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ABSTRACT

Following ideas of van Dijk and Hille we study the link which exists between maximal degenerate representations and Berezin kernels.
We consider the conformal group $\mathrm{Conf}(V)$ of a simple real Jordan algebra V. The maximal degenerate representations π_s ($s \in \mathbb{C}$) we shall study are induced by a character of a maximal parabolic subgroup $\bar{P}$ of $\mathrm{Conf}(V)$. These representations π_s can be realized on a space I_s of smooth functions on V. There is an invariant bilinear form $\mathfrak{B}_s$ on the space I_s. The problem we consider is to diagonalize this bilinear form $\mathfrak{B}_s$, with respect to the action of a symmetric subgroup G of the conformal group $\mathrm{Conf}(V)$. This bilinear form can be written as an integral involving the Berezin kernel B_ν, an invariant kernel on the Riemannian symmetric space G/K, which is a Makarevich symmetric space in the sense of Bertram. Then we can use results by van Dijk and Pevzner who computed the spherical Fourier transform of B_ν. From these, one deduces that the Berezin kernel satisfies a remarkable Bernstein identity:

$$D(\nu)B_\nu = b(\nu)B_{\nu+1},$$

where $D(\nu)$ is an invariant differential operator on G/K and $b(\nu)$ is a polynomial. By using this identity we compute a Hua type integral which gives the normalizing factor for an intertwining operator from I_{-s} to I_s. Furthermore, we obtain the diagonalization of the invariant bilinear form with respect to the action of the maximal compact group U of the conformal group $\mathrm{Conf}(V)$.

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In this section we shall introduce the notion of a Berezin kernel on a symmetric space of a particular type by mean of Jordan algebraic methods.

1.1. Jordan algebras and their conformal groups

A finite-dimensional vector space V on $\mathbb{R}$ or $\mathbb{C}$ is a *Jordan algebra* if it is endowed with a bilinear map $(x, y) \to xy$ from $V \times V$ into V satisfying the two following axioms:

$$\text{(J1)} \quad xy = yx, \quad \forall x, y \in V, \qquad \text{(J2)} \quad x(x^2 y) = x^2(xy), \quad x, y \in V.$$

Let $L(x) \in \mathrm{End}(V)$ denote for every $x \in V$ the linear map defined by $L(x)y = xy$ for every $y \in V$. Let r and n denote respectively the rank and the dimension of the Jordan algebra V. For a regular element x, the minimal polynomial f_x is of degree r,

$$f_x(\lambda) = \lambda^r - a_1(x)\lambda^{r-1} + \cdots + (-1)^r a_r(x).$$

The coefficient a_j is a homogeneous polynomial of degree j, $\Delta(x) = a_r(x)$ is the *Jordan determinant*, and $\mathrm{tr}(x) = a_1(x)$ is the *Jordan trace* of x. (See [8, p. 28] for more details.)

Define the so-called *quadratic representation* of V by: $P(x) = 2L(x)^2 - L(x^2)$, $x \in V$. One checks that $\Delta(x)^2 = \det(P(x))^{r/n}$, where $P(x)$ is seen as an endomorphism of V.

A real Jordan algebra V is said to be *semi-simple* if the bilinear form $\mathrm{Tr}\, L(xy)$ is nondegenerate on V. Such an algebra is unital. Furthermore it is called *Euclidean* if the above bilinear form is positive definite.

An involutive automorphism α of V is called *Euclidean* if the bilinear form $\mathrm{Tr}\, L(\alpha(x)y)$ is positive definite on V. For a semi-simple Jordan algebra such a Euclidean involution always exists.

Assume from now on that V is a simple real Jordan algebra. According to the general construction of Kantor–Koecher–Tits one associates to V a simple Lie group which can be understood as a group of conformal transformations of the corresponding Jordan algebra (in a sense we shall explain).

Let us recall this classical construction. The *structure group* $\mathrm{Str}(V)$ of V can be defined as the subgroup of $GL(V)$ of elements g such that there exists a real number $\chi(g)$ for which

$$(1.1) \qquad \Delta(g.x) = \chi(g)\Delta(x), \quad x \in V.$$

The map $g \to \chi(g)$ is a character of $\mathrm{Str}(V)$ which is a reductive Lie group.

The Jordan algebra V can be identified with the abelian group N of its own translations via the map $y \to n_y$ from V to N, where $n_y(x) = x + y$, $\forall x \in V$. The *conformal group* $\mathrm{Conf}(V)$ (or the Kantor–Köcher–Tits group) of the Jordan algebra V is the group of rational transformations of V generated by translations, elements in $\mathrm{Str}(V)$ and the inversion map $j : x \to -x^{-1}$. It is a simple Lie group. A transformation $g \in \mathrm{Conf}(V)$ is conformal in the sense that, at each point x, where g is well defined, its differential $(Dg)_x$ belongs to the structure group $\mathrm{Str}(V)$.

The subgroup of all affine conformal transformations $P = \mathrm{Str}(V) \ltimes N$ is a maximal parabolic subgroup of $\mathrm{Conf}(V)$. Let σ be the involution of $\mathrm{Conf}(V)$ given by

$$\sigma(g) = j \circ g \circ j, \quad g \in \mathrm{Conf}(V),$$

where $j \in \mathrm{Conf}(V)$ is the inversion map. We define $\overline{N} = \sigma(N)$ and $\overline{P} := \mathrm{Str}(V) \ltimes \overline{N}$.

From the geometric point of view the subgroup $\overline{P}$ can be characterized in the following way:

$$\overline{P} = \left\{ g \in \mathrm{Conf}(V)' \mid g(0) = 0 \right\},$$

where $\mathrm{Conf}(V)'$ is the subset of $\mathrm{Conf}(V)$ of all conformal transformations well-defined at $0 \in V$. It is open and dense in $\mathrm{Conf}(V)$. Moreover, $\mathrm{Conf}(V)' = N\,\mathrm{Str}(V)\overline{N}$. The map $N \times \mathrm{Str}(V) \times \overline{N} \to \mathrm{Conf}(V)'$ is a diffeomorphism. We shall refer to this decomposition as to the *Gelfand–Naimark decomposition* of the conformal group. Furthermore, for every transformation $g \in \mathrm{Conf}(V)$ which is well defined at $x \in V$, the transformation gn_x belongs to $\mathrm{Conf}(V)'$ and its Gelfand–Naimark decomposition is given by:

$$gn_x = n_{g.x} (Dg)_x \bar{n}', \tag{1.2}$$

where $(Dg)_x \in \mathrm{Str}(V)$ is the differential of the conformal map $x \to g.x$ at x and $\bar{n}' \in \overline{N}$ (see [17, Proposition 1.4]).

The flag variety $\mathcal{M} = \mathrm{Conf}(V)/\overline{P}$, which is compact, is the *conformal compactification* of V. In fact the map $x \to (n_x \circ j)P$ gives rise to an embedding of V into $\mathcal{M}$ as an open dense subset, and every transformation in $\mathrm{Conf}(V)$ extends to $\mathcal{M}$.

The Euclidean involution α of V introduced above also defines an involution θ of $\mathrm{Conf}(V)$ by:

$$\theta(g) = \alpha \circ j \circ g \circ j \circ \alpha. \tag{1.3}$$

It turns out that θ is a Cartan involution of $\mathrm{Conf}(V)$ (see [17, Proposition 1.1]). So the fix points subgroup of θ: $U = \mathrm{Conf}(V)^\theta$ is a maximal compact subgroup of $\mathrm{Conf}(V)$.

Let us recall that a simple real Jordan algebra is either a real form of a simple complex Jordan algebra or a simple complex Jordan algebra considered as a real one. We conclude this section with the classification of simple Jordan algebras, together with their conformal groups and maximal compact subgroups U, given in Table 1. We shall refer to the Jordan algebras given in the first and fourth columns as to the *nonsplit case*, and to those of the second and third columns as to the *split case*.

Table 1

	Complex Nonsplit	Euclidean Split	Non-Euclidean Split	Non-Euclidean Nonsplit ($m = 2\ell$)
V	$Sym_m(\mathbb{C})$	$Sym_m(\mathbb{R})$	$\times$	$Sym_{2\ell}(\mathbb{R}) \cap M_\ell(\mathbb{H})$
Conf(V)	$Sp_m(\mathbb{C})$	$Sp_m(\mathbb{R})$	$\times$	$Sp(\ell, \ell)$
U	Sp_m	U_m	$\times$	$Sp_\ell \times Sp_\ell$
V	$M_m(\mathbb{C})$	$Herm_m(\mathbb{C})$	$M_m(\mathbb{R})$	$M_\ell(\mathbb{H})$
Conf(V)	$SL_{2m}(\mathbb{C})$	$SU(m, m)$	$SL_{2m}(\mathbb{R})$	$SL_{2\ell}(\mathbb{H})$
U	SU_{2m}	$S(U_m \times U_m)$	SO_{2m}	$SU_{2\ell}(\mathbb{H})$
V	$Skew_{2m}(\mathbb{C})$	$Herm_m(\mathbb{H})$	$Skew_{2m}(\mathbb{R})$	$\times$
Conf(V)	$SO_{4m}(\mathbb{C})$	SO^*_{4m}	$SO(2m, 2m)$	$\times$
U	SO_{4m}	U_{2m}	$SO_{2m} \times SO_{2m}$	$\times$
V	$\mathbb{C} \times \mathbb{C}^{n-1}$	$\mathbb{R} \times \mathbb{R}^{n-1}$	$\mathbb{R}^p \times \mathbb{R}^q$	$\mathbb{R}^n$
Conf(V)	$SO_{n+2}(\mathbb{C})$	$SO_o(2, n)$	$SO_o(p+1, q+1)$	$SO_o(1, n+1)$
U	SO_{n+2}	$SO_2 \times SO_n$	$SO_{p+1} \times SO_{q+1}$	SO_{n+1}
V	$Herm(3, \mathbb{O})_\mathbb{C}$	$Herm(3, \mathbb{O})$	$Herm(3, \mathbb{O}_s)$	$\times$
Conf(V)	$E_7(\mathbb{C})$	$E_{7(-25)}$	$E_{7(7)}$	$\times$
U	E_7	$E_6 \times SO_2$	SU_8	$\times$
Type	IV	I	II	III

1.2. Makarevich Riemannian symmetric spaces

A Makarevich symmetric space of tube type is a reductive symmetric space which can be realized as an open symmetric orbit in the conformal compactification $\mathcal{M}$ of a simple real Jordan algebra V. We refer the reader to [3,4] and literature cited there for a detailed description. We shall concentrate our interest on Makarevich symmetric spaces carrying an invariant Riemannian metric.

