)/2\}\frac{\mathrm{d}t}{t} \qquad is \ finite \ or \ infinite.$$

Thanks to the time inversion property of Bessel processes, we also have the integral test for large times. To get it, it is enough to replace

$$\int_0 h^\delta(t) \exp\{-h^2(t)/2\}\frac{\mathrm{d}t}{t} \quad by \quad \int^{+\infty} h^\delta(t) \exp\{-h^2(t)/2\}\frac{\mathrm{d}t}{t}.$$

It is important to note that this integral test is also valid for $d \geq 1$. The integral test for transient Bessel process will be extended in section 4 (Theorems 2 and 3). Additionally in section 7, we establish a variant of the Kolmogorov-Dvoretsky-Erdös integral test, (Theorems 11 and 12).

From the Kolmogorov-Dvoretsky-Erdös integral test, we deduce the following laws of the iterated logarithm,

$$\limsup_{t \to 0} \frac{X_t^{(0)}}{\sqrt{2t \log|\log t|}} = 1 \quad and \quad \limsup_{t \to +\infty} \frac{X_t^{(0)}}{\sqrt{2t \log\log t}} = 1, \quad a. \ s. \qquad (1.2)$$

Recall that the future infimum of a pssMp starting at $x \geq 0$ is defined by

$$J_s^{(x)} = \inf_{t \geq s} X_t^{(x)}.$$

Khohsnevisan, Lewis and Wembo [16] studied the asymptotic behaviour of the future infimum of transient Bessel processes. Khoshnevisan et al. [16] established the following law of the iterated logarithm for $J^{(0)}$ and for the Bessel process reflected at its future infimum, $X^{(0)} - J^{(0)}$;

$$\limsup_{t \to +\infty} \frac{J_t^{(0)}}{\sqrt{2t \log|\log t|}} = 1, \qquad almost \ surely, \qquad (1.3)$$

and

$$\limsup_{t \to +\infty} \frac{X_t^{(0)} - J_t^{(0)}}{\sqrt{2t \log|\log t|}} = 1, \qquad almost \ surely. \qquad (1.4)$$

In section 6, we extend these results and we also study the small time behaviour.

When $X^{(0)}$ is a stable Lévy process conditioned to stay positive with no positive jumps and with index $1 < \alpha \leq 2$, we have the following law of the iterated logarithm due to Bertoin [1],

$$\limsup_{t \to 0} \frac{X_t^{(0)}}{t^{1/\alpha}\big(\log|\log t|\big)^{1-1/\alpha}} = c(\alpha) \qquad almost \ surely, \qquad (1.5)$$

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where $c(\alpha)$ is a positive constant.

Recently in [19] we studied the asymptotic behaviour of the upper envelope of the future infimum of pssMp under general hyphotesis. In particular, the author proved that when $X^{(0)}$ is a stable Lévy process conditioned to stay positive with no positive jumps and with index $1 < \alpha \leq 2$, we have

$$\limsup_{t \to 0} \frac{J_t^{(0)}}{t^{1/\alpha} \left(\log |\log t| \right)^{1-1/\alpha}} = c(\alpha), \qquad \text{almost surely,} \qquad (1.6)$$

and that

$$\limsup_{t \to +\infty} \frac{J_t^{(0)}}{t^{1/\alpha} \left(\log \log t \right)^{1-1/\alpha}} = c(\alpha), \qquad \text{almost surely.} \qquad (1.7)$$

In section 6, we will see that under the assumption that $\mathbb{P}(J_1^{(0)} > t)$ is log-regular,i.e.

$$- \log \mathbb{P}\left(J_1^{(0)} > t\right) \sim \lambda t^\beta L(t) \quad \text{as} \quad t \to +\infty,$$

where $\lambda, \beta > 0$ and L is a function which varies slowly at $+\infty$, the upper envelope of $X^{(x)}$ will be described by an explicit law of the iterated logarithm and that it agrees with the upper envelope of its future infimum as we have seen above.

All the asymptotic results presented in section 5 and 6 are consequences of general integral tests which are stated in sections 3 and 4.

The rest of this paper is organized in six sections. Section 2 is devoted to some preliminaries of Lévy processes and pssMp. In section 3, we study the asymptotic properties of the first and last passage time processes of $X^{(0)}$. In section 4, we give the general integral tests for the upper envelope of $X^{(x)}$. Sections 5 and 6 are devoted to applications of the results of sections 3 and 4. In section 7, we will apply our results to Bessel processes and finally in section 8, we will give some examples.

2 Preliminaries.

2.1 Weak convergence and entrance law of pssMp.

Let $\mathcal{D}$ denote the space of Skorokhod of càdlàg paths. We consider a probability measure on $\mathcal{D}$ denoted by $\mathbb{P}$ under which ξ will always be a real Lévy process such that $\xi_0 = 0$. Let Π be the Lévy measure of ξ, that is the measure satisfying

$$\int_{(-\infty,\infty)} (1 \wedge x^2) \Pi(\mathrm{d}x) < \infty,$$

and such that the characteristic exponent Ψ, defined by

$$\mathbb{E}\left(e^{iu\xi_t}\right) = e^{-t\Psi(u)}, \qquad t \geq 0, u \in \mathbb{R}$$

is given, for some $b \geq 0$ and $a \in \mathbb{R}$, by

$$\Psi(u) = iau + \frac{1}{2}b^2 u^2 + \int_{(-\infty,\infty)} \left(1 - e^{iux} + iux \mathbb{1}_{\{|x| \leq 1\}}\right) \Pi(\mathrm{d}x), \quad u \in \mathbb{R}.$$

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Define for $x \geq 0$,

$$\overline{\Pi}^+(x) = \Pi\big((x, \infty)\big), \quad \overline{\Pi}^-(x) = \Pi\big((-\infty, -x)\big), \quad M(x) = \int_0^x \mathrm{d}y \int_y^\infty \overline{\Pi}^-(z)\mathrm{d}z,$$

and

$$J = \int_{[1,\infty)} \frac{x\overline{\Pi}^+(x)}{1 + M(x)}\mathrm{d}x.$$

Then according to Caballero and Chaumont [7], necessary and sufficient conditions for the weak convergence of $X^{(x)}$ on the Skorokhod's space are

$$\textbf{(H)} \quad \xi \text{ is not arithmetic and} \quad \begin{cases} \text{either } 0 < \mathbb{E}(\xi_1) \leq \mathbb{E}(|\xi_1|) < \infty, \\ \text{or } \mathbb{E}(|\xi_1|) < \infty, \mathbb{E}(\xi_1) = 0 \text{ and } J < \infty, \end{cases}$$

and

$$\mathbb{E}\left(\log^+ \int_0^{T_1} \exp\{\alpha\xi_s\}\mathrm{d}s\right) < \infty, \tag{2.8}$$

where T_x is the first passage time above $x \geq 0$, i.e. $T_x = \inf\{t \geq 0 : \xi_t \geq x\}$.

The weak limit found in [7], denoted by $X^{(0)}$, is a pssMp starting from 0 which fulfills the Feller property on $[0, \infty)$ and with the same transition function as $X^{(x)}$, $x > 0$. In all the sequel, we suppose that the Lévy processes considered here satisfy conditions **(H)** and (2.8). We will distinguish the case $0 < \mathbb{E}(\xi_1) \leq \mathbb{E}(|\xi_1|) < \infty$ saying that $m := \mathbb{E}(\xi_1) > 0$.

It is important to note that if $m > 0$, we have that $\mathbb{E}(T_1) < \infty$ and hence condition (2.8) is satisfied. Another example when this condition is satisfied is when ξ has no positive jumps (see section 2 in [7]).

We denote by $\mathbb{P}_x$ the law, under $\mathbb{P}$, of a self-similar Markov process $X^{(x)}$ starting from $x > 0$ and by $\mathbb{P}_0$ the law, under $\mathbb{P}$, of the limiting process $X^{(0)}$. When $m > 0$ (i.e. ξ has positive drift) the entrance law under $\mathbb{P}_0$ has been computed by Bertoin and Caballero [3] and Bertoin and Yor [4] and can be expressed as follows: for every $t > 0$ and every measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\mathbb{E}_0\big(f(X_t)\big) = \frac{1}{m}\mathbb{E}\left(I^{-1}(\hat{\xi})f\left(tI^{-1}(\hat{\xi})\right)\right) \qquad \text{for } t > 0, \tag{2.9}$$

where

$$I(\hat{\xi}) = \int_0^{+\infty} \exp\{\alpha\hat{\xi}_s\}\mathrm{d}s.$$

When $m = 0$ there is no explicit form for the entrance law of $X^{(0)}$ in terms of the underlying Lévy process. However, Caballero and Chaumont [7] proved that it can be obtained as the weak limit of the entrance law for the positive drift case, when the drift tends towards 0. More precisely, for any bounded and continuous function f,

$$\mathbb{E}_0\big(f(X_t)\big) = \lim_{\lambda \to 0} \frac{1}{\lambda}\mathbb{E}\left(I_\lambda^{-1}f\left(tI_\lambda^{-1}\right)\right) \qquad \text{for } t > 0, \tag{2.10}$$

where

$$I_\lambda = \int_0^{+\infty} \exp\Big\{ -\alpha(\xi_s - \lambda s)\Big\} \mathrm{d}s.$$

Now, we introduce the so-called first and last passage times of $X^{(0)}$ by

$$S_y = \inf\Big\{ t \ge 0 : X_t^{(0)} \ge y \Big\} \quad \text{and} \quad U_y = \sup\Big\{ t \ge 0 : X_t^{(0)} \le y \Big\},$$

for $y > 0$. Note that when $m > 0$, i.e. that the process $X^{(0)}$ drifts to $+\infty$, the last passage time U_x is finite a.s. for $x \ge 0$. Moreover, if the process $X^{(0)}$ satisfies the scaling property with some index $\alpha > 0$, then by the definition of S_x and U_x, we deduce that the first passage time process $S = (S_x, x \ge 0)$ and the last passage time process $U = (U_x, x \ge 0)$ are increasing self-similar processes with scaling index α^{-1}. From the path properties of $X^{(0)}$ we easily see that both processes start from 0 and go to $+\infty$ as x increases.
When $m = 0$, the last passage time U_x is no longer a.s. finite but the first passage time process is still self-similar, starts from 0 and goes to $+\infty$ as x increases.

With no loss of generality, we will suppose that $\alpha = 1$. Indeed, we see from the scaling property that if $X^{(x)}$, $x \ge 0$, is a pssMp with index $\alpha > 0$, then $\big(X^{(x)}\big)^\alpha$ is a pssMp with index equal to 1. Therefore, the integral tests and LIL established in the sequel can easily be interpreted for any $\alpha > 0$.

2.2 PssMp with no positive jumps and some path transformations.

Here, we suppose that ξ has no positive jumps. It is known that under this assumption, the process ξ has finite exponential moments of arbitrary positive order (see [2] for background). In particular, we have that

$$\mathbb{E}\big(\exp\{u\xi_t\}\big) = \exp\{t\psi(u)\}, \qquad u \ge 0$$

where ψ is defined by,

$$\psi(u) = au + \frac{1}{2}\sigma^2 u^2 + \int_{]-\infty,0[} \big(e^{ux} - 1 - ux \mathbb{1}_{\{x>-1\}}\big)\Pi(\mathrm{d}x), \quad u \ge 0.$$

It is important to note that the condition that ξ does not derive towards $-\infty$, is equivalent to

$$m = \psi'(0+) \in [0,\infty[.$$

We recall Caballero and Chaumont's construction of $X^{(0)}$, only in this particular case. In this direction, let $T_z = \inf\{s \ge 0 : \xi_s \ge z\}$ be the first time where the process ξ reaches the state $z \ge 0$. Note that due to the absence of positive jumps and since the process ξ does not derive towards $-\infty$, then for all $z \ge 0$

$$T_z < \infty \quad \text{and} \quad \xi_{T_z} = z, \quad \mathbb{P}-\text{a.s.}$$

Let $x_1 \geq x_2 \geq \cdots > 0$, be an infinite decreasing sequence of strictly positive real numbers which converges to 0 and $(\xi^{(n)}, n \geq 1)$ a sequence of random processes which are independent and have the same distribution as ξ.

From the sequences (x_n) and $(\xi^{(n)}, n \geq 1)$, and by Lamperti's transformation (1.1) we may define a sequence of pssMp. More precisely,

$$X_t^{(x_n)} = x_n \exp\left\{\xi_{\tau^{(n)}(t/x_n)}^{(n)}\right\}, \quad t \geq 0, \quad , n \geq 1, \tag{2.11}$$

where

$$\tau^{(n)}(t) = \inf\left\{s \geq 0 : I_s(\xi^{(n)}) > t\right\}, \qquad I_t(\xi^{(n)}) = \int_0^t \exp\left\{\xi_u^{(n)}\right\}\mathrm{d}u.$$

For each $n \geq 2$, we also define the first time in which the process $X^{(x_n)}$ reaches the state x_{n-1}, i.e.

$$S^{(n)} = \inf\left\{t \geq 0 : X_t^{(x_n)} \geq x_{n-1}\right\}.$$

It is clear from (2.11) that $S^{(n)} = x_n I_{T^{(n)}}(\xi^{(n)})$, where $T^{(n)}$ is the first passage time where the process $\xi^{(n)}$ reaches the state $\log(x_{n-1}/x_n)$, i.e.

$$T^{(n)} = \inf\{t \geq 0 : \xi^{(n)} \geq \log(x_{n-1}/x_n)\}.$$

Now from the sequences $(X^{(x_n)}, n \geq 1)$ and $(S^{(n)}, n \geq 2)$, we can construct the process $X^{(0)}$ as the concatenation of the processes $X^{(x_n)}$ on each interval $[0, S^{(n)}]$, i.e.,

$$X_t^{(0)} = \begin{cases} X_{t-\Sigma_2}^{(x_1)} & \text{if } t \in [\Sigma_2, \infty[, \\ X_{t-\Sigma_3}^{(x_2)} & \text{if } t \in [\Sigma_3, \Sigma_2[, \\ \vdots \\ X_{t-\Sigma_{n+1}}^{(x_n)} & \text{if } t \in [\Sigma_{n+1}, \Sigma_n[, \\ \vdots \end{cases}$$

where $\Sigma_n = \sum_{k \geq n} S^{(k)}$. We can deduce, by the definition of the process $X^{(0)}$, that $\Sigma_n = \inf\left\{t \geq 0 : X_t^{(0)} \geq x_{n-1}\right\}$

Caballero and Chaumont proved that this construction makes sense, it does not depend on the sequence (x_n), and $X_0^{(0)} = 0$. As we mentioned before, $X^{(0)}$ is a càdlàg pssMp with the same semi-group as $(X, \mathbb{P}_x)$ for $x > 0$. In the general case such construction is more complicate since it depends also on the overshoots of the sequence of Lévy processes $(\xi^{(n)}, n \geq 1)$.

