The adiabatic invariant of any harmonic oscillator : an unexpected application of Glauber's formalism

Martin Devaud, Valentin Leroy, Jean-Claude Bacri, Thierry Hocquet

The adiabatic invariant of any harmonic oscillator : an unexpected application of Glauber's formalism

M Devaud, V Leroy,† J-C Bacri and T Hocquet

Laboratoire Matière et Systèmes Complexes,
Université Paris 7 and CNRS UMR 7057,
10 rue Alice Domon et Léonie Duquet, 75013 PARIS, France

E-mail: `Thierry.Hocquet@upmc.fr`

Abstract. In this theoretical paper, we propose a general derivation of the adiabatic invariant of the n-degree-of-freedom harmonic oscillator, available whichever the physical nature of the oscillator and of the parametrical excitation it undergoes. This derivation is founded on the use of the classical Glauber variables and ends up with this simplest result: the oscillator's adiabatic invariant is just the sum of all the semiclassical quanta numbers associated with its different eigenmodes.

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1. Introduction

The adiabatic invariance is generally put on short allowance in most elementary or even undergraduate level textbooks and courses of lectures about mechanics. Moreover, when mentioned, it is usually restricted to the mere one-degree-of-freedom harmonic oscillator (HO1) case. The reason for this is at least twofold. First, adiabatic invariance is commonly regarded as but a curiosity and its teaching as a blind-alley occupation. Second, there is no simple and general theory to be applied to any case. Establishing the correct adiabatic invariant for a given harmonic oscillator consequently requires subtle – and often very smart [1] – reasonings that must be started afresh for any other oscillator. The case of many-degree-of-freedom oscillators is even more arduous, of course.

The aim of the present paper is to propose a simple theory of the adiabatic invariance of the harmonic oscillator, in the general n-degree-of-freedom case. The prerequisite for an easy reading of this article is a rudimentary knowledge of hamiltonian (classical) mechanics, nothing more. The paper is organized as follows.

In section 2, we begin with the HO1 case. We introduce the so-called standard variable, its conjugate momentum, and build the classical Glauber variable associated with the HO1. We deliberately use the same notations as in our foregoing paper [2], which was devoted to the particular case of the pendulum. More generally, this section 2 can be regarded as a recall of section 2 and subsection 4.1 in [2], which we include here for the sake of self-consistency. We end up with a most simple and

† Present address: Laboratoire Ondes et Acoustique, ESPCI, 10 rue Vauquelin, 75005 PARIS

general expression of the HO1's adiabatic invariant, as well as a very simple physical interpretation.

Section 3 is devoted to the extension of section 2's results to the HOn case. The notion of standard variables is generalized as well as that of Glauber variables. An unpublished calculation is presented that allows to build the HOn's adiabatic invariant, the physical interpretation of which is straightforward.

Several comments are gathered in the conclusion section 4.

2. The HO1 case

2.1. General definitions

At which condition a one-degree-of-freedom physical system can be regarded as a HO? A simplest answer to this question may be: "there should exist a parametrization with a dynamical variable θ such that the kinetic energy E_{k} and the potential energy E_{p} read

$$E_{\mathrm{k}} = \frac{1}{2} M \dot{\theta}^2, \qquad E_{\mathrm{p}} = \frac{1}{2} K \theta^2, \tag{1}$$

where M and K are *positive* coefficients, respectively accounting for inertia and elasticity. In the widely taught case where θ has the units of a length, M is a mass ans K a stiffness. If θ is an angle, as in the case of the pendulum considered in [2], then M is an inertia momentum and K an angular stiffness. And so on. In fact, the above definition may be extended to nonmechanical systems for which the notion of kinetic or potential energy makes no particular sense. For example, in the case of the self-capacitor (L, C) electric circuit, the dynamical variable can be chosen as the charge Q of the capacitor, and the magnetic $\frac{1}{2} L \dot{Q}^2$ and electric $\frac{1}{2} \frac{Q^2}{C}$ energies respectively stand instead of E_{k} and E_{p}. Nevertheless, whichever the exact physical nature of the HO1 under consideration and of the variable θ chosen to describe its dynamics, the motion equation ruling θ can be derived from the Lagrangian

$$L(\theta, \dot{\theta}) = E_{\mathrm{k}} - E_{\mathrm{p}} = \frac{1}{2} M \dot{\theta}^2 - \frac{1}{2} K \theta^2. \tag{2}$$

If both inertia and elasticity parameters M and K are time-independent, the motion is said to be *free*. If one at least of these parameters is time-dependent, the motion is said to be *parametrically excited*. In the latter case, the Lagrangian explicitly depends on time and should be noted $L(\theta, \dot{\theta}, t)$. However, in both above cases, the Lagrange equation holds and reads

$$\frac{\mathrm{d}}{\mathrm{d}t} (M \dot{\theta}) = -K \theta, \tag{3a}$$

i.e. in fine

$$\ddot{\theta} + \mathit{\Gamma} \dot{\theta} + \omega^2 \theta = 0, \tag{3b}$$

with

$$\mathit{\Gamma}(t) = \frac{\dot{M}}{M}, \qquad \omega^2(t) = \frac{K}{M}. \tag{3c}$$

