Refinment of the "up to a constant" ordering using contructive co-immunity and alike. Application to the Min/Max hierarchy of Kolmogorov complexities.

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January 2, 2008

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Abstract

We introduce orderings $\ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ between total functions $f, g : \mathbb{N} \to \mathbb{N}$ which refine the pointwise "up to a constant" ordering $\leq_{\mathrm{ct}}$ and also insure that $f(x)$ is often much less than $g(x)$. With such $\ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$'s, we prove a strong hierarchy theorem for Kolmogorov complexities obtained with jump oracles and/or Max or Min of partial recursive functions.

We introduce a notion of *second order* conditional Kolmogorov complexity which yields a uniform bound for the "up to a constant" comparisons involved in the hierarchy theorem.

1 Introduction

1.1 Comparing total functions $\mathbb{N} \to \mathbb{N}$

Notation 1.1. Equality, inequality and strict inequality up to a constant between total functions $I \to \mathbb{N}$, where I is any set, are denoted as follows:

$$\begin{array}{rcl} f \ \leq_{\mathrm{ct}} \ g & \Leftrightarrow & \exists c \in \mathbb{N}\ \forall x \in I\ f(x) \leq g(x) + c \\ f \ =_{\mathrm{ct}} \ g & \Leftrightarrow & f \leq_{\mathrm{ct}} g \ \wedge \ g \leq_{\mathrm{ct}} f \\ & \Leftrightarrow & \exists c \in \mathbb{N}\ \forall x \in I\ |f(x) - g(x)| \leq c \\ f \ <_{\mathrm{ct}} \ g & \Leftrightarrow & f \leq_{\mathrm{ct}} g \ \wedge \ \neg(g \leq_{\mathrm{ct}} f) \\ & \Leftrightarrow & f \leq_{\mathrm{ct}} g \ \wedge \ \forall c \in \mathbb{N}\ \exists x \in I\ g(x) > f(x) + c \end{array}$$

Total functions $f, g : \mathbb{N} \to \mathbb{N}$ can be compared in diverse ways. The simplest one is pointwise comparison via the partial ordering relation $\forall x\ f(x) < g(x)$. In case functions are considered up to an additive constant, for instance with Kolmogorov complexity, pointwise comparison has to be replaced by the $\leq_{\mathrm{ct}}$ preordering or the $<_{\mathrm{ct}}$ ordering.

Observe that the $<_{\mathrm{ct}}$ ordering is an infinite intersection:

$$f <_{\mathrm{ct}} g \;\Leftrightarrow\; f \leq_{\mathrm{ct}} g \;\wedge\; \forall c \in \mathbb{N}\; f <_{io} g - c$$

where $<_{io}$ (io stands for "infinitely often") is the non transitive relation

$$f <_{io} g \;\Leftrightarrow\; \{x : f(x) < g(x)\} \text{ is infinite}$$

Relation $<_{io}$ can be much refined via localization: instead of merely demanding $\{x : f(x) < g(x)\}$ to be infinite, one can ask it to have infinite intersection with every infinite set in a family $\mathcal{C}$ of sets.
In case $\mathcal{C}$ is the family of all subsets of $\mathbb{N}$, this gives the relation

$$\{x : f(x) < g(x)\} \text{ is cofinite}$$

which is a partial ordering relation.
In case $\mathcal{C}$ is the family of r.e. sets, this is related to the idea of coimmunity. An instance of such a relation appears in a classical result about Kolmogorov complexity K, due to Barzdins (cf. [9] Thm.2.7.1 iii, p.167, or Zvonkin & Levin, [17] p.92.), which states that, for any total recursive function ϕ which tends to $+\infty$, the set $\{x : K(x) < \phi(x)\}$ meets every infinite r.e. set.

In practice, for simple classes $\mathcal{C}$, an infinite subset of $X \cap \{x : f(x) < g(x)\}$, for X infinite in $\mathcal{C}$, can always be found in a not too complex class $\mathcal{D}$. Which leads to consider the relation $\mathrm{OftLess}^{\mathcal{C},\mathcal{D}}$ such that

$$\begin{aligned} f \;\mathrm{OftLess}^{\mathcal{C},\mathcal{D}}\; g \quad \Leftrightarrow \quad & \forall X \in \mathcal{C}\; \exists Y \in \mathcal{D}\; (X \text{ is infinite } \Rightarrow \\ & \qquad Y \text{ is infinite } \wedge\; Y \subseteq \{x : f(x) < g(x)\}) \end{aligned}$$

If $\mathcal{C} = \mathcal{D}$ then this relation is transitive, hence is a strict partial ordering. However, in case $\mathcal{C} \neq \mathcal{D}$, transitivity may fail (for instance, a counterexample is obtained via Lemma 8.10).

The key observation for the paper is as follows:

For any $\mathcal{C}, \mathcal{D}$, the relation $f \leq_{\mathrm{ct}} g \wedge \forall c\, (f \;\mathrm{OftLess}^{\mathcal{C},\mathcal{D}}\; g - c)$ is transitive, hence is a partial strict ordering refining $<_{\mathrm{ct}}$. In other words, considering $\mathrm{OftLess}^{\mathcal{C},\mathcal{D}}$ up to any constant and mixing it with $\leq_{\mathrm{ct}}$ always leads to an ordering.

If $\mathcal{F}$ is a family of total functions $\mathbb{N} \to \mathbb{N}$ which tend to $+\infty$ and $\mathcal{F}$ is closed by translations (i.e. $\phi \in \mathcal{F}$ implies $\max(0, \phi - c) \in \mathcal{F}$), then the above observation also applies to the relation $f \leq_{\mathrm{ct}} g \wedge \forall \phi \in \mathcal{F}\; f \;\mathrm{OftLess}^{\mathcal{C},\mathcal{D}}\; \phi \circ g$, i.e. the relation

$$\begin{aligned} & f \leq_{\mathrm{ct}} g \;\wedge\; \forall \phi \in \mathcal{F}\; \forall X \in \mathcal{C}\; \exists Y \in \mathcal{D} \\ & \qquad (X \text{ is infinite } \Rightarrow\; Y \text{ is infinite } \wedge\; Y \subseteq \{x : f(x) < g(x)\}) \end{aligned}$$

which is also a partial strict ordering refining the ordering $<_{\mathrm{ct}}$.

Enriching this relation with the requirement that a code for an infinite subset Y of $X \cap \{x : f(x) < \phi(g(x))\}$ can be effectively computed from codes for ϕ and X, we get the relation $\mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ which is the main concern of this paper.

In §2 we review some needed elements of oracular computability. This is done in terms of partial computable functionals so as to get uniformity in the oracle.

In §3 we recall Xiang Li's notion of constructive immunity and introduce the related notions of $(\mathcal{C}, \mathcal{D})$-density and constructive density.

In §4 we introduce the relation $\mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ and its variant $\mathrm{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}$ (where only total monotone increasing functions in $\mathcal{F}$ are considered) and prove that their intersections $\ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ and $\ll_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}$ with $\leq_{\mathrm{ct}}$ are strict orderings refining the ordering $<_{\mathrm{ct}}$.

1.2 Second order Kolmogorov complexity

In relation with the partial computable functional approach to oracular computability (cf. §2), we develop in §5 a *functional version* $\mathcal{K}(x \,||\, A)$ *of Kolmogorov complexity.* This amounts to a simple, seemingly unnoticed, fact:
Oracular Kolmogorov complexity K^A *can be obtained by instantiating to* A *the second order parameter of a variant of conditional Kolmogorov complexity in which the condition is a set of integers rather than an integer. The oracle is thus viewed as a second order conditional parameter.*
The usual proof of the invariance theorem goes through. This second-order conditional complexity allows for a uniform choice of oracular Kolmogorov complexities (this is detailed in §7) since, for any A,

$$\mathcal{K}(\mathtt{x} \,||\, A) =_{\mathrm{ct}} K^A(\mathtt{x})$$

i.e. $\forall A\ \exists c\ \forall \mathtt{x}\ |K^A(\mathtt{x}) - \mathcal{K}(\mathtt{x} \,||\, A)| \leq c$.

A typical benefit of the functional version of K is as follows. Usual properties with K involving equality or inequality "up to a constant" go through oracles. Let c_A be the involved constant for the oracle A version. For a single equality or inequality involving K^A, it may be possible to modify K^A (by an additive constant) so that $c_A = 0$. But this is no more possible for several equalities or inequalities since the needed modifications of K^A may – a priori – be incompatible.
Thus, for a system of equalities or inequalities, there is no a priori A-computable bound of the involved constant c_A for the oracle A version. However, in case (which is also usual) such properties also go through the functional version, the constant bound involved in the functional version is valid for any oracle. In other words, *whereas the oracular version a priori allows no* A*-computable bound of the constant, the functional version does allow a constant bound.*
This fact is applied in §8.6 to get sharper results.

In §6 we recall the variants $K_{\max}, K_{\min}$ of Kolmogorov complexity introduced in our paper [5] and we extend them to functional versions. The precise relation between such functional versions and the oracular $K_{\max}, K_{\min}$ is detailed in §7.

1.3 A strong hierarchy theorem for Kolmogorov complexities

In §8.1 we prove of a version of Barzdins' result cited in §1.1 (cf. also §3.1) with as much effectivity as possible which involves an ordering relation introduced in §4 and can be stated as $K \ll_{\mathrm{PR}}^{\Sigma_1^0,\Sigma_1^0} \log$. Also, the functional versions of Kolmogorov complexity and the functional approach to oracular computability allow to get a functional version of this result, hence to get effectivity relative to the oracle.

We extend this result in §8.2, 8.3, 8.4 and prove that $K, K_{\max}, K_{\min}$ can be compared via the above OftLess and $\ll$ relations, with more complex classes $\mathcal{C}, \mathcal{D}$, namely $\mathcal{C} = \Sigma_1^0 \cup \Pi_1^0$ and $\mathcal{D} = \exists^{<\phi}(\Sigma_1^0 \wedge \Pi_1^0)$ or the variants in which Π_1^0 is constrained with a "recursively bounded growth" condition (cf. Def.8.7). Also, the class $\mathcal{F}$ can be extended to Min_{PR}, i.e. the class of infima of partial recursive sequences of functions.

The above class $\mathcal{D}$ is a subclass of Δ_2^0 which can be obtained via bounded existential quantification over boolean combinations of Σ_1^0 relations. In §8.5, we show that such syntactical complexities naturally appear when comparing $K, K_{\max}, K_{\min}$.

Finally, in §8.6 we prove the main application of the $\ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ and $\ll_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}$ orderings, which is a strong hierarchy theorem for the Kolmogorov complexities $K, K_{\max}, K_{\min}$ and their oracular versions using the successive jumps.

2 Partial computable functionals and oracular recursion theory

2.1 Notations

Notation 2.1.
1. *[Basic sets]* $\mathbb{X}, \mathbb{Y}$ denote products of non empty finite families of spaces of the form $\mathbb{N}$ or $\mathbb{Z}$ or Σ^* where Σ is some finite alphabet.

2.*[Partial recursive functions]* Let $A \subseteq \mathbb{N}$. We denote $PR^{\mathbb{X}\to\mathbb{Y}}$ (resp. $PR^{\mathbb{X}\to\mathbb{Y},A}$) the family of partial recursive (resp. A-recursive) functions between basic sets $\mathbb{X}$ and $\mathbb{Y}$.

3.*[Bijections between basic spaces]* For any basic spaces $\mathbb{X}, \mathbb{Y}$ and $\mathbb{Z}$ we fix some particular total recursive bijection from $\mathbb{X} \times \mathbb{Y}$ to $\mathbb{Z}$ and denote $\langle \mathtt{x}, \mathtt{y} \rangle_{\mathbb{X}\times\mathbb{Y},\mathbb{Z}}$, or simply $\langle \mathtt{x}, \mathtt{y} \rangle$, the image in $\mathbb{Z}$ of the pair $(\mathtt{x}, \mathtt{y})$.

2.2 Some classical results from recursion theory

We shall use the following classical results from computability theory (cf. Odifreddi's book [11] p.372–374, 288–292, or Shoenfield's book [15]).

Proposition 2.2.
1. (Post's Theorem, 1948 [13]) *A set is Σ^0_{n+1} (resp. Δ^0_{n+1}) if and only if it is recursively enumerable (r.e.) (resp. recursive) in oracle $\emptyset^{(n)}$.*

2. (Post, 1944 [12]) *For any oracle A, every infinite A-r.e. set X contains an infinite set Y which is recursive in A. Moreover, one can recursively go from an r.e. code for X to r.e. codes for such a Y and its complement.*

In particular, every infinite Σ^0_{n+1} set X contains an infinite Δ^0_{n+1} subset Y. Also, one can recursively go from a Σ^0_{n+1}-code for X to Σ^0_{n+1}-codes for such a Y and its complement.

3. *Recall that an A-r.e. set $X \subset \mathbb{N}$ is maximal if it is coinfinite and for any A-r.e. set $Y \supseteq X$ either $\mathbb{N} \setminus Y$ is finite or $Y \setminus X$ is finite.*
(Friedberg, 1958 [6]) *There exists maximal A-r.e. sets.*

Remark 2.3.
1. Since every Π^0_n set is Σ^0_{n+1}, point 2 of the above proposition yields that every infinite Π^0_n set contains an infinite Δ^0_{n+1} subset. This cannot be improved: the complement of any maximal recursively enumerable set is an infinite Π^0_1 set which does not contain any infinite recursive set.

2. Any total function ψ with graph in Σ^0_n is in fact $\emptyset^{(n-1)}$-recursive and has graph in Δ^0_n since $y \neq \psi(x) \;\Leftrightarrow\; \exists z \neq y \; z = \psi(x)$.

2.3 Partial computable functionals

Def.2.4 is classical, cf. Rogers [14] p.361, or Odifreddi [11] p.178.

Definition 2.4. A (partial) functional $\mathcal{F} : \mathbb{X} \times P(\mathbb{N}) \to \mathbb{Y}$ is partial computable if there exists an oracle Turing machine $\mathcal{M}$ such that, given $A \in P(\mathbb{N})$ as oracle and $\mathtt{x} \in \mathbb{X}$ as input,
- $\mathcal{M}$ halts and accepts if and only if $\mathcal{F}(A, \mathtt{x})$ is defined,
- if $\mathcal{M}$ halts and accepts then its output is $\mathcal{F}(A, \mathtt{x})$.

The family of partial computable functionals $\mathbb{X} \times P(\mathbb{N}) \to \mathbb{Y}$ is denoted $PC^{\mathbb{X} \times P(\mathbb{N}) \to \mathbb{Y}}$.

The notion of acceptable enumeration of partial recursive functions (cf. Rogers [14] Ex. 2.10 p.41, or Odifrreddi [11], p.215) extends to functionals.

Definition 2.5. We denote $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ some basic sets (cf. Notation 2.1).

1. An enumeration $(\Phi_i)_{i \in \mathbb{N}}$ of partial computable functionals $\mathbb{X} \times P(\mathbb{N}) \to \mathbb{Y}$ is *acceptable* if

i. $(i, \mathtt{x}, A) \mapsto \Phi_i(\mathtt{x}, A)$ is a partial computable functional.

ii. Every partial computable functional $\mathbb{X} \times P(\mathbb{N}) \to \mathbb{Y}$ is enumerated:

$$\forall \Psi \in PC^{\mathbb{X} \times P(\mathbb{N}) \to \mathbb{Y}} \ \exists i \ \Phi_i = \Psi$$

iii. the parametrization (also called s-m-n) property holds: for every basic set $\mathbb{Z}$, there exists a total recursive function $s^{\mathbb{Z}}_{\mathbb{X}} : \mathbb{N} \times \mathbb{Z} \to \mathbb{N}$ such that

$$\forall i \ \forall \mathtt{z} \in \mathbb{Z} \ \forall \mathtt{x} \in \mathbb{X} \ \forall A \subseteq \mathbb{N} \quad \Phi_i(\langle \mathtt{z}, \mathtt{x} \rangle, A) = \Phi_{s^{\mathbb{Z}}_{\mathbb{X}}(i,\mathtt{z})}(\mathtt{x}, A)$$

where $\langle \mathtt{z}, \mathtt{x} \rangle$ is the image of the pair $(\mathtt{z}, \mathtt{x})$ by some fixed total recursive bijection $\mathbb{Z} \times \mathbb{X} \to \mathbb{X}$ (cf. Notation 2.1).

2. An enumeration $(\mathcal{W}_i)_{i \in \mathbb{N}}$ of Σ^0_1 subsets of $\mathbb{X} \times P(\mathbb{N})$ is *acceptable* if there exists an *acceptable* enumeration $(\Phi_i)_{i \in \mathbb{N}}$ of partial recursive functionals such that $\mathcal{W}_i$ is the domain of Φ_i.
In particular, $(\mathcal{W}_i)_{i \in \mathbb{N}}$ is Σ^0_1 as a subset of $\mathbb{N} \times \mathbb{X} \times P(\mathbb{N})$.

Proposition 2.6. *There exists an acceptable enumeration of partial computable functionals* $\mathbb{X} \times P(\mathbb{N}) \to \mathbb{Y}$.

2.4 Uniform relativization

When dealing with oracles A, it is often possible to get results involving *recursive* transfer functions rather than A-recursive ones. To do so, we must consider enumerations of A-r.e. sets and partial A-recursive functions which are obtained from enumerations of *partial computable functionals* by fixing the second order argument A. Such enumerations will be called *uniform enumerations.*
This amounts to consider relative computability as a concept dependent on the prior notion of partial computable functional, though, historically, relative computability came first, cf. Hinman's book [7] 5.15 p.68.

Proposition 2.7.
Let $(\Phi_i)_{i \in \mathbb{N}}$ *be an acceptable enumeration of partial computable functionals* $\mathbb{X} \times P(\mathbb{N}) \to \mathbb{Y}$.
For $A \subseteq \mathbb{N}$*, define* $\varphi^A_i : \mathbb{X} \to \mathbb{Y}$ *and* $W^A_i \subseteq \mathbb{X}$ *from* Φ_i *and* $\mathcal{W}_i$ *by fixing the second order argument as follows:*

$$\varphi^A_i(\mathtt{x}) \ = \ \Phi_i(\mathtt{x}, A) \qquad W^A_i \ = \ domain(\varphi^A_i) \ = \ \{\mathtt{x} : (\mathtt{x}, A) \in \mathcal{W}_i\}$$

Then the sequences $(\varphi^A_i)_{i \in \mathbb{N}}$ *and* $(W^A_i)_{i \in \mathbb{N}}$ *are acceptable enumerations of the family* $PR^{\mathbb{X} \to \mathbb{Y}, A}$*) of partial* A*-recursive functions* $\mathbb{X} \to \mathbb{Y}$ *and that of* A*-r.e. subsets of* $\mathbb{X}$.
Such acceptable enumerations are called uniform enumerations.

Rogers' theorem (cf. Odifreddi [11] p.219) extends to partial computable funtionals, hence to uniform enumerations.