Note that the process $X^{(0)}$ inherits the path properties of the underlying Lévy process, hence $X^{(0)}$ does not have positive jumps and if $m > 0$ it drifts towards $+\infty$. Hence for all $x \geq 0$

$$S_x \text{ and } U_x \text{ are finite} \quad \text{and} \quad X_{S_x}^{(0)} = X_{U_x}^{(0)} = x, \quad \text{almost surely.}$$

From this construction and the path decomposition in Corollary 1 in Chaumont and Pardo [9], we deduce that the first passage time process $S = (S_x, x \geq 0)$ and the last

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passage time process $U = (U_x, x \geq 0)$ are increasing self-similar processes with independent increments, in fact we will prove at the end of this section that these processes are also self-decomposable. This property was studied by the first time by Getoor in [11] for the last passage time of a Bessel process of index $\delta \geq 3$ and later by Jeanblanc, Pitman and Yor [14] for $\delta > 2$. If $m = 0$, the process S is still an increasing self-similar process with independent increments and also self-decomposable.

We are interested in describing the law of the process $(X^{(0)}_{(S_x-t)^-}, 0 \leq t \leq S_x)$ and in obtaining the law of the first passage time in terms of the underlying Lévy process. With this purpose, we now briefly recall the definition of the Lévy process conditioned to stay positive $\xi^\uparrow$ and refer to [8] for a complete account on this subject. The process $\xi^\uparrow$ is an h-process of ξ killed when it first exists $(0, \infty)$, i.e. at time $R = \inf\{t : \xi_t \leq 0\}$. The law of this strong Markov process $\xi^\uparrow$ is defined by its semi-group:

$$\mathbb{P}(\xi^\uparrow_{t+s} \in dy \,|\, \xi^\uparrow_s = x) = \frac{h(y)}{h(x)}\mathbb{P}(\xi_t + x \in dy \,, t < R)\,, \quad s, t \geq 0, \quad x, y > 0$$

and its entrance law:

$$\mathbb{P}(\xi^\uparrow_t \in dx) = h(x)\hat{N}(\xi_t \in dx, \, t < \zeta)\,,$$

where $\hat{N}$ is the excursion measure of the reflected process $\xi - I = (\xi_t - \inf_{s \leq t} \xi_s, t \geq 0)$, ζ is the lifetime of the generic excursion and h is the positive harmonic function (for ξ killed at time R) which is defined by:

$$h(x) = \mathbb{E}\left(\int_0^\infty \mathbb{1}_{\{I_t \geq -x\}}\, dL_t\right)\,, \quad x \geq 0\,,$$

where $I_t = \inf_{s \leq t} \xi_s$ and L is the local time of the reflected process $\xi - I$.
Note that $\xi^\uparrow$ has no positive jumps and that almost surely

$$\lim_{t \downarrow 0} \xi^\uparrow_t = 0, \qquad \lim_{t \uparrow +\infty} \xi^\uparrow_t = +\infty, \quad \text{and} \quad \xi^\uparrow_t > 0 \quad \text{for all } t > 0.$$

The following time reversal property of $\xi^\uparrow$ is an important tool for our next result, its proof can be found in Theorem VII.18 in Bertoin [2]

Lemma 1 *The law of $(x - \xi_t, 0 \leq t \leq T_x)$ is the same as that of the time-reversed process $(\xi^\uparrow_{(\gamma^\uparrow(x)-t)^-}, 0 \leq t \leq \gamma^\uparrow(x))$, where $\gamma^\uparrow(x) = \sup\{t \geq 0, \xi^\uparrow_t \leq x\}$. Moreover, for every $x > 0$, the process $\left(\xi^\uparrow_t, 0 \leq t \leq \gamma^\uparrow(x)\right)$ is independent of $\left(\xi^\uparrow_{\gamma^\uparrow(x)+t} - x, t \geq 0\right)$, and the latter has the same law as that of the process $\xi^\uparrow$.*

Now, for every $y > 0$ let us define

$$\tilde{X}^{(y)}_t = y \exp\left\{-\xi^\uparrow_{\tau^\uparrow(t/y)}\right\} \qquad t \geq 0,$$

where

$$\tau^\uparrow_t = \inf\left\{s \geq 0 : I_s\big(-\xi^\uparrow\big) > t\right\}, \quad \text{and} \quad I_s\big(-\xi^\uparrow\big) = \int_0^s \exp\left\{-\xi^\uparrow_u\right\}du.$$

We denote by $\tilde{\mathbb{P}}_y$, the law of $\tilde{X}^{(y)}$. Since $\xi^\uparrow$ derives towards $+\infty$, we deduce that $\tilde{X}^{(y)}$, reaches 0 at an almost surely finite random time, denoted by $\tilde{\rho}_y = \inf\{t \geq 0, \tilde{X}^{(y)}_t = 0\}$.

Proposition 1 *Suppose $m \geq 0$. The law of the process time-reversed at its first passage time below x, $\big(X^{(0)}_{(S_x - t)-}, 0 \leq t \leq S_x\big)$ is the same as that of the process $(\tilde{X}^{(x)}_t, 0 \leq t \leq \tilde{\rho}_x)$.*

Proof: Let us take any decreasing sequence (x_n) of positive real numbers which converges to 0 and such that $x_1 = x$. By Lemma 1, we can split $(\tilde{X}^{(x)}_t, 0 \leq t \leq \tilde{\rho})$ into the sequence

$$\Big(x_1 \exp\big\{-\xi^{\uparrow}_{\tau^{\uparrow}(t/x_1)}\big\}, x_1 I_{\gamma^{(n)}}\big(-\xi^{\uparrow}\big) \leq t \leq x_1 I_{\gamma^{(n+1)}}\big(-\xi^{\uparrow}\big)\Big), \quad n \geq 1,$$

where $\gamma^{(n)} = \sup\{t \geq 0 : \xi^{\uparrow}_t \leq \log x_n/x\}$.
Then to prove this result, it is enough to show that, for each $n \geq 1$

$$\Big(X^{(0)}_{(S_{x_n}-t)-}, 0 \leq t \leq S^{(n)}\Big) \overset{(d)}{=} \Big(x \exp\big\{-\xi^{\uparrow}_{\tau^{\uparrow}(t/x_1)}\big\}, x I_{\gamma^{(n)}}\big(-\xi^{\uparrow}\big) \leq t \leq x I_{\gamma^{(n+1)}}\big(-\xi^{\uparrow}\big)\Big)$$

where $S^{(n)} = S_{x_n} - S_{x_{n+1}}$.
Fix $n \geq 1$, from the Caballero and Chaumont's construction, we know that the left-hand side of the above identity has the same law as

$$\Big(x_{n+1} \exp\big\{\xi^{(n+1)}_{\tau^{(n+1)}\big(I_{T^{(n+1)}}\big(\xi^{(n+1)}\big)-t/x_{n+1}\big)}\big\}, 0 \leq t \leq x_{n+1} I_{T^{(n+1)}}\big(\xi^{(n+1)}\big)\Big). \tag{2.12}$$

On the other hand by Lemma 1, we know that $(-\xi^{\uparrow}_t, 0 \leq t \leq \gamma^{(n)})$ is independent of $\xi^{\uparrow(n)} = (\log(x/x_n) - \xi^{\uparrow}_{\gamma^{(n)}+t}, t \geq 0)$ and that the latter has the same law as $-\xi^{\uparrow}$. Since

$$\tau^{\uparrow}\Big(I_{\gamma^{(n)}}\big(-\xi^{\uparrow}\big) + t/x\Big) = \gamma^{(n)} + \inf\Big\{s \geq 0 : \int_0^s \exp\big\{-\xi^{\uparrow(n)}_u\big\} du \geq t/x_n\Big\},$$

it is clear that the right-hand side of the above identity in distribution has the same law as,

$$\Big(x_n \exp\big\{-\xi^{\uparrow}_{\tau^{\uparrow}(t/x_n)}\big\}, 0 \leq t \leq x_n I_{\gamma^{\uparrow}(\log(x_n/x_{n+1}))}\big(-\xi^{\uparrow}\big)\Big). \tag{2.13}$$

Therefore, it is enough to show that (2.12) and (2.13) have the same distribution.
Now, let us define the exponential functional of $\big(\xi^{(n+1)}_{(T^{(n+1)}-t)-}, 0 \leq t \leq T^{(n+1)}\big)$ as follows,

$$B^{(n+1)}_s = \int_0^s \exp\big\{\xi^{(n+1)}_{T^{(n+1)}-u}\big\} du \qquad \text{for } s \in [0, T^{(n+1)}],$$

and $\mathcal{H}(t) = \inf\big\{0 \leq s \leq T^{(n+1)}, B^{(n+1)}_s > t\big\}$, the right continuous inverse of the exponential functional $B^{(n+1)}$.
By a change of variable, it is clear that $B^{(n+1)}_s = I_{T^{(n+1)}}\big(\xi^{(n+1)}\big) - I_{T^{(n+1)}-s}\big(\xi^{(n+1)}\big)$, and if we set $t = x_{n+1} B^{(n+1)}_s$, then $s = \mathcal{H}(t/x_{n+1})$ and hence

$$\tau^{(n+1)}\Big(I_{T^{(n+1)}}\big(\xi^{(n+1)}\big) - t/x_{n+1}\Big) = \tau^{(n+1)}\Big(I_{T^{(n+1)}-s}\big(\xi^{(n+1)}\big)\Big)$$
$$= T^{(n+1)} - \mathcal{H}(t/x_{n+1}).$$

Therefore, we can rewrite (2.12) as follows

$$\left(x_{n+1} \exp \left\{ \xi^{(n+1)}_{T^{(n+1)} - \mathcal{H}(t/x_{n+1})} \right\}, 0 \le t \le x_{n+1} B^{(n+1)}_{T^{(n+1)}} \right), \tag{2.14}$$

and applying Lemma 1, we get that (2.14) has the same law as that of the process defined in (2.13). ∎

An important consequence of this Proposition is the following time-reversed identity. For any $y < x$,

$$\left(X^{(0)}_{(S_x - t)-}, S_y \le t \le S_x \right) \overset{(d)}{=} \left(\tilde{X}^{(x)}_t, 0 \le t \le \tilde{U}_y \right),$$

where $\tilde{U}_y = \sup \left\{ t \ge 0, \tilde{X}^{(x)}_t \le y \right\}$.

Four the next result we need to introduce the concept of self-decomposable random variable.

Definition 1 *We say that a random variable X is self-decomposable if for every constant $0 < c < 1$ there exists a variable Y_c which is independent of X and such that $Y_c + cX$ has the same law as X.*

Corollary 1 *Let $m \ge 0$. For every $x > 0$ the law of S_x, the first passage time of the process $X^{(0)}$ above x, has the same law as $xI\left(-\xi^\uparrow \right)$, where*

$$I\left(-\xi^\uparrow \right) = \int_0^\infty \exp\{ -\xi^\uparrow_u \} \mathrm{d}u.$$

Proof: From Proposition 1, we see that S_x and $\tilde{\rho}_x$ have the same law. By the Lamperti representation of $\tilde{X}^{(x)}$, we deduce that $\tilde{\rho}_x = xI\left(-\xi^\uparrow \right)$ and then the identity in law follows.

Now, let $0 < c < 1$. From Lemma 1, we know that the killed process $(\xi^\uparrow_t, 0 \le t \le \gamma^\uparrow(\log(1/c)))$ is independent of $(\xi^\uparrow_{\gamma^\uparrow(\log(1/c))+t} + \log c, t \ge 0)$ and that the latter has the same as $\xi^\uparrow$, then

$$I\left(-\xi^\uparrow \right) = \int_0^{\gamma^\uparrow(\log(1/c))} \exp\{ -\xi^\uparrow_u \} \mathrm{d}u + c \int_0^{+\infty} \exp\left\{ -\xi^\uparrow_{\gamma^\uparrow(\log(1/c))+u} - \log c \right\} \mathrm{d}u,$$

the self-decomposability follows. ∎

Lemma 2 *Let $m > 0$. For every $x > 0$ the law of U_x, the last passage time of the process $X^{(0)}$ below x, has the same law as $xI\left(\hat{\xi} \right)$. Moreover, U_1 is self-decomposable.*

Proof: The first part of this Lemma is consequence of Proposition 1 in [9]. Let $0 < c < 1$. From the Markov property, we know that $(\xi_t, 0 \le t \le T_{\log(1/c)})$ is independent of $(\xi_{T_{\log(1/c)}+t} + \log c, t \ge 0)$ and that the latter has the same distribution as ξ, then

$$I\left(\hat{\xi} \right) = \int_0^{T_{\log(1/c)}} \exp\left\{ -\xi_u \right\} \mathrm{d}u + c \int_0^{+\infty} \exp\left\{ -\xi_{T_{\log(1/c)}+u} - \log c \right\} \mathrm{d}u,$$

the self-decomposability follows. ∎

3 The lower envelope of the first passage time.

In this section, we are interested in describing the lower envelope at 0 and at $+\infty$ of the first passage time of $X^{(0)}$ trough integral tests. We first deal with the case when $X^{(0)}$ has no positive jumps since in this case we may obtain an explicit integral tests in terms of the tail probability of $I(-\xi^{\uparrow})$ and also we may consider the case when $m = 0$. The general case will not be developed in a complete form but we will use the integral tests of the lower envelope of the last passage times due to Pardo [19] (see Theorems 3 and 4) to get the upper bound. In fact, it does not seem easy to determine the law of S_1 in terms of the underlying Lévy process and even establish integral tests in terms of the decomposition of the first passage time in the Caballero and Chaumont's construction since the sequence of the overshoots related to the underlying Lévy process is a Markov chain (see Proposition 2 in [7]). This limited to us to assume that $m > 0$, since the last passage time is not defined on the oscillating case.