Let α be as previously a Euclidean involution of the Jordan algebra V and let

$$V_0 := \{x \in V \mid \alpha(x) = x\}, \qquad V_1 := \{x \in V \mid \alpha(x) = -x\},$$

be the corresponding eigenspaces of α on V. Notice that the set V_0 is a Euclidean Jordan algebra, whose dimension and rank will be denoted by n_0 and r_0. This fact implies that the interior Ω_0 of the set $\{x^2 \mid x \in V_0\}$ of "positive" elements in V_0 is a symmetric cone in V_0. Notice that $r = r_0$ in the split case and $r = 2r_0$ in the nonsplit case. The Jordan algebra $W = V_0 + iV_1$ is a Euclidean real form of the complexification $V^\mathbb{C} = V + iV$. We will denote by Ω the symmetric cone of W. If V is a simple Jordan algebra of type I, II, or III, then W is simple, while, if V is of type IV, $W \simeq V_0 \times V_0$.

According to [3] one introduces two groups:

$$G := \{g \in \mathrm{Conf}(V) \mid (-\alpha) \circ g \circ (-\alpha) = g\}_o \quad \text{and} \quad K := \{g \in G \mid g.e = e\},$$

where the subscript $_o$ stands as usual for the connected component of the identity transformation and e denotes the identity element of the Jordan algebra V. It follows that K is a maximal compact subgroup of G. Moreover, the associated Riemannian symmetric space is a real tube domain:

$$(1.4) \qquad \mathcal{X} := G/K = \Omega_0 + V_1.$$

The set $\mathcal{X}$ is a Riemannian *Makarevich symmetric space of tube type*. We shall refer to the previous description as to the tube realization of $\mathcal{X}$.

Such a Riemannian Makarevich symmetric space can be obtained as a real form of a Hermitian symmetric space of tube type. The transform $x \mapsto u = (x - e)(x + e)^{-1}$ maps the symmetric space $\mathcal{X}$ onto a bounded domain $\mathcal{D}$, which is the unit ball in V with respect to a so-called spectral norm. Its inverse is the *Cayley transform*

$$(1.5) \qquad c : u \mapsto x = c(u) = (e + u)(e - u)^{-1}.$$

If V is a Euclidean Jordan algebra, then $V_1 = \{0\}$, and $\mathcal{X}$ is a symmetric cone. If V has a complex structure, then $\mathcal{X}$ is a Hermitian symmetric tube.

Table 2 gives the classification of Riemannian Makarevich symmetric spaces obtained in this way. The second row of table gives the root system of the pair $(\mathfrak{g}, \mathfrak{a})$, where $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{a}$ is a Cartan subspace.

Example 1. Let $V = M(m, \mathbb{R})$, the space of $m \times m$ real matrices, be equipped with the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. Then the Jordan determinant coincides with the usual matrix determinant: $\Delta(x) = \det x$. The structure group $\mathrm{Str}(V)$ is the group $S\big(GL(m, \mathbb{R}) \times GL(m, \mathbb{R})\big)$, acting on V by $x \mapsto g_1 x g_2^{-1}$ $(g_1, g_2 \in GL(m, \mathbb{R}))$. The conformal group $\mathrm{Conf}(V)$ is the group $SL(2m, \mathbb{R})/\{\pm I\}$ acting on V by

$$x \mapsto (ax + b)(cx + d)^{-1} \quad \text{if } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The differential of a conformal transformation g is given by $(Dg)_x y = h_1(x) y \times h_2(x)$, where, under the condition that $\det c \neq 0$, $h_1(x) = (ac^{-1}d - b)(cx + d)^{-1}c$, $h_2(x) = (cx + d)^{-1}$. Since $\det(ac^{-1}d - b) \det c = \det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 1$ we finally get: $\chi((Dg)_x) = \det(cx + d)^{-2}$. The Euclidean involution α on $V = M(m, \mathbb{R})$ is given by the usual matrix transposition: $\alpha(x) = x^T$, and $V_0 = Sym_m(\mathbb{R})$, $V_1 = Skew_m(\mathbb{R})$. Recall that an element $g \in \mathrm{Conf}(V)$ belongs to the group G if and only if $g \circ (-\alpha) = (-\alpha) \circ g$. If $g.x = (ax + b)(cx + d)^{-1}$, the above condition on g to be in G leads to

$$\big(xc^T + d^T\big)(-ax + b) = -\big(xa^T + b^T\big)(-cx + d),$$

or equivalently, $a^T c + c^T a = 0$, $b^T d + d^T b = 0$, and, for every $x \in V$, $x(a^T + dc^T b) = (d^T a + b^T c)x$. By Schur's lemma this last condition says that there is $\lambda \in \mathbb{R}$ such that $a^T d + c^T b = \lambda\, \mathrm{Id}_m$.

Table 2

	Complex	Euclidean	Non-Euclidean Split	Non-Euclidean Nonsplit ($m = 2\ell$)
	C_m	A_{m-1}	D_m	C_ℓ
V	$Sym_m(\mathbb{C})$	$Sym_m(\mathbb{R})$	$\times$	$Sym_{2\ell}(\mathbb{R}) \cap M_\ell(\mathbb{H})$
G	$Sp_m(\mathbb{R})$	$SL_m(\mathbb{R}) \times \mathbb{R}_+$	$\times$	$Sp_\ell(\mathbb{C})$
K	U_m	SO_m	$\times$	Sp_ℓ
V	$M_m(\mathbb{C})$	$Herm_m(\mathbb{C})$	$M_m(\mathbb{R})$	$M_\ell(\mathbb{H})$
G	$SU(m,m)$	$SL_m(\mathbb{C}) \times \mathbb{R}_+$	$SO(m,m)$	$Sp(\ell,\ell)$
K	$S(U_m \times U_m)$	SU_m	$SO_m \times SO_m$	$Sp_\ell \times Sp_\ell$
V	$Skew_{2m}(\mathbb{C})$	$Herm_m(\mathbb{H})$	$Skew_{2m}(\mathbb{R})$	$\times$
G	SO^*_{4m}	$SL_m(\mathbb{H}) \times \mathbb{R}_+$	$SO_{2m}(\mathbb{C})$	$\times$
K	U_{2m}	Sp_m	SO_{2m}	$\times$
V	$\mathbb{C} \times \mathbb{C}^{n-1}$	$\mathbb{R} \times \mathbb{R}^{n-1}$	$\mathbb{R}^p \times \mathbb{R}^q$	$\mathbb{R}^n$
G	$SO_0(2,n)$	$SO_0(1,n-1) \times \mathbb{R}_+$	$SO_0(1,p) \times SO_0(1,q)$	$SO_0(1,n)$
K	$SO_2 \times SO_n$	SO_{n-1}	$SO_p \times SO_q$	SO_n
V	$Herm(3,\mathbb{O})_\mathbb{C}$	$Herm(3,\mathbb{O})$	$Herm(3,\mathbb{O}_s)$	$\times$
G	$E_{7(-25)}$	$E_{6(-26)} \times \mathbb{R}_+$	SU^*_8	$\times$
K	$E_6 \times SO_2$	F_4	Sp_4	$\times$

Define the matrix

$$\Upsilon = \begin{pmatrix} 0 & \mathrm{Id}_m \\ \mathrm{Id}_m & 0 \end{pmatrix}.$$

Then the element g belongs to G if and only if $g^T \Upsilon g = \lambda \Upsilon$. Since $\det g = 1$, it follows that $\lambda^{2m} = 1$, and since G is connected, $\lambda = 1$.

Finally we have shown that $G = SO_o(\Upsilon)/\pm\mathrm{Id}$, where $SO(\Upsilon)$ stands for the special orthogonal group of the quadratic form defined by the matrix Υ which has signature (m,m). Therefore $G \simeq SO_o(m,m)/\pm\mathrm{Id}$. The conformal compactification $\mathcal{M}$ of V is the Grassmann manifold $\mathcal{M} = \mathrm{Gr}(2m,m)$ of m-dimensional vector subspaces in $\mathbb{R}^{2m}$. The corresponding Riemannian Makarevich symmetric space $\mathcal{X}$ in its tube realization is the set of matrices $x \in M(m,\mathbb{R})$ with a positive definite symmetric part: $x + x^T \gg 0$. Its bounded realization is the ball $\mathcal{D} = \{x \in M(m,\mathbb{R}) \mid \|x\|_{\mathrm{op}} < 1\}$.

