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Abstract

We consider some special classes of Lévy processes with no gaussian component whose Lévy measure is of the type $\pi(dx) = e^{\gamma x}\nu(e^x - 1)\,dx$, where ν is the density of the stable Lévy measure and γ is a positive parameter which depends on its characteristics. These processes were introduced in [10] as the underlying Lévy processes in the Lamperti representation of conditioned stable Lévy processes. In this paper, we compute explicitly the law of these Lévy processes at their first exit time from a finite or semi-finite interval, the law of their exponential functional and the first hitting time probability of a pair of points.

Key words and phrases: Positive self-similar Markov processes, Lamperti representation, conditioned stable Lévy processes, first exit time, first hitting time, exponential functional.

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1 Introduction

In recent years there has been a general recognition that Lévy processes play an ever more important role in various domains of applied probability theory such as financial mathematics, insurance risk, queueing theory, statistical physics or mathematical biology. In many instances there is a need for explicit examples of Lévy processes where

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tractable mathematical expressions in terms of the characteristics of the underlying Lévy process may be used for the purpose of numerical simulation. Depending on the problem at hand, particular functionals are involved such as the first entrance times and overshoot distributions.

In this paper, we exhibit some special classes of Lévy processes for which we can compute explicitly the law of the position at the first exit time of an interval, the two points hitting probability and the exponential functional. Moreover, two new, concrete examples of scale functions for spectrally one sided processes will fall out of our analysis.

Known examples of overshoot distributions concern essentially (some particular classes) of strictly stable processes and processes whose jumps are of a compound Poisson nature with exponential jumps (or slightly more generally whose jump distribution has a rational Fourier transform). For example, let us state the solution of the two sided exit problem for completely asymmetric stable processes. In that case we take $(X, \mathbf{P}_x)$, $x \in \mathbb{R}$, to be a spectrally positive Lévy stable process with index $\alpha \in (1, 2)$ starting from x. Let $\sigma_a^+ = \inf\{t > 0 : X_t > a\}$ and $\sigma_0^- = \inf\{t > 0 : X_t < 0\}$. It is known (cf. Rogozin [27]) that for $y > 0$,

$$\mathbf{P}_x\Big(X_{\sigma_a^+} - a \in \mathrm{d}y; \sigma_a^+ < \sigma_0^-\Big) = \frac{\sin \pi(\alpha - 1)}{\pi} \left(\frac{a - x}{xy}\right)^{(\alpha-1)} \frac{\mathrm{d}y}{(y + a)(y + a - x)}.$$

For the case of processes whose jumps are of a compound Poisson nature with exponential jumps, the overshoot distribution is again exponentially distributed; see Kou and Wang [18]. See also Lewis and Mordecki [21] and Pistorius [24] for the more general case of a jump distribution with a rational Fourier transform and for which the overshoot distribution belongs to the same class as the respective jump distribution of the underlying Lévy process.

The exponential functional of a Lévy process, ξ, i.e.

$$\int_0^\infty \exp\Big\{ - \xi_s \Big\} \, \mathrm{d}s,$$

also appears in various aspects of probability theory, such as: self-similar Markov processes, random processes in random environment, fragmentation processes, mathematical finance, Brownian motion on hyperbolic spaces, to name but a few. In general, the distribution of exponential functionals can be rather complicated. Nonetheless, it is known for the case that ξ is either: a standard Poisson processes, Brownian motion with drift and a particular class of spectrally negative Lévy processes of bounded variation whose Laplace exponent is of the form

$$\psi(q) = \frac{q(q + 1 - a)}{b + q}, \qquad q \geq 0,$$

where $0 < a < 1 < a + b$. See Bertoin and Yor [6] for an overview on this topic.

The class of Lévy processes that we consider in this paper do not fulfill a scaling property and may have two-sided jumps. Moreover, they have no Gaussian component and their Lévy measure is of the type $\pi(dx) = e^{\gamma x}\nu(e^x - 1)\, dx$, where ν is the density of the stable Lévy measure with index $\alpha \in (0,2)$ and γ is a positive parameter which depends on its characteristics. It is not difficult to see that the latter Lévy measure has a density which is asymptotically equivalent to that of an α-stable process for small $|x|$ and has exponential decay for large $|x|$. This implies that such processes have paths which are of bounded or unbounded variation accordingly as $\alpha \in (0,1)$ and $\alpha \in [1,2)$ respectively. Further, they also have exponential moments. Special families of tempered stable processes, also known as CGMY processes, are classes of Lévy processes with similar properties to the aforementioned which have enjoyed much exposure in the mathematical finance literature as instruments for modelling risky assets. See for example Carr et al. [11], Boyarchenko and Levendorskii [8], Cont [14] or Schoutens [29]. Although the Lévy processes presented in this paper are not tempered stable processes, it is intriguing to note that they possess properties which have proved to be popular for financial models but now with the additional luxury that they come with a number of explicit fluctuation identities.

We conclude the introduction with a brief outline of the remainder of the paper. The next section introduces the classes of processes which are concerned in this study. In section 3, we give the law of the position at the first exit time from a (semi-finite) interval. In section 4 we compute explicitly the two point hitting probability and in section 5, we study the law of the exponential functional of Lévy-Lamperti processes.

2 Preliminaries on Lévy-Lamperti processes

Denote by $\mathcal{D}$ the Skorokhod space of $\mathbb{R}$-valued càdlàg paths and by X the canonical process of the coordinates on $\mathcal{D}$. Positive ($\mathbb{R}_+$-valued), self-similar Markov processes $(X, \mathbb{P}_x)$, $x > 0$, are strong Markov processes with paths in $\mathcal{D}$, which fulfill a scaling property, i.e. there exists a constant $\alpha > 0$ such that for any $b > 0$:

$$\textit{The law of } (bX_{b^{-\alpha}t},\ t \geq 0) \textit{ under } \mathbb{P}_x \textit{ is } \mathbb{P}_{bx}. \tag{2.1}$$

We shall refer to these processes as pssMp. According to Lamperti [20], any pssMp up to its first hitting time of 0 may be expressed as the exponential of a Lévy process, time changed by the inverse of its exponential functional. More formally, let $(X, \mathbb{P}_x)$ be a pssMp with index $\alpha > 0$, starting from $x > 0$, set

$$S = \inf\{t > 0 : X_t = 0\}$$

and write the canonical process X in the following form:

$$X_t = x \exp\left\{\xi_{\tau(tx^{-\alpha})}\right\} \qquad 0 \leq t < S\,, \tag{2.2}$$

where for $t < S$,

$$\tau(t) = \inf\left\{s \geq 0 : \int_0^s \exp\{\alpha \xi_u\}\, du \geq t\right\}.$$

Then under $\mathbb{P}_x$, $\xi = (\xi_t,\ t \geq 0)$ is a Lévy process started from 0 whose law does not depend on $x > 0$ and such that:

(*i*) if $\mathbb{P}_x(S=+\infty)=1$, then ξ has an infinite lifetime and $\limsup_{t\to+\infty}\xi_t=+\infty$, $\mathbb{P}_x$-a.s.,

(*ii*) if $\mathbb{P}_x(S<+\infty,\,X(S-)=0)=1$, then ξ has an infinite lifetime and $\lim_{t\to\infty}\xi_t=-\infty$, $\mathbb{P}_x$-a.s.,

(*iii*) if $\mathbb{P}_x(S<+\infty,\,X(S-)>0)=1$, then ξ is killed at an independent exponentially distributed random time with parameter $\lambda>0$.

As it is mentioned in [20], the probabilities $\mathbb{P}_x(S=+\infty)$, $\mathbb{P}_x(S<+\infty,\,X(S-)=0)$ and $\mathbb{P}_x(S<+\infty,\,X(S-)>0)$ are 0 or 1 independently of x, so that the three classes presented above are exhaustive. Moreover, for any $t<\int_0^\infty \exp\{\alpha\xi_s\}\,\mathrm{d}s$,

$$\tau(t)=\int_0^{x^\alpha t}\frac{\mathrm{d}s}{(X_s)^\alpha}\,,\quad \mathbb{P}_x-\text{a.s.} \tag{2.3}$$

Therefore (2.2) is invertible and yields a one to one relation between the class of pssMp's killed at time S and the one of Lévy processes.