3.1 The case with no positive jumps.

Let us define
$$F^{\uparrow}(t) \overset{\text{(def)}}{=} \mathbb{P}\left(I(-\xi^{\uparrow}) < t\right),$$
and we denote by $\mathcal{H}_0^{-1}$, the totality of positive increasing functions $h(x)$ on $(0, \infty)$ that satisfy

i) $h(0) = 0$, and

ii) there exists $\beta \in (0, 1)$ such that $\displaystyle\sup_{t < \beta} \frac{h(t)}{t} < \infty$.

The lower envelope at 0 of the first passage process S is as follows:

Proposition 2 *Let $m \geq 0$ and $h \in \mathcal{H}_0^{-1}$.*

i) *If*
$$\int_{0+} F^{\uparrow}\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} < \infty,$$
then for all $\epsilon > 0$
$$\mathbb{P}\left(S_x < (1-\epsilon)h(x), \ i.o., \ as \ x \to 0\right) = 0.$$

ii) *If*
$$\int_{0+} F^{\uparrow}\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} = \infty,$$
then for all $\epsilon > 0$
$$\mathbb{P}\left(S_x < (1+\epsilon)h(x), \ i.o., \ as \ x \to 0\right) = 1.$$

Let us define $\mathcal{H}_\infty^{-1}$, the totality of positive increasing functions $h(t)$ on $(0, \infty)$ that satisfy

i) $\lim_{t \to \infty} h(t) = +\infty$, and

ii) there exists $\beta \in (1, +\infty)$ such that $\sup\limits_{t > \beta} \dfrac{h(t)}{t} < \infty$.

The lower envelope at $+\infty$ of the first passage process S is as follows:

Proposition 3 *Let $m \geq 0$ and $h \in \mathcal{H}_\infty^{-1}$.*

i) If
$$\int^{+\infty} F^\uparrow\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1-\epsilon)h(x), \ i.o., \ as \ x \to +\infty\Big) = 0.$$

ii) If
$$\int_{+\infty} F^\uparrow\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1+\epsilon)h(x), \ i.o., \ as \ x \to +\infty\Big) = 1.$$

The above Propositions are consequence of Lemmas 3.1 and 3.2 of Watanabe [24] and Corollary 1, this follows from the fact that S is an increasing self-similar process. It is important to note that the above results may be proved using similar arguments as in Theorems 3 and 4 in Pardo [19], it is enough to exchange $I(\hat{\xi})$ by $I(-\xi^\uparrow)$ and note that $\Gamma = 1$.

3.2 The general case.

Let us define
$$G(t) \stackrel{\text{(def)}}{=} \mathbb{P}\big(S_1 < t\big) \quad \text{and} \quad \overline{F}(t) \stackrel{\text{(def)}}{=} \mathbb{P}\big(I(\hat{\xi}) < t\big).$$
The lower envelope at 0 of the first passage process S is as follows:

Proposition 4 *Let $m > 0$ and $h \in \mathcal{H}_0^{-1}$.*

i) If
$$\int_{0+} G\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1-\epsilon)h(x), \ i.o., \ as \ x \to 0\Big) = 0.$$

ii) If
$$\int_{0+} \overline{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\left(S_x < (1+\epsilon)h(x), \ i.o., \ as \ x \to 0\right) = 1.$$

Proof: We first prove the convergent part. Let (x_n) be a decreasing sequence of positive numbers which converges to 0 and let us define the events $A_n = \{S_{x_{n+1}} < h(x_n)\}$. Now, we choose $x_n = r^n$, for $r < 1$. From the first Borel Cantelli's Lemma, if $\sum_n \mathbb{P}(A_n) < \infty$, it follows

$$S_{r^{n+1}} \ge h\left(r^n\right) \qquad \mathbb{P} - \text{ a.s.,}$$

for all large n. Since the function h and the process S are increasing, we have

$$S_x \ge h(x) \qquad \text{for} \quad r^{n+1} \le x \le r^n.$$

On the other hand, from the scaling property, we get that

$$\sum_n \mathbb{P}\left(S_{r^n} < h\left(r^{n+1}\right)\right) \le \int_1^\infty \mathbb{P}\left(r^t S_1 < h\left(r^t\right)\right)\mathrm{d}t$$

$$= -\frac{1}{\log r}\int_0^r G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x}.$$

From our hypothesis, this last integral is finite. Then from the above discussion, there exist x_0 such that for every $x \ge x_0$

$$S_x \ge r^2 h(x), \qquad \text{for all} \quad r < 1.$$

Clearly, this implies that

$$\mathbb{P}_0\left(S_x < r^2 h(x), \ \text{i.o., as } x \to 0\right) = 0,$$

which proves part (i).
The divergent part follows from the integral test for the lower envelope of the last passage time due to Pardo [19], (see Theorem 3, part (ii)). ∎

The lower envelope at $+\infty$ of the first passage process S is as follows:

Proposition 5 *Let $m > 0$ and $h \in \mathcal{H}_\infty^{-1}$.*

i) If
$$\int^{+\infty} G\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\left(S_x < (1-\epsilon)h(x), \ i.o., \ as \ x \to +\infty\right) = 0.$$

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ii) If

$$\int^{+\infty} \overline{F}\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\Big(S_x < (1+\epsilon)h(x), \ i.o., \ as \ x \to +\infty\Big) = 1.$$

Proof: The proof is very similar to that in Proposition 4. We get the integral test following the same arguments for the proof of part (i) and (ii) for the sequence $x_n = r^n$, with $r > 1$. ∎

Note that in the general case, the integral tests for the lower envelope of S no longer depend on $F^\uparrow$ as in the case with no positive jumps. Recall that it does not seem easy to determine the law of S and even have a nice decomposition as for the last passage times that allows us to obtain and integral test which only depends on the law of S_1. We remark that $\overline{F}(t)$ is smaller or equal to $G(t)$, for $t \geq 0$, but for our purpose the integral tests of above will be very useful since we will compare in sections 5 and 6 the behaviour of $\overline{F}$ and G under different conditions.

4 The upper envelope of pssMp.

Here, we are interested in describing the upper envelope at 0 and at $+\infty$ of the pssMp $X^{(x)}$ trough integral tests. By the same reasons as those mentioned in the precedent section, we will first study the case with no positive jumps.

4.1 The case with no positive jumps.

The following theorem means in particular that the upper envelope at 0 of $X^{(0)}$ only depends on the tail behaviour of the law of $I(-\xi^\uparrow)$ and on the additional hypothesis

$$\mathbb{E}\Big(\log^+ I\big(-\xi^\uparrow\big)^{-1}\Big) < \infty. \tag{4.15}$$

Let us recall that

$$F^\uparrow(t) = \mathbb{P}\Big(I\big(-\xi^\uparrow\big) < t\Big),$$

and denote by $\mathcal{H}_0$ the totality of positive increasing functions $h(t)$ on $(0, \infty)$ that satisfy

i) $h(0) = 0$, and

ii) there exists $\beta \in (0,1)$ such that $\displaystyle\sup_{t < \beta} \frac{t}{h(t)} < \infty$.

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Theorem 2 *Let $m \geq 0$ and $h \in \mathcal{H}_0$.*

i) If

$$\int_{0+} F^{\uparrow}\left(\frac{t}{h(t)}\right) \frac{dt}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}_0\left(X_t > (1+\epsilon)h(t), \ i.o., \ as \ t \to 0\right) = 0.$$

ii) Assume that (4.15) is satisfied. If

$$\int_{0+} F^{\uparrow}\left(\frac{t}{h(t)}\right) \frac{dt}{t} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}_0\left(X_t > (1-\epsilon)h(t), \ i.o., \ as \ t \to 0\right) = 1.$$

Proof: Let (x_n) be a decreasing sequence which converges to 0. We define the events $A_n = \left\{\text{There exists } t \in [S_{x_{n+1}}, S_{x_n}] \text{ such that } X_t^{(0)} > h(t)\right\}$. From the fact that S_{x_n} tends to 0, a.s. when n goes to $+\infty$, we see

$$\left\{X_t^{(0)} > h(t), \ i.o., \ as \ t \to 0\right\} = \limsup_{n \to +\infty} A_n.$$

Since h is an increasing function the following inclusions hold

$$\left\{x_n > h\left(S_{x_n}\right)\right\} \subset A_n \subset \left\{x_n > h\left(S_{x_{n+1}}\right)\right\}. \tag{4.16}$$

Now, we prove the convergent part. We choose $x_n = r^n$, for $r < 1$ and $h_r(t) = r^{-2}h(t)$. Since h is increasing, we deduce that

$$\sum_n \mathbb{P}\left(r^n > h_r\left(S_{r^{n+1}}\right)\right) \leq \int_1^{+\infty} \mathbb{P}\left(r^t > h\left(S_{t^r}\right)\right) dt \leq -\frac{1}{\log r} \int_0^r \mathbb{P}\left(t > h\left(S_t\right)\right) \frac{dt}{t}.$$

Replacing h by h_r in (4.16), we see that we can obtain our result if

$$\int_0^r \mathbb{P}\left(t > h\left(S_t\right)\right) \frac{dt}{t} < \infty.$$

From elementary calculations and Corollary 1, we deduce that

$$\int_0^r \mathbb{P}\left(t > h\left(S_t\right)\right) \frac{dt}{t} = \mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{1}_{\left\{t/r < I(-\xi^{\uparrow}) < t/h(t)\right\}} \frac{dt}{t}\right),$$

where $h^{-1}(s) = \inf\{t > 0, h(t) > s\}$, the right inverse function of h. Then, this integral converges if

$$\int_0^{h^{-1}(r)} \mathbb{P}\left(I(-\xi^{\uparrow}) < \frac{t}{h(t)}\right) \frac{dt}{t} < \infty.$$

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This proves part (i).

Next, we prove the divergent case. We suppose that h satisfies

$$\int_{0^+} F^\uparrow\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} = \infty.$$

Take, again, $x_n = r^n$, for $r < 1$ and define

$$B_n \overset{\text{(def)}}{=} \bigcup_{m=n}^{\infty} A_m = \left\{\text{There exists } t \in (0, S_{r^n}] \text{ such that } X_t^{(0)} > h_r(t)\right\}.$$

Note that the family (B_n) is decreasing and

$$B \overset{\text{(def)}}{=} \bigcap_{n \geq 1} B_n = \left\{X_t^{(0)} > h_r(t), \text{ i.o., as } t \to 0\right\},$$

then it is enough to prove that $\lim \mathbb{P}(B_n) = 1$ to obtain our result.

Again replacing h by h_r in inclusion (4.16), we see

$$\mathbb{P}(B_n) \geq 1 - \mathbb{P}\left(r^j \leq h_r\left(S_{r^j}\right)\right), \text{ for all } n \leq j \leq m - 1\Big), \tag{4.17}$$

where m is chosen arbitrarily $m \geq n + 1$.

Now, we define the events

$$C_n \overset{\text{(def)}}{=} \left\{r^n > rh\left(S_{r^n}\right)\right\}.$$

We will prove that $\sum \mathbb{P}(C_n) = \infty$. Since the function h is increasing, from the identity in law of Corollary 1 it is straightforward that

$$\sum_{n \geq 1} \mathbb{P}(C_n) \geq \int_0^{+\infty} \mathbb{P}\left(r^t > h\left(S_{r^t}\right)\right)\mathrm{d}t = -\frac{1}{\log r}\int_0^1 \mathbb{P}\left(t > h\left(tI\left(-\xi^\uparrow\right)\right)\right)\frac{\mathrm{d}t}{t}.$$

Hence, if this last integral is infinite, we get that $\sum \mathbb{P}(C_n) = \infty$. In this direction, we have

$$\int_0^r \mathbb{P}\left(t > h\left(tI\left(-\xi^\uparrow\right)\right)\right)\frac{\mathrm{d}t}{t} = \mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{1}_{\left\{t/r < I(-\xi^\uparrow) < t/h(t)\right\}}\frac{\mathrm{d}t}{t}\right).$$

On the other hand, we see

$$\int_0^{h^{-1}(r)} \mathbb{P}\left(I\left(-\xi^\uparrow\right) < \frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} = \int_0^{h^{-1}(r)} \mathbb{P}\left(\frac{t}{r} < I\left(-\xi^\uparrow\right) < \frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t}$$
$$+ \int_0^{h^{-1}(r)} \mathbb{P}\left(I\left(-\xi^\uparrow\right) < \frac{t}{r}\right)\frac{\mathrm{d}t}{t},$$

and

$$\int_0^{h^{-1}(r)} \mathbb{P}\left(I\left(-\xi^\uparrow\right) < \frac{t}{r}\right)\frac{\mathrm{d}t}{t} \leq \mathbb{E}\left(\log^+ \frac{h^{-1}(r)}{r}I\left(-\xi^\uparrow\right)^{-1}\right)$$

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which is clearly finite from our assumptions. Then, we deduce

$$\mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{1}_{\left\{t/r < I(-\xi^\uparrow) < t/h(t)\right\}} \frac{\mathrm{d}t}{t}\right) = \infty,$$

and hence $\sum \mathbb{P}(C_n) = \infty$.

