# Trace Results on Domains with Self-Similar Fractal Boundaries

Yves Achdou, Nicoletta Tchou

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**Abstract**

This work deals with trace theorems for a family of ramified bidimensional domains $\Omega$ with a self-similar fractal boundary $\Gamma^\infty$. The fractal boundary $\Gamma^\infty$ is supplied with a probability measure $\mu$ called the self-similar measure. Emphasis is put on the case when the domain is not a $\epsilon - \delta$ domain and the fractal is not post-critically finite, for which classical results cannot be used. It is proven that the trace of a square integrable function belongs to $L^p_\mu$ for all real numbers $p \geq 1$. A counterexample shows that the trace of a function in $H^1(\Omega)$ may not belong to $BMO(\mu)$ (and therefore may not belong to $L^\infty_\mu$). Finally, it is proven that the traces of the functions in $H^1(\Omega)$ belong to $H^s(\Gamma^\infty)$ for all real numbers $s$ such that $0 \leq s < d_H/4$, where $d_H$ is the Hausdorff dimension of $\Gamma^\infty$. Examples of functions whose traces do not belong to $H^s(\Gamma^\infty)$ for all $s > d_H/4$ are supplied.
There is an important contrast with the case when $\Gamma^\infty$ is post-critically finite, for which the square integrable functions have their traces in $H^s(\Gamma^\infty)$ for all $s$ such that $0 \leq s < d_H/2$.

## 1 Introduction

This work deals with some properties of $H^1(\Omega)$ for a family of ramified domains $\Omega$ of $\mathbb{R}^2$ with a self-similar fractal boundary called $\Gamma^\infty$ below, see Figure 1. The domain $\Omega$ depends on a parameter $a$, $1/2 \leq a \leq a^*$. The restriction $a \leq a^*$ allows for the construction of $\Omega$ as a union of non-overlapping sub-domains, see (2) below. The inessential condition $1/2 \leq a$ ensures that the Hausdorff dimension of $\Gamma^\infty$ is not smaller than one.
Such a geometry can be seen as a bidimensional idealization of the bronchial tree, for example. Indeed, this work is part of a wider project aimed at simulating the diffusion of medical sprays in lungs. Since the exchanges between the lungs and the circulatory system take place only in the last generations of the bronchial tree (the smallest structures), reasonable models for the diffusion of, e.g., oxygen may involve a non-homogeneous Neumann or Robin condition on the boundary $\Gamma^\infty$. Similarly, the lungs are mechanically coupled to the diaphragm, which also implies non-homogeneous boundary conditions on $\Gamma^\infty$, if one is interested in a coupled fluid-structure model. It is therefore necessary to study traces of functions on $\Gamma^\infty$. In that respect, the present work is a continuation of [2].
Sobolev spaces of functions defined in irregular domains have been widely studied in the literature:

- Jones [6] (and Vodopjanov et al [18] in the case $n = 2$, see also [11, 7]) have studied the open bounded subsets $\Omega$ of $\mathbb{R}^n$ such that there exists a continuous extension operator from

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$W^{\ell,p}(\Omega)$ in $W^{\ell,p}(\mathbb{R}^n)$, for all nonnegative integers $\ell$ and real numbers $p$, $1 \leq p \leq \infty$. Jones has proved that if $\Omega$ is a $\epsilon - \delta$ domain for some parameters $\epsilon, \delta > 0$, see [6, 7] for the definition, then the above extension property is true. Moreover, in dimension two, if the extension property stated above is true, then $\Omega$ is a $\epsilon - \delta$ domain for some parameters $\epsilon, \delta > 0$. In dimension two, the definition of such domains is equivalent to that of quasi-disks, see [11].

- Jonsson and Wallin [7] have considered closed subsets $F$ of $\mathbb{R}^n$ supplied with a Borel measure $\mu$ such that there exists a positive real number $d$ and two positive constants $c_1$ and $c_2$ with
$$c_1 r^d \leq \mu(B(x,r)) \leq c_2 r^d,$$
for all $x \in F$ and $r < 1$ (here $B(x,r)$ is the ball in $F$ with center $x$ and radius $r$, with respect to the Euclidean distance in $\mathbb{R}^n$); in [7], these sets are called $d$-sets. Sobolev and Besov spaces can be defined on the $d$-sets. Using as a main ingredient Whitney extension theory, Jonsson and Wallin have proved extension and trace results for Besov and Sobolev spaces. In particular, see [7] page 103, there exists a continuous trace operator from $W^{\alpha,p}(\mathbb{R}^n)$ onto $W^{\alpha-\frac{n-d}{p},p}(F)$, if $1 \leq p \leq \infty$ and $0 < \alpha - \frac{n-d}{p} < 1$.

- There is also a a growing interest on analysis on self-similar fractal sets, see for instance Kigami [8], Strichartz [16, 17], Mosco[14, 13] and references therein. These works aim at intrinsically defining function spaces using Dirichlet forms and a different metric from the Euclidean one. Most of the results concerning analysis on fractal sets are subject to the important assumption that the set is post-critically finite (or p.c.f.), see [8], page 23 for the definition.

- The question of extensions or traces naturally arises in boundary value or transmission problems in domains with fractal boundaries. Results in this direction have been given in [15, 10, 9] for the Koch flake. Here also, the assumption that the fractal set is p.c.f. is generally made.

Our goal here is to study the traces of functions of $H^1(\Omega)$ on the fractal boundary $\Gamma^\infty$. Note that this is different from considering the traces of functions of $H^1(\mathbb{R}^2)$ on $\Gamma^\infty$.

Following Hutchinson[5] and Kigami[8], $\Gamma^\infty$ is supplied with a probability measure $\mu$ called the self-similar measure and defined in (17) below. It will also be seen that with $\mu$, $\Gamma^\infty$ is a $d_H$-set, where $d_H = -\frac{\log 2}{\log a} \geq 1$, so the results of Jonsson and Wallin can be applied.
On the other hand, it is possible to define a trace operator which maps $H^1(\Omega)$ on $L^2_\mu(\Gamma^\infty)$. This has been done in [2].
We will see that if $1/2 \leq a < a^*$, then $\Omega$ is a $\epsilon - \delta$ domain and $\Gamma^\infty$ is post-critically finite. In this case, the results of [6] and those of [7] can be combined to obtain trace results. This is done in § 4.1 below: we prove that the trace of a function in $H^1(\Omega)$ belongs to $H^s(\Gamma^\infty)$ for $0 \leq s \leq d_H/2$.
The main part of the present paper focuses on the case $a = a^*$ for which $\Omega$ is not a $\epsilon - \delta$ domain and $\Gamma^\infty$ is not post-critically finite. Up to our knowledge, this case is not studied in the available literature.
The main results are

- Theorem 4 , which states that if $1/2 \leq a \leq a^*$ then the trace of a function in $H^1(\Omega)$ belongs to $L^p_\mu(\Gamma^\infty)$, for all real numbers $p$ such that $1 \leq p < \infty$.

- Theorem 5 in the case $a = a^*$ which gives an example of a function in $H^1(\Omega)$ whose trace on $\Gamma^\infty$ has not a bounded mean oscillation with respect to $\mu$, (and, as a consequence, does not belong to $L^\infty_\mu(\Gamma^\infty)$).
- Theorem 6 in the case $a = a^*$, which states that the trace of a function in $H^1(\Omega)$ belongs to $H^s(\Gamma^\infty)$ defined in (29) below, for all real numbers $s$ such that $0 \le s < d_H/4$, where $d_H$ is the Hausdorff dimension of $\Gamma^\infty$, and Proposition 5, which states the existence of a function in $H^1(\Omega)$ whose trace does not belong to $H^s(\Gamma^\infty)$ for all $s > d_H/4$. Note the important contrast with the case $a < a^*$ for which the trace of a function in $H^1(\Omega)$ belongs to $H^s(\Gamma^\infty)$ for $0 \le s \le d_H/2$. Similar results concerning the spaces $W^{1,p}(\Omega)$, $2 \le p < \infty$, are given in Remark 11.

The article is organized as follows: the geometry is presented in section 2; in particular, the critical case $a = a^*$ when $\Omega$ is not a quasi-disk and $\Gamma^\infty$ is not a post-critical set is carefully discussed. In section 3, we recall some of the results of [2] on the space $H^1(\Omega)$, concerning Poincaré inequality and the construction of the trace operator. The main part of the paper is Section 4 where the accurate trace results mentioned above are given. Finally, for the ease of the reader, the most technical proofs (involving geometrical lemmas) are postponed to two appendices at the end of the paper.

## 2 The geometry

Let $a$ be a positive parameter. Consider two similitudes $F_i$, $i = 1, 2$ respectively defined by the following:

$$F_i(x) = \begin{pmatrix} (-1)^i\left(1 - \frac{a}{\sqrt{2}}\right) + \frac{a}{\sqrt{2}}\left(x_1 + (-1)^i x_2\right) \\ 1 + \frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}}\left(x_2 + (-1)^{i+1} x_1\right) \end{pmatrix}.$$

The similitude $F_i$ has the dilation ratio $a$ and the rotation angle $(-1)^{i+1}\pi/4$.
Consider also two points in $\mathbb{R}^2$, $P_1 = (-1, 0)$, $P_2 = (1, 0)$, and define $P_3 = F_1(P_1) = (-1, 1)$, $P_4 = F_2(P_2) = (1, 1)$, $P_5 = F_1(P_2) = (-1+a\sqrt{2}, 1+a\sqrt{2})$ and $P_6 = F_2(P_1) = (1-a\sqrt{2}, 1+a\sqrt{2})$. Let $Y^0$ be the open hexagonal subset of $\mathbb{R}^2$ defined as the convex hull of the last six points.

$$Y^0 = \text{Interior}\Big(\ \text{Conv}(P_1, P_2, P_3, P_4, P_5, P_6)\Big).$$

It is easily seen that one must choose $a \le \sqrt{2}/2$ to prevent $F_1(Y^0)$ and $F_2(Y^0)$ from overlapping. For $n \ge 1$, we call $\mathcal{A}_n$ the set containing all the $2^n$ mappings from $\{1, \dots, n\}$ to $\{1, 2\}$. Similarly $\mathcal{A}_\infty = \{1, 2\}^{\mathbb{N}}$. We define

$$\mathcal{M}_\sigma = F_{\sigma(1)} \circ \dots \circ F_{\sigma(n)} \quad \text{for } \sigma \in \mathcal{A}_n, \tag{1}$$

and the ramified open domain, see Figure 1,

$$\Omega = \text{Interior}\left(\overline{Y^0} \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma(\overline{Y^0})\right)\right). \tag{2}$$

Stronger constraints must be imposed on $a$ to prevent the sets $\mathcal{M}_\sigma(\overline{Y^0})$, $\sigma \in \mathcal{A}_n$, $n > 0$, from overlapping. It can be shown by elementary geometrical arguments that the condition is

$$2\sqrt{2}a^5 + 2a^4 + 2a^2 + \sqrt{2}a - 2 \le 0, \tag{3}$$

*i.e.* ,

$$a \le a^* \simeq 0.593465.$$

In all what follows, we will take $1/2 \le a \le a^*$.
We call $\Gamma^\infty$ the self similar set associated to the similitudes $F_1$ and $F_2$, *i.e.* the unique compact subset of $\mathbb{R}^2$ such that

$$\Gamma^\infty = F_1(\Gamma^\infty) \cup F_2(\Gamma^\infty).$$

The Moran condition, see [12, 8], is that there exists a nonempty bounded open subset $O$ of $\mathbb{R}^2$ such that $F_1(O) \cap F_2(O) = \emptyset$ and $F_1(O) \cup F_2(O) \subset O$. In our case, one can take $O = \Omega$. As shown in [12, 8] this condition allows for computing the Hausdorff dimension of $\Gamma^\infty$; we find that

$$\dim_H(\Gamma^\infty) = -\log 2/\log a. \tag{4}$$

For instance, if $a = a^*$, then $\dim_H(\Gamma^\infty) \simeq 1.3284371$.
We split the boundary of $\Omega$ into $\Gamma^\infty$, $\Gamma^0 = [-1,1] \times \{0\}$ and $\Sigma = \partial\Omega \backslash (\Gamma^0 \cup \Gamma^\infty)$. For what follows, it is important to define the polygonal open domain $Y^N$ obtained by stopping the above construction at step $N+1$,

$$Y^N = \text{Interior}\left(\overline{Y^0} \cup \left(\overset{N}{\underset{n=1}{\cup}} \underset{\sigma\in\mathcal{A}_n}{\cup} \mathcal{M}_\sigma(\overline{Y^0})\right)\right). \tag{5}$$

We introduce the open domains $\Omega^\sigma = \mathcal{M}_\sigma(\Omega)$ and $\Omega^N = \cup_{\sigma\in\mathcal{A}_N} \Omega^\sigma = \Omega\backslash\overline{Y^{N-1}}$. We also define the sets $\Gamma^\sigma = \mathcal{M}_\sigma(\Gamma^0)$ and $\Gamma^N = \cup_{\sigma\in\mathcal{A}_N} \Gamma^\sigma$. The one-dimensional Lebesgue measure of $\Gamma^\sigma$ for $\sigma \in \mathcal{A}_N$ and of $\Gamma^N$ are given by

$$|\Gamma^\sigma| = a^N |\Gamma^0| \quad \text{and} \quad |\Gamma^N| = (2a)^N |\Gamma^0|.$$

It is clear that if $a > 1/2$ then $\lim_{N\to\infty} |\Gamma^N| = +\infty$.