Allowing for (3b), the parametrical excitation of a HO1 results in a time-dependent proper angular frequency $\omega(t)$, accompanied with an *effective* viscosity. Note that the effective viscous rate $\mathit{\Gamma}$ is time-dependent and may be negative as well as positive. By smartly playing on the sign of Γ, one can induce a parametrical amplification, like for instance children pumping a swing.‡

If M and/or K are varied *slowly* with respect to the proper oscillation rate ω (more precisely: if all nonzero Fourier components of M and/or K correspond to angular frequencies much smaller than ω), the parametrical excitation is said to be *adiabatic* (in the Ehrenfest sense). In the latter case, it is easy to check that $\dot{M}/M$, $\dot{K}/K$ (and consequently $\dot{\omega}/\omega$) are *small* compared to ω.

It is the aim of this section 2 (and of section 3) to find out a relevant physical quantity of the oscillator which is conserved in the course of time when an adiabatic parametrical excitation is performed. This quantity is referred to as the adiabatic invariant of the HO.

2.2. Standard and Glauber variables

As said above, parameters M and K and variable θ have various units, according to the physical nature of the oscillator. For the sake of universality, it is convenient to introduce a new dynamical variable the unit of which is fixed once for all. Let us define the so-called *standard* variable

$$q = \sqrt{M}\theta. \tag{4}$$

Note that, in the case of a time-dependence of M, variable q depends explicitly on time (through M), and not only implicitly (through θ). As a consequence, the Lagrangian now reads

$$L(q,\dot{q}) = \frac{1}{2}\big(\dot{q}^2 - \Gamma q\dot{q}\big) - \frac{1}{2}\big(\omega^2 - \frac{1}{4}\Gamma^2\big)q^2, \tag{5a}$$

so that the motion equation (3b) becomes

$$\ddot{q} + \omega'^2 q = 0 \tag{5b}$$

with

$$\omega'^2(t) = \omega^2(t) - \frac{1}{2}\dot{\Gamma}(t) - \frac{1}{4}\Gamma^2(t) = \omega^2(t) - \frac{1}{\sqrt{M}}\frac{\mathrm{d}^2\sqrt{M}}{\mathrm{d}t^2}. \tag{5c}$$

It is noteworthy that, using the standard dynamical variable q instead of θ, the motion equation (3b) is simplified in (5b), where all the effect of the parametrical excitation is concentrated in the time-dependence of the effective angular frequency $\omega'(t)$. Nevertheless, although simple, equation (5b) is (as any Lagrange equation) a second-order (time) differential equation. It will turn out in the following to be more convenient to deal with a couple of first-order (time) differential equations. In this prospect, we should build the Hamiltonian H through a Legendre transformation of Lagrangian L. Let σ be the conjugate momentum of variable θ. We have, using (2),

$$\sigma = \frac{\partial L}{\partial\dot{\theta}} = M\dot{\theta} \quad \rightsquigarrow \quad H = \sigma\dot{\theta} - L = \frac{\sigma^2}{2M} + \frac{1}{2}K\theta^2. \tag{6}$$

We can also define p as the conjugate momentum of the standard variable q (using (5a)) and rewrite Hamiltonian (6) :

$$p = \frac{\partial L}{\partial\dot{q}} = \dot{q} - \frac{1}{2}\Gamma q = \sqrt{M}\dot{\theta} = \frac{\sigma}{\sqrt{M}} \quad \rightsquigarrow \quad H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2. \tag{7}$$

‡ A complete theory of the parametrical amplification is presented in [2] (and experimentally illustrated with a pendulum). An extensive discussion of the pumping of a swing can be founded in [3] and [4].

We shall see below the usefulness of the above expressions. We would nevertheless underline that Hamiltonian H is basically a function of variables θ, σ and t §.

Using either of the above both expressions of H, we obtain a set of two coupled first-order equations. Now, Glauber [5, 6] proposed an elegant formalism in which the oscillator's dynamical variable (θ or q) and its conjugate momentum (σ or p) are combined in a unique complex hamiltonian variable α defined as

$$\alpha = \frac{1}{\sqrt{2\hbar}}\Big(\sqrt{M\omega}\theta + \frac{\mathrm{i}}{\sqrt{M\omega}}\sigma\Big) = \frac{1}{\sqrt{2\hbar}}\Big(\sqrt{\omega}q + \frac{\mathrm{i}}{\sqrt{\omega}}p\Big), \tag{8}$$

where $\hbar$ is the usual quantum constant‖. For reasons that will appear hereafter, we deliberately keep this constant in the above definitions, although our problem is utterly classical. Using the Poisson brackets

$$\{\theta,\sigma\} = \{q,p\} = 1, \tag{9a}$$

one easily derives

$$\{\alpha,\alpha^*\} = \frac{1}{\mathrm{i}\hbar}. \tag{9b}$$

On the other hand, the Hamiltonian reads, in the Glauber formalism,

$$H = \hbar\omega|\alpha|^2. \tag{10}$$

It is tempting to propose the following physical interpretation of the above expression: since $\hbar\omega$ is the energy quantum and H the total oscillation energy, then $N = |\alpha|^2$ can be regarded as the (semi-classical) quanta number in the HO1's oscillation.