Now let us consider three particular classes of pssMp. (We refer to [10] for more details in what follows.) The first one is identified as a stable Lévy processes killed when it first exits from the positive half-line. In particular, if $\mathbf{P}_x$ is the law of a stable Lévy process with index α (or α-stable process for short) initiated from $x>0$ with $\alpha\in(0,2]$, then with $T=\inf\{t:X_t\le 0\}$, under $\mathbf{P}_x$, the process

$$X_t\mathbb{I}_{\{t<T\}}$$

is a pssMp which satisfies condition (*ii*) if it has no negative jumps or (*iii*) if it has negative jumps. We call ξ^* the Lévy process (with finite or infinite lifetime) resulting from the Lamperti representation of the killed stable process. The characteristic exponent of ξ^* has been computed in [10] and is given by

$$\Phi^*(\lambda)=ia^*\lambda+\int_{\mathbb{R}}[e^{i\lambda x}-1-i\lambda(e^x-1)\mathbb{I}_{\{|e^x-1|<1\}}]\pi^*(x)\,\mathrm{d}x-c_-\alpha^{-1}\,,\quad \lambda\in\mathbb{R}\,, \tag{2.4}$$

where a^* is a constant,

$$\pi^*(x)=\frac{c_+e^x}{(e^x-1)^{\alpha+1}}\mathbb{I}_{\{x>0\}}+\frac{c_-e^x}{(1-e^x)^{\alpha+1}}\mathbb{I}_{\{x<0\}}\,,$$

and c_-, c_+ are nonnegative constants such that $c_-c_+>0$. Note that the Lévy measure of ξ^* satisfies $\pi^*(x)=e^x\nu(e^x-1)$, where ν is the density of the stable Lévy measure with index α and symmetry parameters c_- and c_+.

The second class is that of stable processes conditioned to stay positive. (See for instance in [12] for an overview of such processes.) A process in this class is the result of a Doob h-transform with $h(x)=x^{\alpha\rho}$ and $\rho=\mathbf{P}_0(X_1<0)$. More precisely, h is invariant for the killed process mentioned above $(X_t\mathbb{I}_{\{t<T\}},\mathbf{P}_x)$ and the law $\mathbb{P}^\uparrow_x$ defined on each σ-field $\mathcal{F}_t$ generated by the canonical process up to time t by

$$\left.\frac{\mathrm{d}\mathbb{P}^\uparrow_x}{\mathrm{d}\mathbf{P}_x}\right|_{\mathcal{F}_t}=\frac{X_t^{\alpha\rho}}{x^{\alpha\rho}}\mathbf{1}_{\{t<T\}} \tag{2.5}$$

is this of a pssMp which derives toward $+\infty$ (in particular it satisfies condition (i)). Then the underlying Lévy process, which will be denoted by $\xi^{\uparrow}$, is such that

$$\lim_{t\to+\infty} \xi_t^{\uparrow} = +\infty, \qquad \text{a.s.},$$

and from [10] its characteristic exponent is

$$\Phi^{\uparrow}(\lambda) = ia^{\uparrow}\lambda + \int_{\mathbb{R}} [e^{i\lambda x} - 1 - i\lambda(e^x - 1)\mathbb{I}_{\{|e^x-1|<1\}}]\pi^{\uparrow}(x)\,\mathrm{d}x\,, \quad \lambda \in \mathbb{R}\,, \tag{2.6}$$

where $a^{\uparrow}$ is a real constant and

$$\pi^{\uparrow}(x) = \frac{c_+ e^{(\alpha\rho+1)x}}{(e^x-1)^{\alpha+1}}\mathbb{I}_{\{x>0\}} + \frac{c_- e^{(\alpha\rho+1)x}}{(1-e^x)^{\alpha+1}}\mathbb{I}_{\{x<0\}}\,.$$

The third class of pssMp that we will consider is that of stable processes conditioned to hit 0 continuously. Processes in this class are again it is defined as a Doob h-transform with respect to the function $h'(x) = \alpha\rho x^{\alpha\rho-1}$ which is also invariant for the killed process $(X_t\mathbb{I}_{\{t<T\}}, \mathbf{P}_x)$. Then the law $\mathbb{P}_x^{\downarrow}$ which is defined on each σ-field $\mathcal{F}_t$ by

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\downarrow}}{\mathrm{d}\mathbf{P}_x}\right|_{\mathcal{F}_t} = \frac{X_t^{\alpha\rho-1}}{x^{\alpha\rho-1}}\mathbf{1}_{\{t<T\}} \tag{2.7}$$

is this of a pssMp who hits 0 in a continuous way, i.e. $(X, \mathbb{P}_x^{\downarrow})$ satisfies condition (ii). Let $\xi^{\downarrow}$ by the underlying Lévy process in the Lamperti representation of this process, then

$$\lim_{t\to+\infty} \xi_t^{\downarrow} = -\infty \qquad \text{a.s.},$$

and the characteristic exponent of $\xi^{\downarrow}$ is given by

$$\Phi^{\downarrow}(\lambda) = ia^{\downarrow}\lambda + \int_{\mathbb{R}} [e^{i\lambda x} - 1 - i\lambda(e^x - 1)\mathbb{I}_{\{|e^x-1|<1\}}]\pi^{\downarrow}(x)\,\mathrm{d}x\,, \quad \lambda \in \mathbb{R}\,, \tag{2.8}$$

where $a^{\downarrow}$ is a constant and

$$\pi^{\downarrow}(x) = \frac{c_+ e^{\alpha\rho x}}{(e^x-1)^{\alpha+1}}\mathbb{I}_{\{x>0\}} + \frac{c_- e^{\alpha\rho x}}{(1-e^x)^{\alpha+1}}\mathbb{I}_{\{x<0\}}\,.$$

Note that the constants a^*, $a^{\uparrow}$ and $a^{\downarrow}$ are computed explicitly in [10] in terms of α, ρ, c_- and c_+. Actually the process $\xi^{\downarrow}$ corresponds to $\xi^{\uparrow}$ conditioned to drift toward $-\infty$ (or equivalently $\xi^{\uparrow}$ is $\xi^{\downarrow}$ conditioned to drift to $+\infty$). We will sometime use this relationship which is stated in a more formal way the next proposition. In the sequel, P will be a reference probability measure on $\mathcal{D}$ under which ξ^*, $\xi^{\uparrow}$ and $\xi^{\downarrow}$ are Lévy processes whose respective laws are defined above.

Proposition 1. *For every $t \geq 0$, and every bounded measurable function f,*

$$E[f(\xi_t^{\uparrow})] = E[\exp(\xi_t^{\downarrow})f(\xi_t^{\downarrow})]\,.$$

In particular, processes $-\xi^{\uparrow}$ and $\xi^{\downarrow}$ satisfy Cramer's condition: $E(\exp -\xi_1^{\uparrow}) = 1$ and $E(\exp \xi_1^{\downarrow}) = 1$.

Proof. Let f be as in the statement. From (2.5) and (2.7), we deduce that for every $\mathbb{P}_x^{\downarrow}$-a.s. finite $(\mathcal{F}_u)$-stopping time U,

$$x\mathbb{E}_x^{\uparrow}[f(X_U)] = \mathbb{E}_x^{\downarrow}[X_U f(X_U)] \,. \tag{2.9}$$

Let $t \geq 0$. By applying (2.9) to the $(\mathcal{F}_u)$-stopping time

$$x^{\alpha} \inf \Big\{ u : \tau(u) > t \Big\},$$

which is $\mathbb{P}_x^{\downarrow}$-a.s. finite, and using (2.2) (note that $\tau(u)$ is continuous and increasing), we obtain

$$E[f(\xi_t^{\uparrow})] = E[\exp(\xi_t^{\downarrow}) f(\xi_t^{\downarrow})] \,,$$

which is the desired result. ■

We refer to Rivero [25], IV.6.1 for a similar discussion on conditioned stable processes considered as pssMp. In the sequel we call ξ^*, $\xi^{\uparrow}$ and $\xi^{\downarrow}$ the *Lévy-Lamperti processes.* We now compute the law of some of their functionals.

3 Entrance laws for Lévy-Lamperti processes: intervals

In this section, by studying the two sided exit problems for $\xi^{\uparrow}$, ξ^* and $\xi^{\downarrow}$, we shall obtain a variety of new identities including the identification of two new scale functions in the case of one-sided jumps.

To this end, we shall start with a generic result pertaining to any positive self-similar Markov process $(X, \mathbb{P}_x)$, for $x > 0$. Recall that P is the reference probability measure on D. Let ξ be a Lévy process starting from 0, under P, with the same law as the underlying Lévy process associated to $(X, \mathbb{P}_x)$. For any $y \in \mathbb{R}$ let

$$T_y^+ = \inf\{t : \xi_t \geq y\} \text{ and } T_y^- = \inf\{t : \xi_t \leq y\} \,,$$

and for any $y > 0$ let

$$\sigma_y^+ = \inf\{t : X_t \geq y\} \text{ and } \sigma_y^- = \inf\{t : X_t \leq y\}.$$

Lemma 1. *Fix $-\infty < v < 0 < u < \infty$. Suppose that A is any interval in $[u, \infty)$ and B is any interval in $(-\infty, v]$. Then,*

$$P\Big(\xi_{T_u^+} \in A; T_u^+ < T_v^-\Big) = \mathbb{P}_1\Big(X_{\sigma_{e^u}^+} \in e^A; \sigma_{e^u}^+ < \sigma_{e^v}^-\Big)$$

and

$$P\Big(\xi_{T_v^-} \in B; T_u^+ > T_v^-\Big) = \mathbb{P}_1\Big(X_{\sigma_{e^v}^-} \in e^B; \sigma_{e^u}^+ > \sigma_{e^v}^-\Big).$$

The proof is a straightforward consequence of the Lamperti representation (2.2) and is left as an exercise. Although somewhat obvious, this lemma indicates that for the three processes $\xi^{\uparrow}$, ξ^{*} and $\xi^{\downarrow}$, we need to understand how, respectively, an α-stable process conditioned to stay positive, an α-stable process killed when it exits the positive half-line and an α-stable process conditioned to hit the origin continuously, exit a positive interval around $x > 0$. Fortunately this is possible thanks to a result of Rogozin [27] who established the following result for α-stable processes.