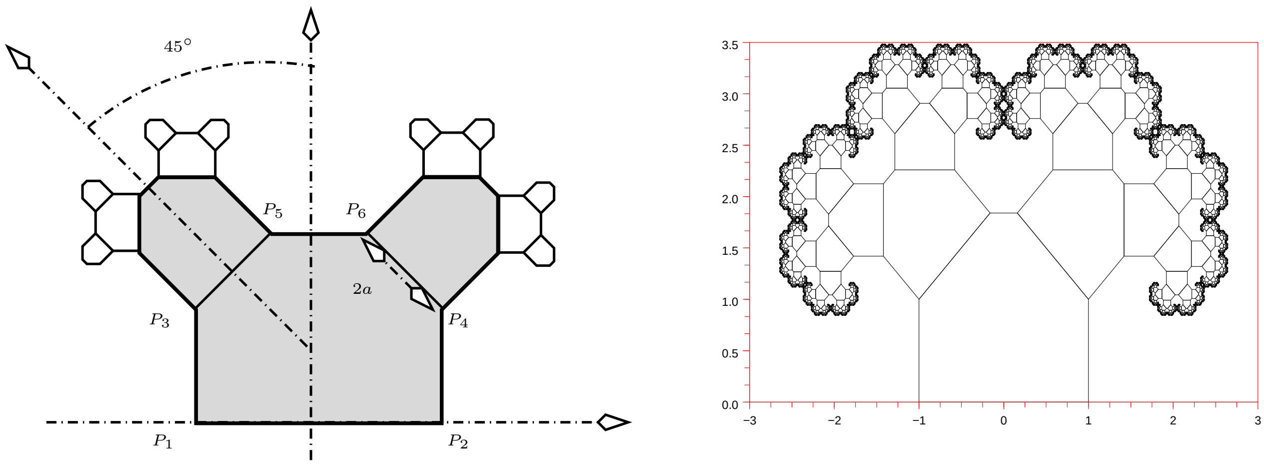


Figure 1: Left, the construction (more exactly $Y^3$). Right, the ramified domain $\Omega$ for the critical value $a = a^*$.

## 2.1 The sub-critical case $a < a^*$

In the case when $1/2 \leq a < a^*$, $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty)$ is empty. As a consequence

**Lemma 1** *If $1/2 \leq a < a^*$, there exists $\epsilon > 0$ and $\delta > 0$ such that $\Omega$ is a $\epsilon - \delta$ domain as defined by Jones [6], see also [7] or in an equivalent manner a quasi-disk, see [11].*

**Proof.** Some of the details will be skipped for brevity. The proof is similar to that given in [11] page 71 for a different domain $\Omega$. We have to show that $\partial\Omega$ is a quasi-circle, i.e. the image of a circle by a quasiconformal mapping of the plane into itself. By Ahlfors theorem, see [3], this is equivalent to the following: there exists $c > 0$ such that for any $x$ and $y$ on $\partial\Omega$, if $\gamma$ is a continuous sub-arc of $\partial\Omega$ which joins $x$ and $y$ with minimal diameter, i.e. $\gamma$ is a continuous function from $[0, 1]$ to $\partial\Omega$ which minimizes $\text{diam}(\gamma[0, 1])$ subject to $\gamma(0) = x$ and $\gamma(1) = y$, then for each $t \in [0, 1]$,

$$|x - \gamma(t)| \leq c|x - y|. \tag{6}$$

This condition mainly stems from the following two facts:

1. there exists a constant $\delta > 0$ such that if $x \in F_1(\Omega)$ and $y \in F_2(\Omega)$ then $|x - y| \geq \delta$: on the other hand, if $\gamma$ is a sub-arc as above, then $\text{diam}(\gamma[0, 1]) \leq \text{diam}(\Omega)$, so (6) is satisfied with $c = \text{diam}(\Omega)/\delta$. By an elementary scaling, the same is true if $x \in \mathcal{M}_\sigma(F_1(\Omega))$ and $y \in \mathcal{M}_\sigma(F_2(\Omega))$.

2. We call $\Sigma_2$ the polygonal line $\Sigma_2 = [P_2, P_4] \cup [P_4, P_6]$ (see the beginning of § 2 for the definition of the points $P_2$, $P_4$ and $P_6$). Take $x \in \Sigma_2$ and $y \in \bigcup_{n\geq 1} F_2^{2n}\Sigma_2$. Then there exists a constant $C > 0$ such that $|x - y| \geq C$, so (6) is satisfied with $c = \text{diam}(\Omega)/C$. The same is true if $x \in \mathcal{M}_\sigma(\Sigma_2)$ and $y \in \mathcal{M}_\sigma(\bigcup_{n\geq 1} F_2^{2n}\Sigma_2)$. By symmetry the same is true if $x \in \mathcal{M}_\sigma(\Sigma_1)$ and $y \in \mathcal{M}_\sigma(\bigcup_{n\geq 1} F_1^{2n}\Sigma_1)$, where $\Sigma_1 = [P_1, P_3] \cup [P_3, P_5]$.

For all the points $x, y$ on $\partial\Omega$, the minimal sub-arc between $x$ and $y$ can be split into $\sum_{i=1}^{p} \gamma_i$ where $p \leq 3$, $\gamma_i$ is a minimal sub-arc joining its endpoints, and $\gamma_i$ is either made of a finite number of straight line segments or falls into the two cases above. This leads to (6). ■

**Remark 1** *If $a < a^*$, the set $\Gamma^\infty$ is a post-critically finite set as defined in [8] page 23, since $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty) = \emptyset$.*

## 2.2 The critical case $a = a^*$

In the critical case when $a = a^*$, it will be proved below that $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty) \subset \Gamma^\infty$ is a non countable set and that $\Omega$ is not a $\epsilon - \delta$ domain.
We introduce the similitudes

$$G_1 = F_1 \circ F_2 \quad \text{and} \quad G_2 = F_2 \circ F_1. \tag{7}$$

It will sometimes be convenient to use the redundant notations

$$G_1^c = G_2 \quad \text{and} \quad G_2^c = G_1, \quad \text{(analogously,} \quad F_1^c = F_2 \quad \text{and} \quad F_2^c = F_1). \tag{8}$$

We need some preliminary lemmas before characterizing

$$\Xi^\infty = F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty) \tag{9}$$

in Proposition 1 below. These lemmas will also be useful in Appendix B.
To simplify the notations, we set $a = a^*$.

**Lemma 2** *Let $H$ be the number*

$$H = \sup_{x\in\Omega} x_2 = (1 + 3a/\sqrt{2})/(1 - a^2).$$

*For all $k \in \mathbb{N}$, for all $n \geq 2k+2$ and all $\sigma \in \mathcal{A}_n$ with*

$$\sigma(2i-1) \neq \sigma(2i), \quad \forall 1 \leq i \leq k, \quad \text{and} \quad \sigma(2k+1) = \sigma(2k+2), \tag{10}$$

*we have*

$$Ha^{2k} \geq \inf_{x\in\Omega^\sigma}(H - x_2) \geq \delta a^{2k} \tag{11}$$

*for $\delta > 0$ independent of $k$, $n$ and $\sigma$.*

**Proof.** Introduce

$$h_1 = \sup_{x\in Y^1} x_2 = 1 + 3a/\sqrt{2}$$

and

$$h_2 = \sup_{x\in F_1\circ F_1(\Omega)} x_2 = \sup_{x\in F_2\circ F_2(\Omega)} x_2.$$

We have $h_1 < h_2 < H = h_1/(1-a^2)$. The bound $h_2 < H$ can be obtained by realizing that

$$h_2 = \sup_{x\in F_1\circ F_1\circ F_2(Y^0)} x_2 + a^4 H,$$

and explicitly computing $\sup_{x\in F_1\circ F_1\circ F_2(Y^0)} x_2$.
Therefore, there exists $\delta > 0$ such that $h_2 < H - \delta$.
Let $\sigma \in \mathcal{A}_n$ satisfy (10). By a simple scaling argument, we see that

$$\sup_{x\in\Omega^\sigma} x_2 = h_1 \sum_{i=0}^{k-1} a^{2i} + h_2 a^{2k} = h_1 \frac{1-a^{2k}}{1-a^2} + h_2 a^{2k} = H(1-a^{2k}) + h_2 a^{2k} \leq H - \delta a^{2k},$$

which yields the lower bound in (11), and

$$\sup_{x\in\Omega^\sigma} x_2 = h_1 \sum_{i=0}^{k-1} a^{2i} + h_2 a^{2k} \geq H(1-a^{2k}),$$

which yields the upper bound in (11). ■

**Lemma 3** *There exists two positive constants $c$ and $C$ such that for all $n > 0$,*

1. *for all $\sigma \in \mathcal{A}_n$, the distance of $G_{\sigma(1)} \circ \cdots \circ G_{\sigma(n)}(\Gamma^0)$ to the vertical axis $\{x_1 = 0\}$ is greater than $c$,*
2. *for all $k \in \{1, \ldots, n-1\}$ and $\sigma, \sigma' \in \mathcal{A}_n$ such that $\sigma(i) = \sigma'(i)$ for all $i < k$ and $\sigma(k) \neq \sigma'(k)$,*

$$ca^{2(k-1)} \leq d(G_{\sigma(1)} \circ \cdots \circ G_{\sigma(n)}(\Gamma^0), G_{\sigma'(1)} \circ \cdots \circ G_{\sigma'(n)}(\Gamma^0)) \leq Ca^{2(k-1)}.$$

*where $d(x,y)$ is the Euclidean distance between the two points $x, y \in \mathbb{R}^2$.*

**Proof.** The quantity $\min_{x\in G_{\sigma(1)}\circ\cdots\circ G_{\sigma(n)}(\Gamma^0)}|x_1|$ can be bounded from below, because its minimum with respect to $\sigma$ is achieved for $\sigma=(2,1,1,1,1,\dots,1)$: this quantity is larger than the abscissa $c$ of $G_2(M)$, where $M$ is the fixed point of $G_1$: elementary calculus yields that $c=(1-\frac{a}{\sqrt{2}}+a^2)\frac{1-2a^2}{1-a^2}>0$.
For proving the second point,

$$\begin{aligned}&d(G_{\sigma(1)}\circ\cdots\circ G_{\sigma(n)}(\Gamma^0),G_{\sigma'(1)}\circ\cdots\circ G_{\sigma'(n)}(\Gamma^0))\\=&a^{2(k-1)}d(G_{\sigma(k)}\circ\cdots\circ G_{\sigma(n)}(\Gamma^0),G_{\sigma'(k)}\circ\cdots\circ G_{\sigma'(n)}(\Gamma^0))\geq ca^{2(k-1)},\end{aligned}$$

where we have used the first point and the fact that $G_{\sigma(k)}\circ\cdots\circ G_{\sigma(n)}(\Gamma^0)$ and $G_{\sigma'(k)}\circ\cdots\circ G_{\sigma'(n)}(\Gamma^0)$ are separated by the axis $\{x_1=0\}$. We have obtained the lower bound. The upper bound comes from the fact that $d(G_{\sigma(k)}\circ\cdots\circ G_{\sigma(n)}(\Gamma^0),G_{\sigma'(k)}\circ\cdots\circ G_{\sigma'(n)}(\Gamma^0))$ is bounded from above by the diameter of $\Omega$. ■

**Definition 1** *For $\sigma\in\mathcal{A}_N$, $N=2(n+2)$, we define the integer prox($\sigma$) by*

$$\begin{cases}\textit{if } \sigma(1,2,3,4)\neq(1,2,2,2) \textit{ and } \sigma(1,2,3,4)\neq(2,1,1,1), \quad \textit{then} \quad \text{prox}(\sigma)=0,\\ \textit{else if } \sigma(5)=\sigma(6), \quad \textit{then } \text{prox}(\sigma)=1,\\ \textit{else } \text{prox}(\sigma) \textit{ is the largest number } k \textit{ s.t.} \begin{cases}\sigma(2i-1) &\neq& \sigma(2i), \quad \forall i,\ 3\leq i\leq k+1,\\ k&\leq& n+1.\end{cases}\end{cases} \tag{12}$$

We will see in Lemma 4 below that the larger prox($\sigma$) is, the closer $\Gamma^\sigma$ to the axis $x_1=0$.

