Pre-compact families of finite sets of integers and weakly null sequences in Banach spaces

Jordi Lopez Abad, S. Todorcevic

▶ To cite this version:

PRE-COMPACT FAMILIES OF FINITE SETS OF INTEGERS AND WEAKLY NULL SEQUENCES IN BANACH SPACES

J. LOPEZ-ABAD AND S. TODORCEVIC

1. INTRODUCTION

The affinities between the infinite-dimensional Ramsey theory and some problems of the Banach space theory and especially those dealing with Schauder basic sequences have been explored for quite some time, starting perhaps with Farahat's proof of Rosenthal's ℓ_1-theorem (see [13] and [19]). The Nash-Williams' theory though implicit in all this was not fully exploited in this context. In this paper we try to demonstrate the usefulness of this theory by applying it to the classical problem of finding unconditional basic-subsequence of a given normalized weakly null sequence in some Banach space E. Recall that Bessaga and Pelczynski [7] have shown that every normalized weakly null sequence in a Banach space contains a subsequence forming a Schauder basis for its closed linear span. However, as demonstrated by Maurey and Rosenthal [16] there exist weakly null sequences in Banach spaces without unconditional basic subsequences. So one is left with a task of finding additional conditions on a given weakly null sequence guaranteeing the existence of unconditional subsequences. One such condition, given by Rosenthal himself around the time of publication of [16] (see also [19]). When put in a proper context Rosenthal's condition reveals the connection with the Nash-Williams theory. It says that if a weakly null sequence (x_n) in some space of the form $\ell_\infty(\Gamma)$ is such that each x_n takes only the values 0 or 1, then (x_n) has an unconditional subsequence. To see the connection, consider the family

$$\mathcal{F} = \{\{n \in \mathbb{N} : x_n(\gamma) = 1\} : \gamma \in \Gamma\}$$

and note that $\mathcal{F}$ is a pre-compact family of finite subsets of $\mathbb{N}$. As pointed out in [19], Rosenthal result is equivalent saying that there is an infinite subset M of $\mathbb{N}$ such that the *trace*

$$\mathcal{F}[M] = \{t \cap M : t \in \mathcal{F}\}$$

is *hereditary*, i.e., it is downwards closed under inclusion. On the other hand, recall that the basic notion of the Nash-Williams' theory is the notion of a *barrier*, which is simply a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ no two members of which are related under the inclusion which has the property that an arbitrary infinite subset of $\mathbb{N}$ contains an initial segment in $\mathcal{F}$. Thus, in particular, $\mathcal{F}$ is a pre-compact family of finite subsets of $\mathbb{N}$. Though the trace of an arbitrary pre-compact family might be hard to visualize, a trace $\mathcal{B}[M]$ of a barrier $\mathcal{B}$ is easily to compute as it is simply equal to the downwards closure of its *restriction*

$$\mathcal{B} \restriction M = \{t \in \mathcal{B} : t \subseteq M\}.$$

A further examination of Rosenthal's result shows that for every pre-compact family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ there is an infinite set M such that the trace $\mathcal{F}[M]$ is actually equal to the downwards closure of a uniform barrier $\mathcal{B}$ on M, or in other words that the $\subseteq$-maximal elements of $\mathcal{F}[M]$ form a uniform barrier on M. As it turns out, this fact holds considerably more information that the conclusion that $\mathcal{F}[M]$ is merely a hereditary family which is especially noticeable if one need to perform further refinements of M while keeping truck on the original family $\mathcal{F}$. This observation was the motivating point for our research which helped us to realize that further extensions of Rosenthal's result require analysis of not only pre-compact families of finite subsets of $\mathbb{N}$ but also maps from barriers into pre-compact families of finite subsets of $\mathbb{N}$, or, more generally, into weakly compact subsets of c_0. We have explained this point in our previous paper [14], where we have presented various results on partial unconditionality such as near-unconditionality or convex-unconditionality as consequences of the structure theory of this kind of mappings. This paper is as a continuation of this line of research. In Section 3 we show how the combinatorics on barriers can be used to prove the c_0-saturation for Banach spaces $C(K)$ when K is a countable compactum. Recall that the c_0-saturation of Banach spaces $C(K)$ over countable compacta K is a result originally due to Pełczyński and Semadeni [21] (see also [5] and [12] for recent accounts on this result.) More particularly, we show that if $(x_i) \subseteq C(K)$ is a normalized weakly-null sequence, then there is $C \geq 1$, some infinite set M, some uniform barrier $\mathcal{B}$ on M of rank at most the Cantor-Bendixson rank of K and some *uniform* assignment $\mu : \mathcal{B} \to c_{00}^+$ with the property that supp $\mu(s) \subseteq s$ for every $s \in \mathcal{B}$, and such that for every block sequence (s_n) of elements of $\mathcal{B}$, the corresponding sequence $(x(s_n))$ of linear combinations,

$$x(s_n) = \sum_{i \in s_n} (\mu(s_n))(i)x_i,$$

is a normalized block sequence C-equivalent to the standard basis of c_0.

The last section concerns the following natural measurement of unconditionality present in a given weakly null sequence (x_n) in a general Banach space E. Given a family $\mathcal{F}$ of finite sets, we say that (x_n) is $\mathcal{F}$-*unconditional* with constant at most $C \geq 1$ iff for every sequence of scalars (a_n),

$$\sup_{s \in \mathcal{F}} \Big\| \sum_{n \in s} a_n x_n \Big\| \leq C \Big\| \sum_{n \in \mathbb{N}} a_n x_n \Big\|.$$

Thus, if for some infinite subset M of $\mathbb{N}$ the trace $\mathcal{F}[M]$ contains the family of all finite subsets of M, the corresponding subsequence $(x_n)_{n \in M}$ is unconditional. Typically, one will not be able to find such a trace, so one is naturally led to study this notion when the family $\mathcal{F}$ is pre-compact, or equivalently, when $\mathcal{F}$ is a barrier. Since for every pair $\mathcal{F}_0$ and $\mathcal{F}_1$ of barriers on $\mathbb{N}$ there is an infinite set M such that $\mathcal{F}_0[M] \subseteq \mathcal{F}_1[M]$ or $\mathcal{F}_1[M] \subseteq \mathcal{F}_0[M]$ and since the two alternatives depend on the ranks of $\mathcal{F}_0$ and $\mathcal{F}_1$, one is also naturally led to the following measurement of unconditionality that refers only to a countable ordinal γ rather than a particular barrier of rank γ. Thus, we say that a normalized basic sequence (x_n) of a Banach space X is γ-*unconditionally saturated* with constant at most $C \geq 1$ if there is an γ-uniform barrier $\mathcal{B}$ on $\mathbb{N}$ such that for every infinite $M \subseteq \mathbb{N}$ there is infinite $N \subseteq M$ such that the corresponding subsequence $(x_n)_{n \in N}$ of (x_n) is $\overline{\mathcal{B} \restriction N}$-unconditional with constant at most C. (Here, $\overline{\mathcal{B} \restriction N}$ denotes the topological closure of the restriction $\mathcal{B} \restriction N$ which in turn is equal to the trace $\mathcal{B}[N]$, a pleasant property of

any barrier.) It turns out that only indecomposable countable ordinals γ matter for this notion. We shall see, extending the well-known example of Maurey-Rosenthal of a normalized weakly-null sequence without unconditional subsequences, that every normalized basic sequence has a subsequence which is ω-unconditionally saturated, and that this cannot be extended further. For example, we show that for every indecomposable countable ordinal $\gamma > \omega$ there is a compactum K of Cantor-Bendixson rank $\gamma + 1$ and a normalized 1-basic weakly-null sequence $(x_n) \subseteq C(K)$ such that (x_n) is β-unconditionally saturated for all $\beta < \gamma$ but not γ-unconditionally saturated. More precisely, the summing basis of c_0 is finitely block-representable in every subsequence of (x_n), and so in particular, no subsequence of (x_n) is unconditional.

2. PRELIMINARIES

Let $\mathbb{N}$ denote the set of all non-negative integers and let FIN denote the family of all finite sets of $\mathbb{N}$. The topology on FIN is the one induced from the Cantor cube $^{\mathbb{N}}2$ via the identification of subsets of $\mathbb{N}$ with their characteristics function. Observe that this topology coincides with the one induced by c_0, the Banach space of sequences converging to zero, with the same identification of finite sets and corresponding characteristic functions. Thus, we say that a family $\mathcal{F} \subseteq$ FIN is *compact* if it is a compact space under the induced topology. We say that $\mathcal{F} \subseteq$ FIN is *pre-compact* if its topological closure $\overline{\mathcal{F}}^{\text{top}}$ taken in the Cantor cube $^{\mathbb{N}}2$ consists only of finite subsets of $\mathbb{N}$. Given $X, Y \subseteq \mathbb{N}$ we write

(1) $X < Y$ iff $\max X < \min Y$. We will use the convention $\emptyset < X$ and $X < \emptyset$ for every X.

(2) $X \sqsubseteq Y$ iff $X \subseteq Y$ and $X < Y \setminus X$.

A sequence (s_i) of finite sets of integers is called a *block sequence* iff $s_i < s_j$ for every $i < j$, and it is called a *Δ-sequence* iff there is some finite set s such that $s \sqsubseteq s_i$ $(i \in \mathbb{N})$ and $(s_i \setminus s)$ is a block sequence. The set s is called the *root* of (s_i). Note that $s_i \to_i s$ iff for every subsequence of (s_i) has a Δ-subsequence with root s. It follows that the topological closure $\overline{\mathcal{F}}$ of a pre-compact family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is included in its downwards closure

$$\overline{\mathcal{F}}^{\subseteq} = \{s \subseteq t \; : \; t \in \mathcal{F}\}$$

with respect to the inclusion relation and also included in its downwards closure

$$\overline{\mathcal{F}}^{\sqsubseteq} = \{s \sqsubseteq t \; : \; t \in \mathcal{F}\}$$

with respect to the relation $\sqsubseteq$. We say that a family $\mathcal{F} \subseteq$ FIN is $\subseteq$-*hereditary* if $\mathcal{F} = \overline{\mathcal{F}}^{\subseteq}$ and $\sqsubseteq$-*hereditary* if $\mathcal{F} = \overline{\mathcal{F}}^{\sqsubseteq}$. The $\subseteq$-hereditary families will simply be called *hereditary families*. We shall consider the following two restrictions of a given family $\mathcal{F}$ of subsets of $\mathbb{N}$ to a finite or infinite subset X of $\mathbb{N}$

$$\mathcal{F} \upharpoonright X = \{s \in \mathcal{F} \; : \; s \subseteq X\},$$
$$\mathcal{F}[X] = \{s \cap X \; : \; s \in \mathcal{F}\}.$$

There are various ways to associate an ordinal index to a pre-compact family $\mathcal{F}$ of finite subsets of $\mathbb{N}$. All these ordinal indices are based on the fact that for $n \in \mathbb{N}$, the index of the family

$$\mathcal{F}_{\{n\}} = \{s \in \text{FIN} \; : \; n < s, \; \{n\} \cup s \in \mathcal{F}\}$$

is smaller or equal from that of $\mathcal{F}$. For example, one may consider the *Cantor-Bendixson index* $r(\mathcal{F})$, the minimal ordinal α for which the iterated Cantor-Bendixson derivative $\partial^{\alpha}(\mathcal{F})$ is equal to $\emptyset$, then clearly $r(\mathcal{F}_{\{n\}}) \leq r(\mathcal{F})$ for all $n \in \mathbb{N}$. Recall that $\partial\mathcal{F}$ is the set of all proper accumulation points of $\mathcal{F}$ and that $\partial^{\alpha}(\mathcal{F}) = \bigcap_{\xi < \alpha} \partial(\partial^{\xi}(\mathcal{F}))$. The rank is well defined since $\overline{\mathcal{F}}$ is countable and therefore a scattered compactum so the sequence $\partial^{\xi}(\mathcal{F})$ of iterated derivatives must vanish. Observe that if $\mathcal{F}$ is a nonempty compact, then necessarily $r(\mathcal{F})$ is a successor ordinal.