Theorem 1 (Rogozin [27]). *Suppose that $(X, \mathbf{P}_x)$ is an α-stable process, initiated from x, which has two sided jumps. Denoting $\rho = \mathbf{P}_0(X_1 < 0)$ we have for $a > 0$ and $x \in (0, a)$,*

$$\mathbf{P}_x\Big(X_{\sigma_a^+} - a \in dy; \sigma_a^+ < \sigma_0^-\Big) = \frac{\sin \pi\alpha(1-\rho)}{\pi}(a-x)^{\alpha(1-\rho)} x^{\alpha\rho} y^{-\alpha(1-\rho)}(y+a)^{-\alpha\rho}(y+a-x)^{-1}\, dy$$

Note that an expression for $\mathbf{P}_x(-X_{\sigma_0^-} \in \mathrm{d}y; \sigma_a^+ > \sigma_0^-)$ can be derived from the above expression by replacing x by $a-x$ and ρ by $1-\rho$.

In the sequel, with an abuse of notation, we will denote by T_y^+ and T_y^- for the first passage times above and below $y \in \mathbb{R}$, respectively, of the processes $\xi^{\uparrow}, \xi^{*}$ or $\xi^{\downarrow}$ depending on the case that we are studying.

We now proceed to split the remainder of this section into three subsections dealing with the two sided exit problem and its ramifications for the three processes $\xi^{\uparrow}, \xi^{*}$ and $\xi^{\downarrow}$ respectively.

3.1 Calculations for $\xi^{\uparrow}$

The two sided exit problem for $\xi^{\uparrow}$ can be obtained from Lemma 1 and Theorem 1 as follows. We give the case for two-sided jumps. Note that this is not a restriction as the two-sided exit functionals we consider are weakly continuous in the Skorokhod space. Therefore by taking limits as $\alpha(1-\rho) \to 1$ or $\alpha\rho \to 1$ we deduce identities for the case that $\xi^{\uparrow}$ is spectrally negative and spectrally positive respectively. Note that necessarily in the spectrally one sided case $\alpha \in (1, 2)$.

Theorem 2. *Fix $\theta \geq 0$ and $-\infty < v < 0 < u < \infty$.*

$$P\Big(\xi^{\uparrow}_{T_u^+} - u \in d\theta; T_u^+ < T_v^-\Big) = \frac{\sin \pi\alpha(1-\rho)}{\pi}(e^u - 1)^{\alpha(1-\rho)}(1-e^v)^{\alpha\rho} \times (e^{u+\theta})^{\alpha\rho+1}(e^{u+\theta} - e^u)^{-\alpha(1-\rho)}(e^{u+\theta} - e^v)^{-\alpha\rho}(e^{u+\theta} - 1)^{-1}\, d\theta$$

and

$$P\Big(v - \xi^{\uparrow}_{T_v^-} \in d\theta; T_u^+ > T_v^-\Big) = \frac{\sin \pi\alpha\rho}{\pi}(1-e^v)^{\alpha\rho}(e^u - 1)^{\alpha(1-\rho)} \times (e^{v-\theta})^{\alpha\rho+1}(e^v - e^{v-\theta})^{-\alpha\rho}(e^u - e^{v-\theta})^{-\alpha(1-\rho)}(1 - e^{v-\theta})^{-1}\, d\theta.$$

Proof. Recall that $(X, \mathbf{P}_1)$ denotes an α-stable process initiated from 1 and that $(X, \mathbb{P}^{\uparrow}_1)$ is an α-stable process conditioned to stay positive initiated from 1. From Lemma 1, we have for $\theta \geq 0$

$$
\begin{aligned}
&P\Big(\xi^{\uparrow}_{T^+_u} \leq u+\theta; T^+_u < T^-_v\Big)\\
&\quad = \mathbb{P}^{\uparrow}_1\Big(X_{\sigma^+_{e^u}} \in [e^u, e^{u+\theta}];\ \sigma^+_{e^u} < \sigma^-_{e^v}\Big)\\
&\quad = \int_0^{e^{u+\theta}-e^u} (y+e^u)^{\alpha\rho}\mathbf{P}_1\Big(X_{\sigma^+_{e^u}} - e^u \in \mathrm{d}y; \sigma^+_{e^u} < \sigma^-_{e^v}\Big)\\
&\quad = \int_0^{e^{u+\theta}-e^u} (y+e^u)^{\alpha\rho}\mathbf{P}_{1-e^v}\Big(X_{\sigma^+_{(e^u-e^v)}} - (e^u - e^v) \in \mathrm{d}y; \sigma^+_{(e^u-e^v)} < \sigma^-_0\Big)\\
&\quad = \frac{\sin \pi\alpha(1-\rho)}{\pi}(e^u-1)^{\alpha(1-\rho)}(1-e^v)^{\alpha\rho}\\
&\qquad\quad \times \int_0^{e^{u+\theta}-e^u} (y+e^u)^{\alpha\rho} y^{-\alpha(1-\rho)}(y+e^u-e^v)^{-\alpha\rho}(y+e^u-1)^{-1}\mathrm{d}y
\end{aligned}
$$

from which the first part of the theorem follows.

The second part of the theorem can be proved in a similar way. Indeed for $\theta \geq 0$

$$
\begin{aligned}
&P\Big(\xi^{\uparrow}_{T^-_v} \geq v-\theta; T^+_u > T^-_v\Big)\\
&\quad = \mathbb{P}^{\uparrow}_1\Big(X_{\sigma^-_{e^v}} \in [e^{v-\theta}, e^v];\ \sigma^+_{e^u} > \sigma^-_{e^v}\Big)\\
&\quad = \int_0^{e^v-e^{v-\theta}} (e^v-y)^{\alpha\rho}\mathbf{P}_1\Big(e^v - X_{\sigma^-_{e^v}} \in \mathrm{d}y; \sigma^+_{e^u} > \sigma^-_{e^v}\Big)\\
&\quad = \int_0^{e^v-e^{v-\theta}} (e^v-y)^{\alpha\rho}\mathbf{P}_{1-e^v}\Big(-X_{\sigma^-_0} \in \mathrm{d}y; \sigma^+_{(e^u-e^v)} > \sigma^-_0\Big)\\
&\quad = \frac{\sin \pi\alpha\rho}{\pi}(1-e^v)^{\alpha\rho}(e^u-1)^{\alpha(1-\rho)}\\
&\qquad\quad \times \int_0^{e^v-e^{v-\theta}} (e^v-y)^{\alpha\rho} y^{-\alpha\rho}(y+e^u-e^v)^{-\alpha(1-\rho)}(y+1-e^v)^{-1}\mathrm{d}y.
\end{aligned}
$$

This completes the proof. ■

Note that since the process $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, is an α-stable process conditioned to stay positive, it follows that, in the case that there are two-sided jumps, there is no creeping out of the interval (v, u) with probability one. That is to say.

$$
P\Big(\xi^{\uparrow}_{T^+_u} = u; T^+_u < T^-_v\Big) = P\Big(\xi^{\uparrow}_{T^-_v} = v; T^+_u > T^-_v\Big) = 0.
$$

Taking $v \downarrow -\infty$ in the first part of the above theorem and $u \uparrow \infty$ in the second part we obtain the solution to the one-sided exit problem as follows.

Corollary 1. *Fix $\theta \geq 0$ and $-\infty < v < 0 < u < \infty$.*

$$P\Big(\xi^{\uparrow}_{T_u^+} - u \in d\theta, T_u^+ < \infty\Big)$$
$$= \frac{\sin \pi\alpha(1-\rho)}{\pi}(e^u - 1)^{\alpha(1-\rho)} e^{u+\theta}(e^{u+\theta} - e^u)^{-\alpha(1-\rho)}(e^{u+\theta} - 1)^{-1} d\theta$$

and

$$P\Big(v - \xi^{\uparrow}_{T_v^-} \in d\theta; T_v^- < \infty\Big)$$
$$= \frac{\sin \pi\alpha\rho}{\pi}(1 - e^v)^{\alpha\rho}(e^{v-\theta})^{\alpha\rho+1}(e^v - e^{v-\theta})^{-\alpha\rho}(1 - e^{v-\theta})^{-1} d\theta.$$

To give some credibility to these identities, and for future reference, let us check that we may recover the identitiy in Caballero and Chaumont [10] for the law of the minimum.