**Remark 2** *Note that, for $k\in\{1,\dots,n\}$, there are $2^{2n-k+1}$ maps $\sigma$ in $\mathcal{A}_N$ such that prox($\sigma$) $=k$, there are $2^{n+1}$ maps $\sigma$ in $\mathcal{A}_N$ such that prox($\sigma$) $=n+1$, and there are $7\cdot 2^{2n+1}$ maps $\sigma$ in $\mathcal{A}_N$ such that prox($\sigma$) $=0$.*

**Lemma 4** *There exists two positive numbers $H'$ and $\delta'$ such that for all $n\geq0$, for all $\sigma\in\mathcal{A}_{2n+4}$, the distance of $\Gamma^\sigma$ to the axis $\{x_1=0\}$ is greater than $\delta' a^{2\text{prox}(\sigma)}$ and smaller than $H' a^{2\text{prox}(\sigma)}$.*

**Proof.** The proof uses Lemma 2 and the fact that the image of the line $\{x_2=H\}$ by $F_1\circ F_2\circ F_2\circ F_2$ or $F_2\circ F_1\circ F_1\circ F_1$ is exactly the axis $\{x_1=0\}$. ■

**Proposition 1** *The set $\Xi^\infty$ given by (9) is contained in the vertical axis $\{x_1=0\}$, and is characterized by*

$$\Xi^\infty=\left\{\lim_{n\to\infty}G_1\circ F_2\circ F_2\circ G_{\sigma(1)}\circ\cdots\circ G_{\sigma(n)}(O);\ \sigma\in\mathcal{A}_\infty\right\} \tag{13}$$

*where $O=(0,0)$ is the origin. Moreover, for $P\in\Xi^\infty$, there exists a unique $\sigma\in\mathcal{A}_\infty$ such that*

$$P=\lim_{n\to\infty}G_1\circ F_2\circ F_2\circ G_{\sigma(1)}\circ\cdots\circ G_{\sigma(n)}(O)=\lim_{n\to\infty}G_2\circ F_1\circ F_1\circ G^c_{\sigma(1)}\circ\cdots\circ G^c_{\sigma(n)}(O). \tag{14}$$

*The Hausdorff dimension of $\Xi^\infty$ is $\dim_H(\Gamma^\infty)/2$.*

**Proof.** Since $F_1(\Gamma^\infty)=\Gamma^\infty\cap\{x_1\leq0\}$ and $F_2(\Gamma^\infty)=\Gamma^\infty\cap\{x_1\geq0\}$, we immediately see that $\Xi^\infty=\Gamma^\infty\cap\{x_1=0\}$. Then (13) is a consequence of Lemma 4. Moreover, if $P=\lim_{n\to\infty}G_1\circ F_2\circ F_2\circ G_{\sigma(1)}\circ\cdots\circ G_{\sigma(n)}(O)$, then by symmetry, we also have $P=\lim_{n\to\infty}G_2\circ F_1\circ F_1\circ G^c_{\sigma(1)}\circ\cdots\circ G^c_{\sigma(n)}(O)$. Finally, for $\sigma,\sigma'\in\mathcal{A}_\infty$ such that $\sigma\neq\sigma'$, there exists $k>0$ such that $\sigma(i)=\sigma'(i)$ for all $i<k$ and $\sigma(k)\neq\sigma'(k)$: from the second statement of Lemma 3, we see that $d(G_1\circ F_2\circ F_2\circ G_{\sigma(1)}\circ\cdots\circ G_{\sigma(n)}(O),G_1\circ F_2\circ F_2\circ G_{\sigma'(1)}\circ\cdots\circ G_{\sigma'(n)}(O)\geq ca^{2k}$, for all

$n > k$. This yields the uniqueness in (14).
The set $\Xi^\infty$ is the image by $G_1 \circ F_2 \circ F_2$ of the self-similar fractal associated with the similitudes $G_1$ and $G_2$, whose dilation ratii are $a^2 < 1/2$. The latter is a Cantor set contained in the horizontal line $x_2 = (1 + 3a/\sqrt{2})(1 - a^2)$: since the Moran condition is satisfied, its Hausdorff dimension is $-\log(2)/\log(a^2) = -\log(2)/(2\log(a)) = 1/2 \dim_H(\Gamma^\infty)$. Therefore, the Hausdorff dimension of $\Xi^\infty$ is $\dim_H(\Gamma^\infty)/2$. ■
To illustrate Proposition 1, we display in Figure 2 the sets $Y^0$, $F_1(Y^0)$, $G_1(Y^0)$, $G_1 \circ F_2(Y^0)$, $G_1 \circ F_2 \circ F_2(Y^0)$, $G_1 \circ F_2 \circ F_2 \circ F_1(Y^0)$, $G_1 \circ F_2 \circ F_2 \circ G_1(Y^0)$, $G_1 \circ F_2 \circ F_2 \circ G_1 \circ F_1(Y^0)$, $G_1 \circ F_2 \circ F_2 \circ G_1 \circ G_1(Y^0)$,....

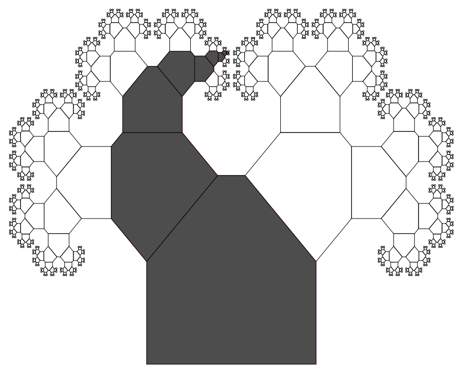

Figure 2: In dark, $Y^0$ and the images of $Y^0$ by $F_1$, $G_1$, $G_1 \circ F_2$, $G_1 \circ F_2 \circ F_2$, $G_1 \circ F_2 \circ F_2 \circ F_1$, $G_1 \circ F_2 \circ F_2 \circ G_1$, $G_1 \circ F_2 \circ F_2 \circ G_1 \circ F_1$, $G_1 \circ F_2 \circ F_2 \circ G_1 \circ G_1$,....

**Remark 3** *A consequence of (14) is that $\Omega$ is not a $\epsilon - \delta$ domain if $a = a^*$.*

**Remark 4** *A consequence of (13) and (14) is that the set $\Gamma^\infty$ is not post-critically finite if $a = a^*$. Note that most of the available results on function spaces on self-similar fractals are valid for post-critically finite sets only.*

**Remark 5** *Other examples of non post-critically finite fractal boundaries may be constructed by taking similitudes with rotation angles $\pm\pi/(2^{\ell+1})$, where $\ell$ is a positive integer, and a suitable dilation factor.*

# 3 The space $H^1(\Omega)$

Let $H^1(\Omega)$ be the space of functions in $L^2(\Omega)$ with first order partial derivatives in $L^2(\Omega)$. We also define $\mathcal{V}(\Omega) = \{v \in H^1(\Omega); v|_{\Gamma^0} = 0\}$ and $\mathcal{V}(Y^n) = \{v \in H^1(Y^n); v|_{\Gamma^0} = 0\}$.
The results stated below are important for the study of elliptic boundary value problems in $\Omega$. We refer to [2] for their proofs.
We will sometimes use the notation $\lesssim$ to indicate that there may arise constants in the estimates, which are independent of the index $n$ in $\Omega^n$ (recall that $\Omega^n$ is the union of all $\mathcal{M}_\sigma(\Omega)$, $\sigma \in \mathcal{A}_n$) or $\Gamma^n$ ( $\Gamma^n$ is the union of all $\mathcal{M}_\sigma(\Gamma^0)$, $\sigma \in \mathcal{A}_n$), $Y^n = Y^0 \cup \bigcup_{1\le p\le n} \bigcup_{\sigma\in\mathcal{A}_p} \mathcal{M}_\sigma(Y^0)$, or the index $\sigma$ in $\Omega^\sigma = \mathcal{M}_\sigma(\Omega)$ or $\Gamma^\sigma = \mathcal{M}_\sigma(\Gamma^0)$.

### 3.1 Poincaré inequality and consequences

**Theorem 1** *There exists a constant $C > 0$, such that*

$$\forall v \in \mathcal{V}(\Omega), \qquad \|v\|^2_{L^2(\Omega)} \le C \|\nabla v\|^2_{L^2(\Omega)}.$$

**Corollary 1** *There exists a positive constant $C$ such that for all $v \in H^1(\Omega)$,*

$$\|v\|^2_{L^2(\Omega)} \le C \left( \|\nabla v\|^2_{L^2(\Omega)} + \|v|_{\Gamma^0}\|^2_{L^2(\Gamma^0)} \right).$$

**Corollary 2** *There exists a positive constant $C$ such that for all integer $n \ge 0$ and for all $\sigma \in \mathcal{A}_n$, for all $v \in H^1(\Omega^\sigma)$,*

$$\|v\|^2_{L^2(\Omega^\sigma)} \le C \left( a^{2n} \|\nabla v\|^2_{L^2(\Omega^\sigma)} + a^n \|v|_{\Gamma^\sigma}\|^2_{L^2(\Gamma^\sigma)} \right),$$

*and for all $v \in H^1(\Omega^n)$*

$$\|v\|^2_{L^2(\Omega^n)} \le C \left( a^{2n} \|\nabla v\|^2_{L^2(\Omega^n)} + a^n \|v|_{\Gamma^n}\|^2_{L^2(\Gamma^n)} \right).$$

We need to estimate $\|v\|^2_{L^2(\Omega^n)}$ when $v \in H^1(\Omega)$:

**Lemma 5** *There exists a positive constant $C$ such that for all $v \in H^1(\Omega)$, for all $n \ge 0$,*

$$\|v\|^2_{L^2(\Omega^n)} \le C\, (2a^2)^n \left( \|\nabla v\|^2_{L^2(\Omega)} + \|v|_{\Gamma^0}\|^2_{L^2(\Gamma^0)} \right). \tag{15}$$

Condition (3) implies $2a^2 < 1$, because $2(a^*)^2 \sim 0.7044022575$. Thus, (15) implies the Rellich type theorem:

**Theorem 2 (Compactness)** *The imbedding of $H^1(\Omega)$ in $L^2(\Omega)$ is compact.*

The following lemma will be useful for defining a trace operator on $\Gamma^\infty$:

**Lemma 6** *There exists a positive constant $C$ such that for all $v \in H^1(\Omega)$, for all integers $p \ge 0$,*

$$\sum_{\sigma \in \mathcal{A}_p} \int_{\Gamma^\sigma} (v|_{\Gamma^\sigma})^2 \le C (2a)^p \left( \|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \right). \tag{16}$$

**Remark 6** *Note that $\frac{|\Gamma^p|}{|\Gamma^0|} = (2a)^p$, so (16) is equivalent to*

$$\frac{1}{|\Gamma^p|} \sum_{\sigma \in \mathcal{A}_p} \int_{\Gamma^\sigma} (v|_{\Gamma^\sigma})^2 \lesssim \|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}.$$

**Corollary 3** *There exists a positive constant $C$ such that for all $v \in H^1(\Omega)$, for all integers $p \ge 0$,*

$$\sum_{\sigma \in \mathcal{A}_p} \int_{\Gamma^\sigma} (v|_{\Gamma^\sigma} - \langle v|_{\Gamma^0} \rangle)^2 \le C (2a)^p \|\nabla v\|^2_{L^2(\Omega)},$$

*where $\langle v|_{\Gamma^0} \rangle$ is the mean value of $v|_{\Gamma^0}$ on $\Gamma^0$.*

### 3.2 A trace operator on $\Gamma^\infty$

For defining traces on $\Gamma^\infty$, we recall the classical result on self-similar measures, see [4]:

**Theorem 3** *There exists a unique Borel regular probability measure $\mu$ on $\Gamma^\infty$ such that for any Borel set $A \subset \Gamma^\infty$,*

$$\mu(A) = \frac{1}{2}\mu\left(F_1^{-1}(A)\right) + \frac{1}{2}\mu\left(F_2^{-1}(A)\right). \tag{17}$$

The measure $\mu$ is called the *self-similar measure defined in the self-similar triplet* $(\Gamma^\infty, F_1, F_2)$.