Theorem 2.8.
1. (Rogers' theorem) *If* $(\Psi_i)_{i\in\mathbb{N}}$ *and* $(\Phi_i)_{i\in\mathbb{N}}$ *are both acceptable enumerations of partial computable functionals* $\mathbb{X}\times P(\mathbb{N})\to\mathbb{Y}$ *, then there exists some recursive bijection* $\theta:\mathbb{N}\to\mathbb{N}$ *such that* $\Psi_i=\Phi_{\theta(i)}$ *for all* $i\in\mathbb{N}$*.*

2. *If* $(\psi_i^A)_{i\in\mathbb{N}}$ *and* $(\varphi_i^A)_{i\in\mathbb{N}}$ *are uniform enumerations of partial A-recursive functions then there exists some* recursive *bijection* $\theta:\mathbb{N}\to\mathbb{N}$ *such that* $\psi_i^A=\varphi_{\theta(i)}^A$ *for all* $i\in\mathbb{N}$*.*

Uniform enumerations allow for *effective* (as opposed to A-effective) closure results for a lot of operations on partial A-recursive functions and A-r.e. sets which correspond to closure properties of partial computable functionals admitting sets and partial functions as arguments, cf. Hinman [7] §II.2, II.4.

2.5 Acceptable enumerations of some subclasses of Δ_2^0

Comparison of K and $K_{\min}, K_{\max}$ in the hierarchy theorem 8.14 involves particular Δ_2^0 sets described in Def.2.12 below. First, we fix a notion of bounded quantification pertinent for our applications.

Definition 2.9.
1. We consider on each basic set a norm such that
- $||x||=|x|$ if $x\in\mathbb{N}$ or $\mathbb{Z}$,
- $||x||=length(x)$ if $x\in\Sigma^*$ where Σ is a finite alphabet,
- $||(x_1,...,x_k)||=\max(||x_1||,...,||x_1||)$.

2. Suppose $\mu:\mathbb{N}\to\mathbb{N}$ is a total function (resp. $\mu:\mathbb{N}\times P(\mathbb{N})\to\mathbb{N}$ is a total functional) which is monotone increasing (resp. with respect to its first argument). Let $\mathbb{X}$ is a basic set.
For $R\subseteq\mathbb{X}\times(\{0,1\}^*)^m$ and $\mathcal{R}\subseteq\mathbb{X}\times(\{0,1\}^*)^m\times P(\mathbb{N})$, we let

$$\begin{array}{rcl}\exists^{\leq\mu}R &=& \{\mathtt{x}:\exists\vec{\mathtt{u}}\ (|\mathtt{u}_1|,\ldots,|\mathtt{u}_m|\leq\mu(||\mathtt{x}||)\ \wedge\ R(\vec{\mathtt{u}},\mathtt{x}))\}\\ \exists^{\leq\mu}\mathcal{R} &=& \{(\mathtt{x},A):\exists\vec{\mathtt{u}}\ (|\mathtt{u}_1|,\ldots,|\mathtt{u}_m|\leq\mu(||\mathtt{x}||)\ \wedge\ \mathcal{R}(\vec{\mathtt{u}},\mathtt{x},A))\}\end{array}$$

If $\mathcal{C}\subseteq P(\mathbb{X})$ (resp. $\mathcal{C}\subseteq P(\mathbb{X}\times P(\mathbb{N}))$, we denote $\exists^{\leq\mu}\mathcal{C}$ the subclass of subsets of $\mathbb{X}$ (resp. $\mathbb{X}\times P(\mathbb{N})$) consisting of all sets $\exists^{\leq\mu}R$ where R is in $\mathcal{C}$.

Note 2.10. In view of applications to Kolmogorov complexity, we choose bounded quantifications over binary words (where the bound applies to the length). Of course, going from μ to 2^μ, we can reduce to bounded quantifications over $\mathbb{N}$.

As is well known, bounded quantification does not increase syntactical complexity of Δ_2^0 sets.

Proposition 2.11. *If $\mu : \mathbb{N} \to \mathbb{N}$ has Σ^0_2 graph then $\exists^{\leq\mu}\Delta^0_2 \subseteq \Delta^0_2$, be it for relations in $\mathbb{X}$ or in $\mathbb{X} \times P(\mathbb{N})$.*

Proof. In case $\mu(x) = x$ this is just the commutation of a bounded quantification with an unbounded one. In general, we have

$$\begin{aligned}
&\exists \vec{\mathtt{u}}\ (\vec{\mathtt{u}} \leq \mu(||\mathtt{x}||)\ \wedge\ R(\mathtt{x}, \vec{\mathtt{u}}))\\
&\quad \Leftrightarrow\ \exists y\ (y = \mu(||\mathtt{x}||)\ \wedge\ \exists \vec{\mathtt{u}}\ (\vec{\mathtt{u}} \leq y\ \wedge\ R(\mathtt{x}, \vec{\mathtt{u}})))\\
&\quad \Leftrightarrow\ \forall y\ (y = \mu(||\mathtt{x}||)\ \Rightarrow\ \exists \vec{\mathtt{u}}\ (\vec{\mathtt{u}} \leq y\ \wedge\ R(\mathtt{x}, \vec{\mathtt{u}})))
\end{aligned}$$

which are respectively $\exists\ (\Sigma^0_2 \wedge \Delta^0_2)$, hence Σ^0_2, and $\forall\ (\Pi^0_2 \vee \Delta^0_2)$, hence Π^0_2. □

Definition 2.12.
Let $\mathcal{C}$ be a syntactical class among

$$\Sigma^0_1 \quad , \quad \Pi^0_1 \quad , \quad \Sigma^0_1 \vee \Pi^0_1 \quad , \quad \exists^{\leq\mu}(\Sigma^0_1 \wedge \Pi^0_1)$$

1. Let $\mathcal{C}[\mathbb{X}]$ be the family of subsets of $\mathbb{X}$ which are $\mathcal{C}$-definable. An acceptable enumeration $(W_i^{\mathcal{C}[\mathbb{X}]})_{i\in\mathbb{N}}$ of $\mathcal{C}[\mathbb{X}]$ is an enumeration obtained from acceptable enumerations $(W_i^{\mathbb{X}\times(\{0,1\}^*)^m})_{i\in\mathbb{N}}$ of r.e. subsets of the $\mathbb{X} \times (\{0,1\}^*)^m$'s as follows:

$$\begin{array}{rcll}
W_i^{\Pi^0_1[\mathbb{X}]} & = & \mathbb{X} \setminus W_i^{\mathbb{X}} & \\
W_i^{\Sigma^0_1\wedge\Pi^0_1[\mathbb{X}]} & = & W_j^{\mathbb{X}} \cap (\mathbb{X} \setminus W_k^{\mathbb{X}}) & \text{where } i = \langle j, k\rangle \\
W_i^{\Sigma^0_1\vee\Pi^0_1[\mathbb{X}]} & = & W_j^{\mathbb{X}} \cup (\mathbb{X} \setminus W_k^{\mathbb{X}}) & \text{where } i = \langle j, k\rangle \\
W_i^{\exists^{\leq\mu}(\Sigma^0_1\wedge\Pi^0_1)[\mathbb{X}]} & = & \exists^{\leq\mu} W_j^{(\Sigma^0_1\wedge\Pi^0_1)[\mathbb{X}\times(\{0,1\}^*)^m]} & \text{where } i = \langle j, m\rangle
\end{array}$$

2. Let $\mathcal{C}[\mathbb{X} \times P(\mathbb{N})]$ be the family of subsets of $\mathbb{X} \times P(\mathbb{N})$ which are $\mathcal{C}$-definable. An acceptable enumeration $(\mathcal{W}_i^{\mathcal{C}[\mathbb{X}]})_{i\in\mathbb{N}}$ of $\mathcal{C}[\mathbb{X} \times P(\mathbb{N})]$ is defined similarly from acceptable enumerations $(\mathcal{W}_i^{\mathbb{X}\times(\{0,1\}^*)^m\times P(\mathbb{N})})_{i\in\mathbb{N}}$ of Σ^0_1 subsets of the $\mathbb{X} \times (\{0,1\}^*)^m$'s.

3. Let $A \subseteq \mathbb{N}$ and $\mathcal{C}^A$ be the A-oracle syntactical class associated to $\mathcal{C}$.
An enumeration $(W_i^{\mathcal{C}^A[\mathbb{X}]})_{i\in\mathbb{N}}$ of $\mathcal{C}^A[\mathbb{X}]$ is uniform if it is obtained from an acceptable enumeration $(\mathcal{W}_i^{\mathcal{C}[\mathbb{X}\times P(\mathbb{N})]})_{i\in\mathbb{N}}$ of $\mathcal{C}[\mathbb{X} \times P(\mathbb{N})]$ by fixing the second order argument. I.e. $W_i^{\mathcal{C}^A[\mathbb{X}]} = \{\mathtt{x} \in \mathbb{X} : (\mathtt{x}, A) \in \mathcal{W}_i^{\mathcal{C}[\mathbb{X}\times P(\mathbb{N})]}\}$.

2.6 The min and max operators

The following definitions and results collect material from [5, 4].

Definition 2.13. Let $\mathbb{X}$ be some basic set. We denote min and max the operators which map partial functions $\varphi : \mathbb{X} \times \mathbb{N} \to \mathbb{N}$ and partial functionals $\Phi : \mathbb{X} \times P(\mathbb{N}) \times \mathbb{N} \to \mathbb{N}$ onto partial functions $\min\varphi, \max\varphi : D \to \mathbb{N}$ and functionals $\min \Phi, \max \Phi : D \to \mathbb{N}$ such that

$$\begin{array}{rcl}(\min \varphi)(\mathtt{x}) & = & \min\{\varphi(\mathtt{x}, t) : t \in \mathbb{N} \ \wedge \ \varphi(\mathtt{x}, t) \text{ is defined}\} \\ (\max \varphi)(\mathtt{x}) & = & \max\{\varphi(\mathtt{x}, t) : t \in \mathbb{N} \ \wedge \ \varphi(\mathtt{x}, t) \text{ is defined}\} \\ (\min \Phi)(\mathtt{x}, A) & = & \min\{\Phi(\mathtt{x}, A, t) : t \in \mathbb{N} \ \wedge \ \phi(\mathtt{x}, t) \text{ is defined}\} \\ (\max \Phi)(\mathtt{x}, A) & = & \max\{\Phi(\mathtt{x}, A, t) : t \in \mathbb{N} \ \wedge \ \phi(\mathtt{x}, t) \text{ is defined}\}\end{array}$$

with the convention that $\min \emptyset$ and $\max \emptyset$ and the max of an infinite set are undefined.

2. We let

$$\begin{array}{rcl} Min_{PR}^{\mathbb{X}\to\mathbb{N}} & = & \{\min \varphi : \varphi \in PR^{\mathbb{X}\times\mathbb{N}\to\mathbb{N}}\} \\ Min_{PR^A}^{\mathbb{X}\to\mathbb{N}} & = & \{\min \varphi : \varphi \in PR^{\mathbb{X}\times\mathbb{N}\to\mathbb{N},A}\} \\ Min_{PC}^{\mathbb{X}\times P(\mathbb{N})\to\mathbb{N}} & = & \{\min \Phi : \Phi \in PC^{P(\mathbb{N})\times\mathbb{X}\times\mathbb{N}\to\mathbb{N}}\}\end{array}$$

The classes $Max_{PR}^{\mathbb{X}\to\mathbb{N}}$, $Max_{PR^A}^{\mathbb{X}\to\mathbb{N}}$ and $Max_{PC}^{\mathbb{X}\times P(\mathbb{N})\to\mathbb{N}}$ are defined similarly from the max operator.

Note 2.14.
1. Simple examples of functions in Min_{PR} are Kolmogorov complexities K and H. Examples of functions in Max_{PR} are the Busy Beaver function and the (partial) function giving the cardinal of W_n (if finite).

2. The functional $\mathcal{K}(\ \|\ )$, defined in §5, is in $Min_{PC}^{P(\mathbb{N})\times\mathbb{N}\to\mathbb{N}}$.

Let's mention an easy result as concerns the syntactical complexity of these functions.

Proposition 2.15. *Any function in $Min_{PR}^{\mathbb{X}\to\mathbb{N}} \cup Max_{PR}^{\mathbb{X}\to\mathbb{N}}$ has $\Sigma^0_1 \wedge \Pi^0_1$ graph. The result extends to functionals and also relativizes.*

Proof. Observe that $y = (\min \varphi)(\mathtt{x})$ can be written

$$(\exists t\ y = \varphi(\mathtt{x}, t)) \ \wedge \ (\forall t\ \forall s\ (\varphi(\mathtt{x}, t) \text{ converges in } s \text{ steps } \Rightarrow \ y \leq \varphi(\mathtt{x}, t)))$$

Idem for $y = (\max \varphi)(\mathtt{x})$ with $\geq$ in place of $\leq$. □

We shall use the following straightforward corollary of the above Proposition.

Proposition 2.16. *All functions in Min_{PR} and Max_{PR} are partial recursive in $\emptyset'$.*

An enumeration theorem holds for the families introduced in Def.2.13.

Proposition 2.17. *There exists an acceptable enumeration* $(\phi_i)_{i\in\mathbb{N}}$ *of* $Min_{PR}^{\mathbb{X}\to\mathbb{N}}$ *(where acceptable means that the analogs of conditions i–iii of Def.2.5 hold. In particular, the function* $(i,\mathtt{x})\mapsto\phi_i(\mathtt{x})$ *is itself in* $Min_{PR}^{\mathbb{N}\times\mathbb{X}\to\mathbb{N}}$*).*
Idem with the class $Max_{PR}^{\mathbb{X}\to\mathbb{N}}$ *and the functional classes* $Min_{PC}^{\mathbb{X}\times P(\mathbb{N})\to\mathbb{N}}$ *and* $Min_{PC}^{\mathbb{X}\times P(\mathbb{N})\to\mathbb{N}}$*.*

The following simple result about Min_{PR} and Max_{PR} will be useful.

Proposition 2.18.
1. *If* $\phi\in Min_{PR}^{\mathbb{X}\to\mathbb{N}}$ *and* $f:\mathbb{Y}\to\mathbb{X}$ *is in* $PR^{\mathbb{Y}\to\mathbb{X}}$ *then* $\phi\circ f\in Min_{PR}^{\mathbb{Y}\to\mathbb{N}}$*.*
Idem with Max_{PR} *in place of* Min_{PR}*.*

2. *If* $\psi\in Min_{PR}^{\mathbb{N}\to\mathbb{N}}$ *is monotone increasing and* $\phi\in Min_{PR}^{\mathbb{X}\to\mathbb{N}}$ *then* $\psi\circ\phi\in Min_{PR}^{\mathbb{X}\to\mathbb{N}}$*.*

Proof. 1. Let $\phi(\mathtt{x})=\min_t\varphi(\mathtt{x},t)$ where φ is partial recursive. Then $\phi(f(\mathtt{y}))=\min_t\varphi(f(\mathtt{y}),t)$ is in $Min_{PR}^{\mathbb{Y}\to\mathbb{N}}$ since $\varphi(f(\mathtt{y}),t)$ is in $PR^{\mathbb{Y}\to\mathbb{N}}$.

2. Let $\phi(\mathtt{x})=\min_t\varphi(\mathtt{x},t)$ and $\psi(x)=\min_u\theta(x,u)$ where φ,θ are partial recursive. Since ψ is monotone increasing, letting $(\pi_1,\pi_2):\mathbb{N}\to\mathbb{N}^2$ be the inverse of Cantor bijection, we have

$$\begin{array}{rcl}\psi(\phi(\mathtt{x})) &=& \psi(\min_t\varphi(\mathtt{x},t))\\ &=& \min_t(\psi(\varphi(\mathtt{x},t)))\\ &=& \min_t(\min_u\theta(\varphi(\mathtt{x},t),u))\\ &=& \min_v\theta(\varphi(\mathtt{x},\pi_1(v)),\pi_2(v))\end{array}$$

which is in $Min_{PR}^{\mathbb{X}\to\mathbb{N}}$ since $\theta(\varphi(\mathtt{x},\pi_1(v)),\pi_2(v))$ is partial recursive. □

3 Coimmunity and density

3.1 Constructive coimmunity and constructive density

A classical result about Kolmogorov complexity K, due to Barzdins (cf. [9] Thm.2.7.1 iii, p.167, or Zvonkin & Levin, [17] p.92.), states that if φ is total recursive and tends to $+\infty$ then

$$\{x : K(x)<\varphi(x)\}$$

is an r.e. set which meets every infinite r.e. set, i.e. $\{x : K(x)<\varphi(x)\}\cap W_i$ is an infinite r.e. set whenever W_i is infinite.
(The case φ is monotone increasing is due to Kolmogorov, cf. [17] p.90, or [9] Thm.2.3.1 iii, p.119–120).
In particular, K has no total recursive unbounded lower bound.

In §8 we extend in various ways this result to sets which are no more r.e. sets and involve Kolmogorov complexities $K_{\min}$ or $K_{\max}$. We also consider effectiveness of such properties in a sense related to the notion of constructive immunity, first considered in Xiang Li, 1983 [10] (cf. Odifreddi's book [11] p.267).

Definition 3.1. Let $(W_i)_{i\in\mathbb{N}}$ be an acceptable enumeration of recursively enumerable subsets of some basic set $\mathbb{X}$.

1i. (Dekker, 1958). A set $X \subseteq \mathbb{X}$ is *immune* if it is infinite and contains no infinite r.e. set.

1ii. (Xiang Li, 1983 [10]). A set $X \subseteq \mathbb{X}$ is *constructively immune* if it is infinite and there exists some partial recursive function $\varphi : \mathbb{N} \to \mathbb{X}$ such that

$$\forall i\ (W_i \textit{ is infinite } \Rightarrow \varphi(i) \textit{ is defined } \wedge\ \varphi(i) \in W_i \setminus X)$$

2i. A set $Z \subseteq \mathbb{X}$ is Σ^0_1-dense if it contains an infinite r.e. subset of any infinite r.e. set included in $\mathbb{X}$.

2ii. A set $Z \subseteq \mathbb{X}$ is constructively Σ^0_1-dense if there exists some total recursive function λ such that

$$\forall i\ (W_i \textit{ is infinite } \Rightarrow\ W_{\lambda(i)} \textit{ is an infinite subset of } Z \cap W_i)$$

Note 3.2. Rogers'Thm.2.8 insures that the above notion of constructive immunity and Σ^0_1-density do not depend on the chosen enumeration of r.e. sets.

Proposition 3.3. *$Z \subseteq \mathbb{X}$ is constructively immune if and only if it is infinite and its complement is constructively Σ^0_1-dense.*

Proof. $\Leftarrow$. Let $\varphi(i)$ be the point which appears first in the enumeration of $W_{\lambda(i)}$ (of course, $\varphi(i)$ is undefined in case $W_{\lambda(i)}$ is empty).