Corollary 2. *Let $\underline{\xi}^{\uparrow}_{\infty} = \inf_{t\geq 0} \xi^{\uparrow}_t$. For $z \geq 0$,*

$$P\Big(-\underline{\xi}^{\uparrow}_{\infty} \leq z\Big) = (1 - e^{-z})^{\alpha\rho}.$$

Proof. The required probability may be identified as equal to $P(T^-_{-z} = \infty)$ and hence, since there is no probability of creeping over the level $-z$,

$$P\Big(-\underline{\xi}^{\uparrow}_{\infty} \leq z\Big)$$
$$= 1 - \frac{\sin \pi\alpha\rho}{\pi}(1 - e^{-z})^{\alpha\rho}\int_0^\infty (e^{-z-\theta})^{\alpha\rho+1}(e^{-z} - e^{-z-\theta})^{-\alpha\rho}(1 - e^{-z-\theta})^{-1}\mathrm{d}\theta$$
$$= 1 - \frac{\sin \pi\alpha\rho}{\pi}(1 - e^{-z})^{\alpha\rho}\int_0^\infty (e^{-\theta})^{\alpha\rho+1}(1 - e^{-\theta})^{-\alpha\rho}(e^{z} - e^{-\theta})^{-1}\mathrm{d}\theta.$$

Next note that the integral in the right hand side satisfies

$$\int_0^\infty (e^{-\theta})^{\alpha\rho+1}(1 - e^{-\theta})^{-\alpha\rho}(e^{z} - e^{-\theta})^{-1}\mathrm{d}\theta$$
$$= \int_1^\infty \frac{e^{-z}}{y(y-1)^{\alpha\rho}(y - e^{-z})}\mathrm{d}y$$
$$= \int_0^\infty \frac{e^{-z}}{(u+1)u^{\alpha\rho}(u + 1 - e^{-z})}\mathrm{d}u$$
$$= \int_0^\infty \left\{\frac{1}{u^{\alpha\rho}(u+1-e^{-z})} - \frac{1}{(u+1)u^{\alpha\rho}}\right\}\mathrm{d}u$$
$$= (1 - e^{-z})^{-\alpha\rho}\int_0^\infty \frac{1}{v^{\alpha\rho}(v+1)}dv - \int_0^\infty \frac{1}{(u+1)u^{\alpha\rho}}\mathrm{d}u$$
$$= [(1 - e^{-z})^{-\alpha\rho} - 1]\int_0^\infty \frac{1}{(u+1)u^{\alpha\rho}}\mathrm{d}u \tag{3.10}$$

where in the first equality we have applied the change of variable $y = e^\theta$, in the second equality $y = u+1$ and in the fourth equality $u = (1-e^{-z})v$. Note also that by writing $w = (u+1)^{-1}$ we also discover that

$$\int_0^\infty \frac{1}{(u+1)u^{\alpha\rho}}\mathrm{d}u = \int_0^1 (1-w)^{-\alpha\rho}w^{\alpha\rho-1}\mathrm{d}w = \Gamma(1-\alpha\rho)\Gamma(\alpha\rho) = \frac{\pi}{\sin\pi\alpha\rho}. \tag{3.11}$$

In conclusion we deduce that

$$\int_0^\infty (e^{-\theta})^{\alpha\rho+1}(1-e^{-\theta})^{-\alpha\rho}(e^z - e^{-\theta})^{-1}\mathrm{d}\theta = [(1-e^{-z})^{-\alpha\rho} - 1]\frac{\pi}{\sin\pi\alpha\rho}$$

and hence the required identity holds. ■

Finally, to complete this subsection, when $(X, \mathbb{P}_1^\uparrow)$ is a spectrally negative process we also gain some information concerning the scale function, $W^{\uparrow,\mathrm{n}}$, of its underlying Lévy process, denoted here by $\xi^{\uparrow,\mathrm{n}}$. Specifically, in that case it is know that $1-\rho = 1/\alpha$ (and $\alpha \in (1,2)$) and that $P(-\underline{\xi}^{\uparrow,\mathrm{n}}_\infty \le x) = mW^{\uparrow,\mathrm{n}}(x)$, where $m = E(\xi_1^{\uparrow,\mathrm{n}})$. This implies

$$W^{\uparrow,\mathrm{n}}(x)(x) = \frac{1}{m}(1-e^{-x})^{\alpha\rho} = \frac{1}{m}(1-e^{-x})^{\alpha-1}.$$

Recall that for a given spectrally negative Lévy process it is known that the Laplace transform of the scale function is given by the inverse of the associated Laplace exponent (see for instance Theorem VII.8 in Bertoin [2]). We can therefore compute the Laplace exponent $\psi^\uparrow(\theta) = \log E(e^{\theta\xi_1^{\uparrow,\mathrm{n}}})$ for $\theta \ge 0$, as follows:

$$\begin{aligned}\psi^\uparrow(\theta) &= m\left(\int_0^\infty e^{-\theta x}(1-e^{-x})^{\alpha-1}\mathrm{d}x\right)^{-1} \\ &= m\left(\int_0^1 u^{\theta-1}(1-u)^{\alpha-1}\mathrm{d}u\right)^{-1} = m\frac{\Gamma(\theta+\alpha)}{\Gamma(\theta)\Gamma(\alpha)}.\end{aligned}$$

The knowledge of the scale function allow us to write a stronger result than that given in Corollary 1 as follows.

Lemma 2. *Let $\underline{\xi}^{\uparrow,\mathrm{n}}_t = \inf_{0\le s\le t} \xi^{\uparrow,\mathrm{n}}_s$. For $v < 0$, $\theta \ge 0, \phi \ge \eta$ and $\eta \in [0,-v]$ we have*

$$\begin{aligned}&P\Big(v - \xi^{\uparrow,\mathrm{n}}_{T_v^-} \in d\theta, \xi^{\uparrow,\mathrm{n}}_{T_v^- -} - v \in d\phi, \underline{\xi}^{\uparrow,\mathrm{n}}_{T_v^- -} - v \in d\eta\Big) \\ &= K^{-1}\,(1-e^{v+\eta})^{\alpha-2}(e^{v+\eta})(e^{-\theta-\phi})^\alpha(1-e^{-\theta-\phi})^{-1-\alpha}\,d\theta\,d\phi\,d\eta,\end{aligned}$$

where

$$K = \frac{e^{(\alpha-2)v}}{\alpha(\alpha-1)}\int_1^{e^{-v}}\frac{(e^{-v}-y)}{y(y-1)^{\alpha-1}}dy - \frac{(1-e^v)^{\alpha-1}}{\alpha(\alpha-1)}\frac{\pi}{\sin\pi(\alpha-1)}.$$

Proof. First recall that the process $\xi^{\uparrow,\mathrm{n}}$ drifts towards $+\infty$ a.s. Taking this account, we have from Example 8 of Doney and Kyprianou [16] that the required probability is proportional to

$$W^{\uparrow,\mathrm{n}}(-v-\mathrm{d}\eta)\pi^\uparrow(-\theta-\phi)\mathrm{d}\theta\mathrm{d}\phi.$$

Hence the triple law of interest has a density with respect to $\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\eta$ which is proportional to

$$(1-e^{v+\eta})^{\alpha-2}(e^{v+\eta})(e^{-\theta-\phi})^{\alpha}(1-e^{-\theta-\phi})^{-1-\alpha}.$$

For convenience let us write the constant of proportionality as K^{-1}. As $(X,\mathbb{P}_1^{\uparrow})$ is derived from a spectrally negative stable process, it cannot creep downwards (cf. p175 of Bertoin [2]). This allows us to compute the unknown constant via the total probability formula and after a straightforward computation, we have

$$\begin{aligned} K &= \int_0^\infty \int_0^\infty \int_0^{-v} (1-e^{v+\eta})^{\alpha-2}(e^{v+\eta})(e^{-\theta-\phi})^{\alpha}(1-e^{-\theta-\phi})^{-1-\alpha}\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\eta \\ &= \frac{e^{(\alpha-2)v}}{\alpha(\alpha-1)}\int_1^{e^{-v}} \frac{(e^{-v}-y)}{y(y-1)^{\alpha-1}}\mathrm{d}y - \frac{(1-e^v)^{\alpha-1}}{\alpha(\alpha-1)}\frac{\pi}{\sin\pi(\alpha-1)} \end{aligned}$$

and the proof is complete. ■

3.2 Calculations for ξ^*

Henceforth we shall assume that $(X,\mathbb{P}_x)$ is an α-stable process killed on first exit of the positive half line starting from $x>0$. As before, unless otherwise stated, we shall assume that there are two-sided jumps, moreover, spectrally one-sided results may be considered as limiting cases of the two sided jumps case. We start with the two- and one-sided exit problems with the latter as a limiting case of the former. We offer no proof as the calculations are essentially the same.