**Proposition 2** *For $1/2 \le a \le a^*$, the measure $\mu$ is d-measure on $\Gamma^\infty$, with $d = -\log 2/\log a$, according to the definition in [7], page 28: there exists two positive constants $c_1$ and $c_2$ such that*

$$c_1 r^d \le \mu(B(x,r)) \le c_2 r^d,$$

*for any $r$ $0 < r < 1$ and $x \in \Gamma^\infty$, where $B(x,r)$ is the ball of $\Gamma^\infty$ centered at $x$ and with radius $r$. In other words the closed set is a d-set, see [7], page 28.*

**Proof.** The proof stems from the Moran condition in § 2. It is due to Moran [12] and has been extended by Kigami, see [8], §1.5, especially Proposition 1.5.8 and Theorem 1.5.7. ■

Let $L^2_\mu$ be the Hilbert space of the functions defined $\mu$ a.e. on $\Gamma^\infty$ that are $\mu$-measurable and square integrable with respect to $\mu$, with the norm $\|v\|_{L^2_\mu} = \sqrt{\int_{\Gamma^\infty} v^2 d\mu}$.

A Hilbertian basis of $L^2_\mu$ can be constructed with e.g. Haar wavelets.

Similarly we define $L^p_\mu$, $p \in [1, +\infty)$ as the space of the measurable functions $v$ on $\Gamma^\infty$ such that $\int_{\Gamma^\infty} v^p d\mu < \infty$, endowed with the norm $\|v\|_{L^p_\mu} = \left(\int_{\Gamma^\infty} v^p d\mu\right)^{1/p}$.

**Remark 7** *It can be proved that, even if $a = a^*$, $\mu(F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty)) = 0$, see [12, 5, 8]. Similarly, introducing the set*

$$\mathcal{N}^\infty = \cup_{n\in\mathbb{N}} \cup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma \left(F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty)\right), \tag{18}$$

*we have*

$$\mu\left(\mathcal{N}^\infty\right) = 0.$$

*For all $x \in \Gamma^\infty \backslash \mathcal{N}^\infty$, there exists a unique sequence $\sigma_x \in \mathcal{A}_\infty$ such that $x = \lim_{n\to\infty} F_{\sigma_x(1)} \circ \dots F_{\sigma_x(n)}(O)$, where $O$ is the origin. The mapping $\aleph : \Gamma^\infty \backslash \mathcal{N}^\infty \to \Gamma^0$ defined by*

$$\aleph(x) = (-1 + 2\sum_{i\ge 1} (\sigma_x(i) - 1)2^{-i}, 0) \tag{19}$$

*is injective and such that the Lebesgue measure of $\aleph(\Gamma^\infty)$ is 2, and we have the formula*

$$\frac{1}{2}\int_{\Gamma^0} f\, dx = \int_{\Gamma^\infty\backslash\mathcal{N}^\infty} f \circ \aleph\, d\mu, \qquad \forall f \in L^1(\Gamma^0). \tag{20}$$

*Formula (20) is first proven for the characteristic functions of the intervals $(-1 + p2^{-n+1}, -1 + (p+1)2^{-n+1})$, $n \in \mathbb{N}$, $p \in \{0, \dots, 2^n - 1\}$, then for $f \in L^1(\Gamma^0)$ by density.*

Consider the sequence of linear operators $\ell^n : H^1(\Omega) \to L^2_\mu$,

$$\ell^n(v) = \sum_{\sigma\in\mathcal{A}_n} \left(\frac{1}{|\Gamma^\sigma|}\int_{\Gamma^\sigma} v\, dx\right) \mathbf{1}_{\mathcal{M}_\sigma(\Gamma^\infty)}, \tag{21}$$

where $|\Gamma^\sigma|$ is the one-dimensional Lebesgue measure of $\Gamma^\sigma$, see [2].

**Proposition 3** *The sequence $(\ell^n)_n$ converges in $\mathcal{L}(H^1(\Omega), L^2_\mu)$ to an operator that we call $\ell^\infty$.*

**Proof.** See [2]. ■

# 4 Finer results on traces

## 4.1 The sub-critical case when $a < a^*$.

In the case when $a < a^*$, the theory of Jones (see [6, 7]), or of Vodopjanov et al [18] see also [11] page 70, can be applied since $\Omega$ is a $\epsilon - \delta$ domain (Lemma 1): it is possible to define a continuous extension operator form $H^1(\Omega)$ to $H^1(\mathbb{R}^2)$. Then, since $\Gamma^\infty$ is $d_H$-set where $d_H = -\frac{\log 2}{\log a}$ (Proposition 2), one can use the trace theorem by Jonsson and Wallin, see [7] Theorem 1 page 103, which states that

$$H^1(\mathbb{R}^2)|_{\Gamma^\infty} = H^{d_H/2}(\Gamma^\infty) = \left\{ v \in L^2_\mu(\Gamma^\infty), \iint_{\Gamma^\infty \times \Gamma^\infty} \frac{(v(x) - v(y))^2}{|x-y|^{2d_H}} d\mu_x d\mu_y < \infty \right\}.$$

Combining the two points above, a function in $H^1(\Omega)$ has a trace in $H^{d_H/2}(\Gamma^\infty)$.

## 4.2 The critical case when $a = a^*$.

We aim at finding more information on the operator $\ell^\infty$. We will successively answer the questions: does the trace of $u \in H^1(\Omega)$ belong to $L^p_\mu(\Gamma^\infty)$, $1 \le p < \infty$, to $BMO(\Gamma^\infty, \mu)$, and finally to the Sobolev spaces $H^s(\Gamma^\infty)$?

### 4.2.1 The trace functions belong to $L^p_\mu$, $1 \le p < \infty$.

The main result is as follows:

**Theorem 4** *For any $p \in [1, \infty)$ and $u \in H^1(\Omega)$,*

$$\ell^\infty(u) \in L^p_\mu,$$

*and there exists a constant $C > 0$ such that*

$$\|\ell^\infty(u)\|_{L^p_\mu} \le C \|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega).$$

*Moreover, for all $u \in H^1(\Omega)$, $\ell^n(u)$ converges to $\ell^\infty(u)$ in $L^p_\mu$ as $n$ tends to $\infty$.*

**Proof.** The proof relies on trace theorems obtained for a different and simpler geometry in [1]. For completeness, we recall the proof of these results in Appendix A. Let $\widetilde{F}_1$ and $\widetilde{F}_2$ be the affine maps in $\mathbb{R}^2$

$$\widetilde{F}_1(x) = (-\frac{3}{2} + \frac{x_1}{2}, 3 + \frac{x_2}{2}), \qquad \widetilde{F}_2(x) = (\frac{3}{2} + \frac{x_1}{2}, 3 + \frac{x_2}{2}). \tag{22}$$

We introduce the points $\widetilde{P}_1 = (-1, 0)$, $\widetilde{P}_2 = (1, 0)$, $\widetilde{P}_3 = \widetilde{F}_1(\widetilde{P}_1) = (-2, 3)$, $\widetilde{P}_4 = \widetilde{F}_2(\widetilde{P}_2) = (2, 3)$, $\widetilde{P}_5 = \widetilde{F}_1(\widetilde{P}_2) = (-1, 3)$ and $\widetilde{P}_6 = \widetilde{F}_2(\widetilde{P}_1) = (1, 3)$. Note that the last four points are aligned. We introduce the trapezoidal domain

$$\widetilde{Y}^0 = \text{Interior}\Big( \text{ Conv}(\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3, \widetilde{P}_4) \Big)$$

and $\widetilde{\Gamma}^0 = [\widetilde{P}_1, \widetilde{P}_2] = \Gamma^0$. As above, for $\sigma \in \mathcal{A}_n$, we define

$$\widetilde{\mathcal{M}}_\sigma = \widetilde{F}_{\sigma(1)} \circ \cdots \circ \widetilde{F}_{\sigma(n)}, \tag{23}$$

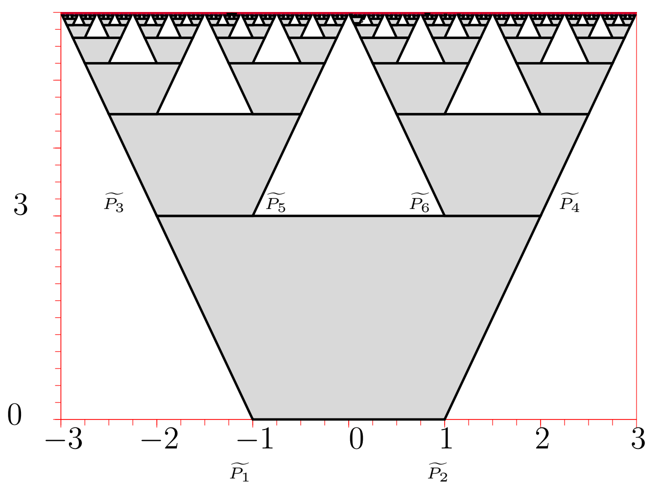


Figure 3: The domain $\widetilde{\Omega}$.

and the open domain

$$\widetilde{\Omega} = \text{Interior}\left(\overline{\widetilde{Y}^0} \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{\sigma\in\mathcal{A}_n} \widetilde{\mathcal{M}}_\sigma(\overline{\widetilde{Y}^0})\right)\right), \tag{24}$$

see Figure 3.

We also define the sets $\widetilde{\Gamma}^\sigma = \widetilde{\mathcal{M}}_\sigma(\widetilde{\Gamma}^0)$ and $\widetilde{\Gamma}^N = \cup_{\sigma\in\mathcal{A}_N}\widetilde{\Gamma}^\sigma$. Finally, we denote by $\widetilde{\Gamma}^\infty$ the self-similar set associated to the similitudes $\widetilde{F}_1$ and $\widetilde{F}_2$, which is the straight line segment between the points $(-3,6)$ and $(3,6)$. It is important to realize that the self-similar measure defined in the self-similar triplet $(\widetilde{\Gamma}^\infty, \widetilde{F}_1, \widetilde{F}_2)$ is $1/6$ times the Lebesgue measure on $\widetilde{\Gamma}^\infty$, i.e. $\widetilde{\mu} = \frac{1}{6}dx$. As above, we introduce the sequence of linear operators $\widetilde{\ell}^n : H^1(\widetilde{\Omega}) \to L^2(\widetilde{\Gamma}^\infty, dx)$ by

$$\widetilde{\ell}^n(v) = \sum_{\sigma\in\mathcal{A}_n} \left(\frac{1}{|\widetilde{\Gamma}^\sigma|}\int_{\widetilde{\Gamma}^\sigma} v(x)\,dx\right) \mathbf{1}_{\widetilde{\mathcal{M}}_\sigma(\widetilde{\Gamma}^\infty)}. \tag{25}$$

A result similar to Proposition 3 holds: the sequence $(\widetilde{\ell}^n)_n$ converges in $\mathcal{L}(H^1(\widetilde{\Omega}), L^2_{\widetilde{\mu}})$ to an operator that we call $\widetilde{\ell}^\infty$.