$\Rightarrow$. Define a partial recursive function $\mu(i,n)$ which satisfies:
- $\mu(i,0) = \varphi(i)$
- $\mu(i,n+1) = \varphi(i_n)$ where i_n is such that $W_{i_n} = W_i \setminus \{\mu(i,m) : m \leq n\}$

Using the parametrization theorem, let λ be total recursive so that $W_{\lambda(i)} = \{\mu(i,m) : m \in \mathbb{N}\}$. If W_i is infinite then all $\mu(i,m)$'s are defined and distinct and belong to $W_i \cap Z$. Thus, $W_{\lambda(i)}$ is an infinite subset of $W_i \cap Z$. □

In case Z is r.e., constructive Σ^0_1-density amounts to say that $Z \cap W_i$ is infinite whenever W_i is infinite.

Barzdin's result gives an instance of a constructively Σ^0_1-dense r.e. set. Other examples are maximal r.e. sets.

Proposition 3.4. *Any maximal r.e. set Z is constructively Σ^0_1-dense.*

Proof. Let $Z \subseteq \mathbb{X}$ be r.e. where $\mathbb{X}$ is some basic set. We prove that for every infinite r.e. set $W_i \subseteq \mathbb{X}$ the intersection $Z \cap W_i$ is also infinite.
In fact, suppose $Z \cap W_i$ is finite. Then $W_i \setminus Z$ is an infinite r.e. set disjoint from Z. Thus, $Z' = Z \cup W_i$ is an r.e. set containing Z such that the

difference $Z' \setminus Z = W_i \setminus Z$ is infinite. Since Z is maximal this implies that Z' is cofinite. Thus,

$$\mathbb{X} \setminus Z = (\mathbb{X} \setminus Z') \cup (W_i \setminus (Z \cap W_i)) = A \cup (W_i \setminus B)$$

where A, B are finite sets. Hence $\mathbb{X} \setminus Z$ is r.e. and, consequently Z is recursive. A contradiction. □

3.2 Uniform constructive density

In order to deal with Kolmogorov complexities $K^{\emptyset'}, K^{\emptyset''}, \ldots$ and their Min/Max versions, we shall consider constructive density for Σ^0_n sets. This will be done through relativization of Σ^0_1-density with respect to jump oracle $\emptyset^{(n-1)}$.

There is two natural ways to relativize Σ^0_1-density to an oracle A :

($*$) Consider the W_i^A's and ask for λ A-recursive.

($**$) Consider the W_i^A's and ask for λ recursive.

The second way, which is the stronger one, will be the one pertinent for applications to Kolmogorov complexities. Of course, to deal with ($**$), we must consider uniform enumerations of A-r.e. sets and partial A-recursive functions (cf. Prop.2.7), i.e. we have to consider the notion of constructive density with functionals. This will, in fact, give *a strong version of* ($**$) *in which λ is a total recursive function which does not depend on A.*

Definition 3.5.
1. Let $\mathcal{Z} \subseteq \mathbb{X} \times P(\mathbb{N})$. For $A \subseteq \mathbb{N}$, let's denote $\mathcal{Z}^A = \{\mathtt{x} \in \mathbb{X} : (\mathtt{x}, A) \in \mathcal{Z}\}$. Consider an acceptable enumeration $(\mathcal{W}_i)_{i \in \mathbb{N}}$ of Σ^0_1 subsets of $\mathbb{X} \times P(\mathbb{N})$ (cf. Def.2.5) and let $W_i^A = \{\mathtt{x} : (\mathtt{x}, A) \in \mathcal{W}_i\}$.
$\mathcal{Z}$ is constructively Σ^0_1-dense if there exists some total *recursive* function λ such that, for all $i \in \mathbb{N}$ and all $A \in P(\mathbb{N})$,

$$(*) \quad \forall i \in \mathbb{N}\ \forall A \in P(\mathbb{N})\ (W_i^A \textit{ is infinite} \Rightarrow W_{\lambda(i)}^A \textit{ is an infinite subset of } W_i^A \cap \mathcal{Z}^A)$$

2. $Z \subseteq \mathbb{X}$ is constructively uniformly $\Sigma^{0,A}_1$-dense if there exists some constructively Σ^0_1-dense set $\mathcal{Z} \subseteq \mathbb{X} \times P(\mathbb{N})$ such that $Z = \mathcal{Z}^A$.
In particular, there exists some total *recursive* function λ such that

$$(**) \quad \forall i \in \mathbb{N}\ (W_i^A \textit{is infinite} \Rightarrow W_{\lambda(i)}^A \textit{is an infinite subset of } W_i^A$$

When $A = \emptyset^{(n-1)}$ we shall also say that Z is constructively uniformly Σ^0_n-dense.

Note 3.6. Thm.2.8 insures that the above notion of constructive uniform $\Sigma^{0,A}_1$-density does not depend on the chosen enumeration of A-r.e. sets, as long as it is uniform, cf. Prop.2.7.

Remark 3.7. Using Point 2 of Prop.2.2, one can suppose that if W_i^A is infinite then $W_{\lambda(i)}^A$ is A-recursive and an A-r.e. code for its complement is given by another total recursive function λ'.

Note 3.8. In the vein of what we mentioned at the start of §3.1, if $\varphi : \mathbb{N} \to \mathbb{N}$ is total A-recursive and tends to $+\infty$ then Lemma.8.1 insures that $\{x : K^A(x) < \varphi(x)\}$ is an A-r.e. set which is uniformly constructively $\Sigma_1^{0,A}$-dense. In case $\varphi(x) <_{\rm ct} \log(x)$, this set is coinfinite since it excludes integers with incompressible binary representations.

Remark 3.9. Immunity can also be relativized according to the different policies $(*)$ and $(**)$. Also, Prop.3.3 admits straightforward extensions to the functional setting and the uniform relativized one.

Finally, let's observe that Prop.3.4 relativizes in the uniform sense.

Proposition 3.10. *Any maximal A-r.e. set Z is uniformly constructively $\Sigma_1^{0,A}$-dense.*

Proof. Let $Z = \{\mathtt{x} : (\mathtt{x}, A) \in \mathcal{Z}\}$ where $\mathcal{Z} \subseteq \mathbb{X} \times P(\mathbb{N})$ is Σ_1^0. There is a total recursive function θ such that, for all A and i, $\mathcal{Z} \cap \mathcal{W}_i = \mathcal{W}_{\theta(i)}$. In particular, $Z \cap W_i^A = W_{\theta(i)}^A$ and the argument of Prop.3.4 goes through. $\square$

3.3 Constructive $(\mathcal{C}, \mathcal{D})$-density

Comparison of K and $K_{\rm min}, K_{\rm max}$ in the hierarchy theorem 8.14 leads to a particular version of constructive density applied to Σ_1^0 and to Π_1^0 sets and involving subclasses $\exists^{\leq \mu}(\Sigma_1^0 \wedge \Pi_1^0)$ of Δ_2^0 sets described in Def.2.9 below.

We now introduce some central notions of this paper.

Definition 3.11. Let $\mathbb{X}$ be a basic set.
1i. Let $\mathcal{S}, \mathcal{T}$ be families of subsets of $\mathbb{X}$. A set $Z \subseteq \mathbb{X}$ is $(\mathcal{S}, \mathcal{T})$-dense if for every infinite set $X \in \mathcal{S}$ the intersection $Z \cap X$ contains an infinite subset Y which is in $\mathcal{T}$.

ii. Let $\mathcal{S}, \mathcal{T}$ be families of subsets of $\mathbb{X} \times P(\mathbb{N})$. A set $\mathcal{Z} \subseteq \mathbb{X} \times P(\mathbb{N})$ is $(\mathcal{S}, \mathcal{T})$-dense if for every $\mathcal{X} \in \mathcal{S}$ there exists $\mathcal{Y} \in \mathcal{T}$ such that, for every A, letting $\mathcal{X}^A = \{\mathtt{x} : (\mathtt{x}, A) \in \mathcal{X}\}$,

$$\mathcal{X}^A \text{ is infinite } \Rightarrow \mathcal{Y}^A \text{ is infinite and included in } \mathcal{X}^A \cap \mathcal{Z}^A$$

2. Let $\mathcal{C}, \mathcal{D}$ be syntactical classes as in Def.2.12.

i. Z is constructively $(\mathcal{C}, \mathcal{D})$-dense if it is $(\mathcal{C}[\mathbb{X}], \mathcal{D}[\mathbb{X}])$-dense in the sense of 1i above and, moreover, a $\mathcal{D}$-code for Y can be recursively obtained from a $\mathcal{C}$-code for X. In other words, there exists some total recursive function $\lambda : \mathbb{N} \to \mathbb{N}$ such that, for all i

$$W_i^{\mathcal{C}[\mathbb{X}]} \text{ is infinite } \Rightarrow W_{\lambda(i)}^{\mathcal{D}[\mathbb{X}]} \text{ is infinite and included in } W_i^{\mathcal{C}[\mathbb{X}]} \cap Z$$

ii. A set $\mathcal{Z} \subseteq \mathbb{X} \times P(\mathbb{N})$ is constructively $(\mathcal{C}, \mathcal{D})$-dense if it is $(\mathcal{C}[\mathbb{X}], \mathcal{D}[\mathbb{X}])$-dense in the sense of 1ii above and, moreover, an $\mathcal{D}$-code for $\mathcal{Y}$ can be recursively obtained from a $\mathcal{C}$-code for $\mathcal{X}$. In other words, there exists some total recursive function $\lambda : \mathbb{N} \to \mathbb{N}$ such that, for all i

$$\begin{aligned} &(\mathcal{W}_i^A)^{\mathcal{C}[\mathbb{X} \times P(\mathbb{N})]} \text{ is infinite} \\ &\quad \Rightarrow (\mathcal{W}_{\lambda(i)}^A)^{\mathcal{D}[\mathbb{X} \times P(\mathbb{N})]} \text{ is infinite and included in } (\mathcal{W}_i^A)^{\mathcal{C}[\mathbb{X} \times P(\mathbb{N})]} \cap \mathcal{Z}^A) \end{aligned}$$

Note 3.12.
1. Clearly, (constructive) (Σ_1^0, Σ_1^0)-density is exactly (constructive) Σ_1^0-density in the sense of Def.3.5.

2. See Lemmas 8.6, 8.8 for examples of constructive $(\Sigma_1^0, \exists^{\leq\mu}(\Sigma_1^0 \wedge \Pi_1^0))$-density and $(\Pi_1^0, \exists^{\leq\mu}(\Sigma_1^0 \wedge \Pi_1^0))$-density.

Let's state a simple result about $(\mathcal{C}, \mathcal{D})$-density.

Proposition 3.13.
1. *The family of (constructively) $(\mathcal{C}, \mathcal{D})$-dense subsets of $\mathbb{X}$ (resp. $\mathbb{X} \times P(\mathbb{N})$) is superset closed.*

2. *Let $Z_1, Z_2 \subseteq \mathbb{X}$. If Z_1 is (constructively) $(\mathcal{C}, \mathcal{D})$-dense and Z_2 is (constructively) $(\mathcal{D}, \mathcal{E})$-dense then $Z_1 \cap Z_2$ is (constructively) $(\mathcal{C}, \mathcal{E})$-dense.*
Idem for $\mathcal{Z}_1, \mathcal{Z}_2 \subseteq \mathbb{X} \times P(\mathbb{N})$.

Proof. Point 1 is obvious. As for point 2, let X be an infinite set in $\mathcal{C}[\mathbb{X}]$. Using $(\mathcal{C}, \mathcal{D})$-density of Z_1 we (recursively) get (a code for) an infinite $X_1 \subseteq X \cap Z_1$ in $\mathcal{D}$. Then, using $(\mathcal{D}, \mathcal{E})$-density of Z_2, we (recursively) get (a code for) an infinite $X_2 \subseteq X_1 \cap Z_2 \subseteq X \cap (Z_1 \cap Z_2)$ in $\mathcal{E}$.
For $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{X} \subseteq \mathbb{X} \times P(\mathbb{N})$, fix the second order argument A and argue similarly with $\mathcal{Z}_1^A, \mathcal{Z}_2^A, \mathcal{X}^A$. □

4 The OftLess relations and the $\ll$ orderings

In this § we introduce the central notions of this paper to compare the growth of total functions $f, g : \mathbb{N} \to \mathbb{N}$.

4.1 Relations $\mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$, $\mathrm{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}$ on maps $\mathbb{N} \to \mathbb{N}$

Definition 4.1. Let $\mathcal{C}, \mathcal{D}$ be syntactical classes (cf. Def.2.12) and $\mathcal{F}$ be a countable family of functions $\mathbb{N} \to \mathbb{N}$ and $(\phi_i)_{i\in\mathbb{N}}$ be a (non necessarily injective) enumeration of $\mathcal{F}$ (in §8, $\mathcal{F}$ will be PR or Min_{PR}, cf. Def.2.13).

We let $f \ \mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}} \ g$ (resp. $f \ \mathrm{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}} \ g$) be the relation between total functions $f, g : \mathbb{N} \to \mathbb{N}$ defined by the following conditions:

i. For every total (resp. and monotone increasing) function $\phi : \mathbb{N} \to \mathbb{N}$ in $\mathcal{F}$ which tends to $+\infty$, the set $\{x : f(x) < \phi(g(x))\}$ is constructively $(\mathcal{C}, \mathcal{D})$-dense.

ii. The constructive $(\mathcal{C}, \mathcal{D})$-density in condition i is uniform in ϕ : There exists some total recursive $\lambda : \mathbb{N}^2 \to \mathbb{N}$ such that, for all i, j,

ϕ_i is total (resp. and monotone increasing) and tends to $+\infty$
$\wedge\ W_j^{\mathcal{C}}$ is infinite

$$\Rightarrow\ W_{\lambda(i,j)}^{\mathcal{D}} \text{ is an infinite subset of } W_j^{\mathcal{C}} \cap \{x : f(x) < \phi_i(g(x))\}$$

Remark 4.2.
1. The notation OftLess stresses the fact that f is often much smaller than g : consider functions ϕ which are much smaller than the identity function, e.g. $\max(0, z - c)$, $\lfloor z/c \rfloor$, $\lfloor \log(z) \rfloor$, $\log^*(z)$,...

2. $\mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ carries the contents, reformulated in terms of uniform constructive $(\mathcal{C}, \mathcal{D})$-density, of Barzdins result cited above, and that of adequate variants that we shall prove about $K_{\max}$ and $K_{\min}$ (cf. Lemmas 8.1, 8.6, 8.8).

3. Suppose $\mathcal{F}$ contains all translation functions $z \mapsto \max(0, z - c)$. If $f\ \mathrm{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}\ g$ (a fortiori if $f\ \mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}\ g$) then g is necessarily unbounded. Else, if c is a bound for g, consider $\phi(z) = \max(0, z - c)$ to get a contradiction.

4. $\mathrm{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}$ is an extension of $\mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ which has much better properties (cf. Thm.4.4).

4.2 Monotonicity versus recursive lower bound

In case $\mathcal{F} = PR$, the monotonicity condition can be put in another equivalent form.

Proposition 4.3. *Relation $f\ \mathrm{OftLess}_{PR\uparrow}^{\mathcal{C},\mathcal{D}}\ g$ holds if and only if conditions i, ii in Def.4.1 hold for every total functions $\phi, \phi_i : \mathbb{N} \to \mathbb{N}$ which recursively tend to $+\infty$, i.e. there are recursive growth modulus $\xi, \xi_i : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall N\ \forall n \geq \xi(N)\ \phi(n) \geq N \quad , \quad \forall N\ \forall n \geq \xi_i(N)\ \phi_i(n) \geq N$$

Proof. $\Rightarrow$. If ϕ is total recursive and monotone increasing and tends to $+\infty$ then it tends recursively to $+\infty$: a possible recursive growth modulus is

$$\xi(N) = \text{least } x \text{ such that } \phi(x) \geq N$$

$\Leftarrow$. Observe that any total $\phi \in PR$ which tends recursively to $+\infty$ has a total recursive minorant ψ which also tends to $+\infty$, namely

$$\psi(0) = \varphi(0) \quad , \quad \psi(N+1) = \varphi(\xi(1 + \psi(N)))$$

where ξ is a recursive growth modulus of φ.
Of course, if true for ψ, conditions i, ii are also true for ϕ. $\square$

4.3 Transitivity

It is clear that if $\mathcal{C} \neq \mathcal{D}$ then $\text{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ and $\text{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}$ may not be transitive, hence may not be orderings. However, we have the following result.

Theorem 4.4 (Transitivity theorem).
1. *Let $\mathcal{B},\mathcal{C},\mathcal{D}$ be syntactical classes and $\mathcal{F},\mathcal{G}$ be countable classes of functions containing the identity function $Id : \mathbb{N} \to \mathbb{N}$. Then,*

$$\textit{i.}\quad e \ \ \text{OftLess}_{\mathcal{F}}^{\mathcal{B},\mathcal{C}} \ \ f \ \ \text{OftLess}_{\mathcal{G}}^{\mathcal{C},\mathcal{D}} \ \ g \quad \Longrightarrow \quad e \ \ \text{OftLess}_{\mathcal{G}}^{\mathcal{B},\mathcal{D}} \ \ g$$

$$\textit{ii.}\quad e \ \ \text{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{B},\mathcal{C}} \ \ f \ \ \text{OftLess}_{\mathcal{G}\uparrow}^{\mathcal{C},\mathcal{D}} \ \ g \quad \Longrightarrow \quad \begin{cases} e \ \ \text{OftLess}_{\mathcal{G}\uparrow}^{\mathcal{B},\mathcal{D}} \ \ g \\ e \ \ \text{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{B},\mathcal{D}} \ \ g \end{cases}$$

In case $\mathcal{F}$ is recursively closed by negative translation of the output, i.e.

$$\phi \in \mathcal{F} \ \Rightarrow \ \forall c \ \max(0, \phi - c) \in \mathcal{F}$$

and there exists a total recursive function $\theta : \mathbb{N}^2 \to \mathbb{N}$ such that

$$\max(0, \phi_i - c) = \phi_{\theta(i,c)}$$

then

$$\textit{iii.}\quad e \ \leq_{\text{ct}} \ f \ \ \text{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}} \ \ g \quad \Longrightarrow \quad e \ \ \text{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}} \ \ g$$

In case $\mathcal{F}$ is recursively closed by negative translation of the output and also by negative translation of the input, i.e.

$$\phi \in \mathcal{F} \ \Rightarrow \ \forall c \ x \mapsto \phi(\max(0, x - c)) \in \mathcal{F}$$

and there exists a total recursive function $\zeta : \mathbb{N}^2 \to \mathbb{N}$ such that, for all x,

$$\phi_i(\max(0, x - c)) = \phi_{\zeta(i,c)}(x)$$

then

$$\textit{iv.}\quad e \ \leq_{\text{ct}} \ f \ \ \text{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}} \ \ g \ \leq_{\text{ct}} \ h \quad \Longrightarrow \quad e \ \ \text{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}} \ \ h$$

2. Case $\mathcal{C} = \mathcal{D}$. *Relations* $\text{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{C}}$ *and* $\text{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{C}}$ *are strict orderings.*

Proof. 1i. Suppose $e \ \text{OftLess}_{\mathcal{F}}^{\mathcal{B},\mathcal{C}} \ f \ \text{OftLess}_{\mathcal{G}}^{\mathcal{C},\mathcal{D}} \ g$. Observe that

$$\{x : e(x) < f(x)\} \ \cap \ \{x : f(x) < \phi(g(x))\} \ \subseteq \ \{x : e(x) < \phi(g(x))\}$$

Since $Id \in \mathcal{F}$, the sets on the left are respectively constructively $(\mathcal{B},\mathcal{C})$-dense and $(\mathcal{C},\mathcal{D})$-dense, uniformly in ϕ. Applying Prop.3.13, we see that $\{x : e(x) < \phi(g(x))\}$ is constructively $(\mathcal{B},\mathcal{D})$-dense, whence $e \ \text{OftLess}_{\mathcal{G}}^{\mathcal{B},\mathcal{D}} \ g$.