We are now ready to introduce the basic combinatorial concepts of this section. For this we need the following piece of notation, where X and Y are subsets of $\mathbb{N}$

$$_{*}X = X \setminus \{\min X\} \text{ and } X/Y = \{m \in X \ : \ \max Y < m\}$$

The set $_{*}X$ is called the *shift* of X. Given integer $n \in \mathbb{N}$, we write X/n to denote $X/\{n\} = \{m \in X \ : \ m > n\}$. The following notions have been introduced by Nash-Williams.

Definition 2.1. ([15]) Let $\mathcal{F} \subseteq \mathrm{FIN}$.
(1) $\mathcal{F}$ is called *thin* if $s \not\sqsubseteq t$ for every pair s, t of distinct members of $\mathcal{F}$.
(2) $\mathcal{F}$ is called *Sperner* if $s \not\subseteq t$ for every pair $s \neq t \in \mathcal{F}$.
(3) $\mathcal{F}$ is called *Ramsey* if for every finite partition $\mathcal{F} = \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_k$ there is an infinite set $M \subseteq \mathbb{N}$ such that at most one of the restrictions $\mathcal{F}_i \restriction M$ is non-empty.
(4) $\mathcal{F}$ is called a *front* on M if $\mathcal{F} \subseteq \mathcal{P}(M)$, it is thin, and for every infinite $N \subseteq M$ there is some $s \in \mathcal{F}$ such that $s \sqsubseteq N$.
(5) $\mathcal{F}$ is called a *barrier* on M if $\mathcal{F} \subseteq \mathcal{P}(M)$, it is Sperner, and for every infinite $N \subseteq M$ there is some $s \in \mathcal{F}$ such that $s \sqsubseteq N$.

Clearly, every barrier is a front but not vice-versa. For example, the family $\mathbb{N}^{[k]}$ of all k-element subsets of $\mathbb{N}$ is a barrier. The basic result of Nash-Williams [15] says that every front (and therefore every barrier) is Ramsey. Since as we will see soon there are many more barriers than those of the form $\mathbb{N}^{[k]}$ this is a far reaching generalization of the classical result of Ramsey. To see a typical application, let $\mathcal{F}$ be a front on some infinite set M and consider its partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, where $\mathcal{F}_0$ is the family of all $\subseteq$-minimal elements of $\mathcal{F}$. Since $\mathcal{F}$ is Ramsey there is an infinite $N \subseteq M$ such that one of the restrictions $\mathcal{F}_i \restriction M$ is empty. Note that $\mathcal{F}_1 \restriction N$ must be empty. Since $\mathcal{F}_0 \restriction N$ is clearly a Sperner family, it is a barrier on N. Thus we have shown that every front has a restriction that is a barrier. Since barrier are more pleasant to work with one might wonder why introducing the notion of front at all. The reason is that inductive constructions lead more naturally to fronts rather than barriers. To get an idea about this, it is instructive to consider the following notion introduced by Pudlak and Rödl.

Definition 2.2. ([22]) For a given countable ordinal α, a family $\mathcal{F}$ of finite subsets of a given infinite set M is called α-*uniform on* M provided that:
(a) $\alpha = 0$ implies $\mathcal{F} = \{\emptyset\}$,
(b) $\alpha = \beta + 1$ implies that $\mathcal{F}_{\{n\}}$ is β-uniform on M/n,
(c) $\alpha > 0$ limit implies that there is an increasing sequence $\{\alpha_n\}_{n \in M}$ of ordinals converging to α such that $\mathcal{F}_{\{n\}}$ is α_n-uniform on M/n for all $n \in M$.
$\mathcal{F}$ is called uniform on M if it is α-uniform on M for some countable ordinal α.

REMARK 2.3. (a) If $\mathcal{F}$ is a front on M, then $\overline{\mathcal{F}} = \overline{\mathcal{F}}^{\sqsubseteq}$.

(b) If $\mathcal{F}$ is uniform on M, then it is a front (though not necessarily a barrier) on M.

(c) If $\mathcal{F}$ is α-uniform (front, barrier) on M and $\Theta : M \to N$ is the unique order-preserving onto mapping between M and N, then $\Theta"\mathcal{F} = \{\Theta"s : s \in \mathcal{F}\}$ is α-uniform (front, barrier) on M.

(d) If $\mathcal{F}$ is α-uniform (front, barrier) on M then $\mathcal{F} \upharpoonright N$ is α-uniform (front, barrier) on N for every $N \subseteq M$.

(e) If $\mathcal{F}$ is uniform (front, barrier) on M, then for every $s \in \overline{\mathcal{F}}^{\sqsubseteq}$ the family

$$\mathcal{F}_s = \{t : s < t \text{ and } s \cup t \in \mathcal{F}\}$$

is uniform (front, barrier) on M/s.

(e) If $\mathcal{F}$ is α-uniform on M, then $\partial^\alpha(\overline{\mathcal{F}}) = \{\emptyset\}$, hence $r(\mathcal{F}) = \alpha + 1$. (Hint: use that $\partial^\beta(\mathcal{F}_{\{n\}}) = (\partial^\beta(\mathcal{F}))_{\{n\}}$ for every β and every compact family $\mathcal{F}$).

(f) It is easy to prove by induction on n that every n-uniform family on M is of the form $M^{[n]}$. This is not the case in general.

(g) An important example of a ω-uniform barrier on $\mathbb{N}$ is the family $\mathcal{S} = \{s : |s| = \min(s) + 1\}$. We call $\mathcal{S}$ a *Schreier barrier* since its downwards closure is commonly called a *Schreier family*. Indeed, it can be proved a $\mathcal{B}$ is a ω-uniform family on M iff there is an unbounded mapping $f : M \to \omega$ such that $\mathcal{B} = \{s \subseteq M : |s| = f(\min s) + 1\}$.

The following result based on Nash-Williams' extension of Ramsey's theorem explains the relationship between the concepts introduced above (see [4] for proofs and fuller discussion).

Proposition 2.4. *The following are equivalent for a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$:*

(a) $\mathcal{F}$ is Ramsey.

(b) There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is Sperner.

(c) There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is either empty or uniform on M.

(d) There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is either empty or a front on M.

(e) There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is either empty or a barrier on M.

(f) There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is thin.

(g) There is an infinite $M \subseteq \mathbb{N}$ such that for every infinite $N \subseteq M$ the restriction $\mathcal{F} \upharpoonright N$ cannot be split into two disjoint families that are uniform on N. $\qquad\square$

In this kind of Ramsey theory one frequently performs diagonalisation arguments that can be formalized using the following notion.

Definition 2.5. An infinite sequence $(M_k)_{k \in \mathbb{N}}$ of infinite subsets of $\mathbb{N}$ is called a *fusion sequence of subsets of $M \subseteq \mathbb{N}$* if for all $k \in \mathbb{N}$:

(a) $M_{k+1} \subseteq M_k \subseteq M$,

(b) $m_k < m_{k+1}$, where $m_k = \min M_k$.

The infinite set $M_\infty = \{m_k\}_{k \in \mathbb{N}}$ is called the *fusion set (or limit)* of the sequence $(M_k)_{k \in \mathbb{N}}$.

We have also the following simple facts connecting these combinatorial notions with the topological concepts considered at the beginning of this section.

Proposition 2.6. *Fix a family $\mathcal{F} \subseteq \text{FIN}$.*

(a) If $\mathcal{F}$ is a barrier on M then $\overline{\mathcal{F}}^{\subseteq} = \overline{\mathcal{F}^{\subseteq}} = \overline{\mathcal{F}}$, and hence $\overline{\mathcal{F}}^{\subseteq}$ is a compact family.

(b) If $\mathcal{F}$ is a barrier on M then for every $N \subseteq M$, $\overline{\mathcal{F} \upharpoonright N}^{\subseteq} = \overline{\mathcal{F}}^{\subseteq} \upharpoonright N$.

(c) Suppose that $\mathcal{F}$ is a barrier on M. Then for every $N \subseteq M$ such that $M \setminus N$ is infinite we have that $\mathcal{F}[N] = \overline{\mathcal{F} \upharpoonright N}^{\subseteq}$, and in particular $\mathcal{F}[N]$ is downwards closed.

(d) A family $\mathcal{F} \subseteq M^{[<\infty]}$ is the topological closure of a barrier on M iff $\mathcal{F}^{\subseteq-\max} = \mathcal{F}^{\subseteq-\max}$ is a barrier on M.

Barriers describe small families of finite sets, as it is shown in the following.

Theorem 2.7. [14] *Let $\mathcal{F} \subseteq \mathrm{FIN}$ be an arbitrary family. Then there is an infinite set $M \subseteq \mathbb{N}$ such that either*

(a) $\mathcal{F}[M]$ is the closure of a uniform barrier on M, or

(b) $M^{[\infty]} \subseteq \overline{\mathcal{F}}^{\subseteq}$.

Note that it follows that if $\mathcal{F}$ is pre-compact then condition (a) must hold.

We shall follow standard terminology and notation when dealing with sequences in Banach spaces (see [13]). We recall now few standard definitions we are going to use along this paper.

Definition 2.8. Let (x_i) be a sequence in a Banach space E.

(a) (x_i) is called *weakly-null* iff for every $x^* \in E^*$, the sequence of scalars $(x^*(x_i))_i$ tends to 0.

(b) (x_i) is called a *Schauder basis* of E iff for every $x \in E$ there is a unique sequence of scalars (a_i) such that $x = \sum_i a_i x_i$. This is equivalent to say that $x_i \neq 0$ for every i, the closed linear span of (x_i) is X, and there is a constant $\theta \geq 1$ such that for every sequence of scalars (a_i), and every interval $I \subseteq \mathbb{N}$,

$$\|\sum_{i \in I} a_i x_i\| \leq \theta \|\sum_{i \in \mathbb{N}} a_i x_i\|. \tag{1}$$

(c) (x_i) is called a basic sequence iff it is a Schauder basis of its closed linear span, i.e., $x_i \neq 0$ for every i, and there is $\theta \geq 1$ such that for every sequence of scalars (a_i), and every interval $I \subseteq \mathbb{N}$, $\|\sum_{i \in I} a_i x_i\| \leq \theta \|\sum a_i x_i\|$. The infimum of those constants θ is called the basic constant of (x_i).

(d) (x_i) is called θ-*unconditional* $(\theta \geq 1)$ iff for every sequence of scalars (a_i), and every subset $A \subseteq \mathbb{N}$,

$$\|\sum_{i \in A} a_i x_i\| \leq \theta \|\sum_{i \in \mathbb{N}} a_i x_i\|. \tag{2}$$

(x_i) is called unconditional if it is θ-unconditional for some $\theta \geq 1$.

Given two basic sequences $(x_i)_{i \in M}$ and $(y_i)_{i \in N}$ of some Banach spaces E and F, indexed by the infinite sets $M, N \subseteq \mathbb{N}$, we say that $(x_i)_{i \in M} \subseteq E$ and $(y_i)_{i \in N} \subseteq F$ are θ-equivalent, denoted by $(x_i)_{i \in M} \sim_\theta (y_i)_{i \in N}$, if the order preserving bijection Φ between the two index-sets M and N lifts naturally to an isomorphism between the corresponding closed linear spans of these sequences sending x_i to $y_{\Phi(i)}$.

The *sequence of evaluation functionals* of c_0 is the biorthogonal sequence (p_i) of the natural basis (e_i) of c_0, i.e. if $x = \sum_i a_i e_i \in c_0$, then $p_i(x) = a_i$. Note that weakly compact subsets K of c_0 are characterized by the property that every sequence in K has a pointwise converging subsequence to an element of K. It is clear that for every weakly-compact subset $K \subseteq c_0$ the

restrictions of evaluation mappings (p_i) to K is weakly-null in $C(K)$. The sequence of restrictions will also be denoted by (p_i). Observe that (p_i) as a sequence in the Banach space $C(K)$ is a monotone basic sequence iff K is closed under restriction to initial intervals.