Example 2. Let V be the space $\mathbb{R} \times \mathbb{R}^{n-1}$. One writes $x = (x_0, x_1)$, $x_0 \in \mathbb{R}$, $x_1 \in \mathbb{R}^{n-1}$. Then the Jordan product is defined as follows: $z = x \circ y$ if $z_0 = x_0 y_0 - (x_1|y_1)$, $z_1 = x_0 y_1 + y_0 x_1$, where $(x_1|y_1)$ is the usual inner product on

$\mathbb{R}^{n-1}$. The Jordan determinant is then given by $\Delta(x) = x_0^2 + \|x_1\|^2$. (This Jordan algebra is actually a Jordan field.) The corresponding structure group is the group $\mathrm{Str}(V) = \mathbb{R}_+ \times O(n)$, and the conformal group $\mathrm{Conf}(V)$ is equal to $O(1, n+1)$. The conformal compactification of V is the n-sphere S^n. Consider the involution $\alpha : (x_0, x_1) \mapsto (x_0, -x_1)$. The corresponding groups G and K are isomorphic respectively to $SO_0(1, n)$ and $SO(n)$. The Riemannian symmetric space $\mathcal{X}$ is the real hyperbolic space of dimension n, realized as a half-space

$$\mathcal{X} = \{(x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_0 > 0\}.$$

The bounded realization of this symmetric space is isomorphic to the Euclidean ball:

$$\mathcal{D} = \{x \in \mathbb{R}^n \mid \|x\| < 1\}.$$

1.3. Iwasawa decomposition and spherical Fourier transform

According to (1.3) the Cartan involution of the group G is given by

$$\theta(g) = (-j) \circ g \circ (-j), \quad g \in G.$$

The Lie algebra $\mathfrak{g}$, which is an algebra of quadratic vector fields on V, decomposes under $d\theta$ into a direct sum of two eigenspaces: $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of the group K introduced above. Let us fix a Jordan frame $\{c_1, \ldots, c_{r_0}\}$ in V_0. Then the space $\mathfrak{a}$ of the linear vector fields

$$\xi(x) = \sum_{j=1}^{r_0} t_j L(c_j)x \quad (t_j \in \mathbb{R}),$$

is a Cartan subspace in $\mathfrak{p}$. The root system of the pair $(\mathfrak{g}, \mathfrak{a})$ is of type A, C, or D. We choose the Weyl chamber $\mathfrak{a}^+$ defined by $t_1 < \cdots < t_{r_0}$ in case of type A, or $0 < t_1 < \cdots < t_{r_0}$ in case of type C or D.

For type A the positive roots are: $\{\frac{t_j - t_i}{2}, i < j\}$, for type D: $\{\frac{t_j - t_i}{2}, i < j, \frac{t_j + t_i}{2}, i \neq j\}$ and for type C: $\{\frac{t_j - t_i}{2}, i < j, \frac{t_j + t_i}{2}, i \neq j, t_i\}$.

We define $A = \exp \mathfrak{a}$, and, as usual the nilpotent subgroup N. It can be written $N = N_0 \ltimes N_1$, where N_0 is a triangular subgroup in $\mathrm{Str}(V_0)$, and N_1 is the group of translations

$$x \mapsto x + v \quad \text{with } v \in V_1.$$

The *Iwasawa decomposition* can be described as follows: every $x \in \mathcal{X}$ can be uniquely written

$$x = na \cdot e = n_0 a \cdot e + v,$$

with $a \in A$, $n_0 \in N_0$, $v \in V_1$. One writes $a = \exp \mathcal{A}(x)$ with $\mathcal{A}(x) \in \mathfrak{a}$. If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we write, for $\xi(x) = \sum_j t_j L(c_j)x$,

$$\langle \lambda, \xi \rangle = \sum_{j=1}^{r_0} \lambda_j t_j.$$

Then $e^{\langle \lambda, \mathcal{A}(x) \rangle} = \Delta_\lambda(x_0)$, where Δ_λ is the power function of the symmetric cone Ω_0 in the Euclidean Jordan algebra V_0, and x_0 denotes the V_0-component of x: $x = x_0 + x_1$, $x_0 \in V_0$, $x_1 \in V_1$.

The *spherical functions* φ_λ for $\mathcal{X} = G/K$ are given by

$$\varphi_\lambda(x) = \int_K \Delta_{\rho-\lambda}\big((k \cdot x)_0\big)\, dk, \tag{1.6}$$

where dk is the normalized Haar measure of K, and ρ is the half-sum of the positive roots:

$$\langle \rho, \xi \rangle = \frac{1}{2} \operatorname{Tr}(\operatorname{ad} \xi)\Big|_{Lie(N)} = \frac{1}{2} \operatorname{Tr}(\operatorname{ad} \xi)\Big|_{Lie(N_0)} + \frac{1}{2} \operatorname{Tr}(\operatorname{ad} \xi)\Big|_{Lie(N_1)}.$$

One obtains $\rho_j = \frac{d_0}{4}(2j - r_0 - 1) + \frac{n_1}{2r_0}$, where the integer d_0 is defined by $n_0 = r_0 + \frac{d_0}{2} r_0(r_0 - 1)$.

The *spherical Fourier transform* of a K-invariant integrable function f on $\mathcal{X}$ is defined on $i\mathfrak{a}^*$ by

$$\mathcal{F}f(\lambda) = \int_{\mathcal{X}} \varphi_{-\lambda}(x) f(x) \mu(dx) = \int_{\mathcal{X}} \Delta_{\rho+\lambda}(x_0) f(x) \mu(dx),$$

where $\mu(dx) = \Delta(x_0)^{-n/r}\, dx$ is a G-invariant measure on $\mathcal{X}$. Here and elsewhere further the measure dx denotes the Euclidean measure associated to the Euclidean structure defined on V by $(x|y) = \operatorname{tr}(x\alpha(y))$.

1.4. Berezin kernels and the Kantor cross ratio

The *Kantor cross ratio* of four points of a simple Jordan algebra V is the rational function

$$\{x_1, x_2, x_3, x_4\} = \frac{\Delta(x_1 - x_3)}{\Delta(x_2 - x_3)} : \frac{\Delta(x_1 - x_4)}{\Delta(x_2 - x_4)}. \tag{1.7}$$

It is invariant under conformal transformations and extends to the conformal compactification $\mathcal{M}$ of V.

The invariance by translations is obvious. The invariance under $\operatorname{Str}(V)$ comes from the fact that Δ is semi-invariant under $\operatorname{Str}(V)$. The invariance under the inverse $j : x \mapsto -x^{-1}$ follows from the formula [8, Lemma X.4.4]:

$$\Delta\big(j(x) - j(y)\big) = \frac{\Delta(y - x)}{\Delta(x)\Delta(y)}.$$

Furthermore, it is proved that a local transformation which preserves the cross ratio is the restriction of an element of $\mathrm{Conf}(V)$ (cf. [11, Theorem 6]).

Being given the Kantor cross ratio (1.7) we shall introduce the kernel F on $\mathcal{X}$ by

$$F(x,y) = \{x, y, -\alpha(x), -\alpha(y)\} = \frac{\Delta(x+\alpha(x))\Delta(y+\alpha(y))}{\Delta(x+\alpha(y))\Delta(y+\alpha(x))}. \tag{1.8}$$

This function is invariant under G and positive. The *Berezin kernel* B_ν can be defined for every $\nu \in \mathbb{C}$ as

$$B_\nu(x,y) = F(x,y)^{\frac{r_0}{r}\nu} = \begin{cases} F(x,y)^\nu & \text{in the split case,} \\ F(x,y)^{\frac{\nu}{2}} & \text{in the nonsplit case.} \end{cases} \tag{1.9}$$

In virtue of the G-invariance of the kernel B_ν the *Berezin function*

$$\psi_\nu(x) = B_\nu(x,e) = \left(\frac{\Delta(x_0)}{\Delta\left(\frac{x+e}{2}\right)^2}\right)^{\frac{r_0}{r}\nu}$$

is K-invariant. In this formula x_0 denotes as before the V_0-component of x in the decomposition $V = V_0 + V_1$, $x = x_0 + x_1$.

If $\{c_1, \dots, c_{r_0}\}$ is a Jordan frame of V_0, and if $x = \sum_{j=1}^{r_0} e^{t_j} c_j$, then

$$\psi_\nu(x) = \prod_{j=1}^{r_0} \left(\cosh\frac{t_j}{2}\right)^{-2\nu}.$$

Theorem 1.1. *Assume $\Re\nu > \frac{n}{r_0} - 1$. Then the function ψ_ν is integrable and its spherical Fourier transform is given by*

$$\mathcal{F}\psi_\nu(\lambda) = \frac{P(\lambda,\nu)P(-\lambda,\nu)}{Q(\nu)}, \tag{1.10}$$

with

$$P(\lambda,\nu) = \prod_{j=1}^{r_o} \Gamma\left(\frac{1}{2} + \nu - \delta + \lambda_j\right), \quad \delta = \frac{n}{2r_0}, \qquad Q(\nu) = c\prod_{j=1}^{2r_o} \Gamma(\nu - \beta_j)$$

where the constant c[1] *and the real numbers β_j's depend on the Jordan algebra V. In particular, for ν real and $\lambda \in i\mathfrak{a}^*$, then $\bar{\lambda} = -\lambda$ and*

$$\mathcal{F}\psi_\nu(\lambda) = \frac{|P(\lambda,\nu)|^2}{Q(\nu)} \geqslant 0,$$

[1] We denote by c throughout this paper different constants depending only on V.