Thanks to (9b), and using the Ehrenfest theorem, one immediately gets α's motion equation

$$\dot\alpha = \frac{\partial\alpha}{\partial t} + \{\alpha, H\} = -\,\mathrm{i}\omega\alpha + f\alpha^*, \tag{11a}$$

where

$$f(t) = \frac{1}{2M\omega}\frac{\mathrm{d}(M\omega)}{\mathrm{d}t} = \frac{\mathrm{d}\ln\sqrt{KM}}{\mathrm{d}t}. \tag{11b}$$

Looking for an exact solution of the above equation (11a), one can set

$$\alpha(t) = \mathrm{e}^{-\mathrm{i}\varphi(t)}A(t), \tag{12a}$$

where $A(t)$ is a complex amplitude and $\varphi(t)$ a phase defined as

$$\varphi(t) = \int_{-\infty}^{t}\omega(t')\mathrm{d}t'. \tag{12b}$$

One then gets

$$\dot A = f(t)A^*\mathrm{e}^{2\mathrm{i}\varphi(t)}, \tag{12c}$$

which is equivalent to (11a).

§ If one chooses expression (7) of H with variables (q,p), the Hamilton equations should be written $\dot q = \frac{\partial q}{\partial t} + \frac{\partial H}{\partial p}$ and $\dot p = \frac{\partial p}{\partial t} - \frac{\partial H}{\partial q}$.

‖ Glauber introduced his formalism in the quantum mechanics domain, in order to describe the quasi-classical ("coherent") states of the HO. Hence the presence of $\hbar$ in the above definition (8) of α. More details can be found in [7, 8].

2.3. Adiabatic invariant of the HO1

In the case of an *adiabatic* parametric excitation of the oscillator, and according with the above-recalled definition of the Ehrenfest adiabaticity, f is *small* compared to ω, so that

$$\left|\frac{\dot{A}}{A}\right| = |f| \ll \omega, \tag{13}$$

which means that the amplitude $A(t)$ itself varies slowly with respect to $\varphi(t)$. Integrating (12c) we have

$$A(t) = A(0) + \int_0^t \mathrm{d}t'\, f(t')A^*(t')\mathrm{e}^{2\mathrm{i}\varphi(t')}. \tag{14a}$$

In the integral on the right-hand side of the above expression, the product fA^* varies slowly, so that its contribution to the time evolution of amplitude $A(t)$ is averaged out by the brisk oscillation of the phase factor $\mathrm{e}^{2\mathrm{i}\varphi(t')}$. One easily convinces oneself that (14a) reads

$$A(t) = A(0)\left(1 + o\left(\frac{f}{\omega}\right)\right). \tag{14b}$$

The so-called Secular Approximation (SA) consists in neglecting the $o\left(\frac{f}{\omega}\right)$ term. At this approximation, we are left with $A(t)$ = constant. In the complex plane of the Glauber variable α, the trajectory of the representative point (the so-called "phase portrait") is then a circle, with centre at the origin of the coodinates and (non uniformly) followed clockwise. The adiabatic invariant of the HO1 is thus simply the radius of this circle, or equivalently the quanta number $N = |\alpha|^2$. Allowing for (10), it is then straightforward to calculate the power P needed to vary the oscillator's parameters M and/or K. If the latter variation is adiabatic, we have N = constant, and consequently

$$P = P_{\mathrm{ad}} = N\hbar\dot{\omega}, \tag{15a}$$

the physical interpretation of which is straightforward. Coming back to the initial variable θ, an approximate solution of equation (3b) reads

$$\theta(t) = \Theta(t)\cos\varphi(t), \tag{15b}$$

where $\Theta(t)$ is a slowly varying amplitude and $\varphi(t)$ the phase defined in (12b). The oscillation energy is consequently

$$\mathcal{E}(t) = \frac{1}{2}K(t)\Theta^2(t) = N\hbar\omega(t), \tag{15c}$$

hence the adiabatic invariant

$$\frac{K(t)}{\omega(t)}\Theta^2(t) = \sqrt{K(t)M(t)}\Theta^2(t) = \text{constant}, \tag{15d}$$

which can be derived as well by trying to satisfy equation (3b) with solution (15b) (the slowly varying envelop approximation to be made is, under the circumstances, equivalent to the SA).

In the next section, we shall show that the above results and considerations can be transposed to the HOn case.