There are two particularly important examples of weakly-compact subsets of c_0 naturally associated to a normalized weakly null sequence $(x_i)_{i \in M}$ of a Banach space E:
(a) the set

$$R_E((x_i)_{i \in M}) = \{(x^*(x_i))_{i \in M} \in c_0 \ : \ x^* \in B_{E^*}\}$$

is symmetric, 1-bounded and weakly-compact subset of c_0.
(b) If $E = C(K)$, K compactum, then the set

$$R_K((x_i)_{i \in M}) = \{(x_i(c))_{i \in M} \in c_0 \ : \ c \in K\}$$

is also 1-bounded and weakly-compact.

In both cases one has that $(x_i)_{i \in M}$ is 1-equivalent to the evaluation mapping sequences of $C(R_E((x_i)_{i \in M}))$ and $C(R_K((x_i)_{i \in M}))$.

We say that a subset X of c_0 is *weakly pre-compact* if its closure relative to the weak topology of c_0 is weakly compact. We have then the following, not difficult to prove.

Proposition 2.9. *(a) $\mathcal{F} \subseteq \mathrm{FIN}$ is pre-compact iff the set $\{\chi_s \ : \ s \in \mathrm{FIN}\} \subseteq c_0$ of characteristic functions of sets in $\mathcal{F}$ is weakly-pre-compact.*
(b) For every weakly-pre-compact subset X of c_0 and every $\varepsilon > 0$ one has that

$$\mathrm{supp}_\varepsilon X = \{\{n \in \mathbb{N} \ : \ |\xi(n)| \geq \varepsilon\} \ : \ \xi \in X\} \ \text{is pre-compact.}$$

Finally, we introduce few combinatorial notions concerning mappings from families of finite sets of integers into c_0. For more details see [14].

Definition 2.10. *([14]) Let $\mathcal{F} \subseteq \mathrm{FIN}$ be an arbitrary family, and let $f : \mathcal{F} \to c_0$.*
(a) f is internal if for every $s \in \mathcal{F}$ one has that $\mathrm{supp}\, f(s) \subseteq s$.
(b) f is uniform if for every $t \in \mathrm{FIN}$ one has that

$$|\{\varphi(s)(\min(s/t)) \ : \ t \sqsubseteq s, s \in \mathcal{F}\}| = 1$$

(c) f is Lipschitz if for every $t \in \mathrm{FIN}$ one has that

$$|\{\varphi(s) \upharpoonright t \ : \ t \sqsubseteq s, s \in \mathcal{F}\}| = 1$$

(d) f is called a U-mapping if $\mathcal{F}$ if it is internal and uniform.
(e) f is called a L-mapping if $\mathcal{F}$ if it is internal and Lipschitz.

REMARK 2.11. (a) Every uniform mapping is Lipschitz, but the reciprocal is in general false. For example, the mapping $f : \mathrm{FIN} \to c_0$ defined by $f(s)(i) = i$ if $i \in s$ and $f(s)(i) = 0$ is Lipschitz but not uniform.
(b) Every L-mapping $f : \mathcal{F} \to c_0$ can be naturally extended to a continuous mapping $f' : \overline{\mathcal{F}}^{\sqsubseteq} \to c_0$ by setting $f'(t) = f(s) \upharpoonright t$ for (any) $s \in \mathcal{F}$ such that $t \sqsubseteq s$.
(c) The importance of internal mappings can be seen, for example, by the well-known result of Pudlak-Rödl [22] stating that if $f : \mathcal{B} \to X$ is a function defined on a barrier $\mathcal{B}$ on M then

there is $N \subseteq M$, a barrier $\mathcal{C}$ on N, and an internal mapping $g : \mathcal{B} \restriction N \to \mathcal{C}$ such that for every $s, t \in \mathcal{B} \restriction N$ one has that $f(s) = f(t)$ iff $g(s) = g(t)$.

(d) U-mappings were used in [14] to produce some weakly-null sequences playing important role in the better understanding of an abstract concept of unconditionality (see [14] for more details).

The main result on mappings defined on barriers is the following:

Theorem 2.12. [14] *Suppose that $\mathcal{B}$ is a barrier on M, $K \subseteq c_0$ is weakly-compact and suppose that $f : \mathcal{B} \to K$. Then for every $\varepsilon > 0$ there is $N \subseteq M$ and there is a U-mapping $g : \mathcal{B} \restriction N \to c_{00}$ such that for every $s \in \mathcal{B} \restriction N$ one has that*

$$\|f(s) \restriction N - g(s)\|_{\ell_1} \leq \varepsilon.$$

Corollary 2.13. *Suppose that $f : \mathcal{B} \to c_0$ is an internal mapping defined on a barrier $\mathcal{B}$. Suppose that in addition f is bounded, i.e. there is C such that for every $s \in \mathcal{B}$ one has that $\|f(s)\|_\infty \leq C$. Then for every $\varepsilon > 0$ there exists is a U-mapping $g : \mathcal{B} \restriction N \to c_{00}$ such that for every $s \in \mathcal{B} \restriction N$ one has that*

$$\|f(s) - g(s)\|_{\ell_1} \leq \varepsilon.$$

PROOF. Let us prove first that the image of f is weakly-pre-compact: For suppose that $(f(s_n))_n$ is an arbitrary sequence. Let $M \subseteq \mathbb{N}$ be such that $(\operatorname{supp} f(s_n))_{n \in M}$ converges to some $s \in \overline{\mathcal{B}}^{\sqsubseteq}$. This is possible because f is internal. Since f is bounded, we can find $N \subseteq M$ such that $(f(s_n))_{n \in N}$ is weak-convergent in c_0.

Now the desired result follows from 2.12 by using that f is in addition internal. $\square$

3. c_0-SATURATION OF $C(K)$ FOR A COUNTABLE COMPACTUM K

Recall the result of Pelczynski and Semadeni [21] which says that every Banach space of the form $C(K)$ for K a countable compactum is c_0-saturated in the sense that every of its closed infinite-dimensional subspaces contains an isomorphic copy of c_0. The purpose of this section is to examine the c_0-saturation using the theory of mappings on barriers developed above in Section 3. We start with a convenient reformulation of the problem. We start with a definition.

Definition 3.1. For a given subset X of c_0, let $\operatorname{supp} X = \{\{i \in \mathbb{N} : \xi(i) \neq 0\} : \xi \in X\}$ be the *support set* of X. We say that a weakly compact subset K of c_0 is *supported by a barrier on M* if its support set $\operatorname{supp} K$ is the is the closure of a uniform barrier on M.

Lemma 3.2. *Suppose that K is a countable compactum. Suppose that $(x_i) \subseteq C(K)$ is a normalized weakly null sequence. Then for every $\varepsilon > 0$ there is subsequence $(x_i)_{i \in M}$ and a weakly-compact subset $L \subseteq c_0$ supported by a barrier on $\mathbb{N}$ of rank not bigger than the Cantor-Bendixson rank of K such that $(x_i)_{i \in M}$ and the evaluation mapping $(p_i)_{i \in \mathbb{N}}$ of $C(L)$ are $(1 + \varepsilon)$-equivalent.*

PROOF. Fix $\varepsilon > 0$. Find first an strictly decreasing sequence (ε_i) such that $\sum_i \varepsilon_i \leq \varepsilon$ and such that

$$\{\varepsilon_i : i \in \mathbb{N}\} \cap \{|x_i(c)| : c \in K\} = \emptyset. \tag{3}$$

This is possible because K is countable. Now define $\varphi : K \to \mathcal{P}(\mathbb{N})$ by $\varphi(c) = \{i \in \mathbb{N} : |x_i(c)| \geq \varepsilon_i\}$. Note that (3) implies that φ is a continuous function. Enumerate $K = \{c_k\}_{k \in \mathbb{N}}$.

Since (x_i) is weakly-null we can find a fusion sequence (M_k) such that for every k and every $i \in M_k$ one has that $|x_i(c_k)| < \varepsilon_k$. Now if we set N to be the corresponding fusion set then for every k one has that $\{i \in M \, : \, |x_i(c_k)| \geq \varepsilon_i\} \subseteq \{n_0, \ldots, n_{k-1}\}$. This means that the mapping $\psi = \xi_M \cdot \psi$ is continuous with image included in FIN. Set $N = {}_*M$ and denote the immediate predecessor of $i \in N$ in M by i^-. Since K is a zero-dimensional compactum, we can find clopen sets $C_i \subseteq K$ $(i \in N)$ such that

$$K \setminus x_i^{-1}((-\varepsilon_{i-}, \varepsilon_{i-})) \subseteq C_i \subseteq K \setminus x_i^{-1}([-\varepsilon_i, \varepsilon_i]) \text{ for every } i \in N.$$

Set $y_i = \chi_{C_i} x_i$ for each $i \in N$. So one has
(i) $\|x_i - y_i\|_K < \varepsilon_{i-}$, so $(x_i)_{i \in N}$ and $(y_i)_{i \in N}$ are $1 + \varepsilon$-equivalent, and
(ii) for every $c \in K$ and every $i \in N$, if $|y_i(c)| \leq \varepsilon_i$, then $y_i(c) = 0$.

Since for every $c \in K$, by (ii) above, one has that

$$\{i \in N \, : \, y_i(c) \neq 0\} = \{i \in N \, : \, c \in C_i \text{ and } |x_i(c)| \geq \varepsilon_i\} = \psi(c),$$

it follows that the support set $\mathcal{F}$ of $R_K((y_i)_{i \in N})$ coincide with the image of ψ, so it is a compact family of $\mathbb{N}$. We use now Theorem 2.7 to find $P \subseteq N$ such that $\mathcal{F}[P]$ is the closure of a uniform barrier on P. This implies that $R_K((y_i)_{i \in P})$ is supported by a barrier $\mathcal{B}$ on P. Let θ be the unique order preserving mapping from $\mathbb{N}$ onto P, and let $\Theta : c_0 \restriction P = \{\xi \in c_0 \, : \, \operatorname{supp} \xi \subseteq P\} \subseteq c_0 \to c_0$ be defined by $\Theta(\xi)(n) = \xi(\theta(n))$. This is an homeomorphism between $c_0 \restriction P$ and c_0, both with the weak topology, so $L = \Theta'' R_K((y_i)_{i \in P})$ is a weakly-compact subset of c_0 and supported by the barrier $\theta^{-1}\mathcal{B} = \{\theta^{-1}s \, : \, s \in \mathcal{B}\}$ on $\mathbb{N}$. Now it is easy to see that the evaluation mapping $(p_i)_{i \in \mathbb{N}}$ of $C(L)$ is a normalized weakly-null sequence $1 + \varepsilon$-equivalent to $(x_i)_{i \in P}$. $\qquad\square$

Theorem 3.3. *Suppose that $(x_i) \subseteq C(K)$ is a normalized weakly-null sequence for a countable compactum K. Then there is a constant $C \geq 1$, an infinite set M, a uniform barrier $\mathcal{B}$ on M whose rank is at most the Cantor-Bendixson rank of K, and some U-mapping $\mu : \mathcal{B} \to c_{00}^+$ such that for every block sequence $(s_n) \subseteq \mathcal{B}$ the corresponding sequence of linear combinations $\left(\sum_{i \in s_n}(\mu(s_n))(i)x_i\right)_n$ is a normalized block sequence C-equivalent to the unit vector basis of c_0.*

PROOF. The proof is by induction on the Cantor-Bendixson rank of K. First of all, by Lemma 3.2 we may assume that K is a weakly-compact subset of c_0 supported by a barrier $\mathcal{B}$ on $\mathbb{N}$ and that the normalized weakly null sequence (x_i) is the corresponding evaluation mapping sequence $(p_i)_{i \in \mathbb{N}}$. If $\alpha = 1$, then $\mathcal{B} = \mathbb{N}^{[1]}$ and clearly (p_i) is equivalent to the unit vector basis of c_0. So assume that $\alpha > 1$. By going to a subsequence of (p_i) if needed, we may also assume in this case that $|s| \geq 2$ for every $s \in \mathcal{B}$. For each integer n set $\mathcal{F}_n = \overline{\bigcup_{m \leq n} \mathcal{B}_{\{m\}}}$. Since $\mathcal{B}$ is a α-uniform family, we have that for every n, $\partial^\alpha \mathcal{F}_n = \emptyset$, so its Cantor-Bendixson rank is strictly smaller than $\alpha + 1$. For each $n \in \mathbb{N}$, let

$$K_n = \{f \restriction s \, : \, s \in \mathcal{F}_n\}.$$

This is a compactum whose support is $\mathcal{F}_n$ and whose rank is strictly smaller than $\alpha + 1$. So, the evaluation mapping sequence (p_i) is a weakly-null sequence of $C(K_n)$ for every n. Observe that for every sequence of scalars (a_i) we have that

$$\left\| \sum_i a_i p_i \right\|_n = \left\| \sum_i a_i p_i \right\|_{K_n} = \sup\left\{ \left\| \sum_{i \in s} a_i p_i \right\|_K \, : \, s \in \mathcal{F}_n \right\}. \tag{4}$$

Using the fact that the family $\mathcal{F}_n$ is hereditary, we obtain that (p_i) is 1-unconditional. Since we assume that all the singletons $\{i\}$ belong to $\mathcal{F}_n$, it follows that $(p_i)_{i \geq 1}$ is indeed a 1-unconditional normalized weakly null sequence in $C(K_n)$.