Recall that the Gindikin Γ-function of a symmetric cone Ω is defined by

$$\Gamma_\Omega(\nu) = \int_\Omega e^{-\operatorname{tr}(x)} \Delta(x)^{\nu - \frac{n}{r}}\, dx.$$

If the cone Ω is irreducible (the corresponding Euclidean Jordan algebra W is simple), then

$$\Gamma_\Omega(\nu) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\left(\nu - (j-1)\frac{d}{2}\right),$$

where the integer d is defined by $n = r + \frac{d}{2} r(r-1)$.

For types I and II: $Q(\nu) = c 2^{-2r\nu} \Gamma_\Omega(2\nu)$. For type III: $Q(\nu) = c\Gamma_\Omega(\nu)$. For type IV: $\Gamma_\Omega(\nu) = (\Gamma_{\Omega_0}(\nu))^2$, notice that in this case the Jordan algebra W is not simple anymore: $W \simeq V_0 \times V_0$. The above theorem was proved by different methods in [25,29,21].

Let $h(z, w)$ stand for the so-called *canonical polynomial* defined on $V^{\mathbb{C}} \times V^{\mathbb{C}}$ by the following conditions (see [8, p. 262]):

- It is holomorphic in the first variable and anti-holomorphic in the second one.
- For every $g \in \operatorname{Str}(V^{\mathbb{C}})$ one has $h(gz, w) = h(z, g^* w)$.
- For every $x \in W$ one has $h(x, x) = \Delta(e - x^2)$. Recall that the Bergman kernel of the Hermitian symmetric space, whose $\mathcal{X}$ is a real form, is given by $h(z, w)^{-\frac{2n}{r}}$.

Example 1. For $V = M(m, \mathbb{R})$ the canonical polynomial is given by $h(x, x) = \det(I - xx^T)$.

Example 2. For $V = \mathbb{R} \times \mathbb{R}^{n-1}$ the canonical polynomial is given by $h(x, x) = (1 - \|x\|^2)^2$.

Recall that the Cayley transform (1.5) maps the bounded domain $\mathcal{D}$ onto the tube $\mathcal{X} = \Omega_0 + V_1$. Define $\widetilde{F}(u, v) = F(c(u), c(v))$. Since $-\alpha(c(u)) = c(\alpha(u^{-1}))$ we obtain

$$\widetilde{F}(u, v) := \{u, v, \alpha(u^{-1}), \alpha(v^{-1})\} = \frac{\Delta(u - \alpha(u^{-1}))\Delta(v - \alpha(v^{-1}))}{\Delta(u - \alpha(v^{-1}))\Delta(v - \alpha(u^{-1}))}.$$

Furthermore, $\widetilde{F}(u, 0) = \lim_{v \to 0} \widetilde{F}(u, v) = \Delta(\alpha(u^{-1}) - u)\Delta(\alpha(u))$, and one can show that $\widetilde{F}(u, 0) = h(u, u)$. Define similarly

$$\widetilde{B}_\nu(u, v) = B_\nu(c(u), c(v)),$$
$$\widetilde{\psi}_\nu(u) = \psi_\nu(c(u)).$$

Then

$$\widetilde{\psi}_\nu = \begin{cases} h(u, u)^\nu & \text{in the split case,} \\ h(u, u)^{\frac{\nu}{2}} & \text{in the nonsplit case.} \end{cases}$$

Consider the integral

$$I(\nu) = \int_{\mathcal{D}} \widetilde{\psi}_\nu(u) h(u,u)^{-\frac{n}{r}}\, du.$$

It can be written as

$$\int_{\mathcal{D}} h(u,u)^{\frac{r_0}{r}\nu - \frac{n}{r}}\, du.$$

Since $h(u,u)^{-\frac{n}{r}}\, du$ is an invariant measure on $\mathcal{D}$ the last integral equals

$$I(\nu) = c \int_{\mathcal{X}} \psi_\nu(x)\mu(dx) = c\mathcal{F}\psi_\nu(\rho).$$

Therefore we obtain

Corollary 1.2.

$$I(\nu) = \operatorname{vol}\mathcal{D} \frac{Q(\frac{n}{r})}{P(\rho, \frac{n}{r})P(-\rho, \frac{n}{r})} \cdot \frac{P(\rho,\nu)P(-\rho,\nu)}{Q(\nu)}.$$

Eventually we may notice that according to the Jordan algebra's type some simplifications in the above formula, due to explicit expressions of Γ-factors, are possible, see examples below.

Example 1. Let $V = M(m,\mathbb{R})$. Then $V_0 = Sym_m(\mathbb{R})$, $V_1 = Skew_m(\mathbb{R})$, the invariant measure on $\mathcal{X}$ is given by $\det(x_0)^{-m} dx_0 dx_1$, and the Berezin function is

$$\psi_\nu(x) = 4^{m\nu}\left[\frac{\det x_0}{\det(e+x)^2}\right]^\nu, \quad \text{with } \phi_\nu(e) = 1.$$

Its push-forward to $\mathcal{D} = \{u \in V \mid \|u\|_{\mathrm{op}} < 1\}$ equipped with the invariant measure $\det(e - uu^T)^{-m} du$ is given by $\widetilde{\psi}_\nu(u) = \det(e - uu^T)^\nu$.

The spherical Fourier transform of ψ_ν is the given by

$$\begin{aligned}\mathcal{F}\psi_\nu(\lambda) &= \int_{\omega_0+V_1} \left[\frac{\det x_0}{\det(e+x)^2}\right]^\nu \Delta^o_{\rho-\lambda}(x_0)\Delta^o(x_0)^{-m}\, dx_0\, dx_1 \\ &= c\frac{\prod_{j=1}^m \Gamma(\nu - \frac{m-1}{2} + \lambda_j)\Gamma(\nu - \frac{m-1}{2} - \lambda_j)}{4^{-m\nu}\prod_{j=1}^m \Gamma(2\nu - j + 1)}.\end{aligned}$$

Notice that in this case

$$\begin{aligned}Q(\nu) &= c4^{-m\nu}\Gamma_\Omega(2\nu) = c'\Gamma_{\Omega_0}(\nu)\Gamma_{\Omega_0}\left(\nu + \frac{1}{2}\right) \\ &= c'' \prod_{j=1}^m \Gamma\left(\nu - \frac{j-1}{2}\right) \prod_{j=1}^m \Gamma\left(\nu + \frac{1}{2} - \frac{j-1}{2}\right).\end{aligned}$$

Moreover,

$$I(\nu) = \frac{\Gamma_{\Omega_0}(\nu - \frac{m-1}{2})}{\Gamma_{\Omega_0}(\nu + \frac{1}{2})}.$$

Example 2. In the case when $V = \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ the Jordan determinant is given by $\Delta(x) = x_0^2 + \|x_1\|^2$ and the Berezin function is given by

$$\psi_\nu(x) = \left[\frac{4x_0}{(1+x_0)^2 + \|x_1\|^2}\right]^\nu.$$

Its spherical Fourier transform is equal to

$$\mathcal{F}\psi_\nu(\lambda) = c\frac{\Gamma(\nu + \lambda - \frac{n-1}{2})\Gamma(\nu - \lambda - \frac{n-1}{2})}{\Gamma(\nu)\Gamma(\nu + \frac{1}{2} - \frac{n-1}{2})}.$$

And finally,

$$I(\nu) = c\frac{\Gamma(\nu - n + 1)}{\Gamma(\nu + \frac{1}{2} - \frac{n-1}{2})}.$$

1.5. A Bernstein identity

An interesting consequence of Theorem 1.1 is a Bernstein identity. Let $D(\nu)$ be the invariant differential operator on $\mathcal{X} = G/K$ whose symbol, i.e. its image through the Harish-Chandra isomorphism

$$\gamma : \mathbb{D}(\mathcal{X}) \to S(\mathbb{C}^{r_0}), \quad D \mapsto \gamma_D(\lambda),$$

is

$$\gamma_{D(\nu)}(\lambda) = \gamma_\nu(\lambda) = \prod_{j=1}^{r_0}\left(\nu + \frac{1}{2} - \delta + \lambda_j\right)\left(\nu + \frac{1}{2} - \delta - \lambda_j\right).$$

Corollary 1.3. *The Berezin kernel* (1.9) *satisfies the following identity*:

$$D(\nu)B_\nu = b(\nu)B_{\nu+1},$$

where $b(\nu)$ is the polynomial of degree $2r_0$ given by:

$$b(\nu) = \frac{Q(\nu+1)}{Q(\nu)} = \prod_{j=1}^{2r_o}(\nu - \beta_j).$$

In the split case (types I and II): $b(\nu) = \prod_{j=1}^{r_0}(\nu - \frac{d}{4}(j-1))(\nu + \frac{1}{2} - \frac{d}{4}(j-1))$. In case when V is of type III ($r = 2r_0$), $b(\nu) = \prod_{j=1}^{2r_0}(\nu - \frac{d}{2}(j-1))$, where the integer

d is defined through $n = r + \frac{d}{2}r(r-1)$. In case when V is complex (type IV): $b(\nu) = \prod_{j=1}^{r_0}(\nu - \frac{d_0}{2}(j-1))^2$.