1ii. The above argument also gives $e \ \text{OftLess}_{\mathcal{G}\uparrow}^{\mathcal{B},\mathcal{D}} \ g$. To get $e \ \text{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{B},\mathcal{D}} \ g$, argue as above and observe that

$$\{x : e(x) < \phi(f(x))\} \ \cap \ \{x : f(x) < g(x)\} \ \subseteq \ \{x : e(x) < \phi(g(x))\}$$

whenever ϕ is monotone increasing.

1iii. Let c be such that $e(x) \leq f(x)+c$ for all x.
If $\phi \in \mathcal{F}$ is total and tends to $+\infty$, so is its negative output translation

$$\widehat{\phi}_c(z) = \max\{0, \phi(z) - c\}$$

Suppose $f(x) < \widehat{\phi}_c(g(x))$. Then $\widehat{\phi}_c(g(x)) > 0$ so that

$$\begin{array}{ccccc} & & \widehat{\phi}_c(g(x)) & = & \phi(g(x)) - c \\ & & f(x) & < & \phi(g(x)) - c \\ e(x) & \leq & f(x)+c & < & \phi(g(x)) \end{array}$$

This proves the following inclusion

$$\{x : f(x) < \widehat{\phi}_c(g(x))\} \subseteq \{x : e(x) < \phi(g(x)\} \tag{1}$$

Relation $f \ \mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}} \ g$ insures that $\{x : f(x) < \widehat{\phi}_c(g(x))\}$ is constructively $(\mathcal{C},\mathcal{D})$-dense. Inclusion (1) implies that the same is true with $\{x : e(x) < \phi(g(x))\}$. Since a code for $\widehat{\phi}_c$ is recursively obtained from a code for ϕ, this proves $e \ \mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}} \ g$.

1iv. Let c be now such that $e(x) \leq f(x)+c$ and $g(x) \leq h(x)+c$ for all x.
If $\phi \in \mathcal{F}$ is total, monotone increasing and tends to $+\infty$, so is its negative input and output translation

$$\widetilde{\phi}_c(z) = \max\{0, \phi(\max(0, z-c)) - c\}$$

Suppose $f(x) < \widetilde{\phi}_c(g(x))$. Then $\widetilde{\phi}_c(g(x)) > 0$ so that

$$\begin{array}{ccccc} & & \widetilde{\phi}_c(g(x)) & = & \phi(\max(0, g(x)-c) - c \\ & & f(x) & < & \phi(\max(0, g(x)-c) - c \\ e(x) & \leq & f(x)+c & < & \phi(\max(0, g(x)-c) \end{array}$$

Now, $g(x) \leq h(x)+c$ and ϕ is monotone increasing, hence

$$e(x) < \phi(\max(0, g(x)-c) \leq \phi(\max(0, h(x)) = \phi(h(x))$$

This proves inclusion

$$\{x : f(x) < \widetilde{\phi}_c(g(x))\} \subseteq \{x : e(x) < \phi(h(x)\} \tag{2}$$

Relation $f \ \mathrm{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}} \ g$ insures that $\{x : f(x) < \widetilde{\phi}_c(g(x))\}$ is constructively $(\mathcal{D},\mathcal{E})$-dense. Inclusion (2) implies that the same is true with $\{x : e(x) < \phi(h(x))\}$. Since a code for $\widetilde{\phi}_c$ is recursively obtained from a code for ϕ, this proves $e \ \mathrm{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}} \ h$.

2. Transitivity of $\mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{C}}$ and $\mathrm{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{C}}$ is an obvious consequence of 1i–ii. As for irreflexivity, arguing with $\phi = Id$ (which is in $\mathcal{F}$), we see that $f(x) < \phi(f(x))$ is impossible, so that $f \ \mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{C}} \ f$ and $f \ \mathrm{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{D},\mathcal{C}} \ f$ are always false. Thus, $\mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{C}}$ and $\mathrm{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{C}}$ are *strict* orderings. $\square$

4.4 Orderings $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}}$, $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}\uparrow}$ on maps $\mathbb{N} \to \mathbb{N}$

Points iii-iv of the above theorem show that taking intersection with the "up to a constant" ordering $\leq_{\mathrm{ct}}$ transforms the relations $\mathrm{OftLess}^{\mathcal{C},\mathcal{D}}_{\mathcal{F}}$ and $\mathrm{OftLess}^{\mathcal{C},\mathcal{D}}_{\mathcal{F}\uparrow}$ into strict orderings $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}}$ and $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}\uparrow}$.

Definition 4.5. $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}}$ and $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}\uparrow}$ are the intersections of the $\mathrm{OftLess}^{\mathcal{C},\mathcal{D}}_{\mathcal{F}}$ and $\mathrm{OftLess}^{\mathcal{C},\mathcal{D}}_{\mathcal{F}\uparrow}$ relations with the $\leq_{\mathrm{ct}}$ ordering on total maps $\mathbb{N} \to \mathbb{N}$.

Theorem 4.6 ($\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}}$ and $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}\uparrow}$ are strict orderings).
Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be syntactical classes and let $\mathcal{F}, \mathcal{G}$ be countable classes of functions $\mathbb{N} \to \mathbb{N}$ which contain Id and which are recursively closed by output and input translation (cf. Thm.4.4) relative to some enumerations of $\mathcal{F}, \mathcal{G}$. Then

$$
\begin{array}{ll}
i. & e \ll^{\mathcal{B},\mathcal{C}}_{\mathcal{F}} f \ll^{\mathcal{C},\mathcal{D}}_{\mathcal{G}} g \quad \Rightarrow \quad e \ll^{\mathcal{B},\mathcal{D}}_{\mathcal{G}} g \\
ii. & e \ll^{\mathcal{B},\mathcal{C}}_{\mathcal{F}\uparrow} f \ll^{\mathcal{C},\mathcal{D}}_{\mathcal{G}\uparrow} g \quad \Rightarrow \quad \left\{ \begin{array}{l} e \ll^{\mathcal{B},\mathcal{D}}_{\mathcal{F}\uparrow} g \\ e \ll^{\mathcal{B},\mathcal{D}}_{\mathcal{G}\uparrow} g \end{array} \right. \\
iii. & e \leq_{\mathrm{ct}} f \ll^{\mathcal{C},\mathcal{D}}_{\mathcal{G}} g \quad \Rightarrow \quad e \ll^{\mathcal{C},\mathcal{D}}_{\mathcal{G}} g \\
iv. & e \leq_{\mathrm{ct}} f \ll^{\mathcal{C},\mathcal{D}}_{\mathcal{G}\uparrow} g \leq_{\mathrm{ct}} h \quad \Rightarrow \quad e \ll^{\mathcal{C},\mathcal{D}}_{\mathcal{G}\uparrow} h
\end{array}
$$

In particular, properties iii and iv can be applied with $\leq_{\mathrm{ct}}$ replaced by $\ll^{\mathcal{A},\mathcal{B}}_{\mathcal{F}}$ or $\ll^{\mathcal{A},\mathcal{B}}_{\mathcal{F}\uparrow}$.

Relations $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}}$ and $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}\uparrow}$ are strict orderings such that

$$v. \quad f \ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}} g \ \Rightarrow \ f \ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}\uparrow} g \ \Rightarrow \ f <_{\mathrm{ct}} g$$

Proof. Conditions i–iv are straightforward consequences of the similar conditions in Thm.4.4.

Condition i–ii yields transitivity of $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}}$ and $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}\uparrow}$.

Implication $f \ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}} g \Rightarrow f \ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}\uparrow} g$ is trivial. Let's prove that $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}}$ refines $<_{\mathrm{ct}}$ (and not merely $\leq_{\mathrm{ct}}$).
Suppose $f \ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}}$ g. Then $f \leq_{\mathrm{ct}} g$. Also, letting $\psi(z) = \max(0, z-c)$, we see that $\{x : f(x) < \psi(g(x)\} = \{x : f(x) < g(x) - c\}$ is infinite, hence the condition $\forall x \ g(x) \leq f(x) + c$ is impossible, whatever be c. Thus, $f <_{\mathrm{ct}} g$. ☐

The above theorem shows that composition of the orderings $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}}$ and $\ll^{\mathcal{C},\mathcal{D}}_{\mathcal{F}\uparrow}$ is remarkably flexible. In particular,

Corollary 4.7. *If* $1 \leq i \leq n$ *and* $1 \leq j < k \leq m \leq n$ *then*

$$f_0 \ll_{\mathcal{F}_1\uparrow}^{\mathcal{D}_1,\mathcal{E}_1} \ldots \ll_{\mathcal{F}_n\uparrow}^{\mathcal{D}_n,\mathcal{E}_n} f_n \Rightarrow f_0 \ll_{\mathcal{F}_i\uparrow}^{\mathcal{D}_i,\mathcal{E}_i} f_n$$
$$f_0 \ll_{\mathcal{F}_2\uparrow}^{\mathcal{C}_1,\mathcal{C}_2} \ldots \ll_{\mathcal{F}_n\uparrow}^{\mathcal{C}_{n-1},\mathcal{C}_n} f_n \Rightarrow f_0 \ll_{\mathcal{F}_k\uparrow}^{\mathcal{C}_j,\mathcal{C}_m} f_n$$

4.5 Left composition and $\ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$

Def.4.1, 4.5 compare total functions $f, g : \mathbb{N} \to \mathbb{N}$ via the associated sets $\{x : f(x) < \phi(g(x))\}$ for $\phi \in \mathcal{F}$. One could also compare f, g via the sets $\{x : \phi(f(x)) < g(x)\}$ for $\phi \in \mathcal{F}$. Similar properties could be derived.

Though we shall not use it in the sequel, there is a property of $\ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ and $\ll_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}$ which is interesting on its own and gives an alternative definition of $\ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ and $\ll_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}$ where the inequality $f(x) < \phi(g(x))$ gets a symmetric form $\psi(f(x)) < \phi(g(x))$ involving functions ψ, ϕ on both sides of the inequality. We prove it in case $\mathcal{F}$ is PR or Min_{PR}.

Proposition 4.8. *Let* $\mathcal{C}, \mathcal{D}$ *be syntactical classes and* $\mathcal{F}$ *be* PR *or* Min_{PR}. *Let* $\psi : \mathbb{N} \to \mathbb{N}$ *be a total recursive function. Then*

$$f \ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}} g \Rightarrow \psi \circ f \ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}} g$$

$$f \ll_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}} g \Rightarrow \psi \circ f \ll_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}} g$$

Moreover, the constructive density afferent to the relations $\psi \circ f \ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ g *and* $\psi \circ f \ll_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}$ g *is uniform in* ψ.

Proof. 1. Let $\phi : \mathbb{N} \to \mathbb{N}$ be a total function in $\mathcal{F}$ which tend to $+\infty$. We prove that $\{x : \psi(f(x)) < \phi(g(x))\}$ is constructively $(\mathcal{C}, \mathcal{D})$-dense.
Set $\psi'(z) = \max(z, max\{\psi(u) : u \leq z\})$. Then $\psi' \geq Id$ is total recursive, monotone increasing and unbounded. Since $\psi \leq \psi'$, we have

$$\{x : \psi'(f(x)) < \phi(g(x))\} \subseteq \{x : \psi(f(x)) < \phi(g(x))\} \tag{3}$$

2. Define $\alpha, \zeta : \mathbb{N} \to \mathbb{N}$ as follows:

$$\begin{aligned} \alpha(z) &= \text{largest } u \text{ such that } \psi'(u) \leq \phi(z) \\ \zeta(z) &= \text{smallest } s \text{ such that } \psi' \text{ is constant on } [s, \alpha(z)] \end{aligned}$$

Since $\phi(z)$ and $\psi'(z)$ tend to $+\infty$ so do $\alpha(z)$ and ζ. Also,

$$\forall u < \zeta(z) \;\; \psi'(u) < \psi'(\zeta(z)) = \psi'(\alpha(z)) \leq \phi(z) \tag{4}$$

Finally, α and ζ are in $\mathcal{F}$. If $\mathcal{F} = PR$, this is trivial. If $\mathcal{F} = Min_{PR}$ and $\phi(x) = \min_t \phi_t(x)$ then observe that $\alpha(x) = \min_t \alpha_t(x)$ and $\zeta(x) = \min_t \zeta_t(x)$ (where α_t, ζ_t are defined from ψ, ϕ_t as are α, ζ from ψ, ϕ).

3. Condition (4) applied to $z = g(x), u = f(x)$ insures

$$f(x) < \zeta(g(x)) \;\Rightarrow\; \psi'(f(x)) < \phi(g(x))$$

whence

$$\{x : f(x) < \zeta(g(x))\} \;\;\subseteq\;\; \{x : \psi'(f(x)) < \phi(g(x))\} \tag{5}$$

Condition f $\mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ g applied to ζ insures that $\{x : f(x) < \zeta(g(x))\}$ is constructively $(\mathcal{C},\mathcal{D})$-dense. Using inclusions (5) and (3), we see that so is $\{x : \psi(f(x)) < \phi(g(x))\}$.

4. In case f $\mathrm{OftLess}_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}$ g, then ϕ is monotone increasing. Since ψ is also monotone increasing, so are α, ζ and we get $\psi \circ f$ $\mathrm{OftLess}_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ g.

5. Finally, observe that all the construction is uniform in ψ and ϕ. □

5 Functional Kolmogorov complexity

The purpose of this section is to reconsider the oracular version of Kolmogorov complexity. We shall view the oracle as a parameter in a second order variant of conditional Kolmogorov complexity.

5.1 Kolmogorov complexity of a functional

Definition 5.1. Let $\mathbb{X}$ be a basic set.
The Kolmogorov complexity $\mathcal{K}_F : \mathbb{X} \times P(\mathbb{N}) \to \mathbb{N}$ associated to a partial functional $F : \{0,1\}^* \times P(\mathbb{N}) \to \mathbb{X}$ is defined as follows:

$$\mathcal{K}_F(\mathtt{x} \,||\, A) \quad = \quad \text{smallest } |\mathtt{p}| \text{ such that } (F(\mathtt{p}, A) = \mathtt{x})$$

Note 5.2.
1. Forgetting the A, we get the classical notion $K_F(\mathtt{x})$ with $F : \{0,1\}^* \to \mathbb{X}$. Freezing the A also leads to the classical oracular notion. This is the contents of the next obvious proposition and of Thm.7.1 below.

2. The double bar $||$ is used so as to get no confusion with usual conditional Kolmogorov complexity where the condition is a first-order object.

3. The above definition can obviously be extended to conditional Kolmogorov complexity $\mathcal{K}_F(\mathtt{x} \,|\, \mathtt{y} \,||\, A)$ where $F : \{0,1\}^* \times \mathbb{Y} \times P(\mathbb{N}) \to \mathbb{X}$.

Proposition 5.3. *Let F be as in Def.5.1. For $A \in P(\mathbb{N})$, denote*

$$F^A : \{0,1\}^* \to \mathbb{X}$$

the function such that $F^A(\mathtt{p}) = F(\mathtt{p}, A)$. Then, for all $\mathtt{x} \in \mathbb{X}$,

$$K_{F^A}(\mathtt{x}) = \mathcal{K}_F(\mathtt{x} \,||\, A)$$

5.2 Functional invariance theorem

The usual proof of the invariance theorem (Kolmogorov, 1965 [8]) extends easily when considering partial computable functionals $\{0,1\}^* \times P(\mathbb{N}) \to \mathbb{N}$ in place of partial recursive functions $\{0,1\}^* \to \mathbb{N}$, leading to what we call *functional Kolmogorov complexity* and denote $\mathcal{K}(\mathtt{x} \,||\, A)$.

Theorem 5.4 (Functional Invariance Theorem).
1. *Let $\mathcal{F}$ be the family of partial computable functionals $\{0,1\}^* \times P(\mathbb{N}) \to \mathbb{X}$. When F varies in $\mathcal{F}$, there is a least K_F up to an additive constant:*

$$\exists F \in \mathcal{F} \quad \forall G \in \mathcal{F} \quad \mathcal{K}_F \leq_{\mathrm{ct}} \mathcal{K}_G$$

Such an F is said to be optimal in $\mathcal{F}$. We let $\mathcal{K}(\ ||\ )$ be $\mathcal{K}_F$ where F is some fixed optimal functional.

2. *Let $(F_k)_{k\in\mathbb{N}}$ be a partial computable enumeration of $PC^{\{0,1\}^* \times P(\mathbb{N}) \to \mathbb{X}}$. Let $\mathcal{U} : \{0,1\}^* \times P(\mathbb{N}) \to \mathbb{X}$ be such that*

$$\mathcal{U}(0^k 1\mathtt{p}, A) = F_k(\mathtt{p}, A) \qquad \mathcal{U}(0^k) = 0$$

Then $\mathcal{U}$ is optimal in $PC^{\{0,1\}^* \times P(\mathbb{N}) \to \mathbb{X}}$.

Proof. It clearly suffices to prove Point 2. The usual proof of the classical invariance theorem gives indeed the functional version stated above.

$$\begin{aligned}
\mathcal{K}_{F_k}(\mathtt{x} \,||\, A) &= \min\{|\mathtt{p}| : F_k(\mathtt{p}, A) = \mathtt{x}\} \\
&= \min\{|\mathtt{p}| : \mathcal{U}(0^k 1\mathtt{p}, A) = \mathtt{x}\} \\
&= \min\{|0^k 1\mathtt{p}| - k - 1 : \mathcal{U}(0^k 1\mathtt{p}, A) = \mathtt{x}\} \\
&\geq \min\{|\mathtt{q}| - k - 1 : \mathcal{U}(\mathtt{q}, A) = \mathtt{x}\} \\
&= \min\{|\mathtt{q}| : \mathcal{U}(\mathtt{q}, A) = \mathtt{x}\} - k - 1 \\
&= \mathcal{K}_{\mathcal{U}}(\mathtt{x} \,||\, A) - k - 1
\end{aligned}$$

Whence $\mathcal{K}_{\mathcal{U}} \leq \mathcal{K}_{F_k} + k + 1$ and therefore $\mathcal{K}_{\mathcal{U}} \leq_{\mathrm{ct}} \mathcal{K}_{F_k}$. □

Remark 5.5.
1. Obviously, $\mathcal{K}_F(\mathtt{x} \,||\, A)$ does depend on A. For example, if $\mathtt{x} \in \mathbb{N}$ is incompressible then $\mathcal{K}_F(\mathtt{x} \,||\, \emptyset) =_{\mathrm{ct}} \log(\mathtt{x})$ whereas $\mathcal{K}_F(\mathtt{x} \,||\, \{\mathtt{x}\}) =_{\mathrm{ct}} 0$.
The contents of the functional invariance theorem is that, for some F's (the optimal ones) the number

$$\max\{\mathcal{K}_F(\mathtt{x} \,||\, A) - \mathcal{K}_G(\mathtt{x} \,||\, A) : \mathtt{x} \in \mathbb{N}, A \in P(\mathbb{N})\}$$

is finite for any given G.