Fix $\varepsilon > 0$, and let $(\varepsilon_n)_n$ be a summable sequence with $\sum_n \varepsilon_n < \varepsilon/2$. By the Ramsey property of the uniform barrier $\mathcal{B}$, we can find a fusion sequence $(M_k)_k$ such that, setting $n_k = \min M_k$ for each $k \in \mathbb{N}$, we have that for every k the following dichotomy holds:

(I) Either for every $s \in \mathcal{B} \upharpoonright M_k$ there is some $\mu_k(s) \in c_{00}$ with $\operatorname{supp} \mu_k(s) \subseteq s$, $0 \leq \mu_k(s)(i) \leq 1$ for every $i \in s$, and such that for every such that $\| \sum_{i \in s} \mu_k(s)(i) p_i \|_K = 1$ while $\| \sum_{i \in s} \mu_k(s)(i) p_i \|_{n_k} < \varepsilon_k$, or else

(II) $\| \sum_{i \in s} a_i p_i \|_K \leq 2\varepsilon_k^{-1} \| \sum_{i \in s} a_i p_i \|_{n_k}$ for every $s \in \mathcal{B} \upharpoonright M_k$ and every $(a_i)_{i \in s}$.

Suppose first that (I) holds for every k. Let $M_\infty = \{n_k\}$ be the corresponding fusion set. Then let $\mathcal{C} = \mathcal{B} \upharpoonright M_\infty$. For $s \in \mathcal{C}$, define $\mu(s) = \mu_k(s)$, where $n_k = \min s$. This is well defined since $s \in \mathcal{B} \upharpoonright M_k$. For a given $s \in \mathcal{C}$, let

$$x(s) = \sum_{i \in s} \mu(s)(i) p_i.$$

Our intention is to show that for every block sequence $(s_i)_i$ in $\mathcal{C}$ one has that $(x(s_i))_i$ is $2 + \varepsilon$-equivalent to the c_0-basis. So fix such sequence (s_i) and let $(b_i)_{i \in \mathbb{N}}$ be a sequence of scalars with $|b_i| \leq 1$ for every integer i. Since each $x(s_i)$ is normalized and since (p_i) is monotone, we obtain that

$$\| \sum_i b_i x(s_i) \|_K \geq (1/2) \| \sum_i b_i e_i \|_\infty.$$

Suppose that $\xi \in K$, and let $i_0 = \min\{i : s_i \cap \operatorname{supp} \xi \neq \emptyset\}$. Fix $i > i_0$, and let k_i be such that $n_{k_i} = \min s_i$. Since $\operatorname{supp} \xi \cap s_i \in \mathcal{F}_{\max s_{i_0}}$ we have that

$$|x(s_i)(\xi)| \leq \| \sum_{j \in s_i \cap \operatorname{supp} f} a_i^{(k_i)} p_i \|_{\max s_{i_0}} < \varepsilon_{k_i}. \tag{5}$$

It follows that

$$| \sum_i b_i x(s_i)(\xi)| \leq |b_{i_0}| + \sum_{i > i_0} |b_i| |x(s_i)(\xi)| \leq |b_{i_0}| + \frac{\varepsilon}{2}. \tag{6}$$

So, $\| \sum_i b_i x(s_i) \|_K \leq (1 + \varepsilon/2) \| \sum b_i e_i \|_\infty$. Finally use Corollary 2.13 to perturb μ and make it U-mapping.

Suppose now that k_0 is the first k such that (II) holds for k. Set $M = M_k$. It readily follows that for every x in the closed linear span of $(p_i)_{i \in M}$ one has that $\|x\|_K \leq \varepsilon_{k_0}^{-1} \|x\|_{n_{k_0}}$. By inductive hypothesis applied to $(p_i) \subseteq C(K_{n_{k_0}})$, there is some $C \geq 1$, some uniform barrier $\mathcal{C}$ on some $N \subseteq M$ of rank not bigger than the one of $K_{n_{k_0}}$ and some μ fulfilling the conclusions of the Lemma. Fix $s \in \mathcal{C}$. Then $\|\mu(s)\|_{n_{k_0}} = 1$, so we can find some $t_s \subseteq s$ such that $1 = \|\mu(s)\|_{n_{k_0}} = \|\mu(s) \upharpoonright t_s\|_K$. Observe that, by 1-unconditionality of $\| \cdot \|_{n_{k_0}}$, $\|\mu(s) \upharpoonright t\|_{n_{k_0}} = 1$. Define $\nu : \mathcal{C} \to c_{00}$ by $\nu(s) = \mu(s) \upharpoonright t_s$. Finally, let us check that $(x(s_i)) \subseteq C(K)$ is $C\varepsilon_{n_k}^{-1}$-equivalent to the c_0-basis for every block sequence $(s_i)_i$ in $\mathcal{C}$. Fix scalars (a_i), $|a_i| \leq 1$ ($i \in \mathbb{N}$). We obtain the inequality $\| \sum_i a_i \nu(s_i) \|_K \geq (1/2) \| \sum_i a_i e_i \|_\infty$ by the monotonicity of the basic

sequence (p_i). Now,

$$\|\sum_i a_i \nu(s_i)\|_K \leq \frac{1}{\varepsilon_{n_{k_0}}} \|\sum_i a_i \nu(s_i)\|_{n_{k_0}} \leq \frac{1}{\varepsilon_{n_{k_0}}} \|\sum_i a_i \mu(s_i)\|_{n_{k_0}} \leq \frac{C}{\varepsilon_{n_{k_0}}} \|\sum_i a_i e_i\|_\infty. \quad (7)$$

$\square$

4. CONDITIONALITY

We start with the following natural slightly variation on the notion of S_ξ-unconditionality from [3], and which is a generalization of unconditionality (see Definition 2.8 (d)).

Definition 4.1. Let $\mathcal{F}$ be a family of finite sets of integers. A normalized basic sequence (x_n) of a Banach space E is called $\mathcal{F}$-*unconditional* with constant at most $C \geq 1$ iff for every sequence of scalars (a_n),

$$\sup_{s \in \mathcal{F}} \|\sum_{n \in s} a_n x_n\| \leq C \|\sum_{n \in \mathbb{N}} a_n x_n\|.$$

This generalizes the notion of unconditionality covered by the case of $\mathcal{F} = \mathrm{FIN}$. The question is whether every normalized weakly-null sequence has a $\mathcal{F}$-unconditional subsequence. Observe that the subsequence $(x_n)_{n \in M}$ is $\mathcal{F}$-unconditional iff it is $\mathcal{F}[M]$-unconditional, so the existence of an $\mathcal{F}$-unconditional subsequence is closely related to the form of the traces $\mathcal{F}[M]$. If we assume that in addition the family $\mathcal{F}$ is hereditary, then, by the Theorem 2.7, two possibilities can occur: The first one is that some trace of $\mathcal{F}$ consists on all finite subsets of some infinite set M. In this case, for subsequences of $(x_n)_{n \in M}$ the $\mathcal{F}$-unconditionality coincides with the unconditionality. The second case is when some trace of $\mathcal{F}$ is the closure of a uniform barrier. So one is naturally led to examining the standard compact families of finite subsets of $\mathbb{N}$. We begin with the following positive result announced in [16] and first proved by E. Odell [20] concerning the Schreier family $\overline{\mathcal{S}} = \{s \subseteq \mathbb{N} : |s| \leq \min(s) + 1\}$.

Theorem 4.2. *Suppose that (x_n) is a normalized weakly-null sequence of a Banach space E. For every $\varepsilon > 0$ there is a $\overline{\mathcal{S}}$-unconditional subsequence with constant $2 + \varepsilon$.* $\square$

Recall that if $\mathcal{F}$ is a barrier on some set M then its trace $\mathcal{F}[N]$ on any co-infinite subset N of M is hereditary and that for every pair $\mathcal{F}_0$ and $\mathcal{F}_1$ of barriers on the same domain M there is an infinite set $N \subseteq M$ such that $\mathcal{F}_0[N] \subseteq \mathcal{F}_1[N]$ or $\mathcal{F}_1[N] \subseteq \mathcal{F}_0[N]$. Since the two alternatives are dependent on the ranks of $\mathcal{F}_0$ and $\mathcal{F}_1$, one is naturally led to the following measurement of unconditionality.

Definition 4.3. Suppose that γ is a countable ordinal. A normalized basic sequence (x_n) of a Banach space E is called γ-*unconditionally saturated* with constant at most $C \geq 1$ if for every γ-uniform barrier $\mathcal{B}$ on $\mathbb{N}$ and for every infinite M there is infinite $N \subseteq M$ such that the corresponding subsequence $(x_n)_{n \in N}$ of (x_n) is $\overline{\mathcal{B}}$-unconditional with constant at most C.

We say that $(x_n)_n$ is γ-unconditionally saturated if it is γ-unconditionally saturated with constant C for some $C \geq 1$.

REMARK 4.4. (a) A sequence $(x_n)_n$ is γ-unconditionally saturated iff given a γ-uniform barrier $\mathcal{B}$ every subsequence of $(x_n)_n$ has a further $\overline{\mathcal{B}}$-unconditional subsequence. The reason for this is

that given two γ-uniform barriers $\mathcal{B}$ and $\mathcal{C}$ on a set M we have that there is $N \subseteq M$ such that either $\overline{\mathcal{B} \upharpoonright N} \subseteq \overline{\mathcal{C} \upharpoonright N} \subseteq \overline{\mathcal{B} \upharpoonright N} \oplus N^{[\leq 1]}$ or the symmetric situation holds, where $\mathcal{F} \oplus \mathcal{G} = \{s \cup t : s \in \mathcal{G}, t \in \mathcal{F} \text{ and } s < t\}$ (see [4]).

(b) It follows from Theorem 4.2 that every normalized weakly null sequence is ω-unconditionally saturated. Since the ω-uniform barriers are of the form $\{s \in \text{FIN} : |s| = f(\min s) + 1\}$ for some unbounded mapping $f : M \to \mathbb{N}$ one can easily modify the proof of Theorem 4.2 to prove that every normalized weakly-null sequence is ω-unconditionally saturated with constant at most $2 + \varepsilon$.

(c) If the normalized basic sequence (x_n) is monotone, then it is $\overline{\mathcal{B}}$-unconditional iff it is $\mathcal{B}$-unconditional for every uniform barrier $\mathcal{B}$ on $\mathbb{N}$.

(d) An analysis of the Maurey-Rosenthal [16] example of a weakly-null sequence (x_n) with no unconditional basic subsequence (see Example 4.5 below) reveals an ω^2-uniform barrier $\mathcal{B}_{\text{MR}}$ such that no infinite subsequence $(x_n)_{n \in M}$ is $\mathcal{B}_{\text{MR}}$-unconditional with any finite constant C. So this is an example of a normalized weakly-null sequence with no ω^2-unconditionally saturated subsequence.