This identity has been established for V complex by Engliš [5], also in a slightly different form by Unterberger and Upmeier [21], and has been generalized in [6].

In general we do not know any explicit expression for the differential operator $D(\nu)$. However an explicit formula has been obtained in the special case of a simple complex Jordan algebra, and for $\nu = \delta$ [12]:

$$D(\delta) = \Delta(y)^{1+\frac{n}{r}} \Delta\left(\frac{\partial}{\partial z}\right)\Delta\left(\frac{\partial}{\partial \bar{z}}\right)\Delta(y)^{1-\frac{n}{r}},$$

where $z = x + iy$.

Observe that, for $V = \mathbb{C}$ we get: $D(\delta) = y^2 \frac{\partial^2}{\partial z \partial \bar{z}}$ is nothing but the Laplace–Beltrami operator of the upper hyperbolic upper half-plane.

Example 1. When $V = M(m, \mathbb{R})$ the Bernstein polynomial and the Harish-Chandra symbol are respectively given by

$$b(\nu) = \prod_{j=1}^{m}\left(\nu - \frac{j-1}{2}\right)\left(\nu + \frac{1}{2} - \frac{j-1}{2}\right),$$

$$\gamma_\nu(\lambda) = \prod_{j=1}^{m}\left(\nu - \frac{1}{2}(m-1) + \lambda_j\right)\left(\nu - \frac{1}{2}(m-1) - \lambda_j\right).$$

Example 2. When $V = \mathbb{R} \times \mathbb{R}^{n-1}$

$$b(\nu) = \nu\left(\nu - \frac{n}{2} + 1\right), \qquad \gamma_\nu(\lambda) = \left(\nu + \lambda - \frac{n-1}{2}\right)\left(\nu - \lambda - \frac{n-1}{2}\right).$$

1.6. Hua integral

Let us introduce the compact dual U/K of the symmetric space $\mathcal{X} = G/K$ in the complexification $\mathcal{X}^{\mathbb{C}} = G^{\mathbb{C}}/K^{\mathbb{C}}$ of the symmetric space $\mathcal{X}$.

Define $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ and let U be the analytic subgroup of $G^{\mathbb{C}}$ with the Lie algebra $\mathfrak{u}$. The compact symmetric space $\mathcal{Y} = U/K$ is isomorphic to the conformal compactification $\mathcal{M}$ of the Jordan algebra V. The set $\mathcal{Y}'$ of invertible elements y in $V^{\mathbb{C}}$ such that $\bar{y} = y^{-1}$ is open and dense in $\mathcal{Y}$. For $\nu = -\kappa$ ($\kappa \in \mathbb{N}$) the function $\psi_{-\kappa}$ extends as a meromorphic function on $\mathcal{X}^{\mathbb{C}}$. For $y = \sum_{j=1}^{r_0} e^{i\theta_j} c_j$ we have

$$\psi_{-\kappa}(y) = \prod_{j=1}^{r_0}\left(\cos^2\frac{\theta_j}{2}\right)^{\kappa}.$$

This shows that this function is well defined on $\mathcal{Y}$ for $\kappa \in \mathbb{C}$, and bounded for $\Re\kappa \geqslant 0$.

Denote by μ_o the normalized invariant measure on $\mathcal{Y}$ and define

$$J(\kappa) = \int_{\mathcal{Y}} \psi_{-\kappa}(y)\mu_o(dy). \tag{1.11}$$

Theorem 1.4. *For $\Re\kappa \geqslant 0$,*

$$J(\kappa) = c\frac{\prod_{j=1}^{2r_o}\Gamma(\kappa+1+\beta_j)}{\prod_{j=1}^{r_o}\Gamma(\kappa+\frac{1}{2}+\delta-\rho_j)\Gamma(\kappa+\frac{1}{2}+\delta+\rho_j)}.$$

Proof. Assume first that $\kappa \in \mathbb{N}$. The Bernstein identity (1.3) implies that

$$D_{-\kappa}\psi_{-\kappa} = b(-\kappa)\psi_{-\kappa+1}.$$

By integrating this identity on $\mathcal{Y}$ we obtain

$$b(-\kappa)J(\kappa+1) = \int_{\mathcal{Y}} D_{-\kappa}\psi_{-\kappa}\mu_o(dy) = \int_{\mathcal{Y}} \psi_{-\kappa}(y)D_{-\kappa}1\mu_o(dy),$$

since the operator $D_{-\kappa}$ is self adjoint. The constant term of the differential operator $D_{-\kappa}$ is given by $D_{-\kappa}1 = \gamma_{-\kappa}(\rho)$.

We finally get

$$\gamma_{-\kappa}(\rho)J(\kappa) = b(-\kappa)J(\kappa-1). \tag{1.12}$$

On the other hand, the sequence

$$\phi(\kappa) = \frac{\prod_{j=1}^{2r_o}\Gamma(\kappa+1+\beta_j)}{\prod_{j=1}^{r_o}\Gamma(\kappa+\frac{1}{2}+\delta-\rho_j)\Gamma(\kappa+\frac{1}{2}+\delta+\rho_j)}$$

satisfies the same recursion relation. Indeed,

$$\phi(\kappa) = \frac{\prod_{j=1}^{2r_o}(\kappa+\beta_j)}{\prod_{j=1}^{r_o}(\kappa+\frac{1}{2}+\delta-\rho_j)(\kappa+\frac{1}{2}+\delta+\rho_j)}\phi(\kappa-1).$$

Therefore $J(\kappa) = c\phi(\kappa)$ and because of the "initial" condition $J(0) = 1$, we finally get

$$J(\kappa) = \frac{\phi(\kappa)}{\phi(0)}.$$

The functions $\phi(\kappa)$ and $J(\kappa)$ are holomorphic for $\Re\kappa > 0$ and grow polynomially:

$$\left|\phi(\kappa)\right| \leqslant c_1\left(1+|\kappa|\right)^{N_1},$$
$$\left|J(\kappa)\right| \leqslant J(0) = 1.$$

Indeed, $0 \leqslant \psi_{-\kappa}(y) \leqslant 1$. Therefore, the Carlson's theorem implies that $J(\kappa) = \frac{\phi(\kappa)}{\phi(0)}$, for every $\kappa \in \mathbb{C}$ such that $\Re\kappa > 0$. $\square$

By writing $y = c(iv)$, $J(\kappa)$ becomes an integral over iV. One shows that

$$\widetilde{\psi}_{-\kappa}(iv) := \psi_{-\kappa}\big(c(iv)\big) = h(v, -v)^{-\frac{r_0}{r}\kappa}.$$

Therefore,

$$J(\kappa) = \int_V h(v, -v)^{-\frac{r_0}{r}\kappa - \frac{n}{r}}\, dv.$$

In fact, in this realization an invariant measure μ_o on $\mathcal{Y}$ is given by $h(v, -v)^{-\frac{n}{r}} dv$.

The above integral has been considered and computed by Hua in [10, Chapter II] in several special cases.

2. BEREZIN KERNELS AND REPRESENTATIONS

Following ideas developed in [22,23,9] we shall now give an alternative approach to the theory of Berezin kernels considering intertwining operators for the maximal degenerate series representations of a conformal group.

2.1. Maximal degenerate series representations

The character χ of the structure group $\mathrm{Str}(V)$ (cf. (1.1)) can be trivially extended to the whole parabolic subgroup $\overline{P}$ by $\chi(h\bar{n}) := \chi(h)$ for every $h \in \mathrm{Str}(V)$, $\bar{n} \in \overline{N}$.

For every $s \in \mathbb{C}$ we define a character χ_s of $\overline{P}$ by $\chi_s(\bar{p}) := |\chi(\bar{p})|^{s+\frac{n}{2r}}$.

The induced representation $\widetilde{\pi}_s = \mathrm{Ind}_{\overline{P}}^{\mathrm{Conf}(V)}(\chi_s)$ of the conformal group acts on the space

$$\widetilde{I}_s := \big\{ f \in C^\infty\big(\mathrm{Conf}(V)\big) \bigm| f(h\bar{p}) = \chi_s(\bar{p}) f(h),\ \forall h \in \mathrm{Str}(V),\ \bar{p} \in \overline{P} \big\},$$

by left translations. A pre-Hilbert structure on $\widetilde{I}_s$ is given by $\|f\|^2 = \int_U |f(u)|^2\, du$, where U is the maximal compact subgroup of $\mathrm{Conf}(V)$ associated with the Cartan involution θ, and du is the normalized Haar measure of U.

According to the Gelfand–Naimark decomposition a function $f \in \widetilde{I}_s$ is determined by its restriction $f_V(x) = f(n_x)$ on $N \simeq V$. Let I_s be the subspace of $C^\infty(V)$ of functions f_V with $f \in \widetilde{I}_s$. The conformal group acts on I_s by:

$$\pi_s(g) f(x) = \big|A(g, x)\big|^{s+\frac{n}{2r}} f\big(g^{-1}.x\big), \quad g \in \mathrm{Conf}(V),\ x \in V, \tag{2.1}$$

where $A(g, x) := \chi((Dg^{-1})_x)$. This action is usually called the *even maximal degenerated series* representation of $\mathrm{Conf}(V)$.