3. The HOn case

3.1. Matricial notations

A n-degree-of-freedom physical system can be regarded as a HOn if there exists a parametrization with dynamical variables $(\theta_1, \theta_2, \ldots, \theta_i, \ldots, \theta_n)$ such that, setting

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_i \\ \vdots \\ \theta_n \end{bmatrix}, \tag{16}$$

its Lagrangian reads

$$L = \frac{1}{2}({}^t\dot{\theta} M \dot{\theta} - {}^t\theta K \theta), \tag{17}$$

where superscript t indicates matricial transposition and M and K stand for the effective mass and effective stiffness matrices. M and K are $n \times n$ symmetrical matrices of "positive type", *i.e.* associated with positive quadratic forms. M and K can be diagonalized. Their eigenvalues are real and positive, associated with orthogonal real eigenvectors. Of course, if a parametric excitation of the HOn is performed, M and K may depend on time.

3.2. Standard, normal and Glauber variables

Let us consider in particular the effective mass matrix M. As for any positive-type symmetrical matrix, there exists an orthogonal matrix P_d diagonalizing M, *i.e.* such that

$$P_d^{-1} M P_d = D = \begin{pmatrix} m_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & m_j & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & m_n \end{pmatrix}, \tag{18}$$

with all the m_j positive. Setting

$$\sqrt{D} = \begin{pmatrix} \sqrt{m_1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \sqrt{m_j} & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \sqrt{m_n} \end{pmatrix}, \tag{19a}$$

we can then write:

$$M = P_d D P_d^{-1} = S^2, \qquad \text{with } S = P_d \sqrt{D} P_d^{-1}. \tag{19b}$$

Since $P_d^{-1} = {}^tP_d$, S is itself a $n \times n$ positive-type symmetrical matrix. The above equation (19b) is known as the Cholewski factorization of matrix M; matrix S can thus be regarded as M's square root. Using matrix S, one can generalize definition (4) in

$$q = S\theta, \tag{20a}$$

where

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_j \\ \vdots \\ q_n \end{bmatrix} \tag{20b}$$

is the generalized set of *standard* variables. In these new variables, Lagrangian L reads

$$\begin{aligned} L &= \frac{1}{2}{}^t\big(\dot{\overbrace{S^{-1}q}}\big)M\big(\dot{\overbrace{S^{-1}q}}\big) - \frac{1}{2}{}^t qS^{-1}KS^{-1}q \\ &= \frac{1}{2}{}^t\dot{q}\dot{q} + {}^t\dot{q}S\dot{\overbrace{S^{-1}}}q - \frac{1}{2}{}^t q\big(\Omega^2 - \dot{\overbrace{S^{-1}}}M\dot{\overbrace{S^{-1}}}\big)q\,, \end{aligned} \tag{21a}$$

with

$$\Omega^2 = S^{-1}KS^{-1}. \tag{21b}$$

Note that the $n \times n$ matrix Ω^2 is symmetrical and associated with a positive quadratic form (*i.e.* of positive type); it is *a priori* time-dependent and it generalizes quantity ω^2 defined in (3c). Note too that (21a) is the generalization of (5a).

Let $\sigma = \frac{\partial L}{\partial \dot{\theta}} = M\dot{\theta}$ (see (17)) be the conjugate momenta of variables θ, and p the conjugate momenta of variables q. We have

$$\begin{aligned} p = \frac{\partial L}{\partial \dot{q}} &= \dot{q} + S\dot{\overbrace{S^{-1}}}q \\ &= \dot{q} - \dot{S}S^{-1}q \quad (\text{since } SS^{-1} = \mathbb{I} = \text{cst.}) \\ &= \dot{\overbrace{S\theta}} - \dot{S}\theta = S\dot{\theta} \\ &= S^{-1}M\dot{\theta} = S^{-1}\sigma\,, \end{aligned} \tag{22}$$

which is just the generalization of (7). Then, performing a Legendre transformation of L, we obtain the hamiltonian H which equally reads

$$H = \frac{1}{2}\big({}^t\sigma M^{-1}\sigma + {}^t\theta K\theta\big) = \frac{1}{2}\big({}^t pp + {}^t qB^2q\big)\,, \tag{23}$$

thus generalizing expressions (6) and (7).

We shall now introduce the *normal* variables of the oscillator. Let P_e be the orthogonal passage matrix that diagonalizes Ω^2. We thus have

$$P_\mathrm{e}^{-1}\Omega^2 P_\mathrm{e} = \Omega_\mathrm{e}^2, \tag{24a}$$

with

$$\Omega_\mathrm{e} = \begin{pmatrix} \omega_{\mathrm{e}1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \omega_{\mathrm{e}m} & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \omega_{\mathrm{e}n} \end{pmatrix}, \tag{24b}$$

where ω_{em} ($m \in [1, n]$) is the (eigen)angular frequency associated with *eigenmode* m, hence the subscript e to avoid any confusion. The normal variables are then defined by

$$q = P_{\mathrm{e}}\varphi, \qquad p = P_{\mathrm{e}}\pi. \tag{25}$$

It is noteworthy that the above variable change is canonical: indices m and m' referring to the normal variables (*i.e.* eigenmodes) and indices j and j' to the standard variables, with understood summation over repeated indices and using $\{q_j, p_{j'}\} = \delta_{jj'}$, one calculates the Poisson brackets

$$\begin{aligned}\{\varphi_m, \pi_{m'}\} &= \{(P_{\mathrm{e}}^{-1})_{mj}q_j, (P_{\mathrm{e}}^{-1})_{m'j'}p_{j'}\} \\ &= (P_{\mathrm{e}}^{-1})_{mj}(P_{\mathrm{e}}^{-1})_{m'j'}\delta_{jj'} \\ &= (P_{\mathrm{e}}^{-1})_{mj}(P_{\mathrm{e}})_{jm'} \quad (\text{since } P_{\mathrm{e}}^{-1} = {}^tP_{\mathrm{e}}) \\ &= \delta_{mm'}. \qquad\qquad \text{Q.E.D.}\end{aligned} \tag{26}$$