Next, for $n \leq m-1$, we define

$$H(n,m) \overset{\text{(def)}}{=} \mathbb{P}\left(r^j \leq rh\left(S_{r^j} - S_{r^m}\right), \text{ for all } n \leq j \leq m-1\right),$$

and we will prove that there exist (n_l) and (m_l), two increasing sequences such that $0 \leq n_l \leq m_l - 1$, and n_l, m_l go to $+\infty$ and $H(n_l, m_l)$ tends to 0 as l goes to infinity. We suppose the contrary, i.e., there exist $\delta > 0$ such that $H(n,m) \geq \delta$ for all sufficiently large integers m and n. Hence from the independence of the increments of S,

$$1 \geq \mathbb{P}\left(\bigcup_{m=n+1}^{\infty} C_m\right) \geq \sum_{m=n+1}^{\infty} \mathbb{P}\left(C_m \cap \left(\bigcap_{j=n}^{m-1} C_j^c\right)\right)$$

$$\geq \sum_{m=n+1}^{\infty} \mathbb{P}\left(r^m > rh\left(S_{r^m}\right)\right) H(n,m) \geq \delta \sum_{m=n+1}^{\infty} \mathbb{P}(C_m),$$

but since $\sum \mathbb{P}(C_n)$ diverges, we see that our assertion is true.

Now, we define

$$\rho_{n_l,m_l}(x) \overset{\text{(def)}}{=} \mathbb{P}\left(r^j \leq rh\left(S_{r^j} - S_{r^{m_l-1}} + x\right) \text{ for, } n_l \leq j \leq m_l - 2\right), \qquad x \geq 0,$$

and

$$G(n_l, m_l) \overset{\text{(def)}}{=} \mathbb{P}\left(r^j \leq rh\left(S_{r^j}\right) \text{ for, } n_l \leq j \leq m_l - 1\right).$$

Since h is increasing, we see that $\rho_{n_l,m_l}(x)$ is increasing in x.

If we denote by μ and $\bar{\mu}$ the laws of S_1 and $S_1 - S_r$ respectively, by the scaling property we may express $H(n_l, m_l)$ and $G(n_l, m_l)$ as follows

$$H(n_l, m_l) = \int_0^{+\infty} \bar{\mu}(\mathrm{d}x) \mathbb{1}_{\left\{h(r^{m_l-1}x) \geq r^{m_l}\right\}} \rho_{n_l,m_l}(r^{m_l-1}x) \quad \text{and,}$$

$$G(n_l, m_l) = \int_0^{+\infty} \mu(\mathrm{d}x) \mathbb{1}_{\left\{h(r^{m_l-1}x) \geq r^{m_l}\right\}} \rho_{n_l,m_l}(r^{m_l-1}x).$$

In particular, we get that for l sufficiently large

$$H(n_l, m_l) \geq \rho_{n_l,m_l}(N) \int_N^{+\infty} \bar{\mu}(\mathrm{d}x) \qquad for \quad N \geq rC,$$

where $C = \sup_{x \leq \beta} x/h(x)$.

Since $H(n_l, m_l)$ converges to 0, as l goes to $+\infty$ and $\bar{\mu}$ does not depend on l, then

$\rho_{n_l,m_l}(N)$ also converges to 0 when l goes to $+\infty$, for every $N \geq rC$. On the other hand, we have

$$G(n_l, m_l) \leq \rho_{n_l,m_l}(N) \int_0^N \mu(\mathrm{d}x) + \int_N^\infty \mu(\mathrm{d}x),$$

then letting l and N go to infinity, we get that $G(n_l, m_l)$ goes to 0. Then, by (4.17) we get that $\lim \mathbb{P}(B_n) = 1$ and with this we finish the proof. ∎

For the integral tests at $+\infty$, we define $\mathcal{H}_\infty$, the totality of positive increasing functions $h(t)$ on $(0, \infty)$ that satisfy

i) $\lim_{t \to \infty} h(t) = \infty$, and

ii) there exists $\beta > 1$ such that $\sup_{t > \beta} \dfrac{t}{h(t)} < \infty.$

The upper envelope of $X^{(x)}$ at $+\infty$ is given by the following result.

Theorem 3 *Let $m \geq 0$ and $h \in \mathcal{H}_\infty$.*

i) If

$$\int^{+\infty} F^\uparrow\left(\frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} < \infty,$$

then for all $\epsilon > 0$ and for all $x \geq 0$,

$$\mathbb{P}_x\Big(X_t > (1+\epsilon)h(t), \ i.o., \ as \ t \to +\infty\Big) = 0.$$

ii) Assume that (4.15) is satisfied. If

$$\int^{+\infty} F^\uparrow\left(\frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} = \infty,$$

then for all $\epsilon > 0$ and for all $x \geq 0$

$$\mathbb{P}_x\Big(X_t > (1-\epsilon)h(t), \ i.o., \ as \ t \to +\infty\Big) = 1.$$

Proof: We first consider the case where $x = 0$. In this case the proof of the tests at $+\infty$ is almost the same as that of the tests at 0. It is enough to apply the same arguments to the sequence $x_n = r^n$, for $r > 1$.

Now, we prove (i) for any $x > 0$. Let $h \in \mathcal{H}_\infty$ such that $\int^{+\infty} F^\uparrow\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t}$ is finite. Let $x > 0$ and $S_x = \inf\{t \geq 0 : X_t^{(0)} \geq x\}$. Since clearly

$$\int^{+\infty} F^\uparrow\left(\frac{t}{h(t - S_x)}\right) \frac{\mathrm{d}t}{t} < \infty,$$

from the Markov property at time S_x, we have for all $\epsilon > 0$

$$\mathbb{P}_0\Big(X_t > (1+\epsilon)h(t-S_x), \ \text{i. o., as } t \to \infty\Big) = \mathbb{P}_x\Big(X_t > (1+\epsilon)h(t), \ \text{i. o., as } t \to \infty\Big) = 0,$$

which proves part (i).

Part (ii) can be proved in the same way. ∎

4.2 The general case.

Proposition 6 *Let $m > 0$ and $h \in \mathcal{H}_0$.*

i) If
$$\int_{0+} G\left(\frac{t}{h(t)}\right) \frac{dt}{t} < \infty,$$

then for all $\epsilon > 0$
$$\mathbb{P}\left(X_t^{(0)} > (1+\epsilon)h(t), \ i.o., \ as \ t \to 0\right) = 0.$$

ii) If
$$\int_{0+} \overline{F}\left(\frac{t}{h(t)}\right) \frac{dt}{t} = \infty,$$

then for all $\epsilon > 0$
$$\mathbb{P}\left(X_t^{(0)} < (1-\epsilon)h(t), \ i.o., \ as \ t \to 0\right) = 1.$$

Proof: Let (x_n) be a decreasing sequence which converges to 0. We define the events $A_n = \left\{\text{There exists } t \in [S_{x_{n+1}}, S_{x_n}) \text{ such that } X_t^{(0)} > h(t)\right\}$. From the fact that S_{x_n} tends to 0, a.s. when n goes to $+\infty$, we see

$$\left\{X_t^{(0)} > h(t), \ i.o., \ as \ t \to 0\right\} = \limsup_{n \to +\infty} A_n.$$

Since h is an increasing function the following inclusion hold

$$A_n \subset \left\{x_n > h\left(S_{x_{n+1}}\right)\right\}. \tag{4.18}$$

Now, we prove the convergent part. We choose $x_n = r^n$, for $r < 1$ and $h_r(t) = r^{-2}h(t)$. Since h is increasing, we deduce

$$\sum_n \mathbb{P}\left(r^n > h_r\left(S_{r^{n+1}}\right)\right) \leq -\frac{1}{\log r} \int_0^r \mathbb{P}\left(t > h(S_t)\right) \frac{dt}{t}.$$

Replacing h by h_r in (4.18), we see that we can obtain our result if

$$\int_0^r \mathbb{P}\left(t > h(S_t)\right) \frac{dt}{t} < \infty.$$

From elementary calculations, we get

$$\int_0^r \mathbb{P}\left(t > h(S_t)\right) \frac{dt}{t} = \mathbb{E}\left(\int_0^{h^{-1}(r)} \mathbb{1}_{\left\{t/r < S_1 < t/h(t)\right\}} \frac{dt}{t}\right),$$

where $h^{-1}(s) = \inf\{t > 0, h(t) > s\}$, the right inverse function of h. Then, this integral converges if

$$\int_0^{h^{-1}(r)} \mathbb{P}\left(S_1 < \frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} < \infty.$$

This proves part (i).

The divergent part follows from the integral test for the upper envelope of the future infimum of pssMp due to Pardo [19], (see Theorem 1, part (ii)). ∎

Proposition 7 *Let $m > 0$ and $h \in \mathcal{H}_\infty$.*

   i) If

$$\int^{+\infty} G\left(\frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} < \infty,$$

   then for all $\epsilon > 0$ and for all $x \geq 0$

$$\mathbb{P}\left(X_t^{(x)} > (1 + \epsilon)h(t), \ i.o., \ as \ t \to +\infty\right) = 0.$$

   ii) If

$$\int^{+\infty} \overline{F}\left(\frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t} = \infty,$$

   then for all $\epsilon > 0$ and for all $x \geq 0$

$$\mathbb{P}\left(X_t^{(x)} < (1 - \epsilon)h(t), \ i.o., \ as \ t \to +\infty\right) = 1.$$

Proof: We first consider the case where $x = 0$. In this case the proof of the tests at $+\infty$ is almost the same as that of the tests at 0. It is enough to apply the same arguments to the sequence $x_n = r^n$, for $r > 1$.

Now, we prove (i) for any $x > 0$. Let $h \in \mathcal{H}_\infty$ such that $\int^{+\infty} G\left(\frac{t}{h(t)}\right) \frac{\mathrm{d}t}{t}$ is finite. Let $x > 0$ and S_x and note by μ_x the law of $X_{S_x}^{(0)}$. Since clearly

$$\int^{+\infty} G\left(\frac{t}{h(t - S_x)}\right) \frac{\mathrm{d}t}{t} < \infty,$$

from the Markov property at time S_x, we have for all $\epsilon > 0$

$$\mathbb{P}_0\left(X_t > (1 + \epsilon)h(t - S_x), \ \text{i. o., as } t \to \infty\right)$$
$$= \int_{[x, +\infty)} \mathbb{P}_y\left(X_t > (1 + \epsilon)h(t), \ \text{i. o., as } t \to \infty\right)\mu_x(\mathrm{d}y) = 0. \tag{4.19}$$

If x is an atom of μ_x, then equality (4.19) shows that

$$\mathbb{P}\left(X_t^{(x)} > (1 + \epsilon)h(t), \ \text{i. o., as } t \to \infty\right) = 0$$

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and the result is proved. Suppose that x is not an atom of μ_x. From Theorem 1 in [7], we know that $X_{S_x}^{(0)} \overset{(d)}{=} xe^{\theta}$, where θ is a positive r.v. such that

$$\xi_{T_z} - z \xrightarrow[z \to +\infty]{(w)} \theta.$$

Then from section 2 in [7], the law of θ is given by

$$\mathbb{P}(\theta > t) = \mathbb{E}(\sigma_1) \int_{(t,\infty)} s\nu(\mathrm{d}s), \quad t \geq 0,$$

where σ is the upward ladder height process associated with ξ and ν its Lévy measure. Hence, $\mathbb{P}(e^{\theta} > z) > 0$ for $z > 1$, and for any $\alpha > 0$, $\mu_x(x, x + \alpha) > 0$. Hence (4.19) shows that there exists $y > x$ such that

$$\mathbb{P}\Big(X_t^{(y)} > (1 + \epsilon)h(t), \text{ i. o., as } t \to \infty\Big) = 0,$$

for all $\epsilon > 0$. The previous allows us to conclude part (i).
Part (ii) can be proved in the same way. ∎

5   The regular case.

In this section, we will assume that $m > 0$. According to Chaumont and Pardo [9] the law of U_1, the last passage time below level 1, is the same as $\nu I(\hat{\xi})$ where ν is a positive random variable bounded above by 1 and independent of the exponential functional $I(\hat{\xi})$. Hence, we have the following inequality $\nu I(\hat{\xi}) \leq I(\hat{\xi})$ a.s.
Now, let us define
$$\overline{F}_\nu(t) := \mathbb{P}\big(\nu I(\hat{\xi}) < t\big),$$

and suppose that

$$ct^{\beta}L(t) \leq \overline{F}(t) \leq \overline{F}_\nu(t) \leq Ct^{\beta}L(t) \qquad \text{as} \quad t \to 0, \tag{5.20}$$

where $\beta > 0$, c and C are two positive constants such that $c \leq C$ and L is a slowly varying function at 0. An important example included in this case is when $\overline{F}$ and $\overline{F}_\nu$ are regularly varying functions at 0.

Proposition 8 *Under condition (5.20), we have that*

$$ct^{\beta}L(t) \leq G(t) \leq C_\epsilon t^{\beta}L(t) \qquad as \quad t \to 0,$$

where C_ϵ is a positive constant bigger than C.