(e) Recall that an ordinal γ is called indecomposable if for every $\beta < \gamma$, $\beta\omega \leq \gamma$. Equivalently, $\gamma = \omega^\beta$ for some β. Suppose that γ is the maximal indecomposable ordinal smaller than some fixed ordinal α. Then a normalized basic sequence (x_n) is α-unconditionally saturated if and only it is γ-unconditionally saturated.

EXAMPLE 4.5. First of all, for a fixed $0 < \varepsilon < 1$ choose a fast increasing sequence (m_i) such that

$$\sum_{i=0}^{\infty} \sum_{j \neq i} \min\{(\frac{m_i}{m_j})^{1/2}, (\frac{m_j}{m_i})^{1/2}\} \leq \frac{\varepsilon}{2}. \tag{8}$$

Let $\text{FIN}^{[<\infty]}$ be the collection of all finite block sequences $E_0 < E_1 < \cdots < E_k$ of nonempty finite subsets of $\mathbb{N}$. Now choose a $1 - 1$ function

$$\sigma : \text{FIN}^{[<\infty]} \to \{m_i\} \tag{9}$$

such that $\varphi((s_i)_{i=0}^n) > s_n$ for all $(s_i) \in \text{FIN}^{[<\infty]}$ Now let $\mathcal{B}_{\text{MR}}$ be the family of unions $s_0 \cup s_1 \cup \cdots \cup s_n$ of finite sets such that

(a) (s_i) is block and $s_0 = \{n\}$.

(b) $|s_i| = \sigma(s_0, \ldots, s_{i-1})$ $(1 \leq i \leq n)$.

It turns out that $\mathcal{B}_{\text{MR}}$ is a ω^2-uniform barrier on $\mathbb{N}$ (see Proposition 4.11 below), hence $\overline{\mathcal{B}_{\text{MR}}} = \overline{\mathcal{B}_{\text{MR}}}^{\sqsubseteq}$ is a compact family with rank $\omega^2 + 1$. Observe that by definition, every $s \in \mathcal{B}_{\text{MR}}$ has a unique decomposition $s = \{n\} \cup s_1 \cup \cdots \cup s_n$ satisfying (a) and (b) above. Now define the mapping $\Phi : \mathcal{B}_{\text{MR}} \to c_{00}$,

$$\Phi(s) = e_n + \sum_{i=1}^{n} \frac{1}{|s_i|^{\frac{1}{2}}} \sum_{k \in s_i} e_k. \tag{10}$$

It follows that Φ is a U-mapping defined on the barrier $\mathcal{B}_{\text{MR}}$. Now we can define the Banach space $\mathfrak{X}_{\text{MR}}$ as the completion of c_{00} under the norm

$$\|x\|_{\text{MR}} = \sup\{|\langle \Phi(s), x \rangle| : s \in \mathcal{B}_{\text{MR}}\}.$$

The natural Hamel basis (e_n) of c_{00} is now a normalized weakly-null monotone basis of $\mathfrak{X}_{MR}$ without unconditional subsequences. Indeed, without ω^2-unconditionally saturated subsequences. Moreover this weakly-null sequence has the property that the summing basis (S_i) of c, the Banach space of *convergent sequences of reals*, is finitely-block representable in the linear span of every subsequence of (e_i) (and so the summing basis of c_0), more precisely, for every M, every $n \in \mathbb{N}$ and every $\varepsilon > 0$ there is a normalized block subsequence $(x_i)_{i=0}^{n-1}$ of $(e_i)_{i \in M}$ such that for every sequence of scalars $(a_i)_{i=0}^{n-1}$,

$$\max\{|\sum_{i=0}^{m} a_i| \, : \, m < n\} \leq \|\sum_{i=0}^{n-1} a_i x_i\|_{C(K)} \leq (1+\varepsilon)\max\{|\sum_{i=0}^{m} a_i| \, : \, m < n\}.$$

On the other hand, by Proposition 4.2 the sequence (p_i) is ω-unconditionally saturated with constant ~ 2.

Another presentation of this space is the following: Since Φ is uniform, it is Lipschitz, so there is a unique extension $\Phi : \overline{\mathcal{B}_{MR}} \to c_{00}$, naturally defined by $\Phi(s) = \Phi(t) \restriction t$, where $t \in \mathcal{B}_{MR}$ is (any) such that $s \sqsubseteq t$. Now define $K = \Phi''\mathcal{B}_{MR} \subseteq c_{00}$. This is a weakly-compact subset of c_{00} whose rank the same than $\overline{\mathcal{B}_{MR}}$, i.e., $\omega^2 + 1$. Then the corresponding evaluation sequence $(p_i) \subseteq C(K)$ is 1-equivalent to the basis $(e_i)_i$ of $\mathfrak{X}_{MR}$.

Building on the idea of Example 4.5, we are now going to find, for every countable indecomposable ordinal γ, a U-sequence with no unconditional subsequences but β-unconditionally saturated for every $\beta < \gamma$. Before embarking into the construction, we need to recall a localized version of Pták's Lemma. For this we need the following notation: Given a family $\mathcal{F}$, and $n \in \mathbb{N}$, let

$$\mathcal{F} \otimes n = \{s_0 \cup \cdots \cup s_{n-1} \, : \, (s_i)_{i=0}^{n-1} \subseteq \mathcal{F} \text{ is block}\}.$$

It can be shown that $\mathcal{F} \otimes n$ is a αn-uniform family if $\mathcal{F}$ is an α-uniform family.

Given $\xi \in c_{00}$ we will write $\xi^{1/2}$ to denote $(\xi(i)^{1/2})$. Given $\xi \in c_{00}$ and a finite set s, let $\langle \xi, s \rangle = \langle \xi, \chi_s \rangle = \sum_{i \in s} \xi(i)$.

Definition 4.6. A *mean* is an element $\mu \in c_{00}^+$ with the property that $\sum_{i \in \mathbb{N}} \mu(i) = 1$. We say that $\mu : \mathcal{B} \to c_{00}^+$ is a *U-mean-assignment* if μ is a U-mapping such that for every $s \in \mathcal{B}$ one has that $\mu(s)$ is a mean.

Lemma 4.7. *Suppose that $\mathcal{B}$ is an α-uniform barrier on M, $\alpha \geq 1$. Let $\gamma = \gamma(\alpha)$ be the maximal indecomposable ordinal not bigger than α, and let $n = n(\alpha) \in \mathbb{N}$, $n \geq 1$, be such that $\gamma n \leq \alpha < \gamma(n+1)$. Then for every $k \in \mathbb{N}$, $k > 1$, every $\varepsilon > 0$, and every β-uniform barrier $\mathcal{C}$ on M with $\beta > \alpha k$ there $N \subseteq M$ and U-mean-assignment $\mu : \mathcal{C} \restriction N \to c_{00}^+$ such that*

$$\sup\{\langle \mu(s)^{\frac{1}{2}}, t \rangle \, : \, t \in \mathcal{B}\} \leq \frac{(1+\varepsilon)(n+1)}{(nk)^{\frac{1}{2}}} \tag{11}$$

for every $s \in \mathcal{C} \restriction N$.

PROOF. The proof is by induction on α. Fix $\varepsilon > 0$ and $k > 1$. Let $\mathcal{C}$ be an β-uniform family on M such that $\beta > \alpha k$.

Notice that if we prove that for every $N \subseteq M$ there is one mean μ with support in $\mathcal{C} \restriction N$ such that (11) holds, then the Ramsey property of the uniform barrier $\mathcal{C}$ gives the existence for some $N \subseteq M$ of a mean-assignment $\mu : \mathcal{C} \restriction N \to c_{00}$ such that $\mu(s)$ has the property (11) for every $s \in \mathcal{C} \restriction N$. Then Corollary 2.13 gives the desired U-mapping.

Let $\mathcal{D}$ be a γ-uniform barrier on M (if $n = 1$ we take $\mathcal{D} = \mathcal{B}$), and fix $N \subseteq M$. Find first $P \subseteq N$ be such that $(\mathcal{D} \otimes nk) \restriction P \subseteq \overline{\mathcal{C}}$ as well as $\mathcal{B} \restriction P \subseteq \overline{\mathcal{D} \otimes (n+1)}$. Consider $(\gamma_i)_{i \in P}$ such that $\mathcal{D}_{\{i\}} \restriction P$ is γ_i-uniform on P/i. Observe that for every $i \in P$ we have that $\gamma_i < \gamma$, so, since γ is indecomposable, $\gamma_i \omega \leq \gamma$. Let μ_0 be any mean such that $\operatorname{supp} \mu_0 \in \mathcal{B} \restriction P$. By inductive hypothesis applied to appropriate α_i's, we can find a block sequence $(\mu_j)_{j=0}^{nk-1}$ of means with support in $\mathcal{B} \restriction P$ such that for every $1 \leq j \leq nk - 1$,

$$\sup\{\langle \mu_j^{\frac{1}{2}}, t \rangle : t \in \mathcal{D}, \text{ and } \min t \leq \max \operatorname{supp} \mu_{j-1}\} < \frac{\varepsilon}{2^{j+1}}. \tag{12}$$

Let $\nu = (1/(nk)) \sum_{j=0}^{nk-1} \mu_j$. Observe that $\operatorname{supp} \nu \in \overline{(\mathcal{D} \otimes (nk)) \restriction P} \subseteq \overline{\mathcal{C}}$. Then, for every $t \in \mathcal{B}$, by (12),

$$\langle \nu^{\frac{1}{2}}, t \rangle = \frac{1}{(nk)^{\frac{1}{2}}} \sum_{j=0}^{k-1} \sum_{i \in t} \mu_j(i)^{\frac{1}{2}} \leq \frac{1 + \frac{\varepsilon}{2}}{(nk)^{\frac{1}{2}}}. \tag{13}$$

Let us point out that $\operatorname{supp} \nu$ is, possibly, not a set in $\mathcal{C}$. However it is easy to slightly perturb ν to a newer mean with support in $\mathcal{C}$ and satisfying (13) for every $t \in \mathcal{B}$: Let $s \in \mathcal{C}$ be such that $\operatorname{supp} \nu \sqsubseteq s$, and set $u = s \setminus \operatorname{supp} \nu$. Let $\delta > 0$ be such that

$$\left(1 + \frac{\varepsilon}{2}\right)(1 - \delta)^{1/2} + (nk\delta|u|)^{1/2} \leq 1 + \varepsilon. \tag{14}$$

Now set

$$\mu = (1 - \delta)\nu + \frac{\delta}{|u|}\chi_u. \tag{15}$$

μ is a mean whose support is $s \in \mathcal{C}$. It can be shown now that for every $t \in \mathcal{B}$,

$$\sum_{i \in t} \mu(i)^{\frac{1}{2}} \leq \frac{1 + \varepsilon}{(nk)^{\frac{1}{2}}}, \tag{16}$$

by the choice of δ. Finally, let $t \in \mathcal{B}$ and let us compute $\sum_{i \in t}(\mu(i))^{1/2}$: First of all we have that $\sum_{i \in t}(\mu(i))^{1/2} = \sum_{i \in u}(\mu(i))^{1/2}$, where $u = t \cap P$. Now, since $u \in \overline{\mathcal{B} \restriction P} \subseteq \overline{\mathcal{D} \otimes (n+1)}$, we can find $t_0 < \cdots < t_n$ in $\mathcal{D}$ such that $u \sqsubseteq t_0 \cup \cdots \cup t_n$, and hence

$$\langle \mu^{1/2}, t \rangle = \sum_{j=0}^{n} \langle \mu^{1/2}, t_j \rangle \leq \frac{(n+1)(1 + \varepsilon)}{(nk)^{\frac{1}{2}}}, \tag{17}$$

as promised. $\qquad\square$

Corollary 4.8. *Suppose that $\mathcal{B}$ is an α-uniform barrier on M, $\alpha \geq 1$. Then for every $\varepsilon > 0$ there is some $k = k(\alpha, \varepsilon)$ such that for every β-uniform barrier on M with $\beta > \alpha k$ there $N \subseteq M$ and some U-mean-assignment $\mu : \mathcal{C} \restriction N \to c_{00}^+$ such that,*

$$\sup\{\langle \mu(s)^{1/2}, t \rangle : t \in \mathcal{B}\} \leq \varepsilon \tag{18}$$

for every $s \in \mathcal{B} \restriction N$. $\qquad\square$

Lemma 4.9. *Fix an indecomposable countable α and a sequence (ε_n) of positive reals. Then:*
(a) there is a collection $(\mathcal{B}_n)$ of α_n-uniform barriers on $\mathbb{N}/n$ and a corresponding sequence of U-mean-assignments $\mu_n : \mathcal{B}_n \to c_{00}^+$ with the following properties:
(a.1) $\alpha_n > 0$, $\sup_n \alpha_n = \alpha$,
(a.2) for every $m < n$ and every $s \in \mathcal{B}_n$

$$\sup\{\langle \mu_n(s)^{\frac{1}{2}}, t \rangle : t \in \mathcal{B}_m\} < \varepsilon_n. \tag{19}$$