Using the variable change (25) to rewrite Hamiltonian (23), one gets

$$H = \frac{1}{2}\left({}^t\pi\pi + {}^t\varphi\Omega_{\mathrm{e}}^2\varphi\right). \tag{27}$$

It is remarkable that, in the above expression, H reads as the sum of n independent HO1's Hamiltonians :

$$H = \sum_{m=1}^{n} \frac{1}{2}\left(\pi_m^2 + \omega_{\mathrm{e}m}^2\varphi_m^2\right) = \sum_{m=1}^{n} H_m. \tag{28}$$

Then, we can define the Glauber variables of the *eigenmodes* as in (8) by

$$\alpha_{\mathrm{e}m} = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\omega_{\mathrm{e}m}}\varphi_{\mathrm{e}m} + \frac{\mathrm{i}}{\sqrt{\omega_{\mathrm{e}m}}}\pi_{\mathrm{e}m}\right), \tag{29a}$$

or equivalently in matricial form, with Ω_{e} displayed in (24b) and with

$$\alpha_{\mathrm{e}} = \begin{bmatrix} \alpha_{\mathrm{e}1} \\ \vdots \\ \alpha_{\mathrm{e}m} \\ \vdots \\ \alpha_{\mathrm{e}n} \end{bmatrix}, \tag{29b}$$

by

$$\begin{aligned}\alpha_{\mathrm{e}} &= \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\Omega_{\mathrm{e}}}\varphi + \frac{\mathrm{i}}{\sqrt{\Omega_{\mathrm{e}}}}\pi\right) \\ &= \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\Omega_{\mathrm{e}}}P_{\mathrm{e}}^{-1}S\theta + \frac{\mathrm{i}}{\sqrt{\Omega_{\mathrm{e}}}}P_{\mathrm{e}}^{-1}S^{-1}\sigma\right).\end{aligned} \tag{29c}$$

We finally get the following expression of the Hamiltonian:

$$H = \sum_m \hbar\omega_{\mathrm{e}m}|\alpha_{\mathrm{e}m}|^2 = {}^t\alpha_{\mathrm{e}}^*\hbar\Omega_{\mathrm{e}}\alpha_{\mathrm{e}} = \alpha_{\mathrm{e}}^\dagger\hbar\Omega_{\mathrm{e}}\alpha_{\mathrm{e}}, \tag{30}$$

superscript † standing for hermitic conjugation (h.c.).

Next, using (26) and (29a), one easily derives

$$\{\alpha_{\mathrm{e}m}, \alpha_{\mathrm{e}m'}\} = \frac{1}{\mathrm{i}\hbar}\delta_{mm'} \tag{31a}$$

(which generalizes (9b)), so that the motion equations read, in matricial form

$$\dot{\alpha_{\mathrm{e}}} = \{\alpha_{\mathrm{e}}, H\} + \frac{\partial \alpha_{\mathrm{e}}}{\partial t} = -\,\mathrm{i}\Omega_{\mathrm{e}}\alpha_{\mathrm{e}} + \frac{\partial \alpha_{\mathrm{e}}}{\partial t}. \tag{31b}$$

At this step of the calculation, we are faced with an interesting feature. Matrix Ω_{e} is, by construction, diagonal. Therefore, in any *free* motion, the time-evolutions of the eigenmodes Glauber variables $\alpha_{\mathrm{e}m}$ are uncoupled. But things are not that simple in a *parametrically excited* motion, because the $\frac{\partial \alpha_{\mathrm{e}}}{\partial t}$ term in the right-hand side of the above equation couples the $\alpha_{\mathrm{e}m}$ with one another. In the n-degree-of-freedom case, the calculation of this term requires some care. This calculation may be omitted in a first reading of this paper, with a jump to equation (35).