Proof: The lower bound is clear since $\overline{F}(t) \leq G(t)$, for all $t \geq 0$ and our assumption. Now, let us define $M_t^{(0)} = \sup_{0 \leq s \leq t} X_s^{(0)}$ and fix $\epsilon > 0$. Then, by the Markov property

and the fact that $J^{(x)}$ is an increasing process, we have

$$\mathbb{P}_0\left(J_1 > \frac{1-\epsilon}{t}\right) \geq \mathbb{P}_0\left(J_1 > \frac{1-\epsilon}{t}, M_1 \geq \frac{1}{t}\right)$$

$$= \mathbb{E}\left(S_{1/t} \leq 1, \mathbb{P}_{X^{(0)}_{S_{1/t}}}\left(J_{1-S_{1/t}} > \frac{1-\epsilon}{t}\right)\right)$$

$$\geq \mathbb{E}\left(S_{1/t} \leq 1, \mathbb{P}_{X^{(0)}_{S_{1/t}}}\left(J_0 > \frac{1-\epsilon}{t}\right)\right).$$

Since $X^{(0)}_{S_{1/t}} \geq 1/t$ a.s., and the Lamperti representation (1.1), we deduce that

$$\mathbb{E}\left(S_{1/t} \leq 1, \mathbb{P}_{X^{(0)}_{S_{1/t}}}\left(J_0 > \frac{1-\epsilon}{t}\right)\right) \geq \mathbb{P}\left(S_{1/t} < 1\right)\mathbb{P}\left(\inf_{s \geq 0} \xi_s > \log(1-\epsilon)\right).$$

On the other hand, under the assumption that ξ drifts towards $+\infty$, we know from Section 2 of Chaumont and Doney [8] (see also Proposition VI.17 in [2]) that for all $\epsilon > 0$

$$K_\epsilon := \mathbb{P}\left(\inf_{s \geq 0} \xi_s > \log(1-\epsilon)\right) > 0.$$

Hence

$$K_\epsilon^{-1} \mathbb{P}_0\left(J_1 > \frac{1-\epsilon}{t}\right) \geq \mathbb{P}\left(S_1 < t\right)$$

which implies that

$$CK_\epsilon^{-1}\left(\frac{t}{1-\epsilon}\right)^\beta L(t) \geq K_\epsilon^{-1}\mathbb{P}\left(U_1 < \frac{t}{1-\epsilon}\right) \geq \mathbb{P}\left(S_1 < t\right), \qquad \text{as} \quad t \to 0,$$

then the proposition is proved. ∎

The next result give us integral tests for the lower envelope of S at 0 and at ∞, under condition (5.20).

Theorem 4 *Under condition (5.20), the lower envelope of S at 0 and at $+\infty$ is as follows:*

i) Let $h \in \mathcal{H}_0^{-1}$, such that either $\lim_{x \to 0} h(x)/x = 0$ or $\liminf_{x \to 0} h(x)/x > 0$, then

$$\mathbb{P}\left(S_x < h(x), \ i.o., \ as \ x \to 0\right) = 0 \ or \ 1,$$

according as

$$\int_{0+} \overline{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x} \qquad \text{is finite or infinite.}$$

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ii) *Let* $h \in \mathcal{H}_\infty^{-1}$, *such that either* $\lim_{x \to +\infty} h(x)/x = 0$ *or* $\liminf_{x \to +\infty} h(x)/x > 0$, *then*

$$\mathbb{P}\Big(S_x < h(x), \ i.o., \ as \ x \to \infty\Big) = 0 \ or \ 1,$$

according as

$$\int^{+\infty} \overline{F}\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} \qquad is \ finite \ or \ infinite.$$

Proof: First let us check that under condition (5.20) we have

$$\int_0^\lambda \overline{F}\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} < \infty \quad \text{if and only if} \quad \int_0^\lambda G\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} < \infty. \tag{5.21}$$

Since $\overline{F}(t) \leq G(t)$ for all $t \geq 0$, it is clear that we only need to prove that

$$\int_0^\lambda \overline{F}\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} < \infty \quad \text{implies that} \quad \int_0^\lambda G\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} < \infty.$$

From the hypothesis, either $\lim_{x \to 0} h(x)/x = 0$ or $\liminf_{x \to 0} h(x)/x > 0$. In the first case, from condition (5.20) there exists $\lambda > 0$ such that, for every $x < \lambda$

$$c\left(\frac{h(x)}{x}\right)^\beta L\left(\frac{h(x)}{x}\right) \leq \overline{F}\left(\frac{h(x)}{x}\right) \leq C\left(\frac{h(x)}{x}\right)^\beta L\left(\frac{h(x)}{x}\right).$$

Since, we suppose that $\int_0^\lambda \overline{F}\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x}$ is finite, then

$$\int_0^\lambda \left(\frac{h(x)}{x}\right)^\beta L\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} < \infty,$$

hence from Proposition 8, we get that $\int_0^\lambda G\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x}$ is also finite. In the second case, since for any $0 < \delta < \infty$, $\mathbb{P}\big(I < \delta\big) > 0$, and $\liminf_{x \to 0} h(x)/x > 0$, we have for any y

$$0 < \mathbb{P}\left(I < \liminf_{x \to 0} \frac{h(x)}{x}\right) < \mathbb{P}\left(I < \frac{h(y)}{y}\right). \tag{5.22}$$

Hence, since for every $t \geq 0$, $\overline{F}(t) \leq G(t)$, we deduce that

$$\int_0^\lambda \overline{F}\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} = \int_0^\lambda G\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} = \infty.$$

Now, let us check that for any constant $\beta > 0$,

$$\int_0^\lambda \overline{F}\left(\frac{h(x)}{x}\right) \frac{\mathrm{d}x}{x} < \infty \quad \text{if and only if} \quad \int_0^\lambda \overline{F}\left(\frac{\beta h(x)}{x}\right) \frac{\mathrm{d}x}{x} < \infty, \tag{5.23}$$

Again, from the hypothesis either $\lim_{x \to 0} h(x)/x = 0$ or $\liminf_{x \to 0} h(x)/x > 0$. In the first case, we deduce (5.23) from (5.20). In the second case, from (5.22) both of the

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integrals in (5.23) are infinite.

Next, it follows from Proposition 4 part (i) and (5.21) that if $\int_{0^+} \overline{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x}$ is finite, then for all $\epsilon > 0$,

$$\mathbb{P}\big(S_x < (1-\epsilon)h(x), \text{ i.o., as } x \to 0\big) = 0.$$

If $\int_{0^+} \overline{F}\left(\frac{h(x)}{x}\right)\frac{\mathrm{d}x}{x}$ diverges, then from Proposition 4 part (ii) that for all $\epsilon > 0$,

$$\mathbb{P}\big(S_x < (1+\epsilon)h(x), \text{ i.o., as } x \to 0\big) = 1.$$

Then (5.23) allows us to drop ϵ in this implications.

The tests at $+\infty$ are proven through the same way. ∎

From the previous Theorem, we deduce that under condition (5.20) the first and the last passage time processes have the same lower functions (see Theorem 5 in [19]).

Theorem 5 *Under condition (5.20), the upper envelope of the pssMp at 0 and at $+\infty$ is as follows:*

i) *Let $h \in \mathcal{H}_0$, such that either $\lim_{t\to 0} t/h(t) = 0$ or $\liminf_{t\to 0} t/h(t) > 0$, then*

$$\mathbb{P}\Big(X_t^{(0)} > h(t), \ i.o., \ as \ t \to 0\Big) = 0 \ or \ 1,$$

according as

$$\int_{0^+} \overline{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} \qquad is \ finite \ or \ infinite.$$

ii) *Let $h \in \mathcal{H}_\infty$, such that either $\lim_{t\to+\infty} t/h(t) = 0$ or $\liminf_{t\to+\infty} t/h(t) > 0$, then for all $x \geq 0$*

$$\mathbb{P}\Big(X_t^{(x)} > h(t), \ i.o., \ as \ t \to \infty\Big) = 0 \ or \ 1,$$

according as

$$\int^{+\infty} \overline{F}\left(\frac{t}{h(t)}\right)\frac{\mathrm{d}t}{t} \qquad is \ finite \ or \ infinite.$$

Proof: We prove this result by following the same arguments as the proof of the previous Theorem. ∎

Note that from this result, we deduce that under condition (5.20) a pssMp and its future infimum have the same upper functions (see Theorem 6 in [19]).

6   The log-regular case.

In this section, we also assume that $m > 0$. Here, we will study to types of behaviour of $\overline{F}$ and $\overline{F}_\nu$, both types of behaviour allow us to obtain laws of the iterated logarithm for the upper envelope of $X^{(0)}$. The first type of behaviour that we will consider is when $\log \overline{F}$ and $\log \overline{F}_\nu$ are regularly varying at 0, i.e.

$$-\log \overline{F}_\nu(1/t) \sim -\log \overline{F}(1/t) \sim \lambda t^\delta L(t), \quad \text{as } t \to +\infty, \tag{6.24}$$

where $\lambda > 0$, $\delta > 0$ and L is a slowly varying function at $+\infty$.
The second type of behaviour that we will consider is when $\log \overline{F}$ and $\log \overline{F}_\nu$ satisfy that

$$-\log \overline{F}_\nu(1/t) \sim -\log \overline{F}(1/t) \sim K(\log t)^\gamma, \quad \text{as } t \to +\infty, \tag{6.25}$$

where K and γ are strictly positive constants.

6.1   Laws of the iterated logarithm for pssMp.

Proposition 9 *Under condition (6.24), the tail probability of S_1 satisfies*

$$-\log G(1/t) \sim \lambda t^\delta L(t) \qquad as \quad t \to +\infty. \tag{6.26}$$

Similarly, under condition (6.25), the tail probability of S_1 satisfies

$$-\log G(1/t) \sim K(\log t)^\gamma \qquad as \quad t \to +\infty. \tag{6.27}$$

Proof: First, we prove the upper bound of (6.26). With the same notation as in the proof of Proposition 8, we see

$$-\log \mathbb{P}\Big(\nu I(\hat{\xi}) < 1/t\Big) = -\log \mathbb{P}_0\Big(J_1 > t\Big) \geq -\log \mathbb{P}_0\Big(M_1 > t\Big),$$

which implies

$$1 \geq \limsup_{t \to \infty} \frac{-\log \mathbb{P}_0\Big(M_1 > t\Big)}{\lambda t^\delta L(t)},$$

and since $\mathbb{P}_0\big(M_1 > t\big) = \mathbb{P}\big(S_1 < 1/t\big)$, we get the upper bound.
Now, fix $\epsilon > 0$. From the proof of Proposition 8, we have

$$\mathbb{P}_0\big(J_1 > (1-\epsilon)t\big) \geq \mathbb{P}\big(S_t < 1\big)\mathbb{P}\Big(\inf_{s \geq 0} \xi_s > \log(1-\epsilon)\Big).$$

On the other hand, we know

$$K_\epsilon := \mathbb{P}\Big(\inf_{s \geq 0} \xi_s > \log(1-\epsilon)\Big) > 0,$$

Hence,

$$-\log \mathbb{P}_0\big(J_1 > (1-\epsilon)t\big) \leq -\log \mathbb{P}\Big(S_1 < 1/t\Big) - \log K_\epsilon,$$

which implies the following lower bound

$$(1 - \epsilon)^\delta \leq \liminf_{t \to \infty} \frac{- \log \mathbb{P}\left(S_1 < 1/t\right)}{\lambda t^\delta L(t)},$$

and since ϵ can be chosen arbitrarily small, (6.26) is proved.

The upper bound of tail behaviour (6.27) is proven through the same way. For the lower bound, we follow the same arguments as above and we get that

$$- \log \mathbb{P}_0\left(J_1 > (1 - \epsilon)t\right) \leq - \log \mathbb{P}\left(S_1 < 1/t\right) - \log K_\epsilon,$$

which implies

$$1 = \liminf_{t \to \infty} \left(\frac{\log(1 - \epsilon)t}{\log t}\right)^\gamma \leq \liminf_{t \to \infty} \frac{- \log \mathbb{P}\left(S_1 < 1/t\right)}{K(\log t)^\gamma},$$

then the proposition is proved. ∎

The following result gives us laws of the iterated logarithm for the first passage time process when condition (6.24) is satisfied.

Define the functions

$$\varphi(x) := \frac{x}{\inf\left\{s : 1/\overline{F}(1/s) > |\log x|\right\}}, \quad x > 0, \quad x \neq 1,$$

and

$$\vartheta(t) := \frac{t^2}{\varphi(t)}, \quad t > 0, \quad t \neq 1.$$

Theorem 6 *Under condition (6.24), we have the following law of the iterated logarithm for S:*

$$\limsup_{x \to 0} \frac{S_x}{\varphi(x)} = 1 \quad and \quad \limsup_{x \to \infty} \frac{S_x}{\varphi(x)} = 1 \quad almost\ surely.$$

The upper envelope of pssMp, under condition (6.24), are described by the following law of the iterated logarithm:

i)

$$\limsup_{t \to 0} \frac{X_t^{(0)}}{\vartheta(t)} = 1, \quad almost\ surely.$$

ii) *For all $x \geq 0$,*

$$\limsup_{t \to +\infty} \frac{X_t^{(x)}}{\vartheta(t)} = 1, \quad almost\ surely.$$

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Proof: This Theorem is a consequence of Propositions 4, 5,6,7 and 9, and it is proven in the same way as Theorem 4 in [9], we only need to emphasize that we can replace $\log G$ by $\log \overline{F}$, since they are asymptotically equivalent. ∎

Note that under condition (6.24) a pssMp and its future infimum satisfy the same law of the iterated logarithm (see Theorem 8 in [19]) but they do not necessarily have the same upper functions.

Similarly, under condition (6.25) we may establish laws of the iterated logarithm for the upper envelope of pssMp and their future infimum. In this direction, let us define

$$\phi(x) := x \exp\left\{ - \left(K^{-1} \log |\log x|\right)^{1/\gamma}\right\}, \quad x > 0, \quad x \neq 1,$$

and

$$\Phi(t) := \frac{t^2}{\phi(t)}, \quad t > 0, \quad t \neq 1.$$

We recall that $J^{(x)} = (J_t^{(x)}, t \geq 0)$ is the future infimum process of $X^{(x)}$, for $x \geq 0$, where $J_t^{(x)} = \inf_{s \geq t} X^{(x)}$.