2. For the functional invariance theorem, we only have to suppose the enumeration $(F_k)k \in \mathbb{N}$ to be partial computable as a functional $\mathbb{N} \times \{0,1\}^* \times P(\mathbb{N}) \to \mathbb{X}$. There is no need that it be acceptable (cf. Def.2.5).

As for the usual Kolmogorov complexity, computable approximation from above is possible.

Proposition 5.6. *There exists a total computable functional*

$$(\mathtt{x}, t, A) \in \mathbb{X} \times P(\mathbb{N}) \times \mathbb{N} \mapsto \mathcal{K}^t(\mathtt{x} \,||\, A)$$

which is decreasing with respect to t *and such that, for all* $\mathtt{x}, A$*,*

$$\mathcal{K}(\mathtt{x} \,||\, A) = \min\{\mathcal{K}^t(\mathtt{x} \,||\, A) : t \in \mathbb{N}\}$$

Proof. Letting $\mathcal{K} = \mathcal{K}_{\mathcal{U}}$ where $\mathcal{K} \in PC^{\{0,1\}^* \to \mathbb{N}}$, set

$$\begin{array}{rcl} B(\mathtt{x}, t, A) & = & \{|\mathtt{p}| : |\mathtt{p}| \leq t \ \wedge \ \mathcal{U}(\mathtt{p}, A) = \mathtt{x} \ \wedge \ \mathcal{U}(\mathtt{p}, A) \text{ halts in } \leq t \text{ steps}\} \\ T(\mathtt{x}, A) & = & \text{smallest } t \text{ such that } B(\mathtt{x}, t, A) \neq \emptyset \\ \mathcal{K}^t(\mathtt{x} \,||\, A) & = & \text{smallest } |\mathtt{p}| \in B(\mathtt{x}, t, A) \cup B(\mathtt{x}, T(\mathtt{x}, A), A) \end{array}$$

□

6 The Min/Max hierarchy of Kolmogorov complexities

Infinite computations in relation with Kolmogorov complexity were first considered in Chaitin, 1976 [3] and Solovay, 1977 [16]. Becher & Daicz & Chaitin, 2001 [1], introduced a variant H^∞ of the prefix version of Kolmogorov complexity by allowing programs leading to possibly infinite computations but finite output (i.e. remove the sole halting condition). This variant satisfies $H^{\emptyset'} <_{\rm ct} H^\infty <_{\rm ct} H$ (cf. [1, 2]).
In [5], 2004, we introduced a machine-free definition $K_{\rm max}$ of the usual (non prefix) Kolmogorov version K^∞, together with a dual version $K_{\rm min}$. The proof in [2] of the above inequalities extends easily to the K setting for $K_{\rm max}$. However, a different argument is required in order to get the $K_{\rm min}$ version (cf. [5]).

6.1 Min/Max Kolmogorov complexities

The following definitions and theorems collects material from [5]. The classical way to define Kolmogorov complexity extends directly to these classes.

Theorem 6.1 (Min/Max Invariance theorem).
1. *Let* $\mathcal{F}$ *be* $Min_{PR}^{\{0,1\}^* \to \mathbb{N}}$ *or* $Max_{PR}^{\{0,1\}^* \to \mathbb{N}}$ *(cf. Def.2.13). When* ϕ *varies in* $\mathcal{F}$ *there is a least* K_ϕ*, up to an additive constant (cf. Notation 2.1):*

$$\begin{array}{lll} \exists \phi \in Min_{PR}^{\{0,1\}^* \to \mathbb{N}} & \forall \psi \in Min_{PR}^{\{0,1\}^* \to \mathbb{N}} & K_\phi \leq_{\rm ct} K_\psi \\ \exists \phi \in Max_{PR}^{\{0,1\}^* \to \mathbb{N}} & \forall \psi \in Max_{PR^A}^{\{0,1\}^* \to \mathbb{N}} & K_\phi \leq_{\rm ct} K_\psi \end{array}$$

Such ϕ's are said to optimal in $\mathcal{F}$.
We let
- $K_{\min}$ denote K_ϕ where ϕ is any function optimal in $Min_{PR}^{\{0,1\}^*\to\mathbb{N}}$,
- $K_{\max}$ denote K_ϕ where ϕ is any function optimal in $Max_{PR}^{\{0,1\}^*\to\mathbb{N}}$.

2. *Suppose $(\phi_k)_{k\in\mathbb{N}}$ is an enumeration of $Min_{PR}^{\{0,1\}^*\to\mathbb{N}}$ such that the function $(k,\mathtt{p})\mapsto\phi_k(\mathtt{p})$ is in $Min_{PR}^{\mathbb{N}\times\{0,1\}^*\to\mathbb{N}}$. Let $U_{\min}$ be such that*

$$U_{\min}(0^k1\mathtt{p})=\phi_k(\mathtt{p}) \qquad U_{\min}(0^k)=\phi_k(\lambda)$$

Then $U_{\min}$ is optimal in $Min_{PR}^{\{0,1\}^*\to\mathbb{N}}$.
Idem with $Max_{PR}^{\{0,1\}^*\to\mathbb{N}}$.

3. *Relativizing to an oracle $A\subseteq\mathbb{N}$, one similarly defines $K_{\min}^A$ and $K_{\max}^A$ and the analog of Point 2 also holds.*

Remark 6.2 ([5]). There exists optimal functions for $Max_{PR}^{\{0,1\}^*\to\mathbb{N}}$ of the form $\max f$ where $f:\{0,1\}^*\times\mathbb{N}\to\mathbb{N}$ is *total recursive.*
This is false for $Min_{PR}^{\{0,1\}^*\to\mathbb{N}}$.

Relativizing to the successive jumps oracles, we get an infinite family of Kolmogorov complexities for which holds a hierarchy theorem.

Theorem 6.3 (The Min/Max Kolmogorov hierarchy, [5]).

$$\log >_{\mathrm{ct}} K >_{\mathrm{ct}} \begin{array}{c}K_{\min}\\K_{\max}\end{array} >_{\mathrm{ct}} K^{\emptyset'} >_{\mathrm{ct}} \begin{array}{c}K_{\min}^{\emptyset'}\\K_{\max}^{\emptyset'}\end{array} >_{\mathrm{ct}} K^{\emptyset''} >_{\mathrm{ct}} \begin{array}{c}K_{\min}^{\emptyset''}\\K_{\max}^{\emptyset''}\end{array} >_{\mathrm{ct}} K^{\emptyset'''}\ldots$$

Strict inequalities $K >_{\mathrm{ct}} K_{\max} >_{\mathrm{ct}} K^{\emptyset'} >_{\mathrm{ct}} K_{\max}^{\emptyset'} >_{\mathrm{ct}} K^{\emptyset''}$ were first proved by Becher & Chaitin, 2001–2002 [1] (for the prefix variants).

The main application of the $\ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ and $\ll_{\mathcal{F}\uparrow}^{\mathcal{C},\mathcal{D}}$ orderings introduced in §4 is a strong improvement of this hierarchy theorem, cf. Thm.8.14.

Finally, we shall need the following result (cf. [5], or [1] as concerns $K_{\max}$).

Theorem 6.4. *$K, K_{\min}, K_{\max}$ are recursive in $\emptyset'$.*

6.2 Functional Min/Max Kolmogorov complexities

The Invariance Theorems for Max_{PR} and Min_{PR} (cf. Thm.6.1) admit functional versions, the proofs of which are exactly the same as that in Thm.5.4.

Theorem 6.5 (Min/Max Functional Invariance Theorem).
1. *When $F:\{0,1\}^*\times P(\mathbb{N})\to\mathbb{N}$ varies over $Min_{PC}^{\{0,1\}^*\times P(\mathbb{N})\to\mathbb{N}}$ or over $Max_{PC}^{\{0,1\}^*\times P(\mathbb{N})\to\mathbb{N}}$, there is a least K_F up to an additive constant:*

$$\exists F\in Min_{PC}^{P(\mathbb{N})\times\{0,1\}^*\to\mathbb{N}} \quad \forall G\in Min_{PC}^{P(\mathbb{N})\times\{0,1\}^*\to\mathbb{N}} \quad \mathcal{K}_F\le_{\mathrm{ct}}\mathcal{K}_G$$
$$\exists F\in Max_{PC}^{P(\mathbb{N})\times\{0,1\}^*\to\mathbb{N}} \quad \forall G\in Max_{PC}^{P(\mathbb{N})\times\{0,1\}^*\to\mathbb{N}} \quad \mathcal{K}_F\le_{\mathrm{ct}}\mathcal{K}_G$$

Such an F is said to be optimal in $Min_{PC}^{P(\mathbb{N})\times\{0,1\}^*\to\mathbb{N}}$ or in $Max_{PC}^{P(\mathbb{N})\times\{0,1\}^*\to\mathbb{N}}$.
We let $K_{\min}(\ \|\ ) = \mathcal{K}_F$ and $K_{\max}(\ \|\ ) = \mathcal{K}_F$ be some fixed such optimal functionals.

2. *Let $(F_k)_{k\in\mathbb{N}}$ be an enumeration of $Min_{PC}^{P(\mathbb{N})\times\{0,1\}^*\to\mathbb{N}}$ which is itself in $Min_{PC}^{\mathbb{N}\times P(\mathbb{N})\times\{0,1\}^*\to\mathbb{N}}$. Let $\mathcal{U}_{\min}$ be such that*

$$\mathcal{U}_{\min}(0^k 1\mathtt{p}, A) = F_k(\mathtt{p}, A) \qquad \mathcal{U}_{\min}(0^k) = 0$$

Then $\mathcal{U}_{\min}$ is optimal in $Min_{PC}^{\{0,1\}^*\to\mathbb{N}}$.
One defines similarly $\mathcal{U}_{\max}$ which is optimal in Max_{PC}.

Remark 6.6.
1. Using the technique of [5], we see that there exists optimal functionals for Max_{PC} of the form $\max F$ where $F : \{0,1\}^* \times \mathbb{N} \times P(\mathbb{N}) \to \mathbb{N}$ is *total recursive*. This is false for Min_{PC}.

2. The inclusions $PC \subseteq Max_{PC} \cap Min_{PC}$ imply that $\mathcal{K}_{\min}(\ \|\ ) \leq_{\rm ct} \mathcal{K}(\ \|\ )$ and $\mathcal{K}_{\max}(\ \|\ ) \leq_{\rm ct} \mathcal{K}(\ \|\ )$. Also, as is well-known, $\mathcal{K}(\ \|\ ) \leq_{\rm ct} \log$. We can choose $\mathcal{K}_{\min}, \mathcal{K}_{\max}$ so that the constant is 0, i.e. for all x and A,

$$\mathcal{K}_{\min}(x\,\|\,A) \leq \mathcal{K}(x\,\|\,A) \leq \log \quad , \quad \mathcal{K}_{\max}(x\,\|\,A) \leq \mathcal{K}(x\,\|\,A) \leq \log$$

3. In fact, the Min/Max hierarchy Theorem 6.3 extends to the functional setting. In §8.6 we shall prove a much stronger result, cf. Thm.8.14.

7 Functional versus oracular

Functional Kolmogorov complexities allow for a uniform choice of oracular Kolmogorov complexities. The benefit of such a uniform choice was developed in §1.2 and is illustrated in the hierarchy theorem in §8.6.

Theorem 7.1. *Denote $K^A, K^A_{\min}, K^A_{\max} : \mathbb{X} \to \mathbb{N}$ the Kolmogorov complexities associated to the families PR^A of partial A-recursive functions and the families Min_{PR^A}, Max_{PR^A} obtained by application of the* min *and* max *operators to $PR^{A,\mathbb{X}\times\mathbb{N}\to\mathbb{N}}$. For all $A \subseteq \mathbb{N}$, we have*

$$K^A =_{\rm ct} \mathcal{K}(\ \|\,A) \quad , \quad K^A_{\min} =_{\rm ct} \mathcal{K}_{\min}(\ \|\,A) \quad , \quad K^A_{\max} =_{\rm ct} \mathcal{K}_{\max}(\ \|\,A)$$

$$\text{i.e. } \forall\ A \in P(\mathbb{N})\ \exists c_A\ \forall \mathtt{x} \left\{ \begin{array}{lcl} |K^A(\mathtt{x}) - \mathcal{K}(\mathtt{x}\,\|\,A)| & \leq & c_A \\ |K^A_{\min}(\mathtt{x}) - \mathcal{K}_{\min}(\mathtt{x}\,\|\,A)|) & \leq & c_A \\ |K^A_{\max}(\mathtt{x}) - \mathcal{K}_{\max}(\mathtt{x}\,\|\,A)|) & \leq & c_A \end{array} \right.$$

Proof. 1. We let $(F_k)_{k\in\mathbb{N}}$ and $\mathcal{U}$ be as in Point 2 of Thm.5.4 and let $F^A_k(\mathtt{x}) = F_k(\mathtt{x}, A)$ and $U^A(\mathtt{x}) = \mathcal{U}(\mathtt{x}, A)$. The sequence $(F^A_k)_{k\in\mathbb{N}}$ is an enumeration of the family $PR^{A,\{0,1\}^*\to\mathbb{N}}$ of partial A-recursive functions $\{0,1\}^* \to \mathbb{X}$, which

is partial A-recursive as a function $\mathbb{N} \times \{0,1\}^* \to \mathbb{X}$. Since $U^A(0^k 1\mathtt{p}) = F_k^A(\mathtt{p})$, the classical invariance theorem, in its relativized version, insures that U^A is optimal in $PR^{A,\{0,1\}^* \to \mathbb{N}}$, whence $K^A =_{\mathrm{ct}} K_{U^A}$.
Now, $\mathcal{U}$ is optimal in $PC^{P(\mathbb{N}) \times \{0,1\}^* \to \mathbb{N}}$, whence $\mathcal{K}(\ ||\ ) =_{\mathrm{ct}} \mathcal{K}_{\mathcal{U}}(\ ||\ )$.
Prop.5.3 insures $K_{U^A}(\mathtt{x}) = \mathcal{K}_{\mathcal{U}}(\mathtt{x}||A)$, whence $K^A =_{\mathrm{ct}} \mathcal{K}(\ ||A)$.

2. The Min and Max cases are similar. □

8 Refining the oracular Min/Max hierarchy with the $\ll, \ll_\uparrow$ orderings

8.1 Barzdins' theorem in a uniform setting

The next lemma is essentially Barzdins' result cited in §3.1. In order (point 2) to get a relativized result with θ recursive rather than merely A-recursive, we shall look at the oracle A as a parameter and use uniform Kolmogorov complexity, cf. §5.2, §7.

Lemma 8.1.
1. *If* $\varphi : \mathbb{N} \to \mathbb{N}$ *is total recursive and tends to* $+\infty$ *then* $\{x : K(x) < \varphi(x)\}$ *is an r.e. set which is constructively* Σ^0_1*-dense.*
Moreover, this result is uniform in φ*. In fact, let* $(\varphi_i)_{i\in\mathbb{N}}$ *and* $(W_i)_{i\in\mathbb{N}}$ *be acceptable enumerations of partial recursive functions* $\mathbb{N} \to \mathbb{N}$ *and r.e. subsets of* $\mathbb{N}$*, there are total recursive functions* $\xi : \mathbb{N} \to \mathbb{N}$ *and* $\theta : \mathbb{N}^2 \to \mathbb{N}$ *such that*

i. $\forall i\ \{x \in domain(\varphi_i) : K(x) < \varphi_i(x)\} = W_{\xi(i)}$

ii. $\forall i, j\ (\varphi_i$ *is unbounded on* $domain(\varphi_i) \cap W_j$
$$\Rightarrow\ (W_{\theta(i,j)} \text{ is infinite } \wedge\ W_{\theta(i,j)} \subseteq W_j \cap \{x : K(x) < \varphi_i(x)\}))$$

2. *Consider second order Kolmogorov complexity* $\mathcal{K}(x\,||\,A)$ *and an acceptable enumeration* $(\Phi_i)_{i\in\mathbb{N}}$ *of partial computable functionals* $\mathbb{N}\times P(\mathbb{N}) \to \mathbb{N}$*. Using Thm.7.1 and Prop.2.7, we shall consider* $\mathcal{K}(x\,||\,A)$ *as uniform Kolmogorov relativization* $K^A(x)$ *and* $\Phi_i(\mathtt{x}, A)$ *as a uniform oracle* A *partial recursive function* $\varphi_i^A(x)$*. We also denote* $W_i^A = domain(\varphi_i^A)$*.*
Point 1 relativizes uniformly, i.e., the above total recursive functions $\xi : \mathbb{N} \to \mathbb{N}$ *and* $\theta : \mathbb{N}^2 \to \mathbb{N}$ *can be taken so as to satisfy* all *possible relativized conditions, i.e.*

i. $\forall i\ \forall A\ \{x \in domain(\varphi_i^A) : K^A(x) < \varphi_i^A(x)\} = W^A_{\xi(i)}$

ii. $\forall i, j\ \forall A\ (\varphi_i^A$ *is unbounded on* $domain(\varphi_i^A) \cap W_j^A$
$$\Rightarrow\ (W^A_{\theta(i,j)} \text{ is infinite } \wedge\ W^A_{\theta(i,j)} \subseteq W_j^A \cap \{x : K^A(x) < \varphi_i^A(x)\}))$$

Note 8.2. Lemma 8.1 is optimal in the sense that there is no possible $(\Pi^0_1, \mathcal{E})$-density result for $\{x : K(x) < \varphi(x)\}$ since this set has Π^0_1 complement.

Proof. Point 1i. Let $K = K_U$ where $U \in PR^{\{0,1\}^* \to \mathbb{N}}$. Then

$$K(x) < \varphi_i(x) \quad \Leftrightarrow \quad \exists \mathtt{p}\ (|\mathtt{p}| < \varphi_i(x)\ \wedge\ U(\mathtt{p}) = x)$$

which is a Σ^0_1 condition. Therefore $\{(i,x) : K(x) < \varphi_i(x)\}$ is r.e. and the parametrization theorem yields the desired total recursive function ξ.