The time-derivation of (29c) yields indeed

$$\frac{\partial \alpha_{\mathrm{e}}}{\partial t} = \frac{1}{\sqrt{2\hbar}} \left\{ \frac{1}{2}\frac{\dot{\Omega}_{\mathrm{e}}}{\Omega_{\mathrm{e}}}\Big(\sqrt{\Omega_{\mathrm{e}}}\varphi - \frac{\mathrm{i}}{\sqrt{\Omega_{\mathrm{e}}}}\pi\Big) + \right.$$
$$\left. \Big(\sqrt{\Omega_{\mathrm{e}}}\dot{\overline{P_e^{-1}}}S\theta + \frac{\mathrm{i}}{\sqrt{\Omega_{\mathrm{e}}}}\dot{\overline{P_e^{-1}}}S^{-1}\sigma\Big) + \Big(\sqrt{\Omega_{\mathrm{e}}}P_e^{-1}\dot{S}\theta + \frac{\mathrm{i}}{\sqrt{\Omega_{\mathrm{e}}}}P_e^{-1}\dot{\overline{S^{-1}}}\sigma\Big)\right\}. \tag{32a}$$

Since $SS^{-1} = \mathbb{I} \rightsquigarrow \dot{\overline{S^{-1}}} = -\,S^{-1}\dot{S}S^{-1}$ and $P_e^{-1}P_e = \mathbb{I} \rightsquigarrow \dot{\overline{P_e^{-1}}}P_e = -\,P_e^{-1}\dot{P}_e$, one finds, all calculations carried out:

$$\frac{\partial \alpha_{\mathrm{e}}}{\partial t} = \frac{\dot{\Omega}_{\mathrm{e}}}{2\Omega_{\mathrm{e}}}\alpha_{\mathrm{e}}^* + \frac{1}{\sqrt{2\hbar}}\left\{\sqrt{\Omega_{\mathrm{e}}}P_e^{-1}\big(\dot{S}S^{-1}P_e - \dot{P}_e\big)\varphi \right.$$
$$\left. - \frac{\mathrm{i}}{\sqrt{\Omega_{\mathrm{e}}}}P_e^{-1}\big(S^{-1}\dot{S}P_e + \dot{P}_e\big)\pi\right\}. \tag{32b}$$

Now, using (29c) to substitute variables α, α^* for φ, π in the above expression and setting

$$T = \sqrt{\Omega_{\mathrm{e}}}P_e^{-1}\big(\dot{S}S^{-1}P_e - \dot{P}_e\big)\frac{1}{\sqrt{\Omega_{\mathrm{e}}}}, \tag{33}$$

we finally obtain

$$\frac{\partial \alpha_{\mathrm{e}}}{\partial t} = \frac{1}{2}\frac{\dot{\Omega}_{\mathrm{e}}}{\Omega_{\mathrm{e}}}\alpha_{\mathrm{e}}^* + \frac{1}{2}(T - {}^tT)\alpha_{\mathrm{e}} + \frac{1}{2}(T + {}^tT)\alpha_{\mathrm{e}}^* = \mathcal{A}\alpha_{\mathrm{e}} + \mathcal{S}\alpha_{\mathrm{e}}^*, \tag{34}$$

where $\mathcal{A} = \frac{1}{2}(T - {}^tT)$ is an antisymmetrical matrix and $\mathcal{S} = \frac{1}{2}\big(\frac{\dot{\Omega}_{\mathrm{e}}}{\Omega_{\mathrm{e}}} + T + {}^tT\big)$ a symmetrical one. Hence the motion equation (see (31b))

$$\dot{\alpha_{\mathrm{e}}} = -\,\mathrm{i}\Omega_{\mathrm{e}}\alpha_{\mathrm{e}} + \mathcal{A}\alpha_{\mathrm{e}} + \mathcal{S}\alpha_{\mathrm{e}}^*, \tag{35}$$

which should be regarded as the generalization of (11a). As announced before, matrices $\mathcal{A}$ and $\mathcal{S}$ are non diagonal. The above equation (35) is in fact the corner-stone of the derivation of the HOn's adiabatic invariant, as will appear below. Exactly as we did in section 2, we can look for a solution of (35) of the form

$$\alpha_{\mathrm{e}}(t) = \mathrm{e}^{-\mathrm{i}\phi_{\mathrm{e}}(t)}\ A_{\mathrm{e}}(t), \tag{36a}$$

where the column-vector $A_{\mathrm{e}}(t) = \begin{bmatrix} A_{\mathrm{e}1}(t) \\ \vdots \\ A_{\mathrm{e}m}(t) \\ \vdots \\ A_{\mathrm{e}n}(t) \end{bmatrix}$ is a complex amplitude and $\phi_{\mathrm{e}}(t)$ the phase matrix defined as

$$\phi_{\mathrm{e}}(t) = \int_{-\infty}^{t} \mathrm{d}t' \Omega_{\mathrm{e}}(t') = \begin{pmatrix} \varphi_{\mathrm{e}1}(t) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \varphi_{\mathrm{e}m}(t) & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \varphi_{\mathrm{e}n}(t) \end{pmatrix}, \tag{36b}$$

with

$$\varphi_{\mathrm{e}m}(t) = \int_{-\infty}^{t} \mathrm{d}t' \omega_{\mathrm{e}m}(t'). \tag{36c}$$