We partition the domains $Y^0$ and $\widetilde{Y}^0$ into six non-overlapping triangles which are numbered as shown in Figure 4, and we call $Q$ (respectively $\widetilde{Q}$) the interior node to $Y^0$ (respectively $\widetilde{Y}^0$). We call $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ the two sets of triangles. There exists a continuous, one to one and piecewise linear function $\psi$ from $Y^0$ onto $\widetilde{Y}^0$, such that

- its restriction to the triangles in $\mathcal{T}$ is linear,
- $\psi$ maps each triangle in $\mathcal{T}$ to the triangle in $\widetilde{\mathcal{T}}$ with the same index,
- the restriction of $\psi$ to $\Gamma^0$ is the identity,
- for $i = 1, 2$,

$$F_i(\psi^{-1}(x)) = \psi^{-1}(\widetilde{F}_i(x)), \quad \forall x \in \widetilde{\Gamma}^0.$$

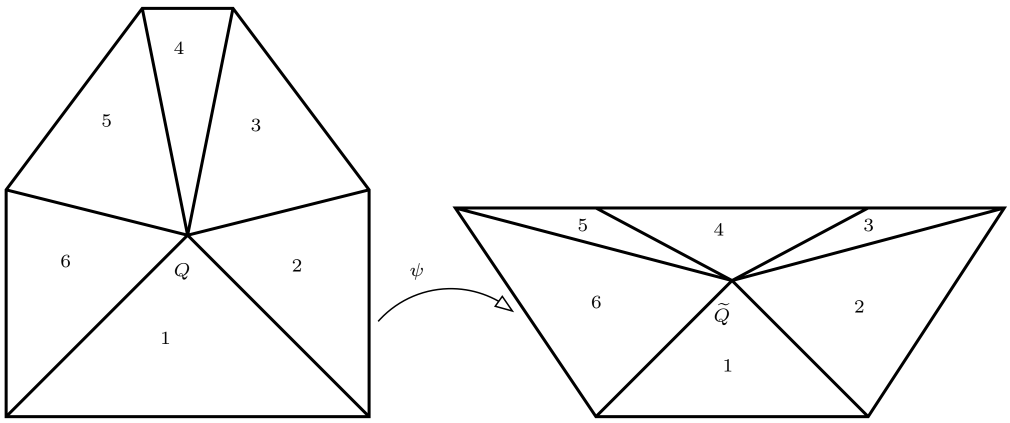


Figure 4: The domains $Y^0$ and $\widetilde{Y}^0$ are partitioned into six triangles

This construction allows for the definition of the continuous linear operator $\Psi : H^1(\Omega) \to H^1(\widetilde{\Omega})$:

$$\begin{aligned} \Psi(u) &= \widetilde{u} \\ \widetilde{u} &= u|_{\mathcal{M}_\sigma(Y^0)} \circ \mathcal{M}_\sigma \circ \psi^{-1} \circ (\widetilde{\mathcal{M}}_\sigma)^{-1} \qquad \text{in} \quad \widetilde{\mathcal{M}}_\sigma(\widetilde{Y}^0). \end{aligned}$$

With $\ell^n$ defined in (21), we use the mapping $\Psi$ for bounding

$$\begin{aligned} \|\ell^n(u)\|^p_{L^p_\mu} &= \int_{\Gamma^\infty} |\ell^n(u)|^p d\mu \\ &= \int_{\Gamma^\infty} \left| \sum_{\sigma \in \mathcal{A}_n} \left( \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} u(x)\, dx \right) \mathbf{1}_{\mathcal{M}_\sigma(\Gamma^\infty)} \right|^p d\mu \\ &= \sum_{\sigma \in \mathcal{A}_n} \frac{1}{2^n} \left| \frac{1}{a^n |\Gamma^0|} \int_{\Gamma^\sigma} u \right|^p \\ &= \sum_{\sigma \in \mathcal{A}_n} \frac{1}{2^n} \left| \frac{1}{a^n |\Gamma^0|} \int_{\Gamma^\sigma} \widetilde{u} \circ \widetilde{\mathcal{M}}_\sigma \circ \psi \circ (\mathcal{M}_\sigma)^{-1} \right|^p . \end{aligned}$$

Therefore,

$$\begin{aligned} \|\ell^n(u)\|^p_{L^p_\mu} &= \sum_{\sigma \in \mathcal{A}_n} \frac{1}{2^n} \left| \frac{1}{|\Gamma^0|} \int_{\Gamma^0} \widetilde{u} \circ \widetilde{\mathcal{M}}_\sigma \right|^p \\ &= \sum_{\sigma \in \mathcal{A}_n} \frac{1}{2^n} \left| \frac{1}{|\widetilde{\Gamma}^\sigma|} \int_{\widetilde{\Gamma}^\sigma} \widetilde{u} \right|^p = 1/6 \|\widetilde{\ell}^n(\widetilde{u})\|^p_{L^p(\widetilde{\Gamma}^\infty, dx)} . \end{aligned}$$

Similarly

$$\|\ell^n(u) - \ell^{n+k}(u)\|^p_{L^p_\mu} = 1/6 \|\widetilde{\ell}^n(\widetilde{u}) - \widetilde{\ell}^{n+k}(\widetilde{u})\|^p_{L^p(\widetilde{\Gamma}^\infty, dx)}, \ \forall n, k \in \mathbb{N}. \tag{26}$$

Finally, in Appendix A, we prove the following result:

**Proposition 4** *For any $p \in [1, \infty)$ and $\widetilde{u} \in H^1(\widetilde{\Omega})$, $\widetilde{\ell}^\infty(\widetilde{u}) \in L^p(\widetilde{\Gamma}^\infty, dx)$, and there exists a constant $C > 0$ such that*

$$\|\widetilde{\ell}^\infty(\widetilde{u})\|_{L^p(\widetilde{\Gamma}^\infty, dx)} \le C \|\widetilde{u}\|_{H^1(\widetilde{\Omega})}, \quad \forall \widetilde{u} \in H^1(\widetilde{\Omega}).$$

*Moreover, for all $\widetilde{u} \in H^1(\widetilde{\Omega})$, $\widetilde{\ell}^n(\widetilde{u})$ converges to $\widetilde{\ell}^\infty(\widetilde{u})$ in $L^p(\widetilde{\Gamma}^\infty, dx)$ as $n$ tends to $\infty$.*

Combining (26) and Proposition 4, it is easy to prove that the sequence $\ell^n(u)$ is a Cauchy sequence in $L^p_\mu$ and to identify its limit as $\ell^\infty(u) \in L^p_\mu$. ∎

**Remark 8** *Note that Theorem 4 holds for bounded domains of the type*

$$\Omega = Interior\left(\overline{Y^0} \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma(\overline{Y^0})\right)\right),$$

*where $F_1$ and $F_2$ are two similitudes, $\mathcal{M}_\sigma$ is defined by (1) and $Y^0$ is a polygonal domain, as soon as the sets $\mathcal{M}_\sigma(Y^0)$, $\sigma \in \mathcal{A}_n$, $n > 0$, do not overlap. The dilation ratii of $F_1$ and $F_2$ may differ from each other.*

### 4.2.2 The trace functions may not belong to $BMO(\Gamma^\infty, \mu)$. A counterexample

We recall the definition of space of the functions with Bounded Mean Oscillation on $\Gamma^\infty, \mu$. This space is referred to as $BMO(\Gamma^\infty, \mu)$ or $BMO$ for brevity.

**Definition 2**

$$BMO(\Gamma^\infty, \mu) = \left\{\phi \in L^1(\Gamma^\infty, \mu) \ s.t. \sup_{r>0, P\in\Gamma^\infty} \langle|\phi - \langle\phi\rangle_{B(P,r),\mu}|\rangle_{B(P,r),\mu} < \infty\right\}, \tag{27}$$

*where $B(P, r)$ is the ball in $\Gamma^\infty$ centered in $P$ and with radius $r$, and where, for a $\mu$-measurable subset $X$ of $\Gamma^\infty$,*

$$\langle v\rangle_{X,\mu} = \frac{1}{\mu(X)} \int_X v d\mu.$$

It is easily seen that $L^\infty(\Gamma^\infty, \mu)$ is a subset of $BMO$.
Our goal is to give an example of a function $u^* \in H^1(\Omega)$ whose trace $\ell^\infty(u^*)$ does not belong to $BMO$. As a consequence, $\ell^\infty(u^*)$ will not belong to $L^\infty(\Gamma^\infty, \mu)$.

- We choose $u^*|_{Y^0} = 0$.
  We also introduce a smooth nonnegative function $\phi_1$ taking the constant values 0 on $\Gamma^0$ and $F_2(\Gamma^0)$, and 1 on $F_1(\Gamma^0)$. We also define the function $\phi_2$ by $\phi_2(x, y) = \phi_1(-x, y)$. We introduce
  $$\mathcal{I} = \int_{Y^0} |\nabla\phi_1|^2.$$
  Remember that for any $\sigma \in \mathcal{A}_n$, if $\phi_i^\sigma$ is the function defined on $\mathcal{M}_\sigma(Y^0)$ by $\phi_i^\sigma = \phi_i \circ (\mathcal{M}_\sigma)^{-1}$, we have
  $$\int_{\mathcal{M}_\sigma(Y^0)} |\nabla\phi_i^\sigma|^2 = \mathcal{I}.$$
- The function $u^*$ will be antisymmetric with respect to the axis $x_1 = 0$, i.e. $\forall(x, y) \in \Omega$, $(-x, y) \in \Omega$ and $u^*(x, y) := -u^*(-x, y)$. Therefore, we will focus on its restriction to $F_1(\Omega)$.
- Let us choose some point $P^* \in \Xi^\infty$, for instance
  $$\begin{aligned} P^* &= \lim_{n\to\infty} G_1 \circ F_2 \circ F_2 \circ (G_2)^n(O) \\ &= \lim_{n\to\infty} F_1 \circ F_2 \circ F_2 \circ F_2 \circ \underbrace{F_2 \circ F_1 \circ F_2 \circ F_1 \circ \cdots \circ F_2 \circ F_1}_{n \text{ times}}(O), \end{aligned}$$
  where $G_1$ and $G_2$ are defined in (7). We denote by $\sigma^*$ the sequence $\sigma^* = (1, 2, 2, 2, 2, 1, 2, 1, \ldots 2, 1 \ldots) \in \mathcal{A}_\infty$ and by $\sigma_n^* \in \mathcal{A}_n$ the truncated sequence such that $\sigma_n^*(j) = \sigma^*(j)$ for any $j \le n$.

- The last ingredient is an arbitrary sequence of positive real numbers $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ such that $\sum_{i\in\mathbb{N}} \alpha_i^2 < +\infty$ and $\sum_{i\in\mathbb{N}} \alpha_i = +\infty$, (for instance take $\alpha_i = \frac{1}{i}$).
- Assume that $u^*|_{Y^{n-1}}$ has already been constructed for some $n \geq 1$, and that $u^*|_{\Gamma^\sigma}$ is constant for all $\sigma \in \mathcal{A}_n$. For $\sigma \in \mathcal{A}_n$ such that $\sigma(1) = 1$, we define $u^*|_{Y^\sigma}$ by the following:

$$\begin{aligned} u^*|_{Y^\sigma} \text{ is constant and take the same value as } u^*|_{\Gamma^\sigma}, \qquad & \text{if } \sigma \neq \sigma_n^*, \\ u^*|_{Y^\sigma} = u^*|_{\Gamma^\sigma} + \alpha_n \phi_{\sigma^*(n)} \circ (\mathcal{M}_\sigma)^{-1}, \qquad & \text{if } \sigma = \sigma_n^*, \end{aligned}$$

where in the last line we have identified $u^*|_{\Gamma^\sigma}$ with its constant value. We have defined $u^*$ in $Y^n \cap F_1(\Omega)$, and we fix $u^*(x,y)$ for $(x,y) \in Y^n \cap F_2(\Omega)$ by $u^*(x,y) = -u^*(-x,y)$. It is easily seen that $u^*|_{\Gamma^\sigma}$ is constant for all $\sigma \in \mathcal{A}_{n+1}$, so the recursion can be continued.