Theorem 7 *Under condition (6.25),we have the following laws of the iterated logarithm:*

i) For the first passage time, we have

$$\limsup_{x \to 0} \frac{S_x}{\phi(x)} = 1, \qquad \limsup_{x \to \infty} \frac{S_x}{\phi(x)} = 1 \quad almost\ surely.$$

ii) For the last passage time, we have

$$\limsup_{x \to 0} \frac{U_x}{\phi(x)} = 1 \quad and \quad \limsup_{x \to +\infty} \frac{U_x}{\phi(x)} = 1 \quad almost\ surely.$$

The upper envelope of pssMp and their future infimum processes, under condition (6.25), are described by the following laws of the iterated logarithm:

iii)

$$\limsup_{t \to 0} \frac{X_t^{(0)}}{\Phi(t)} = 1 \quad and \quad \limsup_{t \to 0} \frac{J_t^{(0)}}{\Phi(t)} = 1 \quad almost\ surely.$$

iv) For all $x \geq 0$,

$$\limsup_{t \to +\infty} \frac{X_t^{(x)}}{\Phi(t)} = 1 \quad and \quad \limsup_{t \to 0} \frac{J_t^{(x)}}{\Phi(t)} = 1 \quad almost\ surely.$$

Proof: We first prove part (i) for small times. Note that it is easy to check that both $\phi(x)$ and $\phi(x)/x$ are increasing in a neighbourhood of 0, moreover the function $\phi(x)/x$ is bounded by 1, for $x \in [0, 1)$.

From condition (6.25) and Proposition 9, we have for all $k_1 < 1$ and $k_2 > 1$ and for all t sufficiently large,

$$k_1 K (\log t)^\gamma \leq - \log G(1/t) \leq k_2 K (\log t)^\gamma,$$

so that for ϕ defined above,

$$k_1 \log |\log t| \leq - \log G \left(\phi(x)/x \right) \leq k_2 \log |\log t|,$$

hence

$$G \left(\frac{\phi(x)}{x} \right) \geq (|\log t|)^{-k_2}.$$

Since $k_2 > 1$, we obtain the convergence of the integral

$$\int_{0+} G \left(\frac{\phi(x)}{x} \right) \frac{dt}{t},$$

which proves that for all $\varepsilon > 0$,

$$\mathbb{P}(S_x < (1 - \varepsilon)\phi(x), \text{i.o., as } x \to 0) = 0$$

from Proposition 4 part (i). The divergent part is proven through the same way so that from Proposition 4 part (ii), one has for all $\varepsilon > 0$,

$$\mathbb{P}(S_x < (1 + \varepsilon)\phi(x), \text{i.o., as } x \to 0) = 1$$

and the conclusion follows.

Condition (6.25) implies that $\phi(x)$ is increasing in a neighbourhood of $+\infty$ whereas $\phi(x)/x$ is decreasing in a neighbourhood of $+\infty$. Hence, the proof of the result at $+\infty$ is done through the same way as at 0, by using Proposition 5.

The parts (ii), (iii) and (iv) can be proved following the same arguments, it is enough to specify that the laws of the iterated logarithm of the last passage process and the future infimum process will use the integral tests found by Pardo [19], (see Theorems 1,2,3 and 4 in [19]) ∎

6.2 The case with no positive jumps.

Here, we suppose that the pssMp $X^{(x)}$ has no positive jumps. Our next result is a remarkable asymptotic property of this type of pssMp. This Theorem means in particular that if there exists a positive function that describes the upper envelope of $X^{(x)}$ by a law of the iterated logarithm then the same function describes the upper envelope of the future infimum of $X^{(x)}$ and the pssMp $X^{(x)}$ reflected at its future infimum.

Theorem 8 *Let us suppose that*

$$\limsup_{t \to 0} \frac{X_t^{(0)}}{\Lambda(t)} = 1 \qquad almost \; surely,$$

where Λ is a positive function such that $\Lambda(0) = 0$, then

$$\limsup_{t \to 0} \frac{J_t^{(0)}}{\Lambda(t)} = 1 \quad and \quad \limsup_{t \to 0} \frac{X_t^{(0)} - J_t^{(0)}}{\Lambda(t)} = 1 \quad almost\ surely.$$

Moreover, if for all $x \geq 0$

$$\limsup_{t \to +\infty} \frac{X_t^{(x)}}{\Lambda(t)} = 1 \quad almost\ surely,$$

where Λ is a positive function such that $\lim_{t \to +\infty} \Lambda(t) = +\infty$, then

$$\limsup_{t \to +\infty} \frac{J_t^{(x)}}{\Lambda(t)} = 1 \quad and \quad \limsup_{t \to +\infty} \frac{X_t^{(0)} - J_t^{(0)}}{\Lambda(t)} = 1 \quad almost\ surely.$$

Proof: First, we prove the result for large times. Let $x \geq 0$. Since $J_t^{(x)} \leq X_t^{(x)}$ for every $t \geq 0$ and our hypothesis, then it is clear

$$\limsup_{t \to +\infty} \frac{J_t^{(x)}}{\Lambda(t)} \leq 1 \quad almost\ surely.$$

Now, fix $\epsilon \in (0, 1/2)$ and define

$$R_n = \inf \left\{ s \geq n : \frac{X_s^{(x)}}{\Lambda(s)} \geq (1 - \epsilon) \right\}.$$

From the above definition, it is clear that $R_n \geq n$ and that R_n diverge a.s. as n goes to $+\infty$. From our hypothesis, we deduce that R_n is finite, a.s.

Now, since $X^{(x)}$ has no positive jumps and applying the strong Markov property and Lamperti representation (1.1), we have

$$\mathbb{P}\left(\frac{J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1 - 2\epsilon)\right) = \mathbb{P}\left(J_{R_n}^{(x)} \geq \frac{(1 - 2\epsilon)X_{R_n}^{(x)}}{(1 - \epsilon)}\right)$$

$$= \mathbb{E}\left(\mathbb{P}\left(J_{R_n}^{(x)} \geq \frac{(1 - 2\epsilon)X_{R_n}^{(x)}}{(1 - \epsilon)} \bigg| X_{R_n}^{(x)}\right)\right)$$

$$= \mathbb{P}\left(\inf_{t \geq 0} \xi \geq \log \frac{(1 - 2\epsilon)}{(1 - \epsilon)}\right) = cW\left(\log \frac{1 - \epsilon}{1 - 2\epsilon}\right) > 0,$$

where $W : [0, +\infty) \to [0, +\infty)$ is the unique absolutely continuous increasing function with Laplace exponent

$$\int_0^{+\infty} e^{-\lambda x} W(x) \mathrm{d}x = \frac{1}{\psi(\lambda)} \quad for\ \lambda > 0,$$

and $c = 1/W(+\infty)$, (see Bertoin [2] Theorem VII.8).
Since $R_n \geq n$,

$$\mathbb{P}\left(\frac{J_t^{(x)}}{\Lambda(t)} \geq (1 - 2\epsilon),\ \text{for some } t \geq n\right) \geq \mathbb{P}\left(\frac{J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1 - 2\epsilon)\right).$$

Therefore, for all $\epsilon \in (0, 1/2)$,

$$\mathbb{P}\left(\frac{J_t^{(x)}}{\Lambda(t)} \geq (1 - 2\epsilon), \text{i.o., as } t \to +\infty\right) \geq \lim_{n \to +\infty} \mathbb{P}\left(\frac{J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1 - 2\epsilon)\right) > 0.$$

The event of the left hand side is in the upper-tail sigma-field $\cap_t \sigma\{X_s^{(x)} : s \geq t\}$ which is trivial, then

$$\limsup_{t \to +\infty} \frac{J_t^{(x)}}{\Lambda(t)} \geq 1 - 2\epsilon \qquad \text{almost surely.}$$

The proof of part (ii) is very similar, in fact

$$\mathbb{P}\left(\frac{X_{R_n}^{(x)} - J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1 - 2\epsilon)\right) = \mathbb{P}\left(J_{R_n}^{(x)} \leq \frac{\epsilon X_{R_n}^{(x)}}{1 - \epsilon}\right)$$

$$= \mathbb{E}\left(\mathbb{P}\left(J_{R_n}^{(x)} \leq \frac{\epsilon X_{R_n}^{(x)}}{1 - \epsilon} \bigg| X_{R_n}^{(x)}\right)\right)$$

$$= \mathbb{P}\left(\inf_{t \geq 0} \xi \leq \log \frac{\epsilon}{1 - \epsilon}\right) = 1 - cW\left(\log \frac{1 - \epsilon}{\epsilon}\right) > 0.$$

Since $R_n \geq n$,

$$\mathbb{P}\left(\frac{X_t^{(x)} - J_t^{(x)}}{\Lambda(t)} \geq (1 - 2\epsilon), \text{ for some } t \geq n\right) \geq \mathbb{P}\left(\frac{X_{R_n}^{(x)} - J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1 - 2\epsilon)\right).$$

Therefore, for all $\epsilon \in (0, 1/2)$,

$$\mathbb{P}\left(\frac{X_t^{(x)} - J_t^{(x)}}{\Lambda(t)} \geq (1 - 2\epsilon), \text{i.o., as } t \to \infty\right) \geq \lim_{n \to +\infty} \mathbb{P}\left(\frac{X_{R_n}^{(x)} - J_{R_n}^{(x)}}{\Lambda(R_n)} \geq (1 - 2\epsilon)\right) > 0.$$

The event of the left hand side of the above inequality is in the upper-tail sigma-field $\cap_t \sigma\{X_s^{(x)} : s \geq t\}$ which is trivial and this establishes part (ii) for large times.

In order to prove the LIL for small times, we now define the following stopping time

$$R_n = \inf\left\{\frac{1}{n} \leq s : \frac{X_s^{(0)}}{\Lambda(s)} \geq (1 - \epsilon)\right\}.$$

Following same arguments as above, we get that for a fixed $\epsilon \in (0, 1/2)$ and n sufficiently large

$$\mathbb{P}\left(\frac{J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1 - 2\epsilon)\right) > 0 \quad \text{and} \quad \mathbb{P}\left(\frac{X_{R_n}^{(0)} - J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1 - 2\epsilon)\right) > 0.$$

Next, we note

$$\mathbb{P}\left(\frac{J_{R_p}^{(0)}}{\Lambda(R_p)} \geq (1 - 2\epsilon), \text{ for some } p \geq n\right) \geq \mathbb{P}\left(\frac{J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1 - 2\epsilon)\right),$$

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and

$$\mathbb{P}\left(\frac{X_{R_p}^{(0)} - J_{R_p}^{(0)}}{\Lambda(R_p)} \geq (1 - 2\epsilon), \text{ for some } p \geq n\right) \geq \mathbb{P}\left(\frac{X_{R_n}^{(0)} - J_{R_n}^{(0)}}{\Lambda(R_n)} \geq (1 - 2\epsilon)\right).$$

Since R_n converge a.s. to 0 as n goes to ∞, the conclusion follows taking the limit when n goes towards to $+\infty$ ∎

Hence, when $F^\uparrow$ satisfies condition (6.24) we have the following laws of the iterated logarithm for the future infimum of $X^{(x)}$ and the pssMp $X^{(x)}$ reflected at its future infimum.

Corollary 2 *Under condition (6.24), we have the following laws of the iterated logarithm:*

i) for all $x \geq 0$

$$\limsup_{t \to 0} \frac{J_t^{(0)}}{\vartheta(t)} = 1 \quad and \quad \limsup_{t \to +\infty} \frac{J_t^{(x)}}{\vartheta(t)} = 1 \quad almost\ surely,$$

ii) for all $x \geq 0$

$$\limsup_{t \to 0} \frac{X_t^{(0)} - J_t^{(0)}}{\vartheta(t)} = 1 \quad and \quad \limsup_{t \to +\infty} \frac{X_t^{(x)} - J_t^{(x)}}{\vartheta(t)} = 1 \quad almost\ surely,$$

7 Bessel processes

Recall that Bessel processes are the only continuous positive self-similar Markov processes. Recall also that a Bessel process of dimension $\delta \geq 0$ with starting point $x \geq 0$ is the diffusion R whose square satisfies the stochastic differential equation

$$R_t^2 = x^2 + 2\int_0^t R_s \mathrm{d}\beta_s + \delta t, \qquad t \geq 0, \tag{7.28}$$

where β is a standard Brownian Motion.

Now, we define $\xi = (2(B_t + at), t \geq 0)$, where B is a standard Brownian motion and $a \geq 0$. By the Lamperti representation, we know that we can define a pssMp starting from $x > 0$, such that

$$X_t^{(x)} = x \exp\{\xi_{\tau(t/x)}\}, \quad t \geq 0.$$

Applying the Itô's formula and Dubins-Schwartz's Theorem (see for instance Revuz and Yor [21]), we see that $X^{(x)}$ satisfy (7.28) with $\delta = 2(a + 1)$. Obviously, ξ satisfies the conditions under which we can define $X^{(x)}$ when $x = 0$. When $a > 0$, it is clear that $X^{(x)}$ is transient and when $a = 0$, it is also clear that the process $X^{(x)}$ oscillates.

Gruet and Shi [12] proved that there exist a finite constant $K > 1$, such that for any $0 < s \leq 2$,

$$K^{-1}s^{1-\delta/2}\exp\left\{-\frac{1}{2s}\right\} \leq \mathbb{P}(S_1 < s) \leq Ks^{1-\delta/2}\exp\left\{-\frac{1}{2s}\right\}. \qquad (7.29)$$

Hence we establish the following integral test for the lower envelope of the first passage time process of the squared Bessel process $X^{(0)}$.