Point 1ii.
In order to prove constructive Σ^0_1-density uniformly in φ, we first define a partial recursive function $\alpha : \mathbb{N}^2 \times \{0,1\}^* \to \mathbb{N}$ such that

> if there exists some $x \in W_j$ such that $\varphi_i(x) \geq 2\,|\mathtt{p}|$ then $\alpha(i,j,\mathtt{p})$ is such an x, else $\alpha(i,j,\mathtt{p})$ is undefined.

Then we shall use the facts that

$$K \leq_{\mathrm{ct}} K_{\mathtt{p} \mapsto \alpha(i,j,\mathtt{p})} \quad , \quad K_{\mathtt{p} \mapsto \alpha(i,j,\mathtt{p})}(\alpha(i,j,\mathtt{p})) \leq_{\mathrm{ct}} |\mathtt{p}|$$

to get an inequality $K(\alpha(i,j,\mathtt{p})) \leq_{\mathrm{ct}} |\mathtt{p}|$ from which $K(\alpha(i,j,\mathtt{p})) < \varphi_i(\mathtt{p})$ can be deduced.

a. The formal definition of α as a partial recursive function is as follows. Denote $W_{j,t}$ the finite subset of W_j obtained after t steps of its standard enumeration. Let $Z_t : \mathbb{N}^2 \times \{0,1\}^* \to P(\mathbb{N})$ be such that

$$Z_t(i,j,\mathtt{p}) = \{x \in W_{j,t} : \varphi_i(x) \text{ halts in } \leq t \text{ steps and is } > 2\,|\mathtt{p}|\}$$

Clearly, $\{(t,i,j,\mathtt{p}) : Z_t(i,j,\mathtt{p}) \neq \emptyset\}$ is a recursive subset of $\mathbb{N}^3 \times \{0,1\}^*$. Thus, we can define the partial recursive function α as follows:

$$\begin{array}{rcl} \mathit{domain}(\alpha) & = & \{(i,j,\mathtt{p}) : \exists t\ Z_t(i,j,\mathtt{p}) \neq \emptyset\} \\ \alpha(i,j,\mathtt{p}) & = & \text{the first element in } Z_t(i,j,\mathtt{p}) \\ & & \text{where } t \text{ is least such that } Z_t(i,j,\mathtt{p}) \neq \emptyset \end{array}$$

Let $(\psi_i)_{i \in \mathbb{N}}$ be an acceptable enumeration of $PR^{\{0,1\}^* \to \mathbb{N}}$. Since α is partial recursive, there exists a total recursive function $\eta : \mathbb{N}^2 \to \mathbb{N}$ such that $\alpha(i,j,\mathtt{p}) = \psi_{\eta(i,j)}(\mathtt{p})$ for all $i,j,\mathtt{p}$. Finally, we let θ be a total recursive function such that

$$W_{\theta(i,j)} = \psi_{\eta(i,j)}(\{\mathtt{p} : |\mathtt{p}| > \eta(i,j)\ \wedge\ (i,j,\mathtt{p}) \in \mathit{domain}(\psi_{\eta(i,j)})\})$$

Since α and $\psi_{\eta(i,j)}$ take values in W_j, we have $W_{\theta(i,j)} \subseteq W_j$ for all i,j.

b. Let $U : \{0,1\}^* \to \mathbb{N}$ be such that $U(0^k 1\mathtt{p}) = \psi_k(\mathtt{p})$ and $U(0^k) = \psi_k(\lambda)$ (where λ is the empty word). The usual invariance theorem insures that U

is optimal. Thus, we can (and shall) suppose that $K = K_U$.
Since $\alpha(i,j,\mathtt{p}) = \psi_{\eta(i,j)}(\mathtt{p})$, we have $\alpha(i,j,\mathtt{p}) = U(0^{\eta(i,j)}1\mathtt{p})$. Thus, for any $(i,j,\mathtt{p}) \in domain(\alpha)$,

$$K(\alpha(i,j,\mathtt{p})) = K_U(U(0^{\eta(i,j)}1\mathtt{p})) \leq |0^{\eta(i,j)}1\mathtt{p}| = |\mathtt{p}| + \eta(i,j) + 1$$

c. Suppose now that φ_i is unbounded on $domain(\varphi_i) \cap W_j$. Then, for all $\mathtt{p}$, the set $Z_t(i,j,\mathtt{p})$ is non empty for t big enough, so that $\alpha(i,j,\mathtt{p}) = \psi_{\eta(i,j)}(\mathtt{p})$ is defined for all $\mathtt{p}$. Also, due to the definition of Z_t, we see that $\alpha(i,j,\mathtt{p})$ tends to $+\infty$ with the length of $\mathtt{p}$. In particular, $W_{\theta(i,j)}$ is infinite.
From the definition of α, we see that $\varphi_i(\alpha(i,j,\mathtt{p})) > 2|\mathtt{p}|$. Using *b*, we see that for all $|\mathtt{p}| > \eta(i,j)$, we have $K(\alpha(i,j,\mathtt{p})) \leq 2\,\mathtt{p} < \varphi(\alpha(i,j,\mathtt{p}))$.
This proves that $W_{\theta(i,j)}$ is included in $\{x : K(x) < \varphi_i(x)\}$.
Thus, $W_{\theta(i,j)}$ is an infinite r.e. set included in $W_j \cap \ \{x : K(x) < \varphi_i(x)\}$.

Point 2i. Let $\mathcal{K}(\ ||\ ) = \mathcal{K}_{\mathcal{U}}$ where $\mathcal{U} \in PC^{P(\mathbb{N})\times\{0,1\}^*\to\mathbb{N}}$. Then

$$K^A(x) < \varphi_i^A(x) \quad \Leftrightarrow \quad \exists \mathtt{p}\ (|\mathtt{p}| < \varphi_i^A x \ \wedge \ \mathcal{U}(\mathtt{p}, A) = x)$$

which is a Σ^0_1 condition. Therefore $\{(i,x,A) : K^A(x) < \varphi_i^A(x)\}$ is Σ^0_1 and the parametrization property (cf. Def.2.5) yields the desired total recursive function ξ.

Point 2ii. The proof is similar to that of Point 1ii. Just add everywhere a second order argument A varying in $P(\mathbb{N})$ and use the parametrization property of Def.2.5. Thus, α is now a partial computable functional

$$\alpha : \mathbb{N}^2 \times \{0,1\}^* \times P(\mathbb{N}) \to \mathbb{N}$$

The enumeration $(\psi_i)_{i\in\mathbb{N}}$ now becomes an enumeration $(\Psi_i)_{i\in\mathbb{N}}$ of the partial computable functionals $\{0,1\}^* \times P(\mathbb{N}) \to \mathbb{N}$. The total recursive functions η, θ are now such that $\alpha(i,j,\mathtt{p},A) = \Psi_{\eta(i,j)}(\mathtt{p},A)$ and

$$W^A_{\theta(i,j)} = \{\Psi_{\eta(i,j)}(\mathtt{p},A) : \mathtt{p} \text{ such that } (i,j,\mathtt{p},A) \in domain(\alpha) \wedge |\mathtt{p}| > \eta(i,j)\}$$

The arguments in *b,c* above go through with the superscript A everywhere and with $\mathcal{U}$ (cf. proof of Point 2i above) in place of U. □

Remark 8.3.
1. Lemma 8.1 still holds for $\phi \in Max_{PR}$ in place of $\varphi \in PR$. However, this does not really add: an easy argument shows that if $\phi \in Max_{PR}$ and $W_j \subseteq domain(\phi)$ is infinite then there exists an infinite $W_k \subseteq W_j$ and $\varphi_i \in PR$ such that $W_k \subseteq domain(\varphi_i)$ and $\varphi_i(x) \leq \phi(x)$ for all $x \in domain(\varphi_i)$. Moreover, k and i can be given by total recursive functions depending on j and a code for ϕ in Max_{PR}.
This also holds uniformly: replace φ by a functional $\Phi \in PC$.

2. Of course, Lemma 8.1 cannot hold for $\phi \in Min_{PR}$ since K is itself in Min_{PR}.

8.2 Comparing K and $K_{\max}$ à la Barzdins

In this subsection and the next one, we now come to central results of the paper, namely,

- *K can be compared to $K_{\max}, K_{\min}$ via the $\ll$ and $\ll_{\uparrow}$ orderings,*
- *$K_{\max}, K_{\min}$ can be compared via the $\mathrm{OftLess}_{\uparrow}$ relation.*

Notation 8.4. We shall write X is $(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{D})$-dense to mean X is $(\mathcal{C}_1, \mathcal{D})$-dense and $(\mathcal{C}_2, \mathcal{D})$-dense.

Remark 8.5. Let $\mathcal{C}_1 \vee \mathcal{C}_2$ be the family of sets $R_1 \cup R_2$ where $R_1 \in \mathcal{C}_1$ and $R_2 \in \mathcal{C}_2$.
If $\mathcal{C}_1, \mathcal{C}_2$ both contain the empty set (which is usually the case), then $\mathcal{C}_1 \cup \mathcal{C}_2 \subseteq \mathcal{C}_1 \vee \mathcal{C}_2$, and therefore $(\mathcal{C}_1 \vee \mathcal{C}_2, \mathcal{D})$-density (resp. constructive density) always implies $(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{D})$-density (resp. constructive density).
Conversely, every infinite set in $\mathcal{C}_1 \vee \mathcal{C}_2$ contains an infinite subset in $\mathcal{C}_1$ or in $\mathcal{C}_2$, so that $(\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{D})$-density implies — hence is equivalent to — $(\mathcal{C}_1 \vee \mathcal{C}_2, \mathcal{D})$-density. However, this is no more true as concerns *constructive density*: if $R_1 \cup R_2$ is infinite one cannot decide (from codes) which one of R_1 and R_2 is infinite.

Lemma 8.6.
1. *Suppose $\phi : \mathbb{N} \to \mathbb{N}$ is a total function in Min_{PR} which is monotone and tends to $+\infty$. Then the set $\{x : K_{\max}(x) < \phi(K(x))\}$ is constructively $(\Sigma^0_1 \cup \Pi^0_1, \exists^{\leq \phi}(\Sigma^0_1 \wedge \Pi^0_1))$-dense (cf. Def.3.11 Point 3).*
Moreover, this result is uniform in ϕ. In fact, let $(\phi_i)_{i \in \mathbb{N}}$ and $(W_i)_{i \in \mathbb{N}}$ be acceptable enumerations of Min_{PR} and r.e. subsets of $\mathbb{N}$. There are total recursive functions $\theta_0, \theta_1 : \mathbb{N}^2 \to \mathbb{N}$ such that, for all i, j, k, with the notations of Def.2.12, if $\phi_i \in Min_{PR}$ is total, monotone and tends to $+\infty$ then

$$\begin{array}{rcl}
W_j \textit{ is infinite} & \Rightarrow & W^{\exists^{\leq \phi_i}(\Sigma^0_1 \wedge \Pi^0_1)}_{\theta_0(i,j)} \textit{ is an infinite subset of} \\
& & W_j \cap \{x : K_{\max}(x) < \phi(K(x))\} \\
\mathbb{N} \setminus W_k \textit{ is infinite} & \Rightarrow & W^{\exists^{\leq \phi_i}(\Sigma^0_1 \wedge \Pi^0_1)}_{\theta_1(i,k)} \textit{ is an infinite subset of} \\
& & (\mathbb{N} \setminus W_k) \cap \{x : K_{\max}(x) < \phi(K(x))\}
\end{array}$$

2. *Consider Kolmogorov relativizations $K^A, K^A_{\max}$ obtained from second order Kolmogorov complexities $\mathcal{K}, \mathcal{K}_{\max}$ (cf. Thm.7.1) and enumerations $(\phi^A_i)_{i \in \mathbb{N}}$ and $(W^A_i)_{i \in \mathbb{N}}$ of Min^A_{PR} and A-r.e. sets which come from acceptable enumerations of functionals in $Min^{\mathbb{N} \times P(\mathbb{N}) \to \mathbb{N}}_{PC}$ and of Σ^0_1 subsets of $\mathbb{N} \times P(\mathbb{N})$ (cf. Prop.2.7). We shall also use notations from Def.2.12.*
Point 1 relativizes uniformly, i.e., the above total recursive functions $\theta_0, \theta_1 : \mathbb{N}^2 \to \mathbb{N}$ can be taken so as to satisfy all *possible relativized conditions. I.e.,*

if $\phi_i^A \in Min_{PR}^A$ *is total, monotone and tends to* $+\infty$ *then*

$$\begin{array}{rcl} W_j^A \text{ is infinite} & \Rightarrow & W_{\theta_0(i,j)}^{\exists^{\leq \phi_i^A}(\Sigma_1^{0,A} \wedge \Pi_1^{0,A})}) \text{ is an infinite subset of} \\ & & W_j^A \cap \{x : K_{\max}^A(x) < \phi^A(K^A(x))\} \\ \mathbb{N} \setminus W_k^A \text{ is infinite} & \Rightarrow & W_{\theta_1(i,k)}^{\exists^{\leq \phi_i^A}(\Sigma_1^{0,A} \wedge \Pi_1^{0,A})} \text{ is an infinite subset of} \\ & & (\mathbb{N} \setminus W_k^A) \cap \{x : K_{\max}^A(x) < \phi^A(K^A(x))\} \end{array}$$

Proof. 1. The strategy.
We essentially keep the strategy of the proof of Lemma 8.1. The idea is, for given i, j, to construct a Max_{PR} function $\alpha : \mathbb{N}^2 \times \{0,1\}^*$ such that $\alpha(i, j, \mathtt{p})$ is in W_j (or in $\mathbb{N} \setminus W_j$) and $\varphi_i(K(\alpha(i, j, \mathtt{p}))) > 2\,|\mathtt{p}|$. Then to use inequalities

$$K_{\max} \leq_{\text{ct}} K_{\mathtt{p} \mapsto \alpha(i,j,\mathtt{p})} \quad , \quad K_{\mathtt{p} \mapsto \alpha(i,j,\mathtt{p})}(\alpha(i, j, \mathtt{p})) \leq_{\text{ct}} |\mathtt{p}|$$

to get an inequality $K_{\max}(\alpha(i, j, \mathtt{p})) \leq_{\text{ct}} |\mathtt{p}|$ from which $K_{\max}(\alpha(i, j, \mathtt{p})) < \varphi_i(K(\alpha(i, j, \mathtt{p})))$ can be deduced.

As we have to deal with Σ_1^0 sets and with Π_1^0 sets, i.e. sets of the form W_j or $\mathbb{N} \setminus W_k$, we shall define two such functions α, namely α_0, α_1.
In order to get these functions in Max_{PR}, we define partial recursive functions $a_0, a_1 : \mathbb{N}^2 \times \{0,1\}^* \times \mathbb{N} \to \mathbb{N}$ and set $\max a_0 = \alpha_0$ and $\max a_1 = \alpha_1$.

2. Approximation of ϕ_i *from above.*
Let $\phi_i(x) = \min_t \varphi_i(x, t)$ where $(\varphi_i)_{i \in \mathbb{N}}$ is an acceptable enumeration of $PR^{\mathbb{N} \times \mathbb{N} \to \mathbb{N}}$.
Using the parametrization theorem, let $\xi : \mathbb{N} \to \mathbb{N}$ be a total recursive function such that $\varphi_{\xi(i)}$ has domain $\{x : \exists u\ \varphi_i(x, t) \text{ does halt}\} \times \mathbb{N}$ and satisfies

$$\begin{array}{rcl} \varphi_{\xi(i)}(x, 0) & = & \varphi_i(x, u) \text{ where } u \text{ is least such that } \varphi_i(x, u) \text{ does halt} \\ \varphi_{\xi(i)}(x, t+1) & = & \min(\{\varphi_{\xi(i)}(x, 0)\} \cup \{\varphi_i(x, v) : v \leq t \text{ and} \\ & & \qquad\qquad\qquad \varphi_i(x, v) \text{ halts in } \leq t \text{ steps}\}) \end{array}$$

Observe that $\varphi_{\xi(i)}(x, t)$ is decreasing in t and $\phi_i(x) = \min_t \varphi_{\xi(i)}(x, t)$, so that $\varphi_{\xi(i)}(x, t)$ is a partial recursive approximation of $\phi_i(x)$ from above.
Also, for any given i, x, either $\phi_i(x)$ is undefined and $\varphi_{\xi(i)}(x, t)$ is defined for no t or $\phi_i(x)$ is defined and $\varphi_{\xi(i)}(x, t)$ is defined for all t.

3. Functions a_ϵ *and* α_ϵ*.*
Denote $W_{j,t}$ the finite subset of W_j obtained after t steps of its standard enumeration. Denote $K^t(x)$ some total, recursive approximation of $K(x)$ from above which is decreasing in t (cf. Prop.5.6).