Theorem 3. *Fix $\theta \geq 0$ and $-\infty < v < 0 < u < \infty$.*

$$\begin{aligned} &P\Big(\xi^*_{T_u^+} - u \in d\theta; T_u^+ < T_v^-\Big) \\ &\quad = \frac{\sin\pi\alpha(1-\rho)}{\pi}(e^u-1)^{\alpha(1-\rho)}(1-e^v)^{\alpha\rho} \\ &\qquad \times(e^{u+\theta})(e^{u+\theta}-e^u)^{-\alpha(1-\rho)}(e^{u+\theta}-e^v)^{-\alpha\rho}(e^{u+\theta}-1)^{-1}d\theta \end{aligned}$$

and

$$\begin{aligned} &P\Big(v - \xi^*_{T_v^-} \in d\theta; T_u^+ > T_v^-\Big) \\ &\quad = \frac{\sin\pi\alpha\rho}{\pi}(1-e^v)^{\alpha\rho}(e^u-1)^{\alpha(1-\rho)} \\ &\qquad \times(e^{v-\theta})(e^v-e^{v-\theta})^{-\alpha\rho}(e^u-e^{v-\theta})^{-\alpha(1-\rho)}(1-e^{v-\theta})^{-1}d\theta. \end{aligned}$$

Corollary 3. *Fix $\theta \geq 0$ and $-\infty < v < 0 < u < \infty$.*

$$\begin{aligned} &P\Big(\xi^*_{T_u^+} - u \in d\theta\Big) \\ &\quad = \frac{\sin\pi\alpha(1-\rho)}{\pi}(e^u-1)^{\alpha(1-\rho)}(e^{u+\theta})^{1-\alpha\rho}(e^{u+\theta}-e^u)^{-\alpha(1-\rho)}(e^{u+\theta}-1)^{-1}d\theta \end{aligned}$$

and

$$P\Big(v-\xi^*_{T_v^-}\in d\theta; T_v^-<\infty\Big)$$
$$=\frac{\sin\pi\alpha\rho}{\pi}(1-e^v)^{\alpha\rho}(e^{v-\theta})(e^v-e^{v-\theta})^{-\alpha\rho}(1-e^{v-\theta})^{-1}d\theta.$$

One may think of computing the distribution of the maximum, $\overline{\xi}^*_\infty$, and the minimum, $\underline{\xi}^*_\infty$, of ξ^* in a similar way to the previous section by integrating out u and v in the above corollary. The law of the minimum was already computed in Caballero and Chaumont (2006) and we refrain from producing the alternative computations here. For the maximum, an easier approach is at hand. Since ξ^* is derived from a stable process killed on first exit of the positive half line one may write

$$P\Big(\overline{\xi}^*_\infty\le z\Big)=P\Big(\exp\{\overline{\xi}^*_\infty\}\le e^z\Big)=\mathbf{P}_1(\sigma^+_{e^z}>\sigma^-_0)=\mathbf{P}_{e^{-z}}(\sigma^+_1>\sigma^-_0).$$

The probability on the right hand side above may be obtained from Theorem 1 by a straightforward integration. The latter calculation has already been performed however in Rogozin [27] and is equal to

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}\int_0^{1-e^{-z}}y^{\alpha\rho-1}(1-y)^{\alpha(1-\rho)-1}\mathrm{d}y$$

Hence, together with the result for the minimum from Caballero and Chaumont (2006) which we include for completeness, we have the following corollary.

Corollary 4. *For $z\ge 0$ we have that*

$$P\Big(\overline{\xi}^*_\infty\in dz\Big)=\frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))}(e^{-z})^{\alpha(1-\rho)}(1-e^{-z})^{\alpha\rho-1}dz$$

and

$$P\Big(-\underline{\xi}^*_\infty\in dz\Big)=\frac{1}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)}(e^z-1)^{\alpha\rho}dz.$$

In the case that ξ^* is spectrally one sided, it seems difficult to use the above result to extract information about any underlying scale functions. The reason for this is that the process ξ^* is exponentially killed at a rate which is intimately linked to its underlying parameters and not at a rate which can be independently varied.

3.3 Calculations for $\xi^\downarrow$

Henceforth we shall assume that $(X,\mathbb{P}^\downarrow_x)$ is an α-stable process conditioned to hit zero continuously starting from $x>0$. Again, unless otherwise stated, we shall assume that there are two-sided jumps, and spectrally one-sided results may be considered as limiting cases of the two sided jumps case. We follow the same programme as the previous two sections dealing with the two- and one-sided exit problems without offering proofs since they follow from the calculations for $\xi^\uparrow$ and Proposition 1

Theorem 4. *Fix $\theta \geq 0$.*

$$
\begin{aligned}
&P\Big(\xi^{\downarrow}_{T_u^+} - u \in d\theta; T_u^+ < T_v^-\Big)\\
&\quad = \frac{\sin \pi\alpha(1-\rho)}{\pi}(e^u - 1)^{\alpha(1-\rho)}(1-e^v)^{\alpha\rho}\\
&\qquad \times (e^{u+\theta})^{\alpha\rho}(e^{u+\theta} - e^u)^{-\alpha(1-\rho)}(e^{u+\theta} - e^v)^{-\alpha\rho}(e^{u+\theta} - 1)^{-1}\, d\theta
\end{aligned}
$$

and

$$
\begin{aligned}
&P\Big(v - \xi^{\downarrow}_{T_v^-} \in d\theta; T_u^+ > T_v^-\Big)\\
&\quad = \frac{\sin \pi\alpha(1-\rho)}{\pi}(e^u - 1)^{\alpha(1-\rho)}(1-e^v)^{\alpha\rho}\\
&\qquad \times (e^{v-\theta})^{\alpha\rho}(e^v - e^{v-\theta})^{-\alpha\rho}(e^u - e^{v-\theta})^{-\alpha(1-\rho)}(1 - e^{v-\theta})^{-1}\, d\theta.
\end{aligned}
$$

Corollary 5. *Fix $\theta \geq 0$.*

$$
\begin{aligned}
&P\Big(\xi^{\uparrow}_{T_u^+} - u \in d\theta; T_u^+ < \infty\Big)\\
&\quad = \frac{\sin \pi\alpha(1-\rho)}{\pi}(e^u - 1)^{\alpha(1-\rho)}(e^{u+\theta} - e^u)^{-\alpha(1-\rho)}(e^{u+\theta} - 1)^{-1}\, d\theta
\end{aligned}
$$

and

$$
\begin{aligned}
&P\Big(v - \xi^{\uparrow}_{T_v^-} \in d\theta; T_v^- < \infty\Big)\\
&\quad = \frac{\sin \pi\alpha\rho}{\pi}(1-e^v)^{\alpha\rho}(e^{v-\theta})^{\alpha\rho}(e^v - e^{v-\theta})^{-\alpha\rho}(1 - e^{v-\theta})^{-1}\, d\theta.
\end{aligned}
$$

From the above corollary we proceed to obtain the law of the maximum of $\xi^{\downarrow}$ (recalling that it is a process with drift to $-\infty$).

Corollary 6. *For $z \geq 0$*

$$
P\Big(\overline{\xi}^{\downarrow}_{\infty} \leq z\Big) = (1 - e^{-z})^{\alpha(1-\rho)}
$$

Proof. Similarly to the calculations in Section 3.1 we make use of the fact that

$$
P\Big(\overline{\xi}^{\downarrow}_{\infty} \leq z\Big) = P(T_z^+ = \infty).
$$

Hence

$$
\begin{aligned}
&P\Big(\overline{\xi}^{\downarrow}_{\infty} \leq z\Big)\\
&\quad = 1 - \frac{\sin \pi\alpha(1-\rho)}{\pi}(e^z - 1)^{\alpha(1-\rho)}\int_0^\infty (e^{z+\theta} - e^z)^{-\alpha(1-\rho)}(e^{z+\theta} - 1)^{-1}\mathrm{d}\theta\\
&\quad 1 - \frac{\sin \pi\alpha(1-\rho)}{\pi}(1 - e^{-z})^{\alpha(1-\rho)}\int_0^\infty (e^{-\theta})^{\alpha(1-\rho)+1}(1 - e^{-\theta})^{-\alpha(1-\rho)}(e^z - e^{-\theta})^{-1}\mathrm{d}\theta.
\end{aligned}
$$

Next note that the integral on the right hand side above has been seen before in (3.10) except for the case that ρ is replaced by $1-\rho$. We thus obtain from (3.10) and (3.11)

$$\int_0^\infty (e^{-\theta})^{\alpha(1-\rho)+1}(1-e^{-\theta})^{-\alpha(1-\rho)}(e^z-e^{-\theta})^{-1}d\theta = [(1-e^{-z})^{-\alpha(1-\rho)}-1]\frac{\pi}{\sin\pi\alpha(1-\rho)}$$

and hence

$$P\Big(\overline{\xi}^{\downarrow}_\infty \le z\Big) = (1-e^{-z})^{\alpha(1-\rho)}$$

as required. ■

Now, we suppose that $(X,\mathbb{P}^{\downarrow}_1)$ has only positve jumps, in which case $\rho = 1/\alpha$. We denote by $\xi^{\downarrow,\mathrm{p}}$ its underlying Lévy process in this particular case. The associated scale function of $\xi^{\downarrow,\mathrm{p}}$ can be identified as

$$E(-\xi^{\downarrow,\mathrm{p}}_1)W^{\downarrow,\mathrm{p}}(x) = P\Big(\overline{\xi}^{\downarrow,\mathrm{p}}_\infty \le x\Big) = (1-e^{-x})^{\alpha-1} = W^{\uparrow,\mathrm{n}}(x)E(\xi^{\uparrow,\mathrm{n}}_1), \tag{3.12}$$

where $W^{\uparrow,\mathrm{n}}$ is the scale function of the spectrally negative Lévy process $\xi^{\uparrow,\mathrm{n}}$.
This observation reflects the duality property for positive self-similar Markov processes in this particular case (see section 2 in Bertoin and Yor [4]). More precisely, we have the duality property between the resolvent operators of $(X,\mathbb{P}^{\uparrow}_x)$, when it is spectrally negative, and $(X,\mathbb{P}^{\downarrow}_x)$, for $x>0$.