(b) Suppose that in addition $\alpha = \omega^\gamma$ with γ limit. Let $\alpha_n \uparrow \alpha$ be any sequence such that $\alpha_n \omega \le \alpha_{n+1}$ ($n \in \mathbb{N}$). Then there is a double sequence $(\mathcal{B}_i^n)$ such that for every integers n and i
(b.1) $\mathcal{B}_i^n$ is an $\alpha_i^{(n)}$-uniform barrier on $\mathbb{N}/(n+i)$, with $\alpha_i^{(n)} > 0$ and $\alpha_i^{(n)} \uparrow_i \alpha_n$.
(b.2) There are U-mean-assignments $\mu_{n,i} : \mathcal{B}_i^n \to c_{00}$ such that for every $s \in \mathcal{B}_i^n$, and every $(m,j) <_{\mathrm{lex}} (n,i)$

$$\sup\{\langle \mu_{n,i}(s)^{\frac{1}{2}}, t \rangle : t \in \mathcal{B}_j^m\} < \varepsilon_{n+i}, \tag{20}$$

where we recall that $<_{\mathrm{lex}}$ denotes the lexicographical order on $\mathbb{N}^2$ defined by $(m,i) <_{\mathrm{lex}} (n,j)$ iff $m < n$, or $m = n$ and $i < j$.

PROOF. (a): Choose $\alpha_n \uparrow_n \alpha$ such that for every $n \in \mathbb{N}$, $\alpha_{n+1} > \alpha_n k(\alpha_n, \varepsilon_n)$, that is is possible since α is indecomposable. Let $\mathcal{C}_n$ be an α_n-uniform family on $\mathbb{N}$ ($n \in \mathbb{N}$). By Corollary 4.8 we can find a fusion sequence (M_n) such that
(c) $\mathcal{C}_m \restriction M_m \subseteq \overline{\mathcal{C}_n}$ if $m \le n$, and
(d) for every $n \in \mathbb{N}$ there is a U-mean-assignment $\nu_n : \mathcal{C}_n \restriction M_n \to c_{00}^+$ such that

$$\sup\{\langle \nu_n(s)^{\frac{1}{2}}, t \rangle : t \in \bigcup_{l < n} \mathcal{C}_l\} < \varepsilon_n \tag{21}$$

for every $s \in \mathcal{C}_n \restriction M_n$. Let $M = \{m_n\}$ be the fusion set of (M_n), and $\Theta : M \to \mathbb{N}$ be the corresponding order preserving onto mapping. It is not difficult to see that $\mathcal{C}_n = (\Theta"\mathcal{B}_n) \restriction (\mathbb{N}/n)$, and $\mu_n : \mathcal{C}_n \to c_{00}$ defined naturally out of $\nu_n \Theta$ fulfils all the requirements.

(b): Suppose that $\alpha = \omega^\gamma$ with γ limit. Let $\alpha_n \uparrow \alpha$ be any sequence such that $\alpha_n \omega \le \alpha_{n+1}$ ($n \in \mathbb{N}$).

Claim. *There is a fusion sequence (M_n), $M_n = \{m_i^{(n)}\}$, a double sequence $(\mathcal{B}_i^n)$ of $\alpha_i^{(n)}$-uniform barriers on $M_n/m_i^{(n)}$ and U-mean-assignments $\mu_{n,i} : \mathcal{B}_i^n \to c_{00}^+$ such that*
(e) $\alpha_i^{(n)} \uparrow_i \alpha_n$ ($n \in \mathbb{N}$), and
(f) for every $(m,j) <_{\mathrm{lex}} (n,i)$, every $s \in \mathcal{B}_i^n$ and every $t \in \mathcal{B}_j^m$, $\langle (\mu_{n,i}(s))^{1/2}, t \rangle < \varepsilon_{n+i}$.

Proof of Claim: First, use Corollary 4.8 applied to α_0 to produce an infinite set $M_0 = \{m_i^{(0)}\}$ and a sequence $(\mathcal{B}_i^0)$ of $\alpha_i^{(0)}$-uniform barriers on $M_0/\{m_i^{(0)}\}$ with $\alpha_i^{(0)} \uparrow \alpha_0$ and U-mean-assignments $\mu_{0,i} : \mathcal{B}_i^0 \to c_{00}$ such that for every i and every $s \in \mathcal{B}_i^0$, $\langle \mu_{0,i}(s)^{1/2}, t \rangle \le \varepsilon_i$ for every $t \in \mathcal{B}_j^0$ with $j < i$. In general, suppose we have found for every $k \le n$ $M_k = \{m_i^{(k)}\} \subseteq M_{k-1}$, $(\mathcal{B}_i^k)$ $\alpha_i^{(k)}$-uniform barriers on $M_k/m_i^{(k)}$ and U-mean-assignments $\mu_{k,i} : \mathcal{B}_i^k \to c_{00}$ such that for every $(k,j) <_{\mathrm{lex}} (m,i)$ every $s \in \mathcal{B}_i^m$ and every $t \in \mathcal{B}_j^k$ $\langle \mu_{m,i}(s)^{1/2}, t \rangle \le \varepsilon_{m+i}$. For each $k \le n$ define the following families

$$\mathcal{B}_k = \{s \subseteq M_k : {}_*s \in \mathcal{B}_{\min s}^k\}. \tag{22}$$

This is clearly an α_k-uniform family on M_k. Since $\alpha_n \omega \leq \alpha_{n+1}$, we can use again Corollary 4.8 and find an infinite subset $M_{n+1} = \{m_i^{(n+1)}\} \subseteq M_n$ and a sequence $(\mathcal{B}_i^{n+1})$ of $\alpha_i^{(n+1)}$-uniform barriers on $M_{n+1}/m_i^{(n+1)}$ and U-mean-assignments $\mu_{n+1,i} : \mathcal{B}_i^{n+1} \to c_{00}$ such that for every $s \in \mathcal{B}_i^{n+1}$,

$$\sup\{\langle (\mu_{n+1,i}(s))^{\frac{1}{2}}, t \rangle : t \in \bigcup_{k \leq n} \mathcal{B}_m \cup \bigcup_{j < i} \mathcal{B}_j^{(n+1)}\} < \varepsilon_{n+i+1}, \tag{23}$$

so, in particular for every $k \leq n$ and every $t \in \mathcal{B}_j^k$, $\langle (\mu_{n+1,i}(s))^{\frac{1}{2}}, t \rangle < \varepsilon_{n+i+1}$. $\qquad \square$

Let M be the fusion set of (M_n), i.e. $M = \{m_0^{(n)}\}$. Observe that $m_0^{(n+i)} \geq m_i^{(n)}$ for every n and i, so $M/m_0^{(n)} \subseteq M_n/m_i^{(n)}$. Set $\mathcal{C}_i^n = \mathcal{B}_i^n \upharpoonright (M/m_0^{(n+i)})$. This is an $\alpha_i^{(n)}$-uniform barrier on $M/m_0^{(n+i)}$. Consider $\nu_{n,i} = \mu_{n,i} \upharpoonright \mathcal{C}_i^n : \mathcal{C}_i^n \to c_{00}$ has the property that for every $(m,j) <_{\text{lex}} (n,i)$, every every $s \in \mathcal{C}_i^n$ and every $t \in \mathcal{C}_j^m$, $\langle (\nu_{n,i}(s))^{1/2}, t \rangle < \varepsilon_{n+i}$. Now use $\Theta : M \to \mathbb{N}$, $\Theta(m_0^{(n)}) = 0$, to define the desired mean-assignments and families. $\qquad \square$

REMARK 4.10. Observe that if $\mathcal{B}$ is α-uniform on M with $\alpha > 0$, then $M^{[1]} \subseteq \overline{\mathcal{B}}$. It readily follows that the mean-assignments μ_n and $\mu_{n,i}$ obtained in Lemma 4.9 have the property that $\|\mu_n(s)^{1/2}\|_\infty \leq \varepsilon_n$ and $\|\mu_{n,i}(s)^{1/2}\|_\infty \leq \varepsilon_{n+i}$ for every s in the corresponding domains.

Proposition 4.11. *(a) Suppose that $\mathcal{C}$ and $\mathcal{B}_i$ are β and α_i-uniform families on M ($i \in \mathbb{N}$) with $\alpha_i \uparrow \alpha$, $\alpha_i, \beta \geq 1$. Let $\sigma : \text{FIN}^{[<\infty]} \to \mathbb{N}$ be 1-1. Then for every $n \in \mathbb{N}$ the family*

$$\mathcal{D} = \{s_0 \cup \cdots \cup s_n : (s_i) \text{ is block}, s_0 \in \mathcal{C} \text{ and } s_i \in \mathcal{B}_{\sigma((s_0,\ldots,s_{i-1}))} \text{ for every } 1 \leq i \leq n-1\}$$

is γ-uniform on M, where $\gamma = \alpha n + \beta^-$ if $1 \leq \beta < \omega$ and $n > 0$, and $\gamma = \alpha n + \beta$ if $\beta \geq \omega$ or $n = 0$.
(b) Suppose that $\mathcal{B}_i$ is α_i-uniform on M ($i \in \mathbb{N}$) with $\alpha_i \uparrow \alpha$. Let $\sigma : \text{FIN}^{[<\infty]} \to \mathbb{N}$ be 1-1. Then the family

$$\mathcal{C} = \{\{n\} \cup s_0 \cup \cdots \cup s_{n-1} : (\{n\}, s_0, \ldots, s_{n-1}) \text{ is block, and}$$
$$s_i \in \mathcal{B}_{\sigma((\{n\}, s_0, \ldots, s_{i-1}))} \text{ for every } 0 \leq i \leq n-1\}$$

is $\alpha\omega$-uniform on M.

PROOF. (a): The proof is by induction on n. If $n = 0$, the result is clear. So suppose that $n > 0$. Now the proof is by induction on β. Suppose first that $\beta = 1$. Then $\mathcal{C} = M^{[1]}$, and so, for every $m \in M$

$$\mathcal{D}_{\{m\}} = \{s_1 \cup \cdots \cup s_n : (s_1, s_2, \ldots, s_n) \text{ is block}, s_1 \in \mathcal{B}_{\sigma((\{m\}))} \text{ and}$$
$$s_i \in \mathcal{B}_{\sigma((\{m\}, s_1, s_2, \ldots, s_{i-1}))} \text{ for every } 2 \leq i \leq n-1\},$$

so, by inductive hypothesis, $\mathcal{D}_{\{m\}}$ is $\alpha(n-1) + \gamma_m$-uniform on M/m, depending whether α_m is finite or infinite, but in any case with $\gamma_m \uparrow \alpha$. Hence $\mathcal{D}$ is αn-uniform on M. The general case for $1 \leq \beta < \omega$ is shown in the same way.

Suppose now that $\beta \geq \omega$. Then for every $m \in M$

$$\mathcal{D}_{\{m\}} = \{t \cup s_1 \cup \cdots \cup s_n : (t, s_1, \ldots, s_n) \text{ is block}, t \in \mathcal{C}_{\{m\}} \text{ and}$$
$$s_i \in \mathcal{B}_{\sigma((\{m\} \cup t, s_1, \ldots, s_{i-1}))} \text{ for every } 1 \leq i \leq n-1\},$$

By inductive hypothesis, $\mathcal{D}_{\{m\}}$ is $\alpha n + \gamma_m$-uniform on M/m, with $\gamma_m \uparrow \beta$, so $\mathcal{D}$ is $\alpha n + \beta$-uniform on M, as desired.