One shows that the norm of a function $f(n_x) = f_V(x) \in I_s$ is given by:

$$\|f\|^2 = \int_V \big|f_V(x)\big|^2 h(x, -x)^{2\Re s}\, dx. \tag{2.2}$$

The formula (2.2) implies that for $\Re s = 0$ the space I_s is contained in $L^2(V)$ and the representation π_s extends as a unitary representation on $L^2(V)$.

In order to address the question of irreducibility and unitarity of these representations we refer the reader to [17], [18,20] and to [27].

Following the standard procedure we introduce an intertwiner between π_s and π_{-s}. Consider the map $\widetilde{A}_s$ defined on $\widetilde{I}_s$ by

$$(2.3) \qquad f \to (\widetilde{A}_s f)(g) := \int_N f(gjn)\, dn, \quad \forall g \in \mathrm{Conf}(V),$$

where dn is a left invariant Haar measure on N. We will see that this integral converges for $\Re s > \frac{n}{2r_0}$.

Proposition 2.1. *For every $f \in \widetilde{I}_s$ the function $\widetilde{A}_s f$ belongs to $\widetilde{I}_{-s}$ and the map $\widetilde{A}_s$ given by* (2.3) *intertwines the corresponding representations of the conformal group*:

$$(2.4) \qquad \widetilde{\pi}_{-s}(g)\big(\widetilde{A}_s f\big) = \widetilde{A}_s\big(\widetilde{\pi}_s(g) f\big), \quad \forall f \in \widetilde{I}_s,\, g \in \mathrm{Conf}(V).$$

Proof. The map $\widetilde{A}_s$ obviously commutes with the left action of $\mathrm{Conf}(V)$. We have to check that $\widetilde{A}_s f$ transforms in an appropriate way under the right action of $\overline{P}$. Notice first that $\widetilde{A}_s f$ is $\overline{N}$ right invariant. Indeed

$$\begin{aligned}\big(\widetilde{A}_s f\big)(g\bar{n}) &= \int_N f(g\bar{n}jn)\, dn = \int_N f(gjn'jjn)\, dn \\ &= \int_V f(gjn'')\, dn'' = \big(\widetilde{A}_s f\big)(g).\end{aligned}$$

For what concerns the action of $\mathrm{Str}(V)$ we have for every $f \in \widetilde{I}_s$:

$$(2.5) \qquad \big(\widetilde{A}_s f\big)(g\ell) = \int_N f(g\ell jn)\, dn = \int_N f\big(gj(\ell^t)^{-1} n\big)\, dn,$$

where h^t denotes the transformation adjoint to $h \in \mathrm{Str}(V)$ with respect to the bilinear form $\mathrm{Tr}\, L(xy)$ on V.

Indeed, one shows, (see [8, Proposition VIII.2.5]) that $j\ell j = (\ell^t)^{-1}$, $\forall \ell \in \mathrm{Str}(V)$. Thus the equality (2.5) implies that

$$\begin{aligned}\big(\widetilde{A}_s f\big)(g\ell) &= \int_N f\big(gj(\ell^t)^{-1} n \ell^t (\ell^t)^{-1}\big)\, dn = |\det \ell|^{\frac{n}{r}} \int_N f\big(gjn'(\ell^t)^{-1}\big)\, dn' \\ &= |\det \ell|^{\frac{n}{r} - s\frac{n}{2r}} \big(\widetilde{A}_s f\big)(g) = \chi_{-s}(\ell)\big(\widetilde{A}_s f\big)(g). \qquad \square\end{aligned}$$

Define the map $A_s : I_s \to I_{-s}$ by $A_s(f_V) = (\widetilde{A}_s f)_V$ $(f \in \widetilde{I}_s)$.

Proposition 2.2. *For every $f \in I_s$ we have,*

$$(A_s f)(x) = \int\limits_V \left|\Delta(x-u)\right|^{2s-\frac{n}{r}} f(u)\, du.$$

Proof. By definition of $\widetilde{A}_s$, we have $(\widetilde{A}_s f)(n_x) = \int_V f(n_x j n_v)\, dv$ $(f \in \widetilde{I}_s)$. According to the Gelfand–Naimark decomposition (1.2) for $g = n_x j$ we have

$$n_x j n_v = n_{(n_x \circ j).v} \big(D(n_x \circ j)\big)_v \bar{n}' = n_{(x - v^{-1})} P\big(v^{-1}\big)\bar{n}'.$$

Thus $(A_s f)(x) = \int_V f(x - v^{-1}) \chi_s(P(v)^{-1})\, dv$. Let $u = x - v^{-1}$. The Jacobian of this transformation being equal to $|\Delta(v)|^{\frac{2n}{r}}$, we finally get:

$$(2.6)\qquad (A_s f)(x) = \int\limits_V f(u) \left|\Delta(x-u)\right|^{2s-\frac{n}{r}} du. \qquad \square$$

The bilinear form on $\widetilde{I}_{-s} \times \widetilde{I}_s$ defined by $\langle f_1, f_2 \rangle = \int_U f_1(u) f_2(u)\, du$ is $\mathrm{Conf}(V)$-invariant. By using the fact that $\langle f_1, f_2 \rangle = c \int_V (f_1)_V(x) (f_2)_V(x)\, dx$ it gives rise to a $\mathrm{Conf}(V)$-invariant bilinear form on $I_s \times I_s$:

Proposition 2.3. *For $\Re s > \frac{n}{2r_0}$, the bilinear form $\mathfrak{B}_s$,*

$$\mathfrak{B}_s(f_1, f_2) = \iint\limits_{V \times V} \left|\Delta(x-y)\right|^{2s-\frac{n}{r}} f_1(x) f_2(y)\, dx\, dy,$$

is well defined on $I_s \times I_s$ and is $\mathrm{Conf}(V)$*-invariant.*

According to the general theory (see for instance [13]) these maximal degenerate series representations π_s are spherical. We shall determine the corresponding U-fixed vector of the representation π_s. Using the Iwasawa decomposition of $\mathrm{Conf}(V)$ we write $g = u\ell(g)\bar{n}$, where $\ell(g) \in \mathrm{Str}(V)$, which is defined up to a multiplication by an element of $U \cap \mathrm{Str}(V)$ on the left. Notice by the way that with the above notation we have $\theta(g^{-1})g = n^{-1}\ell(g)^2\bar{n}$.

Consider $n_x \in N$ admitting the decomposition $n_x = u\ell(n_x)\bar{n}$. Therefore

$$\ell(n_x)^2 = D\big(\theta(n_{-x})n_x\big)(0) = P\big(x^{-1} + \alpha(x)\big)^{-1} P(x)^{-1}.$$

Let $f^o \in \widetilde{I}_s$ be a U-invariant function, with $f^o(e) = 1$. Then

$$f^o(g) = f^o\big(u\ell(g)\bar{n}\big) = \chi_s\big(\ell(g)\big) f^o(u) = \chi_s\big(\ell(g)\big).$$

Eventually we get that the U-fixed vector f_V^o in I_s is given by

$$f_V^o(x) = \left|\Delta\big(x^{-1} + \alpha(x)\big)\Delta(x)\right|^{-(s+\frac{n}{2r})} = h(x, -x)^{-(s+\frac{n}{2r})}.$$

In order to conclude this section we shall discuss conditions under which the intertwining operator A_s is well defined.

Let us evaluate $A_s f^o$ at the identity of the conformal group.

$$\widetilde{A}_s f^o(e) = \int_N f^o(jn)\,dn = \int_N f^o(n)\,dn = \int_V h(x,-x)^{-(s+\frac{n}{2r})}\,dx.$$

Notice that the value $C(s)$ of this integral equals

$$C(s) = J\left(\frac{r}{r_0}\left(s - \frac{n}{2r}\right)\right),$$

where $J(\nu)$ is given by (1.11). For $u \in U$, $f^o(u) \neq 0$. Therefore, for $f \in \widetilde{I}_s$,

$$\left|f(g)\right| \leqslant M\left|f^o(g)\right|,$$

with

$$M = \sup_{u\in U} \frac{|f(u)|}{|f^0(u)|}.$$

Now it is clear that the integral defining A_s is well defined for $\Re s > \frac{n}{2r_0}$. One shows that the map $s \to A_s$ can be actually extended to the whole complex plane as a meromorphic function. Because of the intertwining property we eventually get:

$$A_s \circ A_{-s} = C(s)C(-s)\,\mathrm{Id}_{I_s}.$$

2.2. Restriction of π_s and canonical representations

We shall study the representations π_s when restricted to the subgroup G. The representation space $\widetilde{I}_s$ can be seen as a line bundle over the conformal compactification $\mathcal{M}$. The Makarevich space $\mathcal{X}$ is one of the open orbits of G acting on $\mathcal{M}$. The space $I_s(\mathcal{X})$ of functions in $I(s) = I_s(V)$ supported in the closure $\overline{\mathcal{X}} \subset V$ is then invariant. We shall consider the corresponding representation T_s of G on $I_s(\mathcal{X})$. The problem is to determine for which values of the parameter s the representation T_s of the group G is unitarizable, and then to decompose it into irreducible ones. According to an established terminology taking its roots in [26] one calls such representations *canonical representations* of the group G.