From the identification of the scale function in (3.12) and Lemma 2, it is possible to give a triple law for the process $\xi^{\downarrow,\mathrm{p}}$ at first passage over the level $x>0$.

Lemma 3. *Let* $\overline{\xi}^{\downarrow,\mathrm{p}}_t = \sup_{0\le s\le t}\xi^{\downarrow,\mathrm{p}}_s$. *For* $x>0$, $\theta\ge 0, \phi\ge\eta$ *and* $\eta\in[0,x]$ *we have*

$$\begin{aligned}&P\Big(\xi^{\downarrow,\mathrm{p}}_{T^+_x}-x\in d\theta, x-\xi^{\downarrow,\mathrm{p}}_{T^+_x-}\in d\phi, x-\overline{\xi}^{\downarrow,\mathrm{p}}_{T^+_x-}\in d\eta\Big)\\ &= K^{-1}(1-e^{-x+\eta})^{\alpha-2}e^{-x+\eta}e^{\theta+\phi}(e^{\theta+\phi}-1)^{-1-\alpha}\,d\theta\,d\phi\,d\eta,\end{aligned}$$

where

$$K = \frac{e^{(\alpha-2)v}}{\alpha(\alpha-1)}\int_1^{e^{-v}}\frac{(e^{-v}-y)}{y(y-1)^{\alpha-1}}dy - \frac{(1-e^{v})^{\alpha-1}}{\alpha(\alpha-1)}\frac{\pi}{\sin\pi(\alpha-1)}.$$

In the remainder of this section, we assume that $(X,\mathbb{P}^{\downarrow}_x)$ is spectrally negative and we denote its underlying Lévy process by $\xi^{\downarrow,\mathrm{n}}$ for its underlying Lévy processes. The identification of the scale function of the Lévy process $\xi^{\uparrow,\mathrm{n}}$ and Proposition 1 inspires the following result which identifies the scale function of the Lévy process $\xi^{\downarrow,\mathrm{n}}$. We emphasized that $\xi^{\downarrow,\mathrm{n}}$ is in fact $\xi^{\uparrow,\mathrm{n}}$ conditioned to drift to $-\infty$.

Lemma 4. *The Laplace exponent of* $\xi^{\downarrow,\mathrm{n}}$ *satisfies*

$$\psi^{\downarrow}(\theta) = m\frac{\Gamma(\theta-1+\alpha)}{\Gamma(\theta-1)\Gamma(\alpha)}$$

for $\theta\ge 0$, *and where* $m = E(-\xi^{\downarrow,\mathrm{n}})$. *Moreover, its associated scale function may be identified as*

$$W^{\downarrow,\mathrm{n}}(x) = \frac{1}{m}(1-e^{-x})^{\alpha-1}e^{x}.$$

Proof. From the two-sided exit problem for spectrally negative Lévy processes (see for instance Chapter VII in Bertoin [2]), we know that

$$P\Big(\underline{\xi}^{\uparrow,\mathrm{n}}_{T_y^+} > -x\Big) = \frac{W^{\uparrow,\mathrm{n}}(x)}{W^{\uparrow,\mathrm{n}}(x+y)} \qquad \text{for } \; x, y > 0.$$

On the other hand, from Proposition 1, we get that

$$P\Big(\underline{\xi}^{\uparrow,\mathrm{n}}_{T_y^+} > -x\Big) = e^y P\Big(\underline{\xi}^{\downarrow,\mathrm{n}}_{T_y^+} > -x\Big) = e^y \frac{W^{\downarrow,\mathrm{n}}(x)}{W^{\downarrow,\mathrm{n}}(x+y)}.$$

Hence, from the form of the scale function of $\xi^{\uparrow,\mathrm{n}}$, it follows that

$$P\Big(\underline{\xi}^{\downarrow,\mathrm{n}}_{T_y^+} > -x\Big) = \frac{W^{\downarrow,\mathrm{n}}(x)}{W^{\downarrow,\mathrm{n}}(x+y)} = \frac{e^x(1-e^{-x})^{\alpha-1}}{e^{x+y}(1-e^{-x-y})^{\alpha-1}}.$$

Taking y to ∞, one deduces

$$W^{\downarrow,\mathrm{n}}(x) = \frac{1}{m} e^x (1-e^{-x})^{\alpha-1}.$$

Finally, since

$$\int_0^\infty e^{-\theta x} e^x (1-e^{-x})^{\alpha-1} \mathrm{d}x = \int_0^1 u^{(\theta-1)-1}(1-u)^{\alpha-1}\mathrm{d}u = \frac{\Gamma(\theta-1)\Gamma(\alpha)}{\Gamma(\theta-1+\alpha)},$$

for $\theta > 1$, it is clear from the Laplace transform (cf Chapter 8 of Kyprianou [19]) of the scale function that

$$\psi^{\downarrow}(\theta) = m\frac{\Gamma(\theta-1+\alpha)}{\Gamma(\theta-1)\Gamma(\alpha)},\ \theta \geq 0,$$

is the associated Laplace exponent of $\xi^{\downarrow,\mathrm{n}}$. ■

One may also write down a triple law for the first passage problem of $\xi^{\downarrow,\mathrm{n}}$ as we have seen before for $\xi^{\uparrow,\mathrm{n}}$.

Lemma 5. *Let* $\underline{\xi}^{\downarrow,\mathrm{n}}_t = \inf_{0\leq s\leq t} \xi^{\downarrow,\mathrm{n}}_s$. *For* $v < 0$, $\theta \geq 0, \phi \geq \eta$ *and* $\eta \in [0, -v]$ *we have*

$$\begin{aligned}&P\Big(v - \xi^{\downarrow,\mathrm{n}}_{T_v^-} \in d\theta, \xi^{\downarrow,\mathrm{n}}_{T_v^- -} - v \in d\phi, \underline{\xi}^{\downarrow,\mathrm{n}}_{T_v^- -} - v \in d\eta\Big)\\ &= K^{-1}\, e^{-q(\phi-\eta)}(e^{-(v+\eta)} + \alpha - 2)(1-e^{v+\eta})^{\alpha-2}(e^{-\theta-\phi})^{\alpha}(1-e^{-\theta-\phi})^{-1-\alpha}\, d\theta\, d\phi\, d\eta,\end{aligned}$$

where $q > 0$ *and*

$$K = \frac{1}{\alpha}\int_0^{-v}\int_\eta^\infty e^{-q(\eta-\phi)}(e^{-v-\eta}+\alpha-2)(1-e^{v+\eta})^{\alpha-2}(e^{\phi}-1)^{-\alpha}\, d\phi\, d\eta.$$

Proof. First recall that the process $\xi^{\downarrow,\mathrm{n}}$ drifts towards $-\infty$ a.s. Again, we have from Example 8 of Doney and Kyprianou [16] that the required probability is proportional to

$$e^{-q(\phi-\eta)}W^{\downarrow,\mathrm{n}}(-v-\mathrm{d}\eta)\pi^{\downarrow}(-\theta-\phi)\mathrm{d}\theta\mathrm{d}\phi,$$

where $q>0$ is the killing rate of the descending ladder height process (see for instance Chapter VI in Bertoin [2] for a proper definition) of $\xi^{\downarrow,\mathrm{n}}$.
Hence the triple law of interest has a density with respect to $\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\eta$ which is proportional to

$$e^{-q(\phi-\eta)}(1-e^{v+\eta})^{\alpha-2}(e^{-(v+\eta)}+\alpha-2)(e^{-\theta-\phi})^{\alpha}(1-e^{-\theta-\phi})^{-1-\alpha}.$$

As $(X,\mathbb{P}_1^{\downarrow})$ is derived from a spectrally negative stable process, it cannot creep downwards (cf. p175 of Bertoin [2]). This allows us to compute the unknown constant of proportionality K^{-1} via the total probability formula and after a straightforward computation, we have

$$\begin{aligned} K &= \int_0^{\infty}\int_0^{\infty}\int_0^{-v} e^{-q(\phi-\eta)}(1-e^{v+\eta})^{\alpha-2}(e^{v+\eta})(e^{-\theta-\phi})^{\alpha}(1-e^{-\theta-\phi})^{-1-\alpha}\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\eta \\ &= \frac{1}{\alpha}\int_0^{-v}\int_{\eta}^{\infty} e^{-q(\eta-\phi)}(e^{-v-\eta}+\alpha-2)(1-e^{v+\eta})^{\alpha-2}(e^{\phi}-1)^{-\alpha}\mathrm{d}\phi\mathrm{d}\eta. \end{aligned}$$

and the proof is complete. ■

4 Entrance laws for Lévy-Lamperti processes: points

In this section we explore the two-point hitting problem for the Lévy-Lamperti processes $\xi^{\uparrow}$ and $\xi^{\downarrow}$. There has been little work dedicated to this theme in the past with the paper of Getoor [17] being our principle reference.