(b) follows easily from (a). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The following is a generalization of Maurey-Rosenthal example for arbitrary countable indecomposable ordinal α.

Theorem 4.12. *For every countable indecomposable ordinal α there is a normalized weakly-null sequence which is β-unconditionally saturated for every $\beta < \alpha$ but without unconditional subsequences.*

PROOF. Our example is a slightly modification of a U-sequence introduced in [14]. So, we are going to define a α-uniform barrier $\mathcal{B}$ on $\mathbb{N}$, a U-mean-assignment $\varphi : \mathcal{B} \to c_{00}$ and some $\mathcal{G} \subseteq \mathrm{FIN} \times \mathrm{FIN}$ and then define the norm on c_{00} by

$$\|\xi\| = \max\{\|\xi\|_\infty, \sup\{|\langle \varphi(s) \upharpoonright t, \xi \rangle| : (s,t) \in \mathcal{G}\}\} \qquad (24)$$

where $\mathcal{G} \subseteq \mathrm{FIN} \times \mathrm{FIN}$ is such that its first projection is $\mathcal{B}$. Notice that some sort of restrictions have to be needed in the formula (24), since it is not difficult to see that that for a compact and hereditary family $\mathcal{F}$, a normalized weakly-null sequence $(x_i)_i$ is $\mathcal{F}$-unconditional iff it is equivalent to the evaluation mapping sequence $(p_i)_i$ of a weakly-compact subset $K \subseteq c_0$ that is $\mathcal{F}$-closed, i.e. closed under restriction on elements of $\mathcal{F}$.

Fix $\varepsilon > 0$, and let $\varepsilon_n = \varepsilon/2^{n+3}$. Suppose that $\alpha = \omega^\gamma$. There are two cases to consider. Suppose first that $\gamma = \beta + 1$. We apply Lemma 4.9 (a) to the indecomposable ordinal ω^β and (ε_n) to produce the corresponding sequences of barriers $(\mathcal{C}_n)$ and U-mean-assignments $\nu_n : \mathcal{C}_n \to c_{00}$ ($n \in \mathbb{N}$) satisfying the conclusions (a.1) and (a.2) of the Lemma. If γ is limit, then we use the part (b) of that lemma to produce a double sequence $(\mathcal{B}_i^n)$ and U-mean-assignments $\nu_{n,i} : \mathcal{C}_i^n \to c_{00}$ satisfying (b.1) and (b.2). In order to unify the two cases we set for n, i,

$$\mathcal{B}_i^n = \begin{cases} \mathcal{C}_i & \text{if } \gamma \text{ is successor ordinal} \\ \mathcal{C}_i^n & \text{if } \gamma \text{ is limit ordinal} \end{cases}$$

and

$$\mu_{n,i} = \begin{cases} \nu_i & \text{if } \gamma \text{ is successor ordinal} \\ \nu_{n,i} & \text{if } \gamma \text{ is limit ordinal.} \end{cases}$$

Let $\sigma : \mathrm{FIN}^{[<\infty]} \to \mathbb{N}$ be 1-1 mapping such that $\sigma((s_0, \ldots, s_n)) > \max s_n$ for every block sequence $(s_0, \ldots, s_n)$ of finite sets. For each n define

$$\mathcal{C}_n = \{s_0 \cup \cdots \cup s_{n-1} : (s_i) \text{ is block and } s_i \in \mathcal{B}_{\sigma(\{n\}, s_0, \ldots, s_{i-1})}^n \text{ for every } 0 \leq i \leq n \},$$

So, by Proposition 4.11, if $\alpha = \omega^{\beta+1}$, then $\mathcal{C}_n$ is a $\omega^\beta(n-1) + \zeta$-uniform family on $\mathbb{N}$, where ζ is such that $\mathcal{B}_{\sigma((\{n\}))}^n$ is ζ-uniform; while if $\alpha = \omega^\gamma$ with γ limit, then it is $\alpha_n(n-1) + \zeta$ where ζ is such that $\mathcal{B}_{\sigma((\{n\}))}^n$ is ζ-uniform. Now let

$$\mathcal{C} = \{s \in \mathrm{FIN} : {}_*s \in \mathcal{C}_{\min s}\}. \qquad (25)$$

It turns out that $\mathcal{C}$ is an α-uniform family on $\mathbb{N}$ (so it is a front), not necessarily a barrier. Observe that every $s \in \mathcal{C}$ has a unique decomposition $s = \{n\} \cup s(0) \cup \cdots \cup s(n-1)$ with

$n = \min s$ and $s(i) \in \mathcal{B}_{\sigma(s[i])}$, and where $s[i] = (\{n\}, s_0, \ldots, s_{i-1})$ $(0 \le i \le n - 1)$. For every $s \in \mathcal{C}$ and every $i \le s$, set

$$\xi(s, i) = (\mu_{\min s, \sigma(s[i])}(s(i)))^{1/2}.$$

Define now $\Phi : \mathcal{C} \to c_{00}$ for every $s \in \mathcal{C}$ by

$$\Phi(s) = e_{\min s} + \sum_{i=0}^{n-1} \xi(s, i), \tag{26}$$

It is not difficult to see that $\Phi : \mathcal{C} \to c_{00}$ is a U-mapping. Now define on c_{00} the norm

$$\|\xi\| = \sup\{|\langle \Phi(s) \upharpoonright (s \setminus t), \xi \rangle| : s \in \mathcal{C}, t \subseteq s(i), \text{ for some } i < \min s\} =$$
$$= \sup\{|\langle \Phi(s) \upharpoonright (u \setminus t), \xi \rangle| : u \sqsubseteq s \in \mathcal{C}, t \subseteq s(i), \text{ for some } i < \min s\}, \tag{27}$$

the last equality because Φ is Lipschitz and supported by a front. Let $\mathfrak{X}$ the completion of c_{00} under this norm. Then the Hamel basis $(e_n)_n$ of c_{00} is a normalized basis of $\mathfrak{X}$, moreover monotone (since Φ is Lipschitz with domain a front) and weakly-null: To prove this, it is enough to see that the set

$$L = \{\Phi(s) \upharpoonright (u \setminus t) : s \in \mathcal{B}, u \sqsubseteq s, \text{ and } t \subseteq s(i) \text{ for some } i < \min s\}$$

is weakly-compact. So, let $(\Phi(s_n) \upharpoonright (u_n \setminus t_n))_n$ a typical sequence in L. Since $\mathcal{C}$ is a front, we can find an infinite set M and $u \in \text{FIN}$ such that $(u_n)_{n \in M}$ converges to u and such that (s_n) is a Δ-system with root $u \sqsubseteq r$. Since Φ is Lipschitz de, we obtain that $(\Phi(s_n) \upharpoonright t_n)_{n \in M}$ converges to $\Phi(s_m) \upharpoonright t$ for (any) $m \in M$. If $u = \emptyset$, then $(\Phi(s_n) \upharpoonright (t_n \cup u))_{n \in M}$ converges to 0. Otherwise, let $N \subseteq M$ and $j < \min u$ be such that $t_n \subseteq s_n(j)$ for every $n \in N$. Now $(t_n)_{n \in N}$ is a sequence in the closure of $\mathcal{B}_{\sigma(s[i])}^{\min s}$, hence, we can find $P \subseteq N$ such that $(t_n)_{n \in P}$ is convergent with limit t. It follows that $(\Phi(s_n) \upharpoonright (u_n \setminus t_n))_{n \in P}$ has limit $\Phi(s_n) \upharpoonright (u \setminus t) \in L$, where n is (any) integer in P.

The next is a crucial computation.

Claim. *For every $s, t \in \mathcal{C}$ and every $i \le \min s$ and $j \le \min s$, we have that*

$$0 \le \langle \xi(t, j), \xi(s, i) \rangle \le \begin{cases} \varepsilon_{\max\{\min s, \min t\}} & \text{if } t[j] \ne s[i] \\ 1 & \text{if } t[j] = s[i]. \end{cases}$$

Proof of Claim: Set $n = \min s$, $m = \min t$, and assume that $t[j] \ne s[i]$. Suppose first that $\alpha = \omega^{\beta+1}$. Then, by definition of the mean assignments, $\langle \xi(t, j), \xi(s, i) \rangle \le \varepsilon_{\max\{\sigma(t[j]), \sigma(s[i])\}}$, but $\sigma(u_0, \ldots, u_k) \ge \max u_k$ for every block sequence (u_i), which derives into the desired inequality. Assume now that $\alpha = \omega^\gamma$, γ limit ordinal. If $\min s = \min t$, then $\langle \xi(t, j), \xi(s, i) \rangle \le \varepsilon_{\min s + \max\{\sigma(t[j]), \sigma(s[i])\}} \le \varepsilon_{\min s}$. While if $\min t \ne \min s$, say $\min t < \min s$, then $\langle \xi(t, j), \xi(s, i) \rangle \le \varepsilon_{\min s + \sigma(s[i])} \le \varepsilon_{\min s}$.

If $\sigma(s[i]) = \sigma(t[j]) = l$, then $\min s = \min t = n$, and

$$\langle \xi(s, i), \xi(t, j) \rangle \le \|(\mu_{n,l}(s(i)))^{1/2}\|_{\ell_2} \|(\mu_{n,l}(t(j)))^{1/2}\|_{\ell_2} \le 1, \tag{28}$$

since both are means. $\qquad\square$

Claim. *The summing basis (S_n) of c is finitely block represented in any subsequence of $(e_n)_n$.*

Proof of Claim: Fix an infinite set M of integers, and $l \in \mathbb{N}$. Let $v \in \mathcal{B} \upharpoonright M/l$, $v = \{n\} \cup v(0) \cup \cdots \cup v(n-1)$ its canonical decomposition, and set

$$x_i = \sum_{j \in v(i)} \xi(v,i)(j) e_j. \tag{29}$$

Observe that $\langle \Phi(v), x(v,i) \rangle = \langle \xi(v,i), \xi(v,i) \rangle = 1$, so from the previous claim we obtain that $\|x_i\| = 1$. Now consider scalars $(a_i)_{i \leq n-1}$ with $\|\sum_{i \leq n-1} a_i S_i\|_\infty = 1$. Observe that this implies that $\max_{i \leq n-1} |a_i| \leq 2$. We are going to show that

$$1 \leq \|\sum_{0 \leq i \leq n-1} a_i x_i\| \leq 3 + \varepsilon. \tag{30}$$

To get the left hand inequality, suppose that $1 = \|\sum_{0 \leq i \leq n-1} a_i S_i\|_\infty = |\sum_{i \leq m} a_i|$, where $m \leq n-1$. Let $t = \{n\} \cup s(0) \cup \cdots \cup s(m)$. By (27) it follows that

$$\|\sum_{i \leq n-1} a_i x_i\| \geq \langle \Phi(v) \upharpoonright t, \sum_{i \leq n-1} a_i x_i \rangle = |\sum_{i \leq m} a_i| = 1. \tag{31}$$

Next, fix $s \in \mathcal{C}$ and $t \subseteq s(k)$ for some $k < \min s$. Suppose first that $\min v = \min s$. Let $i_0 = \max\{i \leq n-1 : v(i) = s(i)\}$. If $k > i_0$ then by the previous claim we obtain

$$|\langle \Phi(s) \upharpoonright (s \setminus t), \sum_{i \leq n-1} a_i x_i \rangle| \leq |\sum_{i \leq i_0} a_i| + \sum_{i_0 < i, j \leq n-1} 2|\langle \xi(s,i), \xi(t,j) \rangle| \leq$$

$$\leq \|\sum_{i \leq n-1} a_i S_i\|_\infty + 2n^2 \varepsilon_n \leq (1+\varepsilon)\|\sum_{i \leq n-1} a_i S_i\|_\infty. \tag{32}$$