Theorem 9 *Let $h \in \mathcal{H}_0^{-1}$ and $\delta \geq 2$,*

i) If

$$\int_{0+} \left(\frac{t}{h(t)}\right)^{\frac{\delta-2}{2}} \exp\left\{-\frac{t}{2h(t)}\right\} \frac{dt}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\left(S_t < (1-\epsilon)h(t), \ i.o., \ as \ t \to 0\right) = 0.$$

ii) If

$$\int_{0+} \frac{dt}{t} \left(\frac{t}{h(t)}\right)^{\frac{\delta-2}{2}} \exp\left\{-\frac{t}{2h(t)}\right\} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\left(S_t < (1+\epsilon)h(t), \ i.o., \ as \ t \to 0\right) = 1.$$

Proof: The proof of this Theorem is a simple application of (7.29) to Proposition 2.∎

Similarly, we have the same integral test for large times.

Theorem 10 *Let $h \in \mathcal{H}_\infty^{-1}$ and $\delta \geq 2$,*

i) If

$$\int^{+\infty} \left(\frac{t}{h(t)}\right)^{\frac{\delta-2}{2}} \exp\left\{-\frac{t}{2h(t)}\right\} \frac{dt}{t} < \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\left(S_t < (1-\epsilon)h(t), \ i.o., \ as \ t \to +\infty\right) = 0.$$

ii) If

$$\int_{+\infty} \frac{dt}{t} \left(\frac{t}{h(t)}\right)^{\frac{\delta-2}{2}} \exp\left\{-\frac{t}{2h(t)}\right\} = \infty,$$

then for all $\epsilon > 0$

$$\mathbb{P}\left(S_t < (1+\epsilon)h(t), \ i.o., \ as \ t \to +\infty\right) = 1.$$

From these integral tests, we get the following law of the iterated logarithm

$$\liminf_{t \to 0} S_t \frac{2 \log |\log t|}{t} = 1 \quad \text{and} \quad \liminf_{t \to +\infty} S_t \frac{2 \log \log t}{t} = 1 \quad \text{almost surely.}$$

For the upper envelope of $X^{(0)}$, we have the following integral tests.

Theorem 11 *Let $h \in \mathcal{H}_0$ and $\delta \geq 2$,*

 i) If

$$\int_{0+} \left(\frac{h(t)}{t} \right)^{\frac{\delta-2}{2}} \exp\left\{ -\frac{h(t)}{2t} \right\} \frac{dt}{t} < \infty,$$

 then for all $\epsilon > 0$

$$\mathbb{P}\left(X_t^{(0)} > (1+\epsilon)h(t), \; i.o., \; as \; t \to 0 \right) = 0.$$

 ii) If

$$\int_{0+} \left(\frac{h(t)}{t} \right)^{\frac{\delta-2}{2}} \exp\left\{ -\frac{h(t)}{2t} \right\} \frac{dt}{t} = \infty,$$

 then for all $\epsilon > 0$

$$\mathbb{P}\left(X_t^{(0)} > (1-\epsilon)h(t), \; i.o., \; as \; t \to 0 \right) = 1.$$

Proof: The proof of this Theorem follows from a simple application of (7.29) to Theorem 2. The proof of the additional hypothesis (4.15), is clear from (7.29).   ■

 Similarly, we have the same integral tests for large times.

Theorem 12 *Let $h \in \mathcal{H}_\infty$ and $\delta \geq 2$,*

 i) If

$$\int^{+\infty} \left(\frac{h(t)}{t} \right)^{\frac{\delta-2}{2}} \exp\left\{ -\frac{h(t)}{2t} \right\} \frac{dt}{t} < \infty,$$

 then for all $\epsilon > 0$ and for all $x \geq 0$

$$\mathbb{P}\left(X_t^{(x)} > (1+\epsilon)h(t), \; i.o., \; as \; t \to +\infty \right) = 0.$$

 ii) If

$$\int^{+\infty} \left(\frac{h(t)}{t} \right)^{\frac{\delta-2}{2}} \exp\left\{ -\frac{h(t)}{2t} \right\} \frac{dt}{t} = \infty,$$

 then for all $\epsilon > 0$ and for all $x \geq 0$

$$\mathbb{P}\left(X_t^{(x)} > (1-\epsilon)h(t), \; i.o., \; as \; t \to +\infty \right) = 1.$$

Recall from the Kolmogorov and Dvoretzky-Erdös (KDE for short) integral test (Theorem 1) that for h a nondecreasing, positive and unbounded function as t goes to $+\infty$, the upper envelope of $X^{(0)}$ at 0 may be described as follows:

$$\mathbb{P}\big(X_t^{(0)} > h(t), \text{ i.o., as } t \to 0\big) = 0 \text{ or } 1,$$

according as,

$$\int_0 \left(\frac{h(t)}{t}\right)^{\frac{\delta}{2}} \exp\left\{-\frac{h(t)}{2t}\right\} \frac{\mathrm{d}t}{t} \qquad \text{is finite or infinite.}$$

Note that the class of functions that satisfy the divergent part of Theorems 11 and 12 implies the divergent part of the KDE integral test, hence ϵ can also take the value 0. The convergent part of the KDE integral test obviously implies the convergent part of Theorems 11 and 12.

From these integral tests, we get the following law of the iterated logarithm

$$\limsup_{t \to 0} \frac{X_t}{2t \log |\log t|} = 1 \quad \text{and} \quad \limsup_{t \to +\infty} \frac{X^{(x)}}{2t \log \log t} = 1 \quad \text{almost surely,}$$

for $x \geq 0$.

8 Examples.

Example 1. The first example that we will consider here, is the stable subordinator. Let $X^{(0)}$ be the stable subordinator with index $\alpha \in (0,1)$. From Zolotarev [25], we know that there exists k a positive constant such that

$$\mathbb{P}_0(X_1 > x) \sim kx^{-\alpha} \qquad x \to +\infty,$$

and since a subordinator is an increasing process, then

$$\mathbb{P}(S_1 < x) \sim kx^{\alpha} \qquad x \to 0,$$

From Example 3 in Section 7.1 in [19], we also have that

$$\mathbb{P}\big(I(\hat{\xi}) < x\big) \sim mk\alpha x^{\alpha+1} \qquad x \to 0.$$

Hence from Theorems 4 and 5, we obtain the following corollaries.

Corollary 3 *The lower envelope of S, the first passage time of the stable subordinator $X^{(0)}$ with index $\alpha \in (0,1)$ at 0 and at $+\infty$ is as follows:*

i) Let $h \in \mathcal{H}_0^{-1}$, such that either $\lim_{x \to 0} h(x)/x = 0$ or $\liminf_{x \to 0} h(x)/x > 0$, then

$$\mathbb{P}\big(S_x < h(t), \text{ i.o., as } x \to 0\big) = 0 \text{ or } 1,$$

according as

$$\int_{0+} \left(\frac{h(x)}{x}\right)^{\alpha} \frac{\mathrm{d}x}{x} \quad \text{is finite} \quad \text{or} \quad \int_{0+} \left(\frac{h(x)}{x}\right)^{\alpha+1} \frac{\mathrm{d}x}{x} \quad \text{is infinite.}$$

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ii) *Let* $h \in \mathcal{H}_\infty^{-1}$, *such that either* $\lim_{x \to +\infty} h(x)/x = 0$ *or* $\liminf_{x \to +\infty} h(x)/x > 0$, *then*

$$\mathbb{P}\Big(S_x < h(x), \ i.o., \ as \ x \to \infty\Big) = 0 \ or \ 1,$$

according as

$$\int^{+\infty} \left(\frac{h(x)}{x}\right)^\alpha \frac{\mathrm{d}x}{x} \quad is \ finite \quad or \quad \int^{+\infty} \left(\frac{h(x)}{x}\right)^{\alpha+1} \frac{\mathrm{d}x}{x} \quad is \ infinite.$$

Corollary 4 *The upper envelope of the stable subordinator with index* $\alpha \in (0,1)$ *at* 0 *and at* $+\infty$ *is as follows:*

i) *Let* $h \in \mathcal{H}_0$, *such that either* $\lim_{t \to 0} t/h(t) = 0$ *or* $\liminf_{t \to 0} t/h(t) > 0$, *then*

$$\mathbb{P}\Big(X_t^{(0)} > h(t), \ i.o., \ as \ t \to 0\Big) = 0 \ or \ 1,$$

according as

$$\int_{0^+} \left(\frac{h(x)}{x}\right)^\alpha \frac{\mathrm{d}x}{x} \quad is \ finite \quad or \quad \int_{0^+} \left(\frac{h(x)}{x}\right)^{\alpha+1} \frac{\mathrm{d}x}{x} \quad is \ infinite.$$

ii) *Let* $h \in \mathcal{H}_\infty$, *such that either* $\lim_{t \to +\infty} t/h(t) = 0$ *or* $\liminf_{t \to +\infty} t/h(t) > 0$, *then for all* $x \geq 0$

$$\mathbb{P}\Big(X_t^{(x)} > h(t), \ i.o., \ as \ t \to \infty\Big) = 0 \ or \ 1,$$

according as

$$\int^{+\infty} \left(\frac{h(x)}{x}\right)^\alpha \frac{\mathrm{d}x}{x} \quad is \ finite \quad or \quad \int^{+\infty} \left(\frac{h(x)}{x}\right)^{\alpha+1} \frac{\mathrm{d}x}{x} \quad is \ infinite.$$

Example 2. Let $X^{(0)}$ be a stable Lévy process conditioned to stay positive with no positive jumps and index $1 < \alpha \leq 2$ (See Bertoin [2] for a proper definition).
From Pardo [19], we know that $X^{(0)}$ drifts towards $+\infty$ and that

$$-\log \hat{F}(1/t) \sim \frac{\alpha - 1}{\alpha} \left(\frac{1}{\alpha}\right)^{1/(\alpha-1)} x^{1/\alpha-1} \quad \text{as} \quad t \to +\infty.$$

Then applying Theorems 5 and 6, and Corollary 3, we get the following laws of the iterated logarithm.

Corollary 5 *Let* $X^{(0)}$ *be a stable Lévy process conditioned to stay positive with no positive jumps and* $\alpha > 1$. *Then, the first passage time process satisfies*

$$\liminf_{t \to 0} \frac{S_t \big(\log|\log t|\big)^{\alpha-1}}{t^\alpha} = \frac{1}{\alpha}\left(1 - \frac{1}{\alpha}\right)^{\alpha-1}, \quad almost \ surely.$$

The same law of the iterated logarithm is satisfied for large times.

The processes $X^{(x)}$, $J^{(x)}$ and $X^{(x)} - J^{(x)}$ satisfy the following laws of the iterated logarithm

$$\limsup_{t \to 0} \frac{X_t^{(0)}}{t^{1/\alpha} \left(\log |\log t| \right)^{1-1/\alpha}} = \alpha \, (\alpha - 1)^{-\frac{\alpha-1}{\alpha}}, \qquad almost \ surely,$$

$$\limsup_{t \to 0} \frac{J_t^{(0)}}{t^{1/\alpha} \left(\log |\log t| \right)^{1-1/\alpha}} = \alpha \, (\alpha - 1)^{-\frac{\alpha-1}{\alpha}}, \qquad almost \ surely,$$

$$\limsup_{t \to 0} \frac{X_t^{(0)} - J_t^{(0)}}{t^{1/\alpha} \left(\log |\log t| \right)^{1-1/\alpha}} = \alpha \, (\alpha - 1)^{-\frac{\alpha-1}{\alpha}}, \qquad almost \ surely,$$

and for all $x \geq 0$,

$$\limsup_{t \to +\infty} \frac{X_t^{(x)}}{t^{1/\alpha} \left(\log \log t \right)^{1-1/\alpha}} = \alpha \, (\alpha - 1)^{-\frac{\alpha-1}{\alpha}}, \qquad almost \ surely,$$

$$\limsup_{t \to +\infty} \frac{J_t^{(x)}}{t^{1/\alpha} \left(\log \log t \right)^{1-1/\alpha}} = \alpha \, (\alpha - 1)^{-\frac{\alpha-1}{\alpha}}, \qquad almost \ surely,$$

$$\limsup_{t \to +\infty} \frac{X_t^{(x)} - J_t^{(x)}}{t^{1/\alpha} \left(\log \log t \right)^{1-1/\alpha}} = \alpha \, (\alpha - 1)^{-\frac{\alpha-1}{\alpha}}, \qquad almost \ surely.$$

Example 3. Let ξ be a Lévy process which drifts towards $+\infty$ and with finite exponential moments of arbitrary positive order. Note that this condition is satisfied, for example, when the jumps of ξ are bounded from above by some fixed number, in particular when ξ is a Lévy process with no positive jumps. More precisely, we have

$$\mathbb{E}\left(e^{\lambda \xi_t}\right) = \exp\left\{ t\psi(\lambda) \right\} < +\infty \qquad t, \lambda \geq 0.$$

From Theorem 25.3 in Sato [23], we know that this hypothesis is equivalent to assume that the Lévy measure Π of ξ satisfies

$$\int_{[1,\infty)} e^{\lambda x} \Pi(\mathrm{d}x) < +\infty \qquad \text{for every } \lambda > 0.$$

Under this condition and with the hypothesis that ψ, the Laplace exponent of ξ, varies regularly at $+\infty$ with index $\beta \in (1, 2)$, Pardo [19] gave the following estimates of the tail probabilities of $I(\hat{\xi})$ and $\nu I(\hat{\xi})$,

$$-\log \mathbb{P}\left(\nu I(\hat{\xi}) < 1/x\right) \sim -\log \mathbb{P}\left(I(\hat{\xi}) < 1/x\right) \sim (\beta - 1)\overleftarrow{H}(x) \qquad \text{as} \qquad x \to +\infty,$$

where

$$\overleftarrow{H}(x) = \inf\left\{ s > 0 \, , \, \psi(s)/s > x \right\},$$

is a regularly varying function with index $(\beta - 1)^{-1}$.

Hence, the pssMp associated to ξ satisify condition (6.24). This allow us to obtain laws of iterated logarithm for the first passage time process and for the pssMp in terms of the following function.