Henceforth we shall denote by $(X,\mathbf{P}_x)$ a *symmetric* α-stable process issued from $x>0$ where $\alpha\in(1,2)$. An important quantity in the forthcoming analysis is the resolvent density of the process $(X,\mathbf{P}_x)$ killed on exiting $(0,\infty)$. The latter is known to have a density

$$u(x,y)\mathrm{d}y=\int_0^{\infty}\mathrm{d}t\cdot\mathbf{P}_x(X_t\in dy, t<\sigma_0^-)$$

for $x,y>0$. From Blumenthal et al. [7] we know that

$$\int_0^{\infty}\mathrm{d}t\cdot\mathbb{P}_x(X_t\in\mathrm{d}y,t<\sigma_a^+\wedge\sigma_0^-)=\left\{\frac{|x-y|^{\alpha-1}}{2^{\alpha}\Gamma(\alpha/2)}\int_0^{s(x,y,a)}\frac{u^{\alpha/2-1}}{(u+1)^{1/2}}\mathrm{d}u\right\}\mathrm{d}y$$

where

$$s(x,y,a)=\frac{4xy}{(x-y)^2}\frac{(a-x)(a-y)}{a^2}.$$

It now follows taking limits as $a\uparrow\infty$ that

$$u(x,y)=\frac{1}{2^{\alpha}\Gamma(\alpha/2)}|x-y|^{\alpha-1}\int_0^{4xy/(x-y)^2}\frac{u^{\alpha/2-1}}{(u+1)^{1/2}}\mathrm{d}u.$$

According to the method presented in Getoor [17] one may compute

$$\mathbf{P}_x(X_{\sigma_{\{a,b\}}} = a;\, \sigma_{\{a,b\}} < \sigma_0^-)$$

where $\sigma_{\{a,b\}} = \inf\{t > 0 : X_t = a \text{ or } b\}$ and $a, b > 0$ using the following technique. The two point hitting probability in Getoor [17] is given by the formula

$$\mathbf{P}_x(X_{\sigma_{\{a,b\}}} = a;\, \sigma_{\{a,b\}} < \sigma_0^-) = -\frac{q(x,a)}{q(x,x)}$$

where the $\{x, a, b\} \times \{x, a, b\}$-matrix Q is defined by

$$Q = -U^{-1}$$

and the $\{x, a, b\} \times \{x, a, b\}$-matrix is given by

$$U = \begin{pmatrix} u(x,x) & u(x,a) & u(x,b) \\ u(a,x) & u(a,a) & u(a,b) \\ u(b,x) & u(b,a) & u(b,b) \end{pmatrix}.$$

In particular an easy computation shows that

$$\mathbf{P}_x(X_{\sigma_{\{a,b\}}} = a;\, \sigma_{\{a,b\}} < \sigma_0^-) = \frac{\frac{u(x,a)}{u(b,a)} - \frac{u(x,b)}{u(b,b)}}{\frac{u(a,a)}{u(b,a)} - \frac{u(a,b)}{u(b,b)}} \tag{4.13}$$

Recalling the definitions of $\xi^\uparrow$ and $\xi^\downarrow$ as the Lévy-Lamperti processes associated now with our symmetric stable process conditioned to stay positive and conditioned to be killed continuously at the origin respectively we obtain the following result.

Theorem 5. *Fix $\alpha \in (1,2)$ and $-\infty < v < 0 < u < \infty$. Define*

$$T_{\{v,u\}} = \inf\{t > 0 : \xi_t \in \{v, u\}\}$$

where ξ plays the role of either $\xi^\uparrow$ or $\xi^\downarrow$. We have

$$P\Big(\xi^\uparrow_{T_{\{v,u\}}} = v\Big) = (e^v)^{\alpha/2} f(1, e^v, e^u)$$

and

$$P\Big(\xi^\downarrow_{T_{\{v,u\}}} = v\Big) = (e^v)^{\alpha/2-1} f(1, e^v, e^u)$$

where

$$f(x,a,b) = \frac{\frac{u(x,a)}{u(b,a)} - \frac{u(x,b)}{u(b,b)}}{\frac{u(a,a)}{u(b,a)} - \frac{u(a,b)}{u(b,b)}}.$$

5 Exponential functionals of Lévy-Lamperti processes.

We begin this section by recalling a crucial expression for the entrance law at 0 of pssMp's. In [3, 4], the authors proved that if a non arithmetic Lévy process ξ satisfies $E(|\xi_1|) < \infty$ and $0 < E(\xi_1) < +\infty$, then its corresponding pssMp $(X, \mathbb{P}_x)$ in the Lamperti representation converges weakly as x tends to 0, in the sense of finite dimensional distributions towards a non degenerated probability law $\mathbb{P}_0$. Under these conditions, the entrance law under $\mathbb{P}_0$ is described as follows: for every $t > 0$ and every measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\mathbb{E}_0\big(f(X_t)\big) = \frac{1}{\alpha E(\xi_1)} E\left(I(\xi)^{-1} f\big(tI(\xi)^{-1}\big)\right), \tag{5.14}$$

where $I(\xi)$ is the exponential functional:

$$I(\xi) = \int_0^\infty \exp\{-\alpha \xi_s\} \,\mathrm{d}s.$$

Necessary and sufficient conditions for the weak convergence of $(X, \mathbb{P}_x)$ on the Skorokhod's space were given in [9]. Recall that $(X, \mathbb{P}_x^\uparrow)$ denotes a stable Lévy process conditioned to stay positive as it has been defined in section 2. Then we easily check that $\xi^\uparrow$ satisfies conditions for the weak convergence of $(X, \mathbb{P}_x^\uparrow)$ given in [3, 4, 9]. Note also that in this particular case, the weak convergence of $(X, \mathbb{P}_x^\uparrow)$ had been proved in a direct way in [12]. We denote the limit law by $\mathbb{P}^\uparrow$.

We first investigate the tail behaviour of the law of $I(\xi^\uparrow)$.

Theorem 6. *The law of $I(\xi^\uparrow)$ is absolutely continuous with respect to the Lebesgue measure. The density of $I(\xi^\uparrow)^{-1}$ is given by:*

$$P\Big(I(\xi^\uparrow)^{-1} \in dy\Big) = \alpha E(\xi_1^\uparrow) y^{\alpha\rho - 1} q_1(y)\, dy, \tag{5.15}$$

where q_t is the density of the entrance law of the excursion measure of the reflected process $(X - \underline{X}, \mathbf{P}_0)$, where $\underline{X}_t = \inf_{0\le s\le t} X_s$. Moreover, the law of $I(\xi^\uparrow)$ behaves as

$$P(I(\xi^\uparrow) \ge x) \sim C_1 x^{-\alpha}\,, \quad \text{as } x \to +\infty. \tag{5.16}$$

If X has positive jumps, then

$$P(I(\xi^\uparrow) \le x) \sim C_2 x^{\alpha(\rho-1)-1}\,, \text{ as } x \to 0. \tag{5.17}$$

The constants C_1 and C_2 depend only on α and ρ.

In the case where the process has positive jumps, the law of $I(\xi^\uparrow)$ is given explicitly in the next theorem.

Proof. Let n be the measure of the excursions away form 0 of the reflected process $X-\underline{X}$ under $\mathbf{P}_0$. It is proved in [23] that the entrance law of n is absolutely continuous with respect to the Lebesgue measure. Let us denote by q_t its density. Then from [12], the entrance law of $(X, \mathbb{P}^\uparrow)$ is related to q_1 by

$$\mathbb{P}^\uparrow(X_1 \in \mathrm{d}y) = y^{\alpha\rho} q_1(y)\mathrm{d}y\,, \quad y \geq 0\,.$$

We readily derive (5.15) from identity (5.14). Moreover from (3.18) in [23]:

$$\int_0^x q_1(y)\,\mathrm{d}y \sim C x^{\alpha(1-\rho)+1}\,, \quad \text{as } x \to 0,$$

and from (3.20) of the same paper, if X has positive jumps, then:

$$\int_x^\infty q_1(y)\,\mathrm{d}y \sim C' x^{-\alpha}\,, \quad \text{as } x \to +\infty.$$

This together with (5.14) implies (5.16) and (5.17). The constants C and C' depend only on α and ρ. ■

Another way to prove part (i) of this theorem is to use a result due to Méjane [22] and Rivero [26] which asserts that for a non arithmetic Lévy process ξ, if Cramer's condition is satisfied for $\theta > 0$, i.e. $E(\exp\theta\xi_1) = 1$ and $E(\xi_1^+ \exp\theta\xi_1) < \infty$, then $P(I(\xi) \geq x) \sim Cx^{-\alpha\theta}$. These arguments and Proposition 1 allow us to obtain the asymptotic behaviour at $+\infty$ of $P(I(-\xi^\downarrow) \geq x)$:

Proposition 2. *The law of $I(-\xi^\downarrow)$ behaves as*

$$P(I(-\xi^\downarrow) \geq x) \sim C_3 x^{-\alpha}\,, \quad \textit{as } x \to +\infty. \tag{5.18}$$

The constant C_3 depends only on α and ρ.