We define a_0, a_1 as follows:

$$a_0(i,j,\mathtt{p},0) \quad = \quad \text{the element which appears first in the standard enumeration of } W_j \text{ (hence undefined if } W_j \text{ is empty)}$$

$$a_0(i,j,\mathtt{p},t+1) = \begin{cases} a_0(i,j,\mathtt{p},t) & \text{if } \varphi_{\xi(i)}(K^t(a_0(i,j,\mathtt{p},t)),t) > 2\,|\mathtt{p}| \\ x & \text{if } \varphi_{\xi(i)}(K^t(a_0(i,j,\mathtt{p},t)),t) \leq 2\,|\mathtt{p}| \\ & \text{and } x \text{ is the next element which} \\ & \text{appears in the standard enumeration} \\ & \text{of } W_i \text{ and satisfies } x > a_0(i,j,\mathtt{p},t) \\ \mathit{undefined} & \text{if } \varphi_{\xi(i)}(K^t(a_0(i,j,\mathtt{p},t)),t) \text{ is undefined} \end{cases}$$

and

$$a_1(i,k,\mathtt{p},0) \quad = \quad 0$$

$$a_1(i,k,\mathtt{p},t+1) = \begin{cases} a_1(i,k,\mathtt{p},t) & \text{if } \varphi_{\xi(i)}(K^t(a_1(i,j,\mathtt{p},t)),t) > 2\,|\mathtt{p}| \\ & \text{and } a_1(i,k,\mathtt{p},t) \notin W_{k,t} \\ a_1(i,k,\mathtt{p},t)+1 & \text{if } \varphi_{\xi(i)}(K^t(a_1(i,j,\mathtt{p},t)),t) \leq 2\,|\mathtt{p}| \\ & \text{or } (\varphi_{\xi(i)}(K^t(a_1(i,j,\mathtt{p},t)),t) > 2\,|\mathtt{p}| \\ & \text{and } a_1(i,k,\mathtt{p},t) \in W_{k,t}) \\ \mathit{undefined} & \text{if } \varphi_{\xi(i)}(K^t(a_1(i,k,\mathtt{p},t)),t) \text{ is undefined} \end{cases}$$

Claim. *Suppose $\phi_i \in Min_{PR}$ is total monotone increasing and tends to $+\infty$.*
a. *If W_j is infinite then $(\mathtt{p},t) \mapsto a_0(i,j,\mathtt{p},t)$ and $\mathtt{p} \mapsto \alpha_0(i,j,\mathtt{p})$ are total functions and*

$$\forall \mathtt{p}\ (\alpha_0(i,j,\mathtt{p}) \in W_j\ \wedge\ \phi_i(K(\alpha_0(i,j,\mathtt{p}))) > 2\,|\mathtt{p}|)$$

b. *Function a_1 is always total. If $\mathbb{N} \setminus W_k$ is infinite then $\mathtt{p} \mapsto \alpha_1(i,k,\mathtt{p})$ is a total function and*

$$\forall \mathtt{p}\ (\alpha_1(i,k,\mathtt{p}) \notin W_k\ \wedge\ \phi_i(K(\alpha_1(i,k,\mathtt{p}))) > 2\,|\mathtt{p}|)$$

Proof of Claim.
As seen in 2 above, if ϕ_i is total so is $\varphi_{\xi(i)}$. This insures the total character of a_0 (resp. a_1).
Fix some $\mathtt{p}$. Since $\varphi_{\xi(i)}(x,t) \geq \phi_i(x)$, ϕ_i is monotone increasing and K, ϕ_i are total and tend to $+\infty$, for all large enough x and all t, we have

$$\varphi_{\xi(i)}(K^t(x),t) \geq \phi_i(K^t(x)) \geq \phi_i(K(x)) > 2\,|\mathtt{p}|$$

Suppose W_j is infinite. Then there are elements in W_j which satisfy $\phi_i(K(x)) > 2\,|\mathtt{p}|$. Let $x_0(i,j,\mathtt{p})$ be such an element which appears first in the standard enumeration of W_j. It is easy to see that, for all t large enough, we have $a_0(i,j,\mathtt{p},t) = x_0(i,j,\mathtt{p})$. Thus, $\alpha_0(i,j,\mathtt{p}) = x_0(i,j,\mathtt{p})$ is defined and $\alpha_0(i,j,\mathtt{p}) \in W_j \cap \{\phi_i(K(x)) > 2\,|\mathtt{p}|\}$. Which proves Point a of the Claim.

Suppose $\mathbb{N} \setminus W_k$ is infinite. Then there are elements in $\mathbb{N} \setminus W_k$ which satisfy $\phi_i(K(x)) > 2\,|\mathtt{p}|$. Let $x_1(i,k,\mathtt{p})$ be the least such element. It is easy to see that, for all t large enough (namely, for t such that $W_k \cap [0, x_1(i,k,\mathtt{p})[\subseteq W_{k,t}$), we have $a_1(i,k,\mathtt{p},t) = x_1(i,k,\mathtt{p})$. Thus, $\alpha_1(i,k,\mathtt{p}) = x_1(i,k,\mathtt{p})$ is defined and $\alpha_1(i,k,\mathtt{p}) \in (\mathbb{N} \setminus W_k) \cap \{\phi_i(K(x)) > 2\,|\mathtt{p}|\}$. Which proves Point b of the Claim. $\Box$ *(Claim)*

4. Functions $\eta_\epsilon, \theta_\epsilon$.
Let $(\psi_n)_{n\in\mathbb{N}}$ be an acceptable enumeration of partial recursive functions $\{0,1\}^* \times \mathbb{N} \to \mathbb{N}$. Since $a_0, a_1 : \mathbb{N}^2 \times \{0,1\}^* \times \mathbb{N} \to \mathbb{N}$ are partial recursive, the parametrization property insures that there exists total recursive functions $\eta_0, \eta_1 : \mathbb{N}^2 \to \mathbb{N}$ such that, for all $i, j, k, \mathtt{p}, t$ and $\epsilon = 0, 1$,

$$a_\epsilon(i,j,\mathtt{p},t) = \psi_{\eta_\epsilon(i,j)}(\mathtt{p},t)$$

Taking the max over t, and letting $\alpha_\epsilon = \max a_\epsilon$, we get, for all $i, j, \mathtt{p}$,

$$\alpha_\epsilon(i,j,\mathtt{p},t) = (\max \psi_{\eta_\epsilon(i,j)})(\mathtt{p})$$

For all i, j, k, set

$$Y_\epsilon(i,j) = \{\alpha_\epsilon(i,j,\mathtt{p}) : (i,j,\mathtt{p}) \in \mathit{domain}(\alpha_\epsilon) \ \wedge\ |\mathtt{p}| > \eta_\epsilon(i,j)\}$$

Using the Claim and inequality $K(y) \le y$ (which we always can suppose), observe that

$$\begin{aligned} y \in Y_\epsilon(i,j) &\Leftrightarrow \exists \mathtt{p}\ (2\,|\mathtt{p}| < \phi_i(K(y)) \ \wedge\ |\mathtt{p}| > \eta_\epsilon(i,j) \ \wedge y = \alpha_\epsilon(i,j,\mathtt{p})) \\ &\Leftrightarrow \exists \mathtt{p}\ (|\mathtt{p}| < \phi_i(y) \ \wedge\ |\mathtt{p}| > \eta_\epsilon(i,j) \ \wedge y = \alpha_\epsilon(i,j,\mathtt{p})) \end{aligned}$$

Using Prop.2.15, we see that this is $\exists^{\le \phi_i}(\Sigma^0_1 \wedge \Pi^0_1)$ in i, j, y (cf. Def.2.9). We let $\theta_0, \theta_1 : \mathbb{N}^2 \to \mathbb{N}$ be total recursive functions such that

$$W^{\exists^{\le \phi_i}(\Sigma^0_1 \wedge \Pi^0_1)[\mathbb{N}]}_{\theta_\epsilon(i,j)} = Y_\epsilon(i,j)$$

5. Point 1 of the Lemma.
Let $U : \{0,1\}^* \times \mathbb{N} \to \mathbb{N}$ be such that $U(0^n 1\mathtt{p}, t) = \psi_n(\mathtt{p}, t)$ and $U(0^n, t) = \psi_n(\lambda, t)$ (where λ is the empty word). Taking the max over t, we get $(\max U)(0^n 1\mathtt{p}) = (\max \psi_n)(\mathtt{p})$ and $(\max U)(0^n) = (\max \psi_n)(\lambda)$. Since the $\max \psi_n$'s enumerate $Max^{\{0,1\}^* \to \mathbb{N}}_{PR}$, the invariance theorem 6.1 insures that $\max U$ is optimal in $Max^{\{0,1\}^* \to \mathbb{N}}_{PR}$. Thus, we can (and shall) suppose that $K_{\max} = K_{(\max U)}$.
Since $\alpha_\epsilon(i,j,\mathtt{p}) = (\max \psi_{\eta_\epsilon(i,j)})(\mathtt{p}) = (\max U)(0^{\eta_\epsilon(i,j)} 1\mathtt{p})$, we get

$$\begin{aligned} K_{\max}(\alpha_\epsilon(i,j,\mathtt{p})) &= K_{(\max U)}((\max U)(0^{\eta_\epsilon(i,j)} 1\mathtt{p}) \\ &\le \eta_\epsilon(i,j) + 1 + |\mathtt{p}| \\ &\le 2\,|\mathtt{p}| \qquad \text{in case } |\mathtt{p}| > \eta_\epsilon(i,j) \end{aligned}$$

Suppose ϕ_i is total, monotone and tends to $+\infty$ and W_j (resp. $\mathbb{N} \setminus W_k$) is infinite. Using the last inequality and that from the above Claim relative to $\epsilon = 0$ (resp. $\epsilon = 1$), we see that, for $|\mathtt{p}| > \eta_\epsilon(i,j)$, we have

$$K_{(\max U)}(\alpha_\epsilon(i,j,\mathtt{p})) \leq 2\,|\mathtt{p}| < \phi_i(K(\alpha_\epsilon(i,j,\mathtt{p}))$$

Which proves that $W^{\exists^{\leq \phi_i}(\Sigma^0_1 \wedge \Pi^0_1)[\mathbb{N}]}_{\theta_\epsilon(i,j,k)}$ is included in $\{x : K_{\max}(x) < \phi_i(K(x))\}$. Using the Claim again, this set is also included in W_j (resp. $(\mathbb{N} \setminus W_k)$. This finishes the proof of Point 1 of the Lemma.

6. Point 2 of the Lemma.
The proof is similar to that of Point 1. Just add everywhere a second order argument A varying in $P(\mathbb{N})$ and use the parametrization property of Def.2.5. Thus, a_0, a_1 are now partial computable functionals

$$\mathbb{N}^2 \times \{0,1\}^* \times P(\mathbb{N}) \times \mathbb{N} \to \mathbb{N}$$

The enumeration $(\psi_n)_{n\in\mathbb{N}}$ now becomes an enumeration $(\Psi_n)_{n\in\mathbb{N}}$ of the partial computable functionals $\{0,1\}^* \times P(\mathbb{N}) \to \mathbb{N}$. The total recursive functions $\eta_\epsilon, \theta_\epsilon$ are now such that

$$\begin{array}{rcl}
a_\epsilon(i,j,\mathtt{p},A,t) & = & \Psi_{\eta_\epsilon(i,j)}(\mathtt{p},A,t) \\
\alpha_\epsilon(i,j,\mathtt{p},A) & = & (\max \Psi_{\eta_\epsilon(i,j)})(\mathtt{p},A) \\
W^A_{\theta_\epsilon(i,j,j)} & = & \{\alpha_\epsilon(i,j,\mathtt{p},A) : (i,j,\mathtt{p},A) \in domain(\alpha_\epsilon) \ \wedge \ |\mathtt{p}| > \eta_\epsilon(i,j)\}
\end{array}$$

and U has to be changed to $\mathcal{U} \in PC^{P(\mathbb{N})\times\{0,1\}^* \to \mathbb{N}}$ such that $\mathcal{K}(\ || \ ) = \mathcal{K}_{\mathcal{U}}$. The arguments for the proof of Point 1 above go through with the superscript A everywhere. $\Box$

8.3 Comparing K and $K_{\min}$ à la Barzdins

We shall need the following notion to get an analog of Lemma 8.6 with $K_{\min}$.

Definition 8.7. The growth function of an infinite set $X \subseteq \mathbb{N}$ is defined as

$$growth_X(n) = (n+1)\text{-th point of } X$$

The infinite set X has recursively bounded growth if $growth_X \leq \psi$ for some total recursive function $\psi : \mathbb{N} \to \mathbb{N}$.

Lemma 8.8.
1. *Let's denote $\widetilde{\Pi^{0,A}_1}$ the family of infinite $\Pi^{0,A}_n$ subsets of $\mathbb{N}$ with A-recursively bounded growth.*
Suppose $\phi : \mathbb{N} \to \mathbb{N}$ is a total function in Min_{PR} which is monotone and tends to $+\infty$. Then the set $\{x : K_{\min}(x) < \phi(K(x))\}$ is constructively $(\Sigma^0_1 \cup \widetilde{\Pi^0_1}, \exists^{\leq\phi}(\Sigma^0_1 \wedge \Pi^0_1))$-dense (cf. Def.3.11 Point 3).
Moreover, this result is uniform in ϕ and in a recursive ψ bound for the Π^0_1

set. In fact, let $(\phi_i)_{i\in\mathbb{N}}$*,* $(\psi_m)_{m\in\mathbb{N}}$ *and* $(W_i)_{i\in\mathbb{N}}$ *be acceptable enumerations of* Min_{PR}*,* PR *and of r.e. subsets of* $\mathbb{N}$*. There are total recursive functions* $\theta_0 : \mathbb{N}^2 \to \mathbb{N}$ *and* $\theta_1 : \mathbb{N}^3 \to \mathbb{N}$ *such that, for all* i, j, m, k*, with the notations of Def.2.12, if* $\phi_i \in Min_{PR}$ *and* $\psi_m \in PR$ *are total, monotone and tend to* $+\infty$ *then*

$$W_j \textit{ is infinite } \Rightarrow W_{\theta_0(i,j)}^{\exists^{\leq\phi_i}(\Sigma_1^0\wedge\Pi_1^0)} \textit{ is an infinite subset of } W_j \cap \{x : K_{\min}(x) < \phi_i(K(x))\}$$

$$\mathbb{N}\setminus W_k \textit{ is infinite and } \psi_m \geq growth_{\mathbb{N}\setminus W_k} \Rightarrow W_{\theta_1(i,k,m)}^{\exists^{\leq\phi_i}(\Sigma_1^0\wedge\Pi_1^0)} \textit{ is an infinite subset of } (\mathbb{N}\setminus W_k) \cap \{x : K_{\min}(x) < \phi_i(K(x))\}$$

2. *Consider Kolmogorov relativizations* $K^A, K^A_{\min}$ *and enumerations* $(\phi_i^A)_{i\in\mathbb{N}}$ *and* $(\psi_m^A)_{m\in\mathbb{N}}$ *and* $(W_i^A)_{i\in\mathbb{N}}$ *of* Min^A_{PR}*,* PR^A *and A-r.e. sets as in Point 2 of Lemma 8.6.*
Point 1 relativizes uniformly, i.e., the above total recursive functions $\theta_0, \theta_1 : \mathbb{N}^2 \to \mathbb{N}$ *can be taken so as to satisfy* all *possible relativized conditions. I.e., if* ϕ_i^A, ψ_m^A *are total, monotone and tend to* $+\infty$ *then*

$$W_j^A \textit{ is infinite } \Rightarrow W_{\theta_0(i,j)}^{\exists^{\leq\phi_i^A}(\Sigma_1^{0,A}\wedge\Pi_1^{0,A})} \textit{ is an infinite subset of } W_j^A \cap \{x : K^A_{\min}(x) < \phi_i^A(K^A(x))\}$$

$$\mathbb{N}\setminus W_k^A \textit{ is infinite and } \psi_m^A \geq growth_{\mathbb{N}\setminus W_k^A} \Rightarrow W_{\theta_1(i,k,m)}^{\exists^{\leq\phi_i^A}(\Sigma_1^{0,A}\wedge\Pi_1^{0,A})} \textit{ is an infinite subset of } (\mathbb{N}\setminus W_k^A) \cap \{x : K^A_{\min}(x) < \phi^A(K^A(x))\}$$

Proof. 1. The strategy. The proof follows that of Lemma 8.6 except that now α_ϵ is equal to $\min a_\epsilon$ and that a_1 and α_1 also depend on the index m of the recursive majorant ψ_m of the growth function of the Π_1^0 set.
Since α_0 (resp. α_1) has to be in Min_{PR}, i.e. is to be recursively approximated *from above*, we have to force that, for given $i, j, k, m, \mathtt{p}$, the first defined $a_0(i, j, \mathtt{p}, t)$ (resp. $a_1(i, k, m, \mathtt{p}, t)$) majorizes an element x of W_j (resp. $\mathbb{N}\setminus W_k$) which is such that $K_{\min}(x) < \phi_i(K(x))$.
To insure this, we choose $a_0(i, j, \mathtt{p}, 0)$ (resp. $a_1(i, k, m, \mathtt{p}, 0)$) so that the interval $[0, a_\epsilon(i, j, \mathtt{p}, 0)[$ (resp. $[0, a_\epsilon(i, k, m, \mathtt{p}, 0)[$) contains at least $2^{2|\mathtt{p}|+1}$ points in $\{x : K_{\min}(x) < \phi_i(K(x))\}$.

2. We shall use the partial recursive approximation from above $\varphi_{\xi(i)}(x, t)$ of $\phi_i(x)$ defined in point 2 of the proof of Lemma 8.6.

3. Functions a_ϵ *and* α_ϵ*.*

Let $Z_0(i,j,\mathtt{p})$ be the set of $2^{2|\mathtt{p}|+1}$ distinct elements which appear first in the standard enumeration of W_i. We define a_0 as follows:

$$\begin{array}{rcl}
a_0(i,j,\mathtt{p},0) & = & \text{the largest element of } Z_0(i,j,\mathtt{p}) \\
a_0(i,j,\mathtt{p},t+1) & = & \begin{cases} a_0(i,j,\mathtt{p},t) & \text{if } \varphi_{\xi(i)}(K^t(a_0(i,j,\mathtt{p},t)),t) > 2\,|\mathtt{p}| \\ x & \text{if } \varphi_{\xi(i)}(K^t(a_0(i,j,\mathtt{p},t)),t) \leq 2\,|\mathtt{p}| \\ & \text{and } x \text{ is the largest element of} \\ & Z_0(i,j,\mathtt{p}) \ \cap\ [0, a_0(i,j,\mathtt{p},t)[\\ \mathit{undefined} & \text{if } \varphi_{\xi(i)}(K^t(a_0(i,j,\mathtt{p},t)),t) \text{ is undefined} \end{cases}
\end{array}$$

We now define a_1, using the recursive majorant ψ_m.

$$\begin{array}{rcl}
a_1(i,k,m,\mathtt{p},0) & = & \psi_m(2^{2|\mathtt{p}|+1}) \\
 & & \text{Let } u = \varphi_{\xi(i)}(K^t(a_1(i,k,m,\mathtt{p},t)),t) \\
a_1(i,k,m,\mathtt{p},t+1) & = & \begin{cases} a_1(i,k,m,\mathtt{p},t) & \text{if } u > 2\,|\mathtt{p}| \\ & \text{and } a_1(i,k,m,\mathtt{p},t) \notin W_{k,t} \\ a_1(i,k,m,\mathtt{p},t) - 1 & \text{if } u \leq 2\,|\mathtt{p}| \text{ or } (u > 2\,|\mathtt{p}| \\ & \ \text{and } a_1(i,k,m,\mathtt{p},t) \in W_{k,t}) \\ \mathit{undefined} & \text{if } u \text{ is undefined} \end{cases}
\end{array}$$

Clearly, a_0 and a_1 are partial recursive.