A key observation is the connection between the canonical representation T_s and the Berezin kernel. We shall make this link clear.

Consider the bilinear form on $I_s(\mathcal{X}) \times I_s(\mathcal{X})$ given by

$$\widetilde{\mathfrak{B}}_s^\alpha(f_1, f_2) := \mathfrak{B}_s\big(f_1, f_2 \circ (-\alpha)\big), \tag{2.7}$$

where $\mathfrak{B}_s(f_1, f_2)$ is the bilinear form on $I_s(V) \times I_s(V)$ introduced in Proposition 2.3.

Proposition 2.4. *The bilinear form $\widetilde{\mathfrak{B}}_s^\alpha$ is invariant under the action of the group G.*

The proof uses the invariance property of the cross ratio, and the two following lemmas.

Lemma 2.5. *For every $g \in G$ the following identity holds*:

$$A\big(g, -\alpha(x)\big) = A(g, x).$$

Lemma 2.6. *Let x be an element in V, then for every $g \in G$ the following identity holds*:

$$\Delta\big((gx)_0\big) = A(g, x)\Delta(x_0).$$

(*Recall that the subscript $_0$ means the V_0-component.*)

Let us introduce the multiplication operator M_s on the space $I_s(\mathcal{X})$ by

$$M_s f(x) = \Delta(x_0)^{s+\frac{n}{2r}} f(x).$$

This operator intertwines the canonical representation T_s and the left regular action L of the group G,

$$M_s \circ T_s(g) = L(g) \circ M_s, \quad g \in G,$$

where $(L(g)f)(x) = f(g^{-1} \cdot x)$.

Proposition 2.7. *Let us define the bilinear form $\mathfrak{B}_s^\alpha$ on $\mathcal{C}_c(\mathcal{X}) \times \mathcal{C}_c(\mathcal{X})$ by*

$$\mathfrak{B}_s^\alpha(F_1, F_1) = \mathfrak{B}_s^\alpha(M_s f_1, M_s f_2) = \widetilde{\mathfrak{B}}_s^\alpha(f_1, f_2).$$

Then

$$\mathfrak{B}_s^\alpha(F_1, F_2) = \iint\limits_{\mathcal{X}\times\mathcal{X}} B_\nu(x, y)F_1(x)F_2(y)\mu(dx)\mu(dy), \tag{2.8}$$

where $B_\nu(x, y)$ is the Berezin kernel introduced in (1.9) *and $\nu = -\frac{r}{r_0}(s - \frac{n}{2r})$.*

Therefore the decomposition of the canonical representations reduces now to a classical problem in spherical harmonic analysis.

Let Λ be the set of parameters $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ for which the spherical function φ_λ given by (1.6) is positive definite. Let ψ be a continuous function of positive type on G which is K-biinvariant. By the Bochner–Godement theorem there is a unique bounded positive measure m on Λ such that

$$\psi(x) = \int\limits_\Lambda \varphi_\lambda(x)m(d\lambda).$$

If ψ is integrable, then the measure m is absolutely continuous with respect to the Plancherel measure, with a density given by the Fourier transform of ψ:

$$m(d\lambda) = \mathcal{F}\psi(\lambda)\frac{d\lambda}{|c(\lambda)|},$$

where $c(\lambda)$ is the Harish-Chandra c-function of the symmetric space $\mathcal{X}$.

Therefore the problem is:

(1) Determine the set $\mathcal{W} \subset \mathbb{R}$ of values ν for which the Berezin kernel $\mathfrak{B}_\nu^\alpha$ (or the corresponding K-biinvariant function ψ_ν) is of positive type.

(2) For $\nu \in \mathcal{W}$, determine the positive measure m_ν on Λ such that

$$\psi_\nu(x) = \int_\Lambda \varphi_\lambda(x) m_\nu(d\lambda).$$

These problems have been solved for hyperbolic spaces by van Dijk, Hille, Pasquale [22,24]. Notice however that only real hyperbolic spaces are Makarevich spaces of tube type as introduced above.

The case $G = U(p,q)$ was studied by Hille and Neretin [9,14], and the case $G = SO(p,q)$ by Neretin in [14–16].

One should point out that the first problem is not equivalent to the determination of the so called Wallach set which is the positive definiteness domain of the Bergman kernel. Since a Makarevich symmetric space of tube type is a real form of a Hermitian symmetric space, the set $\mathcal{W}$ does contain the Wallach set related to the associated Bergman kernel. In fact the restriction to a subset of a kernel of positive type is of positive type too. But this inclusion can be strict, as it is the case for real hyperbolic spaces.

For $\nu > \frac{n}{r_0} - 1$ the function ψ_ν is integrable and, for $\lambda \in i\mathfrak{a}^*$:

$$\mathcal{F}\psi_\nu(\lambda) = \frac{|P(\lambda,\nu)|^2}{Q(\nu)} \geqslant 0.$$

It follows that ψ_ν is of positive type, and

$$\psi_\nu(x) = \int_{i\mathfrak{a}^*} \phi_\lambda(x) \frac{|P(\lambda,\nu)|^2}{Q(\nu)} \frac{d\lambda}{|c(\lambda)|^2}.$$

According to Helgason we define the Fourier transform of an integrable function f on $\mathcal{X}$. The Iwasawa decomposition of the group G can be written as $G = NAK$. If $k^{-1}g \in N\exp(H)K$ with $H \in \mathfrak{a}$, one writes $H = \mathcal{A}(x,b)$, $x \in gK$, $b \in kM \in B = K/M$, where M is the centralizer of A in K.

The Fourier transform of f is the function $\hat{f}$ defined on $i\mathfrak{a}^* \times B$ by

$$\hat{f}(\lambda,b) = \int_{\mathcal{X}} f(x) e^{\langle -\lambda+\rho, \mathcal{A}(x,b)\rangle}\, dx.$$

We consider $\hat{f}(\lambda) := \hat{f}(\lambda, \cdot)$ as an element of the space $\mathcal{H}_\lambda \simeq L^2(B)$ which carries a unitary spherical principal series Π_λ of G. Then the map $f \mapsto \hat{f}(\lambda)$ intertwines the left regular representation L and Π_λ.

Theorem 2.8. *For $\nu > \frac{n}{r_0} - 1$, the Berezin form is positive definite. Therefore the representation T_s is unitarizable, decomposes multiplicity free as a direct integral of spherical principal series Π_λ of G, which corresponds to the following Plancherel formula*:

$$\mathfrak{B}_s^\alpha(f, \bar{f}) = \int_{i\mathfrak{a}^*} \|\hat{f}(\lambda)\|^2 \frac{|P(\lambda, \nu)|^2}{Q(\nu)} \frac{d\lambda}{|c(\lambda)|^2},$$

where $\nu = -\frac{r}{r_0}(s - \frac{n}{2r})$ and $f \in C_c(\mathcal{X})$.

2.3. Berezin kernel on the Riemannian compact dual

In order to study the "deformation" of the regular representation of the compact group U we shall investigate the spherical Fourier transform of the Berezin kernel on the compact dual symmetric space $\mathcal{Y}$.

The spherical functions of the symmetric space $\mathcal{Y} = U/K$ are given by

$$\Phi_{\mathbf{m}}(x) = \int_K \Delta_{\mathbf{m}}\big((k.x)_0\big)dk = \varphi_{\mathbf{m}-\rho}(x), \tag{2.9}$$

where the weights $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ are given according to different cases by the following conditions:

- If V is a Euclidean Jordan algebra (root system of type A), then $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n$.
- If V is a non-Euclidean algebra of split type (root system of type D), them $-|m|_1 \geqslant m_2 \geqslant \cdots \geqslant m_n$.
- If V is either a complex or a non-Euclidean Jordan algebra of nonsplit type (root system of type C), then $0 \geqslant m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n$.

The spherical Fourier coefficients of a K-invariant integrable function f on $\mathcal{Y}$ are given by

$$a(\mathbf{m}) = \int_{\mathcal{Y}} f(y)\Phi_{\mathbf{m}}(y)\mu_0(dy).$$

For $\Re\kappa \geqslant 0$, the Berezin function $\psi_{-\kappa}$ is bounded on $\mathcal{Y}$. We will determine its spherical Fourier coefficients

$$a_\kappa(\mathbf{m}) = \int_{\mathcal{Y}} \psi_{-\kappa}(y)\Phi_{\mathbf{m}}(y)\mu_0(dy).$$

Notice that $a_\kappa(0) = J(\kappa)$.

Theorem 2.9. *The Fourier coefficients of the Berezin kernel function $\psi_{-\kappa}$ are given by*

$$a_\kappa(\mathbf{m}) = J(\kappa)\frac{\prod_{j=1}^n \Gamma(\kappa+\frac{1}{2}+\delta-\rho_j)\Gamma(\kappa+\frac{1}{2}+\delta+\rho_j)}{\prod_{j=1}^n \Gamma(\kappa+\frac{1}{2}+\delta-\rho_j+m_j)\Gamma(\kappa+\frac{1}{2}+\delta+\rho_j-m_j)}.$$

Proof. We use the same method as in [28].