**Theorem 5** *The function $u^*$ constructed above satisfies $u^* \in H^1(\Omega)$ and $\ell^\infty(u^*) \notin BMO(\Gamma^\infty, \mu)$.*

**Proof.** By using the bound $\left(\sum_{i=1}^n \alpha_i\right)^2 \leq n \sum_{i=1}^n \alpha_i^2$, it is possible to verify that

$$\|u^*\|^2_{L^2(Y^N)} \leq C \sum_{n=1}^{N-1} \left( a^{2n} n \sum_{i=1}^n \alpha_i^2 \right),$$

and that the quantity in the right hand side is bounded independently of $N$.
Moreover,

$$\|\nabla u^*\|^2_{L^2(Y^N)} = 2\mathcal{I} \sum_{n=1}^{N-1} \alpha_n^2,$$

which is bounded independently of $N$. Therefore $u^* \in H^1(\Omega)$.
We want to prove that $\ell^\infty(u^*) \notin BMO(\Gamma^\infty, \mu)$. For any positive number $r$, we know that

$$\langle \ell^n(u^*) \rangle_{B(P^*,r),\mu} = 0, \quad \forall n \in \mathbb{N},$$

because $u^*$ is antisymmetric with respect to the axis $x_1 = 0$ (note that the ball $B(P^*, r)$ is symmetric w.r.t. the axis $x_1 = 0$). Therefore, we also have

$$\langle \ell^\infty(u^*) \rangle_{B(P^*,r),\mu} = 0.$$

Moreover, from the construction of $u^*$ and since the numbers $\alpha_n$ are positive, we have that

$$|\ell^n(u^*)| \leq |\ell^\infty(u^*)|, \quad \forall n \in \mathbb{N}.$$

Therefore

$$\langle |\ell^\infty(u^*) - \langle \ell^\infty(u^*) \rangle_{B(P^*,r),\mu}| \rangle_{B(P^*,r),\mu} \geq \langle |\ell^n(u^*)| \rangle_{B(P^*,r),\mu}, \quad \forall n \in \mathbb{N}.$$

Symmetry considerations lead to

$$\langle |\ell^n(u^*)| \rangle_{B(P^*,r),\mu} = \frac{1}{\mu(B(P^*,r) \cap F_1(\Gamma^\infty))} \int_{B(P^*,r)\cap F_1(\Gamma^\infty)} \ell^n(u^*) d\mu.$$

Let us now make the important observation that for any $n \in \mathbb{N}$, there exists a positive number $r_n$ such that $B(P^*, r_n) \cap F_1(\Gamma^\infty) \subset \mathcal{M}_{\sigma_n^*}(\Gamma^\infty)$. On the other hand, for any $x \in \mathcal{M}_{\sigma_n^*}(\Gamma^\infty)$,

$$\ell^n(u^*)(x) = \sum_{\sigma \in \mathcal{A}_n} \left( \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} u^*(x)\, dx \right) \mathbf{1}_{\mathcal{M}_\sigma(\Gamma^\infty)}(x) = \frac{1}{|\Gamma^{\sigma_n^*}|} \int_{\Gamma^{\sigma_n^*}} u^*(x)\, dx = \sum_{i=1}^n \alpha_i.$$

This implies that

$$\langle|\ell^n(u^*)|\rangle_{B(P^*,r_n),\mu} = \sum_{i=1}^{n} \alpha_i,$$

thus

$$\langle|\ell^\infty(u^*) - \langle \ell^\infty(u^*)\rangle_{B(P^*,r_n),\mu}|\rangle_{B(P^*,r_n),\mu} \quad \geq \sum_{i=1}^{n} \alpha_i. \tag{28}$$

The right hand side of (28) tends to $+\infty$ as $n \to \infty$. We have proved that $\ell^\infty(u^*) \notin BMO(\Gamma^\infty, \mu)$.
■

**Remark 9** *Note that a similar counterexample can be found for bounded domains of the type*

$$\Omega = \mathit{Interior}\left(\overline{Y^0} \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma(\overline{Y^0})\right)\right),$$

*where $F_1$ and $F_2$ are two similitudes, $\mathcal{M}_\sigma$ is defined by (1) and $Y^0$ is a polygonal domain, as soon as*

- *The sets $\mathcal{M}_\sigma(Y^0)$, $\sigma \in \mathcal{A}_n$, $n > 0$, do not overlap.*
- *The self-invariant set $\Gamma^\infty$ is such that $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty) \neq \emptyset$, (in our setting this occurs for $a = a^*$).*
- *The domain $\Omega$ has a symmetry axis which contains $F_1(\Gamma^\infty) \cap F_2(\Gamma^\infty)$.*

*For instance, the same counterexample can be constructed for the geometry in Figure 3.*
*Moreover it seems that similar counterexamples can be found under relaxed geometrical assumptions, (in particular, the symmetry assumption does not seem crucial).*

#### 4.2.3 Trace theorems in $H^s(\Gamma^\infty)$.

For brevity, let us denote $d_H$ the Hausdorff dimension of $\Gamma^\infty$:

$$d_H = -\log 2 / \log a^*.$$

We define for $0 < s < 1$, the space $H^s(\Gamma^\infty)$ as

$$H^s(\Gamma^\infty) = \left\{ v \in L^2_\mu(\Gamma^\infty) \text{ s.t. } \int_{\Gamma^\infty}\int_{\Gamma^\infty} \frac{|v(x)-v(y)|^2}{d(x,y)^{d_H+2s}} d\mu(x)d\mu(y) < +\infty \right\}, \tag{29}$$

where $d(x, y)$ is the Euclidean distance between the two points $x$ and $y$. Note that the space $H^s(\Gamma^\infty)$ correspond to the space $B^{2,2}_s$ in [7], see page 103.
The space $H^s(\Gamma^\infty)$ is a Hilbert space with the norm

$$\|\cdot\|_{H^s(\Gamma^\infty)} = \sqrt{\|\cdot\|^2_{L^2_\mu} + |\cdot|^2_{H^s(\Gamma^\infty)}},$$

where

$$|v|_{H^s(\Gamma^\infty)} = \left(\int_{\Gamma^\infty}\int_{\Gamma^\infty} \frac{|v(x)-v(y)|^2}{d(x,y)^{d_H+2s}} d\mu(x)d\mu(y)\right)^{1/2}. \tag{30}$$

Our goal is to prove the following

**Theorem 6** *For any $s < d_H/4$,*

$$\ell^\infty(u) \in H^s(\Gamma^\infty), \quad \forall u \in H^1(\Omega),$$

*and there exists a positive constant $C$ such that*

$$\|\ell^\infty(u)\|_{H^s(\Gamma^\infty)} \leq C\|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega). \tag{31}$$

We first state two technical lemmas.
The first lemma is an easy bound (invariant by scaling) on the difference of the mean values of $\ell^\infty(u)$ on $F_1(\Gamma^\infty)$ and $F_2(\Gamma^\infty)$ by the $H^1(\Omega)$ semi-norm of $u$.

**Lemma 7** *There exists a positive constant $C$ such that*

$$|\langle \ell^\infty(u)\rangle_{F_1(\Gamma^\infty),\mu} - \langle \ell^\infty(u)\rangle_{F_2(\Gamma^\infty),\mu}| \leq C\|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H^1(\Omega). \tag{32}$$

**Remark 10** *The bound (32) implies that for all positive integer $p$ and for all $\sigma \in \mathcal{A}_p$,*

$$|\langle \ell^\infty(u)\rangle_{F_1(\mathcal{M}_\sigma(\Gamma^\infty)),\mu} - \langle \ell^\infty(u)\rangle_{F_2(\mathcal{M}_\sigma(\Gamma^\infty)),\mu}| \leq C\|\nabla u\|_{L^2(\mathcal{M}_\sigma(\Omega))}, \quad \forall u \in H^1(\Omega),$$

*where $C$ is exactly the same constant as in (32) (i.e. $C$ does not depend of $\sigma$ and $p$).*

The second lemma is an explicit bound on the $H^s$-semi-norm of the Haar mother wavelet $g^0$ on $\Gamma^\infty$,

$$g^0 = 1_{F_1(\Gamma^\infty)} - 1_{F_2(\Gamma^\infty)}. \tag{33}$$

**Lemma 8** *For all real number $s$ such that $0 < s < \frac{d_H}{4}$,*

$$|g^0|^2_{H^s(\Gamma^\infty)} = 2\int_{F_1(\Gamma^\infty)}\int_{F_2(\Gamma^\infty)} \frac{1}{d(x,y)^{d_H+2s}} d\mu(x)d\mu(y) < \infty. \tag{34}$$

*Moreover, if $s > \frac{d_H}{4}$, then*

$$|g^0|^2_{H^s(\Gamma^\infty)} = 2\int_{F_1(\Gamma^\infty)}\int_{F_2(\Gamma^\infty)} \frac{1}{d(x,y)^{d_H+2s}} d\mu(x)d\mu(y) = \infty.$$

**Proof.** Since it is rather technical, the proof is postponed to the Appendix B. ■
It is easy to construct a function $u$ in $H^1(\Omega)$ whose trace is $g_0$. Therefore, as corollary of Lemma 8, we have the following

**Proposition 5** *There exists a function $u \in H^1(\Omega)$ such that*

$$\ell^\infty(u) \notin H^s(\Gamma^\infty), \quad \forall s > d_H/4.$$

**Proof of Theorem 6** For $s$, $0 < s < d_H/4$, we set $\tau = 2s$ and we have to prove that there exists a constant $C$ such that for all $u \in H^1(\Omega)$,

$$\int_{\Gamma^\infty}\int_{\Gamma^\infty} \frac{|\ell^\infty(u)(x) - \ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x)d\mu(y) \leq C\|u\|^2_{H^1(\Omega)}. \tag{35}$$

Let us decompose the integral in (35) into three terms:

$$\int_{\Gamma^\infty}\int_{\Gamma^\infty} \frac{|\ell^\infty(u)(x) - \ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x)d\mu(y) = I + II + 2III, \tag{36}$$

where

$$\begin{aligned} I &= \int_{F_1(\Gamma^\infty)} \int_{F_1(\Gamma^\infty)} \frac{|\ell^\infty(u)(x) - \ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y), \\ II &= \int_{F_2(\Gamma^\infty)} \int_{F_2(\Gamma^\infty)} \frac{|\ell^\infty(u)(x) - \ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y), \\ III &= \int_{F_1(\Gamma^\infty)} \int_{F_2(\Gamma^\infty)} \frac{|\ell^\infty(u)(x) - \ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y). \end{aligned}$$

We first estimate the term $III$: from the identity

$$\ell^\infty(u)(x) - \ell^\infty(u)(y) = \begin{pmatrix} \ell^\infty(u)(x) - \langle \ell^\infty(u) \rangle_{F_1(\Gamma^\infty),\mu} \\ + \langle \ell^\infty(u) \rangle_{F_1(\Gamma^\infty),\mu} - \langle \ell^\infty(u) \rangle_{F_2(\Gamma^\infty),\mu} \\ + \langle \ell^\infty(u) \rangle_{F_2(\Gamma^\infty),\mu} - \ell^\infty(u)(y) \end{pmatrix},$$

we deduce that

$$\begin{aligned} III &\le 3(III_1 + III_2 + III_3), \\ III_1 &= \int_{F_1(\Gamma^\infty)} \int_{F_2(\Gamma^\infty)} \frac{|\ell^\infty(u)(x) - \langle \ell^\infty(u) \rangle_{F_1(\Gamma^\infty),\mu}|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y), \\ III_2 &= \int_{F_1(\Gamma^\infty)} \int_{F_2(\Gamma^\infty)} \frac{|\langle \ell^\infty(u) \rangle_{F_1(\Gamma^\infty),\mu} - \langle \ell^\infty(u) \rangle_{F_2(\Gamma^\infty),\mu}|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y), \\ III_3 &= \int_{F_1(\Gamma^\infty)} \int_{F_2(\Gamma^\infty)} \frac{|\langle \ell^\infty(u) \rangle_{F_2(\Gamma^\infty),\mu} - \ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y). \end{aligned}$$

We can bound $III_2$ thanks to Lemmas 7 and 8, because

$$\begin{aligned} III_2 &= |\langle \ell^\infty(u) \rangle_{F_1(\Gamma^\infty),\mu} - \langle \ell^\infty(u) \rangle_{F_2(\Gamma^\infty),\mu}|^2 \int_{F_1(\Gamma^\infty)} \int_{F_2(\Gamma^\infty)} \frac{1}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y) \\ &\le C \|u\|^2_{H^1(\Omega)}. \end{aligned}$$