Claim. *Suppose $\phi_i \in Min_{PR}$ is total monotone increasing and tends to $+\infty$.*
a. *If W_j is infinite then $(\mathtt{p},t) \mapsto a_0(i,j,\mathtt{p},t)$ and $\mathtt{p} \mapsto \alpha_0(i,j,\mathtt{p})$ are total functions and*

$$\forall \mathtt{p}\ (\alpha_0(i,j,\mathtt{p}) \in W_j\ \wedge\ \phi_i(K(\alpha_0(i,j,\mathtt{p}))) > 2\,|\mathtt{p}|)$$

b. *Function a_1 is total. If $\mathbb{N} \setminus W_k$ is infinite and ψ_m is a total recursive function such that $\psi_m \geq growth_{\mathbb{N}\setminus W_k}$ then $\mathtt{p} \mapsto \alpha_1(i,k,m,\mathtt{p})$ is a total function and*

$$\forall \mathtt{p}\ (\alpha_1(i,k,m,\mathtt{p}) \notin W_k\ \wedge\ \phi_i(K(\alpha_1(i,k,m,\mathtt{p}))) > 2\,|\mathtt{p}|)$$

Proof of Claim.
As seen in the proof of Lemma 8.6, if ϕ_i is total then so are $\varphi_{\xi(i)}$ and a_0, a_1. Also, for any fixed $\mathtt{p}$, for all large enough x and all t, we have

$$\varphi_{\xi(i)}(K^t(x),t) \geq \phi_i(K^t(x)) \geq \phi_i(K(x)) > 2\,|\mathtt{p}|$$

Suppose W_j is infinite. Then $Z_0(i,j,\mathtt{p})$ contains exactly $2^{2|\mathtt{p}|+1}$ elements. Let $K_{\min} = K_U$ where $U \in Min_{PR}^{\{0,1\}^* \to \mathbb{N}}$. Since there are $2^{2|\mathtt{p}|+1} - 1$ words with length $\leq 2\,|\mathtt{p}|$, there is necessarily some element of $x \in Z_0(i,j,\mathtt{p})$ which is not in $U(\{\mathtt{q} : |\mathtt{q}| \leq 2\,|\mathtt{p}|\})$, hence is such that $K_{\min}(x) = K_U(x) > 2\,|\mathtt{p}|$. Let $x_0(i,j,\mathtt{p})$ be the largest such element. It is easy to see that, for all t large enough, we have $a_0(i,j,\mathtt{p},t) = x_0(i,j,\mathtt{p})$. Thus, $\alpha_0(i,j,\mathtt{p}) = x_0(i,j,\mathtt{p})$ is defined and $\alpha_0(i,j,\mathtt{p}) \in W_j \cap \{\phi_i(K(x)) > 2\,|\mathtt{p}|\}$. Which proves Point a of the Claim.

Suppose $\mathbb{N} \setminus W_k$ is infinite and ψ_m is a total recursive function such that $\psi_m \geq growth_{\mathbb{N}\setminus W_k}$. Then there are $2^{2|\mathtt{p}|+1}$ elements of $\mathbb{N} \setminus W_k$ which are $\leq \psi_m(2^{2|\mathtt{p}|+1})$. As above, there is necessarily some such element x which is not in $U(\{\mathtt{q} : |\mathtt{q}| \leq 2\,|\mathtt{p}|\})$, hence is such that $K_{\min}(x) = K_U(x) > 2\,|\mathtt{p}|$. Let $x_1(i,k,m,\mathtt{p})$ be the largest such element. It is easy to see that, for all t large enough (namely, for t such that $W_k \cap [0, x_1(i,k,m,\mathtt{p})[\subseteq W_{k,t})$, we have $a_1(i,j,m,\mathtt{p},t) = x_1(i,j,m,\mathtt{p})$. Thus, $\alpha_1(i,k,m,\mathtt{p}) = x_1(i,k,m,\mathtt{p})$ is defined and $\alpha_1(i,k,m,\mathtt{p}) \in (\mathbb{N} \setminus W_k) \cap \{\phi_i(K(x)) > 2\,|\mathtt{p}|\}$. Which proves Point b of the Claim. □ *(Claim)*

We conclude the proof of the Lemma as that of Lemma 8.6 with analogous points 4,5,6 : the sole modification is to replace everywhere $K_{\max}$ by $K_{\min}$ and the max operator by the min one. □

8.4 Comparing $K_{\min}$ and $K_{\max}$ à la Barzdins

We shall need the following result from [5] (Thm 7.15).

Proposition 8.9. $K \leq_{\mathrm{ct}} 2\,K_{\min} + K_{\max}$.

Using Prop.8.9, Lemmas 8.6, 8.8 yield the following corollary.

Lemma 8.10.
1. *Let's denote* $\widetilde{\Pi^0_1}$ *the family of infinite* Π^0_1 *subsets of* $\mathbb{N}$ *with recursively bounded growth.*
Suppose $\phi : \mathbb{N} \to \mathbb{N}$ *is a total function in* Min_{PR} *which is monotone and tends to* $+\infty$. *Then*

- *i.* $\{x : K_{\max}(x) < \phi(K_{\min}(x))\}$ *is constructively* $(\Sigma^0_1 \cup \Pi^0_1, \exists^{\leq\phi}(\Sigma^0_1 \wedge \Pi^0_1))$-*dense.*
- *ii.* $\{x : K_{\min}(x) < \phi(K_{\max}(x))\}$ *is constructively* $(\Sigma^0_1 \cup \widetilde{\Pi^0_1}, \exists^{\leq\phi}(\Sigma^0_1 \wedge \Pi^0_1))$-*dense*

Moreover, this result is uniform in ϕ *and, for ii, in a recursive bound for the* Π^0_1 *set, in the sense detailed in Lemmas 8.6,8.8.*

2. *Consider Kolmogorov relativizations* $K^A, K^A_{\min}, K^A_{\max}$ *and enumerations* $(\phi^A_i)_{i\in\mathbb{N}}$ *and* $(\psi^A_m)_{m\in\mathbb{N}}$ *and* $(W^A_i)_{i\in\mathbb{N}}$ *of* Min^A_{PR}, PR^A *and A-r.e. sets as in Point 2 of Lemmas 8.6, 8.8.*
Then Point 1 relativizes uniformly in the sense detailed in Lemmas 8.6,8.8.

Proof. Let $\phi \in Min_{PR}$ be total, monotone increasing and unbounded. Set

$$\theta(x) = \min\{\frac{x}{4}, \phi(\max(0, \lfloor\frac{x-c}{2}\rfloor))\}$$

Then θ is also a total, monotone increasing and unbounded function in Min_{PR}. Also, one can recursively go from a code for ϕ to one for θ. Using Lemmas 8.6, 8.8, it suffices to prove that, for all x,

$$\begin{array}{rcl} K_{\min}(x) < \theta(K(x)) & \Rightarrow & K_{\min}(x) < \phi(K_{\max}(x)) \\ K_{\max}(x) < \theta(K(x)) & \Rightarrow & K_{\max}(x) < \phi(K_{\min}(x)) \end{array}$$

We prove the first implication, the second one being similar.
Applying Prop.8.9, let c be such that, for all x,

$$K(x) < 2\,K_{\min}(x) + K_{\max}(x) + c$$

Suppose $K_{\min}(x) < \theta(K(x))$. Then $K_{\min}(x) < \frac{1}{4}\,K(x)$, so that

$$K(x) \ < \ 2\,K_{\min}(x) + K_{\max}(x) + c \ \leq \ \frac{K(x)}{2} + K_{\max}(x) + c$$

and $K(x) < 2\,(K_{\max}(x) + c)$.
Therefore, $K_{\min}(x) < \theta(K(x)) \leq \theta(2\,(K_{\max}(x) + c)) \leq \phi(K_{\max}(x))$. □

8.5 Syntactical complexity

Whereas $\{x : K(x) < \phi(x)\}$ is r.e. whenever ϕ is partial recursive (cf. Lemma 8.1), the complexity of the sets considered in Lemmas 8.6, 8.8, 8.10 to compare $K, K_{\max}, K_{\min}$ is much higher and does involve bounded quantifications over boolean combinations of Σ^0_1 sets as is the case in the density results obtained in these lemmas.

Proposition 8.11. *Let ϕ be a total function in Min_{PR}. The sets*

$$\begin{array}{ll} \{x : K_{\max}(x) < \phi(K(x))\} & \{x : K_{\max}(x) < \phi(K_{\min}(x))\} \\ \{x : K_{\min}(x) < \phi(K(x))\} & \{x : K_{\min}(x) < \phi(K_{\max}(x))\} \end{array}$$

are all definable by formulas of the form

$$\exists^{\leq \log}\ \forall^{\leq \log}\ (A \wedge B \wedge C)$$

where A, B, C are $\Sigma^0_1 \vee \Pi^0_1$. In particular, theses sets are Δ^0_2 (cf. Prop.2.11).

Proof. Without loss of generality, we can suppose that $K_{\max}(x)$ and $K_{\min}(x)$ are both $\leq \log(x)$ for all x. Let $U : \mathbb{N} \to \mathbb{N}$ and $V, W, \varphi : \mathbb{N}^2 \to \mathbb{N}$ be partial recursive functions such that $K = K_U$ and $K_{\min} = K_\alpha$ and $K_{\max} = K_\beta$ and $\phi(x) = \min_t \varphi(x,t)$ where $\alpha(x) = \min_t V(x,t)$ and $\beta(x) = \min_t W(x,t)$.
Following a usual convention, we shall write $\exists \mathtt{p}^{|\mathtt{p}| \leq x} \ldots$ and $\forall \mathtt{p}^{|\mathtt{p}| \leq x} \ldots$ in place

of $\exists \mathtt{p}\ (|\mathtt{p}| \leq x\ \wedge ...)$ and $\forall \mathtt{p}\ (|\mathtt{p}| \leq x\ \Rightarrow ...)$.
Then $K_{\max}(x) < \phi(K(x))$ if and only if

$$\exists \mathtt{p_1}^{|\mathtt{p_1}|\leq \log(x)}\ \exists \mathtt{p_2}^{|\mathtt{p_2}|\leq \log(x)}\ \forall \mathtt{q_1}^{|\mathtt{q_1}|<|\mathtt{p_1}|}\ \forall \mathtt{q_2}^{|\mathtt{q_2}|<|\mathtt{p_2}|}$$

$$[U(\mathtt{p_1}) = x\ \wedge\ U(\mathtt{q_1}) \neq x$$

$$\wedge\ \exists t\ V(\mathtt{p_2}, t) = x\ \wedge\ \forall t\ (V(\mathtt{p_2}, t) \text{ is undefined or } \leq x)$$

$$\wedge\ (\forall t\ (V(\mathtt{q_2}, t) \text{ is undefined or } \neq x) \vee\ \exists t\ V(\mathtt{q_2}, t) > x)$$

$$\wedge\ \forall t\ (\varphi(|\mathtt{p_1}|, t) \text{ is undefined or } |\mathtt{p_2}| < \varphi(|\mathtt{p_1}|, t))]$$

Which is a formula of the form stated in the Proposition. All three other cases are similar. ☐

Bounded quantifications over boolean combinations of Σ^0_1 sets are also involved for the set of integers with $K, K_{\max}, K_{\min}$ incompressible binary representations.

Proposition 8.12. *The set*

$$I = \{x : \min(K(x), K_{\max}(x), K_{\min}(x)) \geq \lfloor \log(x) \rfloor - 1\}$$

is infinite and is definable by a formula of the form

$$\forall^{\leq \log}\ (A \wedge B)$$

where A, B are $\Sigma^0_1 \vee \Pi^0_1$. In particular, this set is Δ^0_2.

Proof. Without loss of generality we shall suppose that $K \leq K_{\max}$ and $K \leq K_{\min}$. The usual argument to get incompressible integers works: there are $\sum_{i<n} 2^i = 2^n - 1$ programs $\mathtt{p}$ with length $< n$, hence at most $2\,(2^n - 1)$ integers x such that $K_{\max}(x) < n$ or $K_{\min}(x) < n$. Thus, for every n, there exists an integer $x \leq 2^{n+1} - 1$ such that $K_{\max}(x), K_{\min}(x) \geq n$. Observe that such an x is necessarily in I since $\log(x) \leq \log(2^{n+1} - 1) < n + 1$. Which shows that I is infinite.

We let V, W be as in the proof of Prop.8.11. Then $x \in I$ can be written

$$\forall \mathtt{p}^{|\mathtt{p}|<\lfloor \log(x) \rfloor - 1}\ [(\forall t\ (V(\mathtt{p}, t) \text{ is undefined or } \neq x) \vee\ \exists t\ V(\mathtt{p}, t) > x)$$

$$\wedge\ (\forall t\ (W(\mathtt{p}, t) \text{ is undefined or } \neq x) \vee\ \exists t\ W(\mathtt{p}, t) < x)]$$

Which is a formula of the form stated in the Proposition. All three other cases are similar. ☐

Remark 8.13. In case ϕ is small enough (say $\phi(z) \leq z - 1$), the set I is obviously disjoint from all fours sets considered in Prop.8.11.

8.6 The hierarchy theorem

We can now prove the central application of the $\mathrm{OftLess}_{\uparrow}$ relation and the $\ll$ and $\ll_{\uparrow}$ orderings. Namely, a strong hierarchy theorem for $K, K_{\max}, K_{\min}$ and their oracular versions using the successive jumps oracles.
Whereas Thm.6.3 involves the sole $<_{\mathrm{ct}}$ ordering, the refinment obtained in Thm.8.14 below involves a chain of more and more complex orderings which all refine $<_{\mathrm{ct}}$ and are relevant of Thm.4.6 and Cor.4.7.

Theorem 8.14 (The hierarchy theorem). *Let B_n be the subclass of Δ^0_n subsets of $\mathbb{N}$ consisting of sets definable by formulas of the form $\exists^{\leq\mu}(\Sigma^0_n \wedge \Pi^0_n)$ where $\mu : \mathbb{N} \to \mathbb{N}$ is a total function which is recursive in $\emptyset^{(n-1)}$. Let $\widetilde{\Pi^0_n}$ be the set of Π^0_n sets with $\emptyset^{(n)}$-recursively bounded growth (cf. Def.8.7).*
Then

1. $\log \gg^{\Sigma^0_1,\Sigma^0_1}_{\mathrm{PR}} \mathrm{K} \gg^{\Sigma^0_1\cup\Pi^0_1,\mathrm{B}_1}_{\mathrm{Min}_{\mathrm{PR}}\uparrow} \mathrm{K}_{\max} \gg^{\Sigma^0_2,\Sigma^0_2}_{\mathrm{PR}^{\emptyset'}} \mathrm{K}^{\emptyset'} \ldots$

$$\ldots \gg^{\Sigma^0_n,\Sigma^0_n}_{\mathrm{PR}^{\emptyset^{(n-1)}}} \mathrm{K}^{\emptyset^{(n-1)}} \gg^{\Sigma^0_n\cup\Pi^0_n,\mathrm{B}_n}_{\mathrm{Min}_{\mathrm{PR}^{\emptyset^{(n-1)}}}\uparrow} \mathrm{K}^{\emptyset^{(n-1)}}_{\max} \gg^{\Sigma^0_{n+1},\Sigma^0_{n+1}}_{\mathrm{PR}^{\emptyset^{(n)}}} \mathrm{K}^{\emptyset^{(n)}} \ldots$$

2. $\log \gg^{\Sigma^0_1,\Sigma^0_1}_{\mathrm{PR}} \mathrm{K} \gg^{\Sigma^0_1\cup\widetilde{\Pi^0_1},\mathrm{B}_1}_{\mathrm{Min}_{\mathrm{PR}}\uparrow} \mathrm{K}_{\min} \gg^{\Sigma^0_2,\Sigma^0_2}_{\mathrm{PR}^{\emptyset'}} \mathrm{K}^{\emptyset'} \ldots$

$$\ldots \gg^{\Sigma^0_n,\Sigma^0_n}_{\mathrm{PR}^{\emptyset^{(n-1)}}} \mathrm{K}^{\emptyset^{(n-1)}} \gg^{\Sigma^0_n\cup\widetilde{\Pi^0_n},\mathrm{B}_n}_{\mathrm{Min}_{\mathrm{PR}^{\emptyset^{(n-1)}}}\uparrow} \mathrm{K}^{\emptyset^{(n-1)}}_{\min} \gg^{\Sigma^0_{n+1},\Sigma^0_{n+1}}_{\mathrm{PR}^{\emptyset^{(n)}}} \mathrm{K}^{\emptyset^{(n)}} \ldots$$

3. *There is a constant c such that all $>_{\mathrm{ct}}$ inequalities in 1 and 2 (which are inherent to the $\gg$ and $\gg_{\uparrow}$ orderings) are $>$ inequalities up to c.*

4. *Though $K_{\max}$ and $K_{\min}$ are $\leq_{\mathrm{ct}}$ incomparable, we have*

$$K^{\emptyset^{(n-1)}}_{\max} \quad \mathrm{OftLess}^{\Sigma^0_n\cup\Pi^0_n,B_n}_{Min_{PR^{\emptyset^{(n-1)}}}\uparrow} \quad K^{\emptyset^{(n-1)}}_{\min}$$

$$K^{\emptyset^{(n-1)}}_{\min} \quad \mathrm{OftLess}^{\Sigma^0_n\cup\Pi^{0,\leq rec^{\emptyset^{(n-1)}}}_n,B_n}_{Min_{PR^{\emptyset^{(n-1)}}}\uparrow} \quad K^{\emptyset^{(n-1)}}_{\max}$$

Proof. 1. $\leq_{\mathrm{ct}}$ inequalities. Inequality $\log \geq_{\mathrm{ct}} K$ is well-known. The inclusions (cf. Prop.2.16)

$$PR^{\emptyset^{(n)}} \subseteq Min^{\emptyset^{(n)}}_{PR} \subseteq PR^{\emptyset^{(n+1)}} \quad , \quad PR^{\emptyset^{(n)}} \subseteq Max^{\emptyset^{(n)}}_{PR} \subseteq PR^{\emptyset^{(n+1)}}$$

yield inequalities

$$K^{\emptyset^{(n)}} \geq_{\mathrm{ct}} K^{\emptyset^{(n)}}_{\min} \geq_{\mathrm{ct}} K^{\emptyset^{(n+1)}} \quad , \quad K^{\emptyset^{(n)}} \geq_{\mathrm{ct}} K^{\emptyset^{(n)}}_{\max} \geq_{\mathrm{ct}} K^{\emptyset^{(n+1)}}.$$

2. Inequalities ... $\gg$ $\mathrm{K}^{\emptyset^{(i)}}$. Lemma 8.1 with $A = \emptyset$ and $\varphi \circ \log$ in place of φ yields inequality $\log \gg^{\Sigma^0_1,\Sigma^0_1}_{\mathrm{PR}} \mathrm{K}$.
Since $K^A_{\max}$ is recursive in A' (cf. Thm.6.4), Lemma 8.1 with $A = \emptyset^{(n-1)}$ and $\varphi \circ K^{\emptyset^{(n-1)}}_{\max}$ in place of φ yields inequality $K^{\emptyset^{(n-1)}}_{\max} \gg^{\Sigma^0_{n+1},\Sigma^0_{n+1}}_{\mathrm{PR}^{\emptyset^{(n)}}} \mathrm{K}^{\emptyset^{(n)}}$.
Idem with $K_{\min}$.

3. Inequalities $K^{\emptyset^{(i)}} \gg \ldots$. Direct application of Lemmas 8.6, 8.8.

4. Point 3 of the theorem. This is the benefit of the uniform oracular property obtained in Lemmas 8.1, 8.6, 8.8, 8.10.

5. Finally, the OftLess relations (Point 4 of the theorem) are direct application of Lemma 8.10. □

Remark 8.15. The scattered character of comparisons with respect to the $\ll$ orderings is unavoidable since all complexities $K, K_{\max}, K_{\min}, ..., K^{\emptyset^{(n)}}$ are equal up to a constant on the infinite set of integers with $K^{\emptyset^{(n)}}$ incompressible binary representations.

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