Now we consider the exponential functional

$$I(-\xi^*) = \int_0^\infty \exp\{\alpha\xi_s^*\}\,\mathrm{d}s.$$

Recall from section 2 that ξ^* is the Lévy process which is associated to the pssMp $(X_t \mathbb{1}_{\{t<T\}}, \mathbf{P}_x)$ by the Lamperti representation. From this representation, we may check path by path the equality

$$x^\alpha I(-\xi^*) = T\,.$$

Moreover, it follows from Lemma 1 in [10] that when $(X, \mathbf{P}_x)$ has negative jumps, $\mathbf{P}_x(T \leq t) \sim \frac{c_- t}{\alpha x^\alpha}$, as t tends to 0. This result leads to:

Proposition 3. *Suppose that ξ^* has negative jumps, then the law of $I(-\xi^*)$ behaves as*

$$P(I(-\xi^*) \leq x) \sim \frac{c_-}{\alpha} x^{-1}\,, \quad \textit{as } x \to 0. \tag{5.19}$$

In the remainder of this section, we assume that $(X, \mathbb{P}^{\uparrow}_x)$ has no positive jumps.

Theorem 7. *The law of exponential functional* $I(\xi^{\uparrow,\mathrm{n}}) = \int_0^\infty \exp\{-\alpha\xi^{\uparrow,\mathrm{n}}_s\}\, ds$ *is absolutely continuous with respect to the Lebesgue measure and has a continuous density* $p^{\uparrow,\mathrm{n}}(\cdot)$ *which has the following representation by power series*

$$p^{\uparrow,\mathrm{n}}(x) = -\frac{c^{-1}}{\pi x}\sum_{n=1}^{\infty}\Gamma\left(1+\frac{n}{\alpha}\right)\sin\left(\frac{\pi n}{\alpha}\right)\frac{(-x^{-1/\alpha})^n}{n!}, \qquad \text{for} \quad x>0,$$

where $c = c_-\Gamma(2-\alpha)\alpha^{-1}(\alpha-1)^{-1} > 0$.
Moreover the positive entire moments of $(X, \mathbb{P}^{\uparrow})$*, for* $t>0$*, are given by the identity*

$$\mathbb{E}^{\uparrow}\Big(\big(X_t\big)^k\Big) = (mt)^k\frac{\Gamma(\alpha(k+1))}{\Gamma(\alpha)^{(k+1)}k!}, \qquad k\geq 1, \tag{5.20}$$

and its law is absolutely continuous with respect to the Lebesgue measure and has a continuous density $p_t(\cdot)$ *which has the following representation by power series*

$$p_t(x) = -\frac{t^{-1/\alpha}(\alpha-1)}{\Gamma(2-\alpha)c_- m\pi}\sum_{n=1}^{\infty}\Gamma\left(1+\frac{n}{\alpha}\right)\sin\left(\frac{\pi n}{\alpha}\right)\frac{(-x^{1/\alpha})^n}{n!}, \qquad \text{for} \quad x>0.$$

Proof. From Bertoin and Yor [5] , we know that the distribution of the exponential functional of a spectrally negative Lévy process is determined by its negative entire moments. In particular, the exponential functional for $\xi^{\uparrow,\mathrm{n}}$ satisfies

$$E\left(I(\xi^{\uparrow,\mathrm{n}})^{-k}\right) = \alpha m^k\frac{\Gamma(k\alpha)}{\Gamma(\alpha)^k(k-1)!}, \tag{5.21}$$

with the convention that the right-hand side equals αm for $k=1$. In particular, from (5.14) we have that

$$\mathbb{E}^{\uparrow}\Big(\big(X_t\big)^k\Big) = (mt)^k\frac{\Gamma(\alpha(k+1))}{\Gamma(\alpha)^{(k+1)}k!},$$

which proves the identity (5.20).
Now, from the time reversal property of Theorem VII.18 in [2], we deduce that the last passage time of $(X, \mathbb{P}^{\uparrow})$, defined by

$$U_x = \sup\Big\{t\geq 0:\ X_t\leq x\Big\} \qquad \text{for} \quad x\geq 0,$$

is a stable subordinator of index $1/\alpha$. More precisely, its Laplace exponent is given by $\Phi(\lambda) = (\lambda/c)^{1/\alpha}$, where $c = c_-\Gamma(2-\alpha)\alpha^{-1}(\alpha-1)^{-1}$. According to Zolotarev [30], stable subordinators have continuous densities with respect to the Lebesgue measure which may be represented by power series. More precisely, the density of a normalized stable subordinator of index $\beta\in(0,1)$, i.e. $\Phi(\lambda)=\lambda^{\beta}$, is given by

$$\rho_t(x,\beta) = -\frac{t^{-1/\beta}}{\pi x}\sum_{n=1}^{\infty}\Gamma(1+n\beta)\sin(\beta\pi n)\frac{(-x^{-\beta})^n}{n!}, \qquad \text{for} \quad x>0. \tag{5.22}$$

On the other hand from Proposition 1 in [13], we know that U_x has the same law as $x^\alpha I(\xi^{\uparrow,\mathrm{n}})$. Hence, $I(\xi^{\uparrow,\mathrm{n}})$ satisfies that

$$E\Big(\exp\big\{-\lambda I(\xi^{\uparrow,\mathrm{n}})\big\}\Big) = e^{-(\lambda/c)^{1/\alpha}}, \qquad \lambda \geq 0,$$

and its density $p^{\uparrow,\mathrm{n}}(x)$ is given by $\rho_{c^{1/\alpha}}(x, 1/\alpha)$, which proves the first part of the theorem.
Finally from (5.14), we deduce that the density of $X_1^{(0)}$ is given by

$$p_1(x) = -\frac{c^{-1}}{\alpha m\pi}\sum_{n=1}^{\infty}\Gamma\left(1+\frac{n}{\alpha}\right)\sin\left(\frac{\pi n}{\alpha}\right)\frac{(-x^{1/\alpha})^n}{n!}, \qquad \text{for} \quad x>0.$$

The proof is now complete. ■

Theorem 8. *The exponential functional $I(-\xi^*) = \int_0^\infty \exp\{\alpha\xi_s^*\}\,ds$ has a continuous density $p^*(\cdot)$ with respect to the Lebesgue measure which has the following representation by power series*

$$p^*(x) = c^{1/\alpha}\sum_{n=1}^{\infty}\frac{\alpha n - 1}{\Gamma(\alpha n)\Gamma(-n+1+1/\alpha)}x^{\alpha(2-n\alpha)}, \quad \textit{for} \quad x>0,$$

where $c = c_+\Gamma(2-\alpha)\alpha^{-1}(\alpha-1)^{-1} > 0$.

Proof: First, let us define $\hat{X} = -X$ and denote by $\hat{\mathbf{P}}$ for its law starting from 0. Note that the process $(X, \hat{\mathbf{P}})$ is a stable Lévy process with no negative jumps of index $\alpha \in (1,2)$ starting from 0. From the Lamperti representation, it is clear that $T = I^*$ and from the self-similar property, we have

$$\begin{aligned}
\mathbf{P}_1(T>t) &= \mathbf{P}_0(\sigma_{-1}^- > t) = \mathbf{P}_0\left(\inf_{0\leq s\leq t} X_s > -1\right) \\
&= \mathbf{P}_0\left(t^{1/\alpha}\inf_{0\leq s\leq 1} X_s > -1\right) = \hat{\mathbf{P}}\left(t^{1/\alpha}\sup_{0\leq s\leq 1} X_s < 1\right) \\
&= \hat{\mathbf{P}}\left(\left(\frac{1}{\sup_{0\leq s\leq 1} X_s}\right)^\alpha > t\right).
\end{aligned}$$

Hence the exponential functional I^*, under $\mathbf{P}_1$, has the same law as $(\sup_{0\leq s\leq 1} X_1)^{-\alpha}$, under $\hat{\mathbf{P}}$.
Recently, Bernyk, Dalang and Peskir [1] computed the density of the supremum of a stable Lévy process with no negative jumps of index $\alpha \in (1,2)$. More precisely with our notation, the density f of $\sup_{0\leq s\leq 1} X_1$, under $\hat{\mathbf{P}}$, is described as follows

$$f(x) = c^{1/\alpha}\sum_{n=1}^{\infty}\frac{\alpha n - 1}{\Gamma(\alpha n)\Gamma(-n+1+1/\alpha)}x^{n\alpha-2}, \quad \text{for} \quad x>0.$$

Therefore the density p^* of I^* is given by

$$p^*(x) = c^{1/\alpha}\sum_{n=1}^{\infty}\frac{\alpha n - 1}{\Gamma(\alpha n)\Gamma(-n+1+1/\alpha)}x^{\alpha(2-n\alpha)}, \quad \text{for} \quad x>0,$$

which completes the proof. ■

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