Bounds on $III_1$ and $III_3$ are obtained in the same way, so let us only consider $III_1$: we fix $\eta > 0$ such that $(d_H + \tau)(1 + \eta) < 3d_H/2$, which also reads

$$(d_H + \tau)(1 + \eta) - d_H < d_H/2. \tag{37}$$

Since from Theorem 4, $\ell^\infty(u) \in L^p_\mu$, for all $p \in [2, +\infty)$, we have the following Hölder inequality

$$III_1 \le \begin{pmatrix} \left( \int_{F_1(\Gamma^\infty)} \int_{F_2(\Gamma^\infty)} |\ell^\infty(u)(x) - \langle \ell^\infty(u) \rangle_{F_1(\Gamma^\infty),\mu}|^{\frac{2(1+\eta)}{\eta}} d\mu(x) d\mu(y) \right)^{\frac{\eta}{1+\eta}} \\ \left( \int_{F_1(\Gamma^\infty)} \int_{F_2(\Gamma^\infty)} \frac{1}{d(x,y))^{(d_H+\tau)(1+\eta)}} d\mu(x) d\mu(y) \right)^{\frac{1}{1+\eta}} \end{pmatrix},$$

because the second factor is exactly $\left( \frac{1}{2} |g^0|^2_{H^{\frac{(d_H+\tau)(1+\eta)-d_H}{2}}(\Gamma^\infty)} \right)^{\frac{1}{1+\eta}} < \infty$ from (37) and Lemma 8. Therefore, for all $u \in H^1(\Omega)$,

$$\begin{aligned} III_1 &\lesssim \left( \int_{F_1(\Gamma^\infty)} \int_{F_2(\Gamma^\infty)} |\ell^\infty(u)(x) - \langle \ell^\infty(u) \rangle_{F_1(\Gamma^\infty),\mu}|^{\frac{2(1+\eta)}{\eta}} d\mu(x) d\mu(y) \right)^{\frac{\eta}{1+\eta}} \\ &\lesssim \left( \int_{F_1(\Gamma^\infty)} |\ell^\infty(u)(x) - \langle \ell^\infty(u) \rangle_{F_1(\Gamma^\infty),\mu}|^{\frac{2(1+\eta)}{\eta}} d\mu(x) \right)^{\frac{\eta}{1+\eta}} \end{aligned}$$

and from Theorem 4 (in the particular case $p = \frac{2(1+\eta)}{\eta}$),

$$III_1 \lesssim \int_{F_1(\Omega)} |\nabla u(x)|^2 dx.$$

Finally, we have obtained that there exists a positive constant $C$ such that for all $u \in H^1(\Omega)$,

$$III \le C \int_\Omega |\nabla u(x)|^2 dx.$$

We are left with finding an estimate for $I$ and $II$. Since the argument is the same for the two terms, we focus on $I$. Using the change of variables $x = F_1(x')$ and $y = F_1(y')$ and the definition of $d_H$, we obtain that

$$\begin{aligned} I &= \frac{1}{4} \int_{\Gamma^\infty} \int_{\Gamma^\infty} \frac{|\ell^\infty(u \circ F_1)(x') - \ell^\infty(u \circ F_1)(y')|^2}{(a \cdot d(x',y'))^{d_H+\tau}} d\mu(x') d\mu(y') \\ &= 2^{\frac{\tau}{d_H}-1} \int_{\Gamma^\infty} \int_{\Gamma^\infty} \frac{|\ell^\infty(u \circ F_1)(x') - \ell^\infty(u \circ F_1)(y')|^2}{(d(x',y'))^{d_H+\tau}} d\mu(x') d\mu(y'). \end{aligned}$$

This allows for a recursive argument, because we can decompose the last integral into the sum of three terms as in (36). We obtain that

$$I \le 2^{\frac{\tau}{d_H}-1} \left( \begin{array}{l} C \int_{F_1(\Omega)} |\nabla u(x)|^2 dx \\ +2^{\frac{\tau}{d_H}-1} \sum_{i=1}^{2} \int_{\Gamma^\infty} \int_{\Gamma^\infty} \frac{|\ell^\infty(u \circ F_1 \circ F_i)(x'') - \ell^\infty(u \circ F_1 \circ F_i)(y'')|^2}{(d(x'',y''))^{d_H+\tau}} d\mu(x'') d\mu(y'') \end{array} \right),$$

and this implies that

$$\begin{aligned} &\int_{\Gamma^\infty} \int_{\Gamma^\infty} \frac{|\ell^\infty(u)(x) - \ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y) \\ \le &\left( \begin{array}{l} C \left( \int_\Omega |\nabla u(x)|^2 dx + 2^{\frac{\tau}{d_H}-1} \int_{F_1(\Omega)} |\nabla u(x)|^2 dx + 2^{\frac{\tau}{d_H}-1} \int_{F_2(\Omega)} |\nabla u(x)|^2 dx \right) \\ +2^{2(\frac{\tau}{d_H}-1)} \sum_{\sigma \in \mathcal{A}_2} \int_{\Gamma^\infty} \int_{\Gamma^\infty} \frac{|\ell^\infty(u \circ \mathcal{M}_\sigma)(x) - \ell^\infty(u \circ \mathcal{M}_\sigma)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y) \end{array} \right). \end{aligned} \tag{38}$$

Note that the last term in the right hand side of (38) is exactly

$$\sum_{\sigma \in \mathcal{A}_2} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} \int_{\mathcal{M}_\sigma(\Gamma^\infty)} \frac{|\ell^\infty(u)(x) - \ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y).$$

Carrying the recursion, we find that

$$\begin{aligned} &\int_{\Gamma^\infty \times \Gamma^\infty} \frac{|\ell^\infty(u)(x) - \ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y) - \sum_{\sigma \in \mathcal{A}_n} \int_{\mathcal{M}_\sigma(\Gamma^\infty) \times \mathcal{M}_\sigma(\Gamma^\infty)} \frac{|\ell^\infty(u)(x) - \ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y) \\ = &\int_{\Gamma^\infty \times \Gamma^\infty} 1_{\Gamma^\infty \times \Gamma^\infty \setminus \cup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(\Gamma^\infty) \times \mathcal{M}_\sigma(\Gamma^\infty)} \frac{|\ell^\infty(u)(x) - \ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y) \\ \le &C \sum_{m<n} 2^{m(\frac{\tau}{d_H}-1)} \sum_{\sigma \in \mathcal{A}_m} \int_{\mathcal{M}_\sigma(\Omega)} |\nabla u(x)|^2 dx. \end{aligned}$$

From this and Fatou lemma, we obtain that

$$\begin{aligned}
\int_{\Gamma^\infty}\int_{\Gamma^\infty} \frac{|\ell^\infty(u)(x)-\ell^\infty(u)(y)|^2}{d(x,y)^{d_H+\tau}} d\mu(x)d\mu(y) &\leq C\sum_{m\in\mathbb{N}} 2^{m(\frac{\tau}{d_H}-1)} \sum_{\sigma\in\mathcal{A}_m}\int_{\mathcal{M}_\sigma(\Omega)} |\nabla u(x)|^2 dx \\
&= C\sum_{m\in\mathbb{N}}\sum_{0\leq k\leq m} 2^{k(\frac{\tau}{d_H}-1)} \int_{\Omega^m\setminus\Omega^{m-1}} |\nabla u(x)|^2 dx \\
&\leq C\left(\frac{1}{1-2^{(\frac{\tau}{d_H}-1)}}\right)\int_\Omega |\nabla u(x)|^2 dx,
\end{aligned}$$

because $\tau < \frac{d_H}{2}$. This concludes the proof. ■

**Remark 11** *It is easy to extend Theorem 6 and Proposition 5 to the Sobolev space* $W^{1,p}(\Omega)$, $2 \leq p < \infty$:
*For any* $s < 1 - \frac{2}{p} + \frac{d_H}{2p}$,

$$u|_{\Gamma^\infty} \in W^{s,p}(\Gamma^\infty), \quad \forall u \in W^{1,p}(\Omega),$$

*and there exists a positive constant* $C$ *such that*

$$\|u|_{\Gamma^\infty}\|_{W^{s,p}(\Gamma^\infty)} \leq C\|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).$$

*Moreover, there exists a function* $u \in W^{1,p}(\Omega)$ *such that*

$$u|_{\Gamma^\infty} \notin W^{s,p}(\Gamma^\infty), \quad \forall s > 1 - \frac{2}{p} + \frac{d_H}{2p}.$$

# A Proof of Proposition 4

The proof of Proposition 4 uses arguments that have already been written in [1] for a slightly different construction with the same affine maps $\widetilde{F}_i$ (see (22)) but with a different reference domain $\widetilde{Y}^0$.
The first step is an extension result:

**Lemma 9** *Let* $\widehat{\Omega}$ *be the trapezoidal domain whose vertices are* $(-1,0)$, $(1,0)$, $(-3,6)$ *and* $(3,6)$. *With* $\widetilde{\Omega}$ *defined in (24), there exists an extension operator* $\mathcal{J}$ *bounded from* $W^{1,q}(\widetilde{\Omega})$ *to* $W^{1,q}(\widehat{\Omega})$, *for all* $q$, $1 \leq q < 2$.

**Proof.** This is Lemma 11 in [1]. ■
As a consequence we have a density result (Theorem 4 in [1]):

**Proposition 6** *The space* $\mathcal{C}^\infty(\overline{\widetilde{\Omega}})$ *is dense in* $W^{1,q}(\widetilde{\Omega})$, *for all* $q$, $1 \leq q < 2$. *More precisely, for* $q$, $1 \leq q < 2$, *for any* $\widetilde{u} \in W^{1,q}(\widetilde{\Omega})$, *there exists a sequence* $(\widetilde{\phi}_j)_{j\in\mathbb{N}}$, $\widetilde{\phi}_j \in \mathcal{C}^\infty(\overline{\widehat{\Omega}})$ *such that*

$$\lim_{n\to 0} \|\widetilde{u} - \widetilde{\phi}_j\|_{W^{1,q}(\widetilde{\Omega})} = 0.$$

**Remark 12** *Observe that for any function* $\phi \in \mathcal{C}^1(\overline{\widehat{\Omega}})$,

$$\widetilde{\ell}^\infty(\phi \mathbf{1}_{\widetilde{\Omega}}) = \phi \mathbf{1}_{\widetilde{\Gamma}^\infty}. \tag{39}$$

*Indeed, with $\widetilde{\mathcal{M}}_\sigma$ defined in (23),*

$$\|\widetilde{\ell}^n(\phi \mathbf{1}_{\widetilde{\Omega}}) - \phi \mathbf{1}_{\widetilde{\Gamma}^\infty}\|^2_{L^2(\widetilde{\Gamma}^\infty,dx)} = \sum_{\sigma\in\mathcal{A}_n} \int_{\widetilde{\mathcal{M}}_\sigma(\widetilde{\Gamma}^\infty)} \left|\phi(x) - \frac{1}{\widetilde{\Gamma}^\sigma}\int_{\widetilde{\Gamma}^\sigma}\phi\right|^2 dx \le C 2^{-2n}\|\nabla\phi\|^2_{L^\infty(\widehat{\Omega})},$$

*where the last estimate comes from a Taylor expansion of $\phi$. This yields that*

$$\lim_{n\to\infty} \|\widetilde{\ell}^n(\phi \mathbf{1}_{\widetilde{\Omega}}) - \phi \mathbf{1}_{\widetilde{\Gamma}^\infty}\|^2_{L^2(\widetilde{\Gamma}^\infty,dx)} = 0,$$

*and (39).*
*Similarly for any $p \in [1,\infty[$,*

$$\|\widetilde{\ell}^n(\phi \mathbf{1}_{\widetilde{\Omega}}) - \phi \mathbf{1}_{\widetilde{\Gamma}^\infty}\|^p_{L^p(\widetilde{\Gamma}^\infty,dx)} \le C 2^{-pn}\|\nabla\phi\|^p_{L^\infty(\widehat{\Omega})}. \tag{40}$$

*Therefore for all $p \in [1,\infty)$ and $\phi \in \mathcal{C}^1(\overline{\widehat{\Omega}})$,*

$$\lim_{n\to\infty} \|\widetilde{\ell}^n(\phi \mathbf{1}_{\widetilde{\Omega}}) - \phi \mathbf{1}_{\widetilde{\Gamma}^\infty}\|^p_{L^p(\widetilde{\Gamma}^\infty,dx)} = 0.$$