We then get the set of motion equations (equivalent to (35))

$$\dot{A}_{\mathrm{e}} = \mathrm{e}^{\mathrm{i}\phi_{\mathrm{e}}} \mathcal{A} \mathrm{e}^{-\mathrm{i}\phi_{\mathrm{e}}} A_{\mathrm{e}} + \mathrm{e}^{\mathrm{i}\phi_{\mathrm{e}}} \mathcal{S} \mathrm{e}^{\mathrm{i}\phi_{\mathrm{e}}} A_{\mathrm{e}}^{*}, \tag{37a}$$

or equivalently

$$\dot{A}_{\mathrm{e}m} = \sum_{m'=1}^{n} \left\{ \mathrm{e}^{\mathrm{i}(\varphi_{\mathrm{e}m} - \varphi_{\mathrm{e}m'})} \mathcal{A}_{mm'} A_{\mathrm{e}m'} + \mathrm{e}^{\mathrm{i}(\varphi_{\mathrm{e}m} + \varphi_{\mathrm{e}m'})} \mathcal{S}_{mm'} A_{\mathrm{e}m'}^{*} \right\}, \tag{37b}$$

which generalizes (12c).

3.3. *Adiabatic invariant of the HOn*

If the parametric excitation of the HOn (*i.e.* the time variation of whichever oscillator's parameter, M or K) is *slow* with regard to its free dynamics, then matrices $\mathcal{A}$ and $\mathcal{S}$ are *small* compared to Ω_{e} and the complex amplitude A_{e} varies slowly with respect to the phase matrix ϕ_{e}. Consequently, if all nonzero Fourier components of any matrix elements $\mathcal{S}_{mm'}$ correspond to angular frequencies much smaller than $\omega_{\mathrm{e}m} + \omega_{\mathrm{e}m'}$, then the contribution of the second term (with $A_{\mathrm{e}m'}^{*}$) in the right-hand side of equation (37b) is averaged out by the oscillations of the phase factor $\mathrm{e}^{\mathrm{i}\phi_{\mathrm{e}}}$. This term is neglected at the secular approximation, so that (35) is simplified in

$$\dot{\alpha}_{\mathrm{e}} = -\mathrm{i}\Omega_{\mathrm{e}}\alpha_{\mathrm{e}} + \mathcal{A}\alpha_{\mathrm{e}} \quad \text{at the SA.} \tag{38}$$

From the above equation, it is then easy to build the HOn's adiabatic invariant. Introducing the total *eigen* quanta number

$$N_{\mathrm{e}} = \sum_{m=1}^{n} |\alpha_{\mathrm{e}m}|^2 = {}^{t}\alpha_{\mathrm{e}}^{*}\alpha_{\mathrm{e}} = \alpha_{\mathrm{e}}^{\dagger}\alpha_{\mathrm{e}}, \tag{39a}$$

we easily get at the SA

$$\dot{N}_{\mathrm{e}} = \alpha_{\mathrm{e}}^{\dagger}\dot{\alpha}_{\mathrm{e}} + \mathrm{h.c.} = (-\mathrm{i}\alpha_{\mathrm{e}}^{\dagger}\Omega_{\mathrm{e}}\alpha_{\mathrm{e}} + \alpha_{\mathrm{e}}^{\dagger}\mathcal{A}\alpha_{\mathrm{e}}) + \mathrm{h.c.} = 0. \tag{39b}$$

The above result means that the HOn's adiabatic invariant is simply N_{e}.

This result, which generalizes subsection 2.3's conclusion, is, for aught we know, never taught. It should nevertheless be kept in mind that, if the *total* eigen quanta number N_e undergoes no secular evolution under an adiabatic parametric excitation, the *partial* eigen quanta numbers $|\alpha_{\mathrm{e}m}|^2$ may do. From (37*b*), we have indeed at the SA

$$\frac{\mathrm{d}|\alpha_{\mathrm{e}m}|^2}{\mathrm{d}t} = \frac{\mathrm{d}|A_{\mathrm{e}m}|^2}{\mathrm{d}t} = \sum_{m'=1}^{n} \Big(\mathcal{A}_{mm'}\mathrm{e}^{\mathrm{i}(\varphi_{\mathrm{e}m}-\varphi_{\mathrm{e}m'})}A^*_{\mathrm{e}m}A_{\mathrm{e}m'} + \mathrm{c.c.}\Big). \quad (40)$$

In the sum (over m') in the right-hand side of the above equation, any mode m' such that the Bohr angular frequency $\omega_{mm'} = \omega_{\mathrm{e}m} - \omega_{\mathrm{e}m'}$ corresponds to a nonzero Fourier component of the matrix element $\mathcal{A}_{mm'}$ will give a secular contribution to the time evolution of $|\alpha_{\mathrm{e}m}|^2$. This is exactly the classical equivalent to the so-called quantum resonant transition.