Let us define the function

$$f(t) := \frac{\log|\log t|}{\psi(\log|\log t|)} \qquad \text{for} \qquad t > 1, \quad t \neq e.$$

By integration by parts, we can see that the function $\psi(\lambda)/\lambda$ is increasing, hence it is straightforward that the function $tf(t)$ is also increasing in a neighbourhood of ∞.

Corollary 6 *If ψ is regularly varying at $+\infty$ with index $\beta \in (1,2)$, then*

$$\liminf_{x \to 0} \frac{S_x}{xf(x)} = (\beta - 1)^{\beta - 1} \qquad \text{almost surely}$$

and,

$$\liminf_{x \to +\infty} \frac{S_x}{xf(x)} = (\beta - 1)^{\beta - 1} \qquad \text{almost surely.}$$

Let us define

$$g(t) := \frac{\psi(\log|\log t|)}{\log|\log t|} \qquad \text{for} \quad t > 1, \quad t \neq e.$$

Corollary 7 *If ψ is regularly varying at $+\infty$ with index $\beta \in (1,2)$, then*

$$\limsup_{t \to 0} \frac{X_t}{tg(t)} = (\beta - 1)^{-(\beta - 1)} \qquad \text{almost surely}$$

and for all $x \geq 0$,

$$\limsup_{t \to +\infty} \frac{X_t^{(x)}}{tg(t)} = (\beta - 1)^{-(\beta - 1)} \qquad \text{almost surely.}$$

Moreover, if the processes $X^{(x)}$ has no positive jumps, $J^{(x)}$ and $X^{(x)} - J^{(x)}$ satisfy the following laws of the iterated logarithm:

$$\limsup_{t \to 0} \frac{J_t^{(0)}}{tg(t)} = (\beta - 1)^{-(\beta - 1)}, \qquad \text{almost surely,}$$

$$\limsup_{t \to 0} \frac{X_t^{(0)} - J_t^{(0)}}{tg(t)} = (\beta - 1)^{-(\beta - 1)}, \qquad \text{almost surely,}$$

and for all $x \geq 0$,

$$\limsup_{t \to +\infty} \frac{J_t^{(x)}}{tg(t)} = (\beta - 1)^{-(\beta - 1)}, \qquad \text{almost surely,}$$

$$\limsup_{t \to +\infty} \frac{X_t^{(x)} - J_t^{(x)}}{tg(t)} = (\beta - 1)^{-(\beta - 1)}, \qquad \text{almost surely.}$$

Example 4. Sato [22] (see also Sato [23]) studied some interesting properties of positive increasing self-similar processes with independent increments. In particular, the author showed that if $Y = (Y_t, t \geq 0)$ is a process with such characteristics and starting from 0, we can represent its Laplace transform by

$$\mathbb{E}\Big[\exp\big\{-\lambda Y_1\big\}\Big] = \exp\big\{-\kappa(\lambda)\big\} \qquad \text{for } \lambda > 0,$$

where

$$\kappa(\lambda) = c\lambda + \int_0^{+\infty} \big(1 - e^{-\lambda x}\big)\frac{l(x)}{x}\mathrm{d}x,$$

$c \geq 0$ and $l(x)$ is a nonnegative decreasing function on $(0, +\infty)$ with

$$\int_0^{+\infty} \frac{l(x)}{1+x}\mathrm{d}x < +\infty.$$

From its definition, it is clear that the Laplace exponent κ is an increasing continuous function and more precisely a concave function.

Under the assumption that κ varies regularly at $+\infty$ with index $\alpha \in (0, 1)$, we will have the following sharp estimate for the distribution of Y_1.

Let us define the function

$$\rho(t) = \frac{t \log|\log t|}{\overleftarrow{\kappa}(\log|\log t|)}, \qquad \text{for} \quad t \neq e, \quad t > 1,$$

where $\overleftarrow{\kappa}$ is the inverse function of κ.

Proposition 10 *Let $(Y_t, t \geq 0)$ be a positive increasing self-similar processes with independent increments and suppose that κ, its Laplace exponent, varies regularly at $+\infty$ with index $\alpha \in (0, 1)$. Then for every $c > 0$, we have*

$$-\log\mathbb{P}\left(Y_1 \leq \frac{c\rho(t)}{t}\right) \sim (\alpha/c)^{\frac{\alpha}{1-\alpha}}(1-\alpha)\log|\log t| \quad \text{as} \quad t \to 0 \quad (t \to \infty).$$

Proof: From de Brujin's Tauberian Theorem (see for instance Theorem 4.12.9 in [6]), we have that if κ varies regularly at $+\infty$ with index $\alpha \in (0, 1)$ then

$$-\log\mathbb{P}\big(Y_1 \leq x\big) \sim \frac{\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)}{\overleftarrow{\omega}(1/x)}, \qquad \text{for} \quad x \to 0,$$

where $\overleftarrow{\omega}$ is the asymptotic inverse of ω, a regularly varying function at $+\infty$ with index $(\alpha - 1)/\alpha$ and that satisfies

$$\frac{\lambda}{\kappa(\lambda)} \sim \omega\left(\frac{1}{\kappa(\lambda)}\right) \quad \text{for} \quad \lambda \to +\infty. \tag{8.30}$$

Hence, taking $x = c\rho(t)/t$ and $\lambda = \overleftarrow{\kappa}(\log|\log t|)$, and doing some calculations we get the desired result. ∎

This estimate allow us to get the following law of the iterated logarithm.

Corollary 8 *Let $(Y_t, t \geq 0)$ be a positive increasing self-similar processes with independent increments and suppose that κ, its Laplace exponent, satisfies the conditions of the previous Proposition. Then, we have*

$$\liminf_{t \to 0} \frac{Y_t}{\rho(t)} = \alpha(1 - \alpha)^{(1-\alpha)/\alpha}, \qquad \text{almost surely.}$$

The same law of iterated logarithm is satisfied for large times.

Proof: The proof of this Corollary follows from the integral test found by Watanabe [24] and applying the above estimate of the distribution of Y_1. ∎

Now, we denote by κ_1 and κ_2 the Laplace exponents of the last and first passage time processes, respectively. Since $S_1 \leq U_1$, it is clear that $\kappa_2(\lambda) \leq \kappa_1(\lambda)$ for all $\lambda \geq 0$. Let us suppose that κ_1 and κ_2 are regularly varying at $+\infty$ with index α_1 and α_2 respectively, such that $0 < \alpha_2 \leq \alpha_1 < 1$. By Proposition 9 and Theorem 8, we can deduce that κ_1 and κ_2 are asymptotically equivalents and that $\alpha_1 = \alpha_2$. Then by the above Corollary, we have that for

$$h_1(t) := \frac{t \log|\log t|}{\overleftarrow{\kappa}_1(\log|\log t|)} \quad \text{and} \quad h_2(t) := \frac{t \log|\log t|}{\overleftarrow{\kappa}_2(\log|\log t|)}, \qquad \text{for} \quad t \neq e, \quad t > 1,$$

where $\overleftarrow{\kappa}_1$ and $\overleftarrow{\kappa}_2$ are the inverse of κ_1 and κ_2, respectively; the processes U and S satisfy

$$\liminf_{t \to 0} \frac{U_t}{h_1(t)} = \alpha_1(1 - \alpha_1)^{(1-\alpha_1)/\alpha_1} \qquad \text{almost surely,}$$

and

$$\liminf_{t \to 0} \frac{S_t}{h_1(t)} = \alpha_1(1 - \alpha_1)^{(1-\alpha_1)/\alpha_1} \qquad \text{almost surely.}$$

Note that we can replace h_1 by h_2 and that we also have the same laws of the iterated logarithm for large times.
By the sharp estimation in Proposition 10 of the tail probability of S_1, we deduce the following law of the iterated logarithm.
Let us define

$$f_2(t) := \frac{t\overleftarrow{\kappa}_2(\log|\log t|)}{\log|\log t|}, \qquad \text{for} \quad t \neq e, \quad t > 1.$$

Corollary 9 *Let κ_2 be the Laplace exponent of S_1. If κ_2 is regularly varying at $+\infty$ with index $\alpha_2 \in (0,1)$, then*

$$\limsup_{t \to 0} \frac{X_t^{(0)}}{f_2(t)} = \alpha_2^{-1}(1 - \alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \text{almost surely,}$$

and for any $x \geq 0$,

$$\limsup_{t \to +\infty} \frac{X_t^{(x)}}{f_2(t)} = \alpha_2^{-1}(1 - \alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \text{almost surely.}$$

On the other hand, from Theorem 8 we get the following Corollary.

Corollary 10 *Let κ_2 be the Laplace exponent of S_1 and $\overleftarrow{\kappa}_2$ its inverse. If κ_2 is regularly varying at $+\infty$ with index $\alpha_2 \in (0, 1)$, then*

i)

$$\limsup_{t \to 0} \frac{J_t^{(0)}}{f_2(t)} = \alpha_2^{-1}(1 - \alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \textit{almost surely,}$$

and for any $x \geq 0$,

$$\limsup_{t \to +\infty} \frac{J_t^{(x)}}{f_2(t)} = \alpha_2^{-1}(1 - \alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \textit{almost surely.}$$

ii)

$$\limsup_{t \to 0} \frac{X_t^{(0)} - J_t^{(0)}}{f_2(t)} = \alpha_2^{-1}(1 - \alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \textit{almost surely,}$$

and for any $x \geq 0$,

$$\limsup_{t \to +\infty} \frac{X_t^{(x)} - J_t^{(x)}}{f_2(t)} = \alpha_2^{-1}(1 - \alpha_2)^{-(1-\alpha_2)/\alpha_2} \qquad \textit{almost surely.}$$

Now, we will apply these results to the case of transient Bessel process. Here, we employ the usual Bessel functions I_a and K_a, as in Kent [15] and Jeanblanc, Pitman and Yor [14]. It is well known that

$$\mathbb{E}\left(\exp\left\{-\lambda S_1\right\}\right) = \lambda^{a/2}\frac{1}{2^{a/2}\Gamma(a + 1)I_a(\sqrt{2\lambda})}, \qquad \lambda > 0,$$

and

$$\mathbb{E}\left(\exp\left\{-\lambda U_1\right\}\right) = \frac{\lambda^{a/2}}{2^{a/2-1}\Gamma(a)}K_a(\sqrt{2\lambda}), \qquad \lambda > 0,$$

where Γ is the gamma function (see for instance Jeanblanc, Pitman and Yor [14]). Now, we define for $\lambda > 0$

$$\phi_1(\lambda) = \log(2^{a/2-1}\Gamma(a)) - \log K_a(\sqrt{2\lambda}) - \log \lambda^{a/2},$$

$$\phi_2(\lambda) = \log I_a(\sqrt{2\lambda}) + \log(2^{a/2}\Gamma(a + 1)) - \log \lambda^{a/2}.$$

Since, we have the following asymptotic behaviour

$$I_a(x) \sim (2\pi x)^{-1/2}e^x \quad \text{and} \quad K_a(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \quad \text{when } x \to +\infty,$$

(see Kent [15] for instance), we deduce that ϕ_1 and ϕ_2 are regularly varying at $+\infty$ with index $1/2$. From Proposition 9 and Theorem 8, we deduce that they are asymptotically

equivalent.

From the above results, we have that

$$\liminf_{t\to 0} \frac{U_t}{h_1(t)} = 1/4, \qquad \liminf_{t\to 0} \frac{S_t}{h_2(t)} = 1/4 \quad \text{almost surely,}$$

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{f_2(t)} = 4, \qquad \limsup_{t\to 0} \frac{J_t^{(0)}}{f_1(t)} = 4 \quad \text{almost surely,}$$

and

$$\limsup_{t\to 0} \frac{J_t^{(0)}}{f_2(t)} = 4, \qquad \limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{f_1(t)} = 4 \quad \text{almost surely,}$$

where

$$h_1(t) = \frac{t \log|\log t|}{\overleftarrow{\kappa}_1(\log|\log t|)}, \quad h_2(t) = \frac{t \log|\log t|}{\overleftarrow{\kappa}_2(\log|\log t|)}, \quad f_1(t) = \frac{t^2}{h_1(t)}, \quad f_2(t) = \frac{t^2}{h_2(t)}$$

and, $\overleftarrow{\kappa}_1$ and $\overleftarrow{\kappa}_2$ are the inverse functions of κ_1 and κ_2, respectively.

Similarly, we have all these laws of the iterated logarithm for large times.

Example 5. Let $\xi = N$ be a standard Poisson process. From Proposition 3 in Bertoin and Yor [5] and Example 1 in Pardo [19], we know

$$-\log \mathbb{P}\big(I(\hat{\xi}) < t\big) \sim -\log \mathbb{P}\big(\nu I(\hat{\xi}) < t\big) \sim \frac{1}{2}(\log 1/t)^2, \qquad \text{as } t \to 0.$$

Hence, we obtain the following laws of the iterated logarithm. Let us define

$$m(t) := t \exp\left\{-\sqrt{2\log|\log t|}\right\}.$$

Corollary 11 *Let N be a Poisson process, then the pssMp $X^{(x)}$ associated to N by the Lamperti representation satisfies the following law of the iterated logarithm,*

$$\limsup_{t\to 0} \frac{X_t^{(0)} m(t)}{t^2} = 1 \qquad \text{almost surely.}$$

For $x \geq 0$

$$\limsup_{t\to +\infty} \frac{X_t^{(x)} m(t)}{t^2} = 1 \qquad \text{almost surely.}$$

The first passage time process S associated to $X^{(0)}$ satisfies the following law of the iterated logarithm,

$$\liminf_{x\to 0} \frac{S_x}{m(x)} = 1, \quad \text{and} \quad \liminf_{x\to +\infty} \frac{S_x}{m(x)} = 1, \quad \text{almostsurely.}$$

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