(a) We show first that the measure on $\mathcal{Y}$ given by

$$\frac{1}{J(\kappa)}\psi_{-\kappa}(y)\mu_0(dy)$$

converges to the Dirac measure δ at the base point e of $\mathcal{Y}$ in the sense of tight convergence of measures. The proof of this fact is based on the following lemma.

Lemma 2.10. *Let $\mathcal{K}$ be a compact topological space, and μ a positive measure on it such that every non empty open set has a positive measure. Let $q \geqslant 0$ be a continuous function on $\mathcal{K}$ which attains its maximum at only one point x_0. Define, for $n \in \mathbb{N}$,*

$$a_n = \int_{\mathcal{K}} q(x)^n \mu(dx),$$

and, for a continuous function φ on $\mathcal{K}$,

$$L_n(\varphi) = \frac{1}{a_n}\int_{\mathcal{K}} \varphi(x)q(x)^n\mu(dx).$$

Then

$$\lim_{n\to\infty} L_n(\varphi) = \varphi(x_0).$$

Proof. For $0 < \alpha < M = \max q$, there exits a constant C_α such that $a_n \geqslant C_\alpha \alpha^n$. In fact there is a neighborhood $\mathcal{V}$ of x_0 such that $q(x) \geqslant \alpha$ for $x \in \mathcal{V}$, and $a_n \geqslant \mu(\mathcal{V})\alpha^n$. Let $\mathcal{W}$ be a neighborhood of x_0. For $x \in \mathcal{K}\setminus\mathcal{W}$, $q(x) \geqslant \beta < M$. Choose α such that $\beta < \alpha < M$. Then

$$\frac{1}{a_n}\int_{\mathcal{K}\setminus\mathcal{W}} q(x)^n\mu(dx) \leqslant \frac{1}{C_\alpha}\mu(\mathcal{K})\left(\frac{\beta}{\alpha}\right)^n,$$

and

$$\lim_{n\to\infty}\frac{1}{a_n}\int_{\mathcal{K}\setminus\mathcal{W}} q(x)^n\mu(dx) = 0. \qquad \square$$

The function $\psi_{-\kappa}$ attains its maximum $M = 1$ only at $y = e$. Therefore Lemma 2.10 applies. It follows that, for every $\mathbf{m}$,

$$\lim_{\kappa\to\infty} \frac{1}{J(\kappa)} a_\kappa(\mathbf{m}) = 1.$$

(b) Assume that $\kappa \in \mathbb{N}$. From the Bernstein identity (1.3) it follows that the spherical Fourier coefficients $a_\kappa(\mathbf{m})$ satisfy the following recursion relation:

$$\gamma_{-\kappa}(\rho - \mathbf{m}) a_\kappa(\mathbf{m}) = b(-\kappa) a_{\kappa-1}(\mathbf{m}).$$

Since, for $\mathbf{m} = 0$, $\gamma_{-\kappa}(\rho) a_\kappa(0) = b(-\kappa) a_{\kappa-1}(\mathbf{0})$ we obtain

$$\frac{\gamma_{-\kappa}(\rho - \mathbf{m})}{\gamma_{-\kappa}(\rho)} \frac{a_\kappa(\mathbf{m})}{a_\kappa(0)} = \frac{a_{\kappa-1}(\mathbf{m})}{a_{\kappa-1}(0)}.$$

Furthermore, since $a_\kappa(0) = J(\kappa)$, by (a)

$$\lim_{\kappa\to\infty} \frac{a_\kappa(\mathbf{m})}{a_\kappa(0)} = 1.$$

The sequence given by

$$\phi(\kappa) = \frac{\prod_{j=1}^n \Gamma(\kappa + \frac{1}{2} + \delta - \rho_j)\Gamma(\kappa + \frac{1}{2} + \delta + \rho_j)}{\prod_{j=1}^n \Gamma(\kappa + \frac{1}{2} + \delta - \rho_j + m_j)\Gamma(\kappa + \frac{1}{2} + \delta + \rho_j - m_j)},$$

satisfies the same recursion relation as $\frac{a_\kappa(\mathbf{m})}{a_\kappa(0)}$ does. Moreover, from the asymptotic equivalence

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}\left(1 + \frac{1}{2z}(b-a)(b+a-1) + O\left(z^{-2}\right)\right), \quad \text{as } z \to \infty,$$

it follows that $\lim_{\kappa\to\infty} \phi(\kappa) = 1$. Therefore we have proved the theorem for $\kappa \in \mathbb{N}$.

(c) By using Carlson's theorem, as we did in the proof of Theorem 1.4, we conclude that the statement is still valid for $\Re\kappa \geqslant 0$. □

2.4. Restriction of the representation π_s to the compact group U

The compact symmetric space $\mathcal{Y} = U/K$ is also of Makarevich type. Indeed,

$$U = \left\{g \in \mathrm{Conf}(V) \mid \alpha \circ j \circ g \circ j \circ \alpha = g\right\}_o.$$

From the generalized cross ratio (1.7) we define the kernel $F^c(u, v)$ by

$$\begin{aligned} F^c(u,v) &= \left\{u, v, -\alpha\left(u^{-1}\right), -\alpha\left(v^{-1}\right)\right\} \\ &= \frac{\Delta(u + \alpha(u^{-1}))\Delta(v + \alpha(v^{-1}))}{\Delta(u + \alpha(v^{-1}))\Delta(v + \alpha(u^{-1}))} \end{aligned}$$

and $F^c(u, 0) = h(u, -u)$.

Similarly to what we did in Section 2.2 we twist the bilinear form $\mathfrak{B}_s$ by the involution $j \circ \alpha$, in other words we replace y by $\alpha(y^{-1})$, and by introducing multiplication operators

$$M_s^c f(x) = h(x, -x)^{s+\frac{n}{2r}} f(x)$$

we eventually obtain a U-invariant bilinear form on $C^\infty(\mathcal{Y})$:

$$\mathfrak{B}_s^c(F_1, F_2) = \iint_{\mathcal{Y}\times\mathcal{Y}} B_s^c(x, y) F_1(x) F_2(y) \mu_0(dx) \mu_0(dy),$$

where $B_s^c(x, y) = F^c(x, y)^{\frac{r_0}{r}\nu}$.

The spherical dual $\hat{U}_K$ is parameterized by the set of weights $\mathbf{m} = (m_1, \ldots, m_n)$ described in the previous section. To such a weight $\mathbf{m}$ corresponds a (class of) unitary spherical representation $\Pi_{\mathbf{m}}$ on a finite dimensional vector space $\mathcal{H}_{\mathbf{m}}$ of dimension $d_{\mathbf{m}}$. The highest weight μ of $\Pi_{\mathbf{m}}$ is given by $\langle \mu, \xi \rangle = -\sum_{j=1}^{r_0} m_j t_j$, if $\xi(x) = \sum t_j L(c_j) x$. In the space $\mathcal{H}_{\mathbf{m}}$ there is a normalized K-fixed vector $v_{\mathbf{m}}$. The Fourier coefficient $\hat{f}(\mathbf{m})$ of an integrable function f on $\mathcal{Y}$ is the vector in $\mathcal{H}_{\mathbf{m}}$ defined by

$$\hat{f}(\mathbf{m}) = \int_{U/K} \Pi_{\mathbf{m}}(g) v_{\mathbf{m}} f(g)\, dg.$$

The map $f \mapsto \hat{f}(\mathbf{m})$ intertwines the left regular representation of U and $\Pi_{\mathbf{m}}$.

Theorem 2.11. *For $\kappa \in \mathbb{N}$, the Berezin form $\mathfrak{B}_s^c$ is positive definite. Therefore the restriction of π_s to the compact group U decomposes into a direct sum of spherical principal series representations $\Pi_{\mathbf{m}}$ of U according to the following Plancherel formula*:

$$\mathfrak{B}_s^c(f, \bar{f}) = \sum_{\mathbf{m}\in\hat{U}_K} d_{\mathbf{m}} a_\kappa(\mathbf{m}) \|\hat{f}(\mathbf{m})\|^2,$$

where $\kappa = \frac{r}{r_0}(s - \frac{n}{2r})$, $f \in C(U/K)$.

By the first part of the proof of Theorem 2.8, in some sense, the canonical representations of the compact dual of G tend to the left regular representation when the parameter κ goes through the negative integer points. In fact, for $f \in \mathcal{C}_c(\mathcal{Y})$,

$$\lim_{\kappa\to\infty} \frac{1}{J(\kappa)} \int_{\mathcal{Y}\times\mathcal{Y}} \mathfrak{B}_\kappa^c(x, y) f(x) f(y) \mu_0(dx) \mu_0(dy) = \int_{\mathcal{Y}} |f(y)|^2 \mu_0(dy).$$

Similarly, in the noncompact case, as $\nu \to \infty$, the probability measure $I(\nu)^{-1}\psi_\nu(x) \times \mu(dx)$ converges to the Dirac measure δ_e at the identity element e. It follows that, for $f \in \mathcal{C}_c(\mathcal{X})$,

$$\lim_{\nu\to\infty} \frac{1}{I(\nu)} \int\limits_{\mathcal{X}\times\mathcal{X}} \mathcal{B}_\nu^\alpha(x, y) f(x)\overline{f(y)}\mu(dx)\mu(dy) = \int\limits_{\mathcal{X}} \left|f(x)\right|^2 \mu(dx).$$

In some sense, as $\nu \to \infty$, the canonical representation T_ν tends to the regular representation of G on $L^2(\mathcal{X})$.

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