Suppose that $k \leq i_0$. Then

$$|\langle \Phi(s) \upharpoonright (s \setminus t), \sum_{i \leq n-1} a_i x_i \rangle| \leq |\sum_{i \leq i_0, i \neq k} a_i + a_k \langle \xi(v,k), \xi(v,k) \upharpoonright (s(k) \setminus t) \rangle| +$$

$$+ \sum_{i_0 < i, j \leq n-1} 2|\langle \xi(s,i), \xi(t,j) \rangle| \leq$$

$$\leq 3\|\sum_{i \leq n-1} a_i S_i\|_\infty + 2n^2 \varepsilon_n \leq (3+\varepsilon)\|\sum_{i \leq n-1} a_i S_i\|_\infty. \tag{33}$$

Suppose now that $n = \min v \neq \min s$, say $\min s < \min v$. Let $i_0 < n$, if possible, be such that $\min s \in v(i_0)$. Then,

$$|\langle \Phi(s) \upharpoonright (s \setminus t), \sum_{i \leq n-1} a_i x_i \rangle| \leq |a_{i_0}| \|\xi(v, i_0)\|_\infty + 2 \sum_{i_0 \leq i < n} \sum_{0 \leq j < \min t} \langle \xi(t,j), \xi(s,i) \rangle \leq$$

$$\leq 2\varepsilon_n + 2n^2 \varepsilon_n \leq \varepsilon. \tag{34}$$

$\square$

Finally, it rests to show that the sequence (e_n) is β-unconditionally saturated for every $\beta < \alpha$. We consider the two obvious cases:

CASE 1. $\alpha = \omega^{\beta+1}$. Let

$$\mathcal{D} = \{s \subseteq \mathbb{N} : {}_* s \in \mathcal{B}^0_{\min s}\}.$$

This is an ω^β-uniform family on $\mathbb{N}$ since each family $\mathcal{B}^0_m$ is α_m-uniform and $\sup_m \alpha_m = \omega^\beta$. Therefore, the next claim gives that (e_n) is β-unconditionally saturated for every $\beta < \alpha$.

Claim. $(e_n)_n$ *is* $\mathcal{D}$-*unconditional with constant at most* $2 + \varepsilon$.

Proof of Claim: Fix $t \in \mathcal{D}$, and let $(a_i)_{i \in \mathbb{N}}$ be scalars such that $\| \sum_{i \in \mathbb{N}} a_i e_i \| = 1$. Fix also $s \in \mathcal{C}$. Suppose first that $\min s \in t$. Then since $\sigma(s[i]) > \min s \geq \min t$ and $_*t \in \mathcal{B}^0_{\min t}$ we obtain that

$$|\langle \Phi(s), \sum_{i \in t} a_i e_i \rangle| \leq |a_{\min s}| + \varepsilon \leq (1 + \varepsilon)\| \sum_i a_i e_i \|. \tag{35}$$

Now suppose that $\min s \notin t$, but $s \cap t \neq \emptyset$ (otherwise $\langle \Phi(s), \sum_{i \in t} a_i e_i \rangle = 0$). Let

$$i_0 = \min\{i < \min s \, : \, s(i) \cap t \neq \emptyset\}.$$

Then for every $i_0 < i < \min s$ we have that $\sigma(s[i]) > \max s_{i_0} \geq \min t$, so

$$|\sum_{j \in t} a_j \xi(s, i)(j)| < \varepsilon_{\sigma(s[i])}, \tag{36}$$

hence

$$|\langle \Phi(s) \upharpoonright u, \sum_{i \in t} a_i e_i \rangle| \leq | \sum_{j \in t \cap s(i_0)} a_j \xi(s, i_0)(j)| + \sum_{i_0 < i < \min s} | \sum_{j \in t} a_j \xi(s, i)(j)| =$$

$$= |\langle \Phi(s) \upharpoonright (\{n\} \cup s(0) \cup \cdots \cup (s(i_0) \cap t)), \sum_{i \geq \min t} a_i e_i \rangle| +$$

$$+ \sum_{i_0 < i < \min s} | \sum_{j \in t} a_j \xi(s, i)(j)| \leq \| \sum_{i \geq \min t} a_i e_i \| + \varepsilon \| \sum_{i \in \mathbb{N}} a_i e_i \| \leq$$

$$\leq (2 + \varepsilon)\| \sum_i a_i e_i \|, \tag{37}$$

the last inequality because (e_i) is monotone. $\qquad \square$

CASE 2. $\alpha = \omega^\gamma$, γ a countable limit ordinal. The desired result follows from the following fact.

Claim. *For every* $n \in \mathbb{N}$, *the sequence* (e_i) *is* $\mathcal{B}^n_0$-*unconditional with constant at most* $2n + 1$.

Proof of Claim: Fix $n \in \mathbb{N}$ and $t \in \mathcal{B}^n_0$. Let $(a_i)_{i \in \mathbb{N}}$ be scalars such that $\| \sum_{i \in \mathbb{N}} a_i e_i \| = 1$. Fix $s \in \mathcal{C}$. Suppose first that $n \leq \min s$. Then in a similar manner that in CASE 1 one can show that

$$|\langle \Phi(s), \sum_{i \in t} a_i e_i \rangle| \leq |a_{\min s}| + \varepsilon \leq (1 + \varepsilon)\| \sum_i a_i e_i \|. \tag{38}$$

Suppose that $m = \min s < n$, then

$$|\langle \Phi(s), \sum_{i \in t} a_i e_i \rangle| \leq |a_{\min s}| + \sum_{i=0}^{m-1} | \sum_{j \in s(i) \cap t} a_j \xi(s, i)(j)| =$$

$$= |a_{\min s}| + \sum_{i=0}^{m-1} |\langle \Phi(s) \upharpoonright u_i, \sum_{j \geq \min(s(i) \cap t)} a_j \rangle| \leq$$

$$\leq (2m + 1)\| \sum_i a_i e_i \|. \tag{39}$$

where $u_i = s(0) \cup \cdots \cup (s(i) \cap t)$. $\qquad \square$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \square$

Corollary 4.13. *For every indecomposable ordinal α there is a weakly-compact $K \subseteq c_{00}$ such that*

(a) $K \subseteq B_{c_0}$ is point-finite (i.e. $\{\xi(n) : \xi \in K\}$ is finite for every integer n) supported by a α-uniform barrier on $\mathbb{N}$,

(b) the evaluation mapping sequence $(p_i)_i$ of $C(K)$ is a normalized weakly-null monotone basic sequence, and

(c) The summing basis of c is 4-finitely representable in every subsequence of $(p_i)_i$; hence no subsequence of (p_n) is unconditional, but

(d) $(p_i)_i$ is β-unconditionally saturated for every $\beta < \alpha$.

PROOF. Let $\mathcal{C}$ be the α-uniform family on $\mathbb{N}$ and let $\Phi : \mathcal{C} \to c_{00}$ be the U-mapping given in proof of Theorem 4.12. Let $M \subseteq \mathbb{N}$ be such that $\mathcal{C} \upharpoonright N$ is a α-uniform barrier on $\mathbb{N}$. Let θ be the order-preserving mapping from M onto $\mathbb{N}$. Let $\mathcal{B} = \theta"\mathcal{C} = \{\theta"(s) : s \in \mathcal{C}\}$ and let $\varphi : \mathcal{B} \to c_{00}$ be naturally defined by $\varphi(s) = \Phi(\theta^{-1}(s))$. $\mathcal{B}$ is a uniform barrier on $\mathbb{N}$ and φ is a U-mapping. Observe that every $s \in \mathcal{B}$ has a unique decomposition, given by the one of $\theta^{-1}s$. Let

$$K = \{\varphi(s) \upharpoonright (u \setminus t) : u \sqsubseteq s \in \mathcal{B}, \, t \subseteq s(i) \text{ for some } i\}.$$

This is a weakly-compact subset of c_0, and the corresponding evaluation mapping sequence $(p_i)_i$ is 1-equivalent to the subsequence $(e_n)_{n \in M}$ of the weakly-null sequence $(e_i)_i$ given in the proof of Theorem 4.12. So K fulfills all the requirements. $\square$

REFERENCES

[1] D. Alspach and S.A. Argyros, *Complexity of weakly null sequences*, Dissertationes Mathematicae, **321**, (1992), 1–44.

[2] S.A. Argyros, S. Mercourakis and A. Tsarpalias, *Convex unconditionality and summability of weakly null sequences*, Israel J. Math. **107** (1998), 157–193

[3] S. A. Argyros; G. Godefroy; H. P. Rosenthal, *Descriptive set theory and Banach spaces*. Handbook of the geometry of Banach spaces, Vol. 2, 1007–1069, North-Holland, Amsterdam, 2003.

[4] S. A. Argyros and S. Todorcevic, *Ramsey methods in analysis*. Birkhäuser Verlag, Basel 2005.

[5] S. A. Argyros and V. Kanellopoulos, *Determining c_0 in $C(K)$ spaces*, Fund. Math. 187 (2005), no. 1, 61–93.

[6] A. Arvanitakis, *Weakly null sequences with an unconditional subsequence*. Preprint 2004.

[7] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*. Studia Math. **17** 1958 151–164.

[8] C. Bessaga and A. Pełczyński, *Spaces of continuous functions. IV. On isomorphical classification of spaces of continuous functions.* Studia Math. **19** 1960 53–62.

[9] P.G. Casazza and T.J. Shura, *Tsirelson's space*. Lecture Notes in Math., Vol.1363, Springer-Verlag, Berlin 1989.

[10] J. Elton, *Thesis*, Yale University (1978).

[11] I, Gasparis, *A dichotomy theorem for subsets of the power set of the natural numbers*. Proc. Amer. Math. Soc. **129** (2001), no. 3, 759–764.

[12] I. Gasparis, E. Odell and B. Wahl, *Weakly null sequences in the Banach space $C(K)$*. Preprint 2004.

[13] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I. Sequence spaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92.

[14] J. Lopez Abad and S. Todorcevic, *Partial unconditionality of weakly null sequences*. RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 100 (2006), no. 1-2.

[15] C. St. J. A. Nash-Williams, *On well-quasi-ordering transfinite sequences,* Proc. Cambridge Philos. Soc. **61** (1965), 33–39.

[16] B. Maurey and H. P. Rosenthal, *Normalized weakly null sequence with no unconditional subsequence.* Studia Math. 61 (1977), no. 1, 77–98.

[17] B. Maurey, *Une suite faiblement convergente vers zero sans sous-suite inconditionnelle*, Séminaire Maurey-Schwartz 1975-76, exposé IX (1976).

[18] B. Maurey, *Quelques resultats concernant l'inconditionnalite*, Séminaire Maurey-Schwartz 1975-76, exposé XVI (1976).

[19] E. Odell, *Applications of Ramsey theorems to Banach space theory*, Notes in Banach spaces, pp. 379–404, Univ. Texas Press, Austin, Tex., 1980.

[20] E. Odell, On Schreier unconditional sequences, Contemp. Math. **144** (1993), 197-201.

[21] A. Pełczyński and Z. Semadeni, *Spaces of continuous functions III. Spaces $C(\Omega)$ for Ω without perfect subsets*, Studia Math. **18** (1959), 211–222.

[22] P. Pudlak and V. Rödl, *Partition theorems for systems of finite subsets of integers*, Discrete Math., **39**, (1982), 67-73.

[23] H. P. Rosenthal, *The Banach spaces $C(K)$.* Handbook of the geometry of Banach spaces, Vol. 2, 1547–1602, North-Holland, Amsterdam, 2003.

EQUIPE DE LOGIQUE MATHÉMATIQUE, UNIVERSITÉ PARIS 7, 2 PLACE JUSSIEU, 75251 PARIS CEDEX 05, FRANCE.

E-mail address: abad@logique.jussieu.fr

C.N.R.S., U.M.R. 7056 ET UNIVERSITÉ PARIS 7, U.F.R. DE MATHÉMATIQUES, CASE 7012, 2 PLACE JUSSIEU, 75251 PARIS, CEDEX 5, FRANCE.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, CANADA, M5S 3G3

E-mail address: stevo@math.toronto.edu