We would now end this discussion with a last remark. Matrix Ω^2 introduced in (21*b*) if of positive type. Its diagonal terms are therefore positive. Let us set

$$\big(\Omega^2\big)_{jj} = \omega_j^2 \quad (\omega_j > 0). \quad (41a)$$

With the above notation, any element of matrix Ω^2 can read

$$\big(\Omega^2\big)_{jj'} = \kappa_{jj'}\omega_j\omega_{j'}, \quad (41b)$$

with the dimensionless coupling coefficients $\kappa_{jj'}$ satisfying (positive type of Ω^2 again)

$$-1 \leqslant \kappa_{jj'} = \kappa_{j'j} \leqslant 1. \quad (41c)$$

It is possible to define the set of Glauber variables associated with the *standard* variables

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_n \end{bmatrix}, \quad (42a)$$

with

$$\alpha_j = \frac{1}{\sqrt{2\hbar}}\Big(\sqrt{\omega}_j q_j + \frac{\mathrm{i}}{\sqrt{\omega}_j}p_j\Big). \quad (42b)$$

We would nevertheless recall that, despite the simple relation (25) between the standard and normal variables, namely $q = P_\mathrm{e}\varphi$ and $p = P_\mathrm{e}\pi$, the relation between α and α_e is not simple. As already emphasized in the 2-degree-of-freedom case in a foregoing paper (see formulas (20) through (21b) in [9]), one has indeed

$$\alpha = \frac{1}{2}(Q_\mathrm{e} + {}^tQ_\mathrm{e}^{-1})\alpha_\mathrm{e} + \frac{1}{2}(Q_\mathrm{e} - {}^tQ_\mathrm{e}^{-1})\alpha_\mathrm{e}^*, \quad (43a)$$

where Q_e is the $n \times n$ marix defined as

$$Q_{\rm e} = \begin{pmatrix} \sqrt{\omega_1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \sqrt{\omega_j} & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \sqrt{\omega_n} \end{pmatrix} \times P_{\rm e} \times \begin{pmatrix} 1/\sqrt{\omega_{\rm e1}} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & 1/\sqrt{\omega_{{\rm e}m}} & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1/\sqrt{\omega_{{\rm e}n}} \end{pmatrix}. \quad (43b)$$

As in the $n = 2$ case, due to the presence of the different $\sqrt{\omega}$ factors in the above definition, matrix $Q_{\rm e}$ is neither equal to $P_{\rm e}$ nor even orthogonal. It is then easy to check that number $N = |\alpha_1|^2 + \cdots + |\alpha_j|^2 + \cdots + |\alpha_n|^2$ is *not* equal to $N_{\rm e}$, and is consequently *not* conserved in the course of an adiabatically excited motion of the HOn.

4. Conclusion

In this study, we have considered the most general n-degree-of-freedom harmonic oscillator (HOn), as well as the most general parametrical excitation it might undergo. We have introduced its standard variables q and the Glauber variables $\alpha_{\rm e}$ associated with its eigenmodes. In the case of a parametrical excitation of the HOn, we have calculated the *explicit* time variation of $\alpha_{\rm e}$ and shown that it brings in phase-conjugated term, *i.e.* a term proportional to the complex-conjugate Glauber variable $\alpha_{\rm e}^*$. The derivation is easy for $n = 1$, a bit more technical for $n > 1$. If the parametrical excitation is adiabatic (in the Ehrenfest sense), it is shown that the phase-conjugated term can be neglected, for it brings no *secular* contribution to the time evolution of $\alpha_{\rm e}$. Although transitions between eigenmodes may occur if the excitation involves Fourier components in speaking term with the Bohr angular frequencies $\omega_{mm'}$ of the HOn, the total (eigen) quanta number $N_{\rm e} = \alpha_{\rm e}^\dagger \alpha_{\rm e}$, that is the sum of the squared moduli $|\alpha_{{\rm e}m}|^2$ of the different eigenmodes' Glauber variables, is a constant of the motion. We have thus established that the HOn's adiabatic invariant is simply $N_{\rm e}$.

It is our opinion that this result is so simple that it could be at least mentioned in undergraduate level textbooks or lectures about the harmonic oscillator, which suggests of course that the Glauber variables be systematically introduced, even without any further quantum intentions.

References

[1] Lev Davidovich Landau and Evgeny Mikhailovich Lifshitz. *The classical field theory.* Pergamon, Oxford, 1962.

[2] Valentin Leroy, Jean-Claude Bacri, Thierry Hocquet, and Martin Devaud. A hamiltonian approach of the parametric excitation. *Eur. J. Phys.*, 27:469–483, 2006.

[3] William B Case and Mark A Swanson. The pumping of a swing from the seated position. *Am. J. Phys.*, 58(5):463–467, 1990.

[4] William B Case. The pumping of a swing from the standing position. *Am. J. Phys.*, 64(3):215–220, 1996.

[5] Roy J. Glauber. Coherent and incoherent states of the radiation field. *Phys. Rev.*, 131(2766-2788), 1963.

[6] Roy J. Glauber. Optical coherence and photon statistics. In *Quantum optics and electronics*. New York, Gordon and Breach, 1965.

[7] Claude Cohen-Tannoudji, Bernard Diu, and Franck Laloë. *Quantum Mechanics*. New York, John Wiley, 1977.

[8] Eugen Merzbacher. *Quantum mechanics*. John Wiley and Sons, 1998.

[9] Valentin Leroy, Jean-Claude Bacri, Thierry Hocquet, and Martin Devaud. Simulating a one-half spin with two coupled pendula: the free larmor precession. *Eur. J. Phys.*, 27:1363–1383, 2006.