**Lemma 10** *For any $p \in [1,\infty)$ there exists $q \in [1,2)$ and a constant $C$ independent of $n$ such that*

$$\|\widetilde{\ell}^n(\phi)\|_{L^p(\widetilde{\Gamma}^\infty,dx)} \le C\|\phi\|_{W^{1,q}(\widetilde{\Omega})}, \qquad \forall \phi \in W^{1,q}(\widetilde{\Omega}). \tag{41}$$

**Proof.** Call $\widehat{\Gamma}^n$ the smallest straight line segment containing $\widetilde{\Gamma}^n$. For all $p \in [1,\infty)$, there exists $q \in (1,2)$ and a positive constant $c$ independent of $n$ such that

$$\|\psi\|_{L^p(\widehat{\Gamma}^n,dx)} \le c\|\psi\|_{W^{1,q}(\widehat{\Omega})}, \qquad \forall \psi \in W^{1,q}(\widehat{\Omega}). \tag{42}$$

Then, using the extension operator $\mathcal{J}$ mentioned in Lemma 9, we see that for any $p \in [1,\infty)$ there exists $q \in [1,2)$ and a constant $C$ independent of $n$ such that

$$\begin{aligned}
\|\widetilde{\ell}^n(\phi)\|^p_{L^p(\widetilde{\Gamma}^\infty,dx)} &\le 3\int_{\widetilde{\Gamma}^n} |\phi(x)|^p dx = 3\int_{\widetilde{\Gamma}^n} |\mathcal{J}(\phi)(x)|^p dx \\
&\le 3\int_{\widehat{\Gamma}^n} |\mathcal{J}(\phi)(x)|^p dx \\
&\le 3c\|\mathcal{J}(\phi)\|^p_{W^{1,q}(\widehat{\Omega})} \le C\|\phi\|^p_{W^{1,q}(\widetilde{\Omega})}.
\end{aligned}$$

where $c$ is the constant appearing in (42). ■
We are now ready to prove Proposition 4.

**Proof of Proposition 4** Consider $\widetilde{u} \in H^1(\widetilde{\Omega})$, fix $p \in [1,\infty)$ and take $q = q(p) \in [1,2)$ as in Lemma 10. Thanks to Proposition 6, for any $\epsilon > 0$, there exists a function $\phi \in \mathcal{C}^\infty(\overline{\widehat{\Omega}})$ such that

$$\|\widetilde{u} - \phi\|_{W^{1,q}(\widetilde{\Omega})} \le \epsilon. \tag{43}$$

Then for all nonnegative integers $n$ and $k$,

$$\begin{aligned}
&\|\widetilde{\ell}^n(\widetilde{u}) - \widetilde{\ell}^{n+k}(\widetilde{u})\|_{L^p(\widetilde{\Gamma}^\infty,dx)} \\
\le &\|\widetilde{\ell}^n(\widetilde{u} - \phi)\|_{L^p(\widetilde{\Gamma}^\infty,dx)} + \|\widetilde{\ell}^n(\phi) - \widetilde{\ell}^{n+k}(\phi)\|_{L^p(\widetilde{\Gamma}^\infty,dx)} + \|\widetilde{\ell}^{n+k}(\phi - \widetilde{u})\|_{L^p(\widetilde{\Gamma}^\infty,dx)}.
\end{aligned}$$

Using (43) and (41), we obtain that

$$\|\widetilde{\ell}^n(\widetilde{u}) - \widetilde{\ell}^{n+k}(\widetilde{u})\|_{L^p(\widetilde{\Gamma}^\infty,dx)} \le \|\widetilde{\ell}^n(\phi) - \widetilde{\ell}^{n+k}(\phi)\|_{L^p(\widetilde{\Gamma}^\infty,dx)} + 2C\varepsilon. \tag{44}$$

On the other hand, (40) implies that $(\widetilde{\ell}^n(\phi))_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\widetilde{\Gamma}^\infty, dx)$. From this and (44), we see that $(\widetilde{\ell}^n(\widetilde{u}))_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\widetilde{\Gamma}^\infty, dx)$. Finally, it is easy to identify the $L^p(\widetilde{\Gamma}^\infty, dx)$-limit and we obtain the desired result using (41).

# B Proof of Lemma 8.

## B.1 Preliminary definitions and lemmas

Hereafter, in order to make the notation simpler, we write $a$ instead of $a^*$.
Take $N = 2(n+2)$, and consider

$$\begin{aligned} \Xi^{N,1} &= \cup_{\sigma\in\mathcal{A}_n} \Xi^{\sigma,1}, \\ \Xi^{N,2} &= \cup_{\sigma\in\mathcal{A}_n} \Xi^{\sigma,2}, \end{aligned}$$

where

$$\begin{aligned} \Xi^{\sigma,1} &= G_1 \circ F_2 \circ F_2 \circ G_{\sigma(1)} \circ \cdots \circ G_{\sigma(n)}(\Gamma^0), \quad \text{for } \sigma \in \mathcal{A}_n, \\ \Xi^{\sigma,2} &= G_1^c \circ F_2^c \circ F_2^c \circ G_{\sigma(1)}^c \circ \cdots \circ G_{\sigma(n)}^c(\Gamma^0), \quad \text{for } \sigma \in \mathcal{A}_n, \end{aligned}$$

where $G_1$ and $G_2$ are defined in (7) and $G_1^c$ and $G_2^c$ are defined in (8). From these definitions, we see that $\Xi^{N,i}$, $i = 1, 2$, is made of $\#(\mathcal{A}_n) = 2^n$ non-overlapping straight lines and that $\Xi^{N,i} \subset F^i(\Gamma^{N-1}) \subset \Gamma^N$ . The one-dimensional Lebesgue measure of $\Xi^{\sigma,i}$, for $\sigma \in \mathcal{A}_n$ is

$$\left|\Xi^{\sigma,i}\right| = a^{2n+4}|\Gamma^0|.$$

Recalling the definition of prox$(\sigma)$ given in (12) for $\sigma \in \mathcal{A}_N$, we introduce a mapping $\iota$ from $\mathcal{A}_N$ to $\mathbb{N}$ which allows for sorting the $\sigma \in \mathcal{A}_N$ such that prox$(\sigma) =$ k, for any fixed integer $k$, $k = 1, \dots, n+1$.

**Definition 3** *Take $N = 2(n+2)$. For $\sigma \in \mathcal{A}_N$, such that $\sigma(1,2,3,4) = (1,2,2,2)$, let us define the integer $\iota(\sigma)$ by*

$$\begin{cases} \iota(\sigma) = \displaystyle\sum_{q=1}^{\mathrm{prox}(\sigma)} 2^{\mathrm{prox}(\sigma)-\mathrm{q}}(\sigma(2q+3)-1), & \text{if} \quad 1 \le \mathrm{prox}(\sigma) \le \mathrm{n}, \\ \iota(\sigma) = \displaystyle\sum_{q=1}^{n} 2^{n-q}(\sigma(2q+3)-1), & \text{if} \quad \mathrm{prox}(\sigma) = \mathrm{n}+1, \end{cases} \tag{45}$$

*see Figure 5.*
*For $\sigma \in \mathcal{A}_N$, such that $\sigma(1,2,3,4) = (2,1,1,1)$, we choose $\iota(\sigma) = \iota(\sigma^c)$, where $\sigma^c(i) = 3 - \sigma(i)$, for all $i = 1, \dots, N$.*

**Properties of prox and $\iota$**

1. If $1 \le \mathrm{prox}(\sigma) \le \mathrm{n}$, then $\iota(\sigma) \in \{0, \dots, 2^{\mathrm{prox}(\sigma)} - 1\}$, and if $\mathrm{prox}(\sigma) = \mathrm{n}+1$ then $\iota(\sigma) \in \{0, \dots, 2^n - 1\}$.

2. For $\sigma \in \mathcal{A}_N$, there exists $2^{2(n-\min(\mathrm{prox}(\sigma),\mathrm{n}))}$ maps $\eta \in \mathcal{A}_N$ such that

$$\begin{aligned} \eta(1,2,3,4) &= \sigma(1,2,3,4), \\ \mathrm{prox}(\eta) &= \mathrm{prox}(\sigma), \\ \iota(\eta) &= \iota(\sigma). \end{aligned}$$

   These maps satisfy $\eta(j) = \sigma(j)$ for all $j \in \{1, \dots, 4 + 2\min(\mathrm{prox}(\sigma), \mathrm{n})\}$.

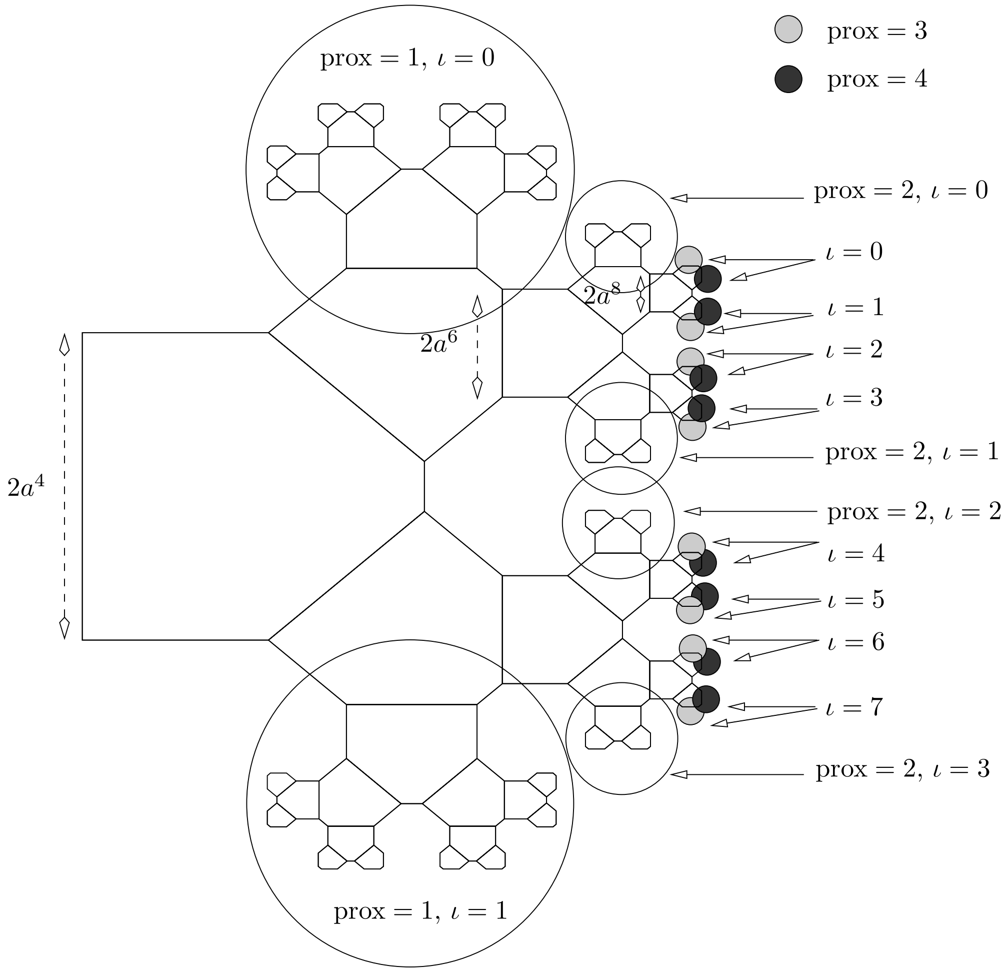


Figure 5: Zoom on $F_1 \circ F_2 \circ F_2 \circ F_2(Y^5)$. The values of $\text{prox}(\sigma)$ and of $\iota(\sigma)$ for $n = 3$, i.e. $N = 10$.

3. For $\sigma \in \mathcal{A}_N = \mathcal{A}_{2n+4}$, and $m < n$, the mapping $\sigma' \in \mathcal{A}_{2m+4}$ defined by $\sigma'(i) = \sigma(i)$, $i = 1, \dots, 2m+4$ is such that

$$\begin{aligned} \mathrm{prox}(\sigma') &= \min(\mathrm{prox}(\sigma), \mathrm{m}+1), \\ \iota(\sigma') &= \left[ \frac{\iota(\sigma)}{2^{\min(\mathrm{prox}(\sigma),\mathrm{n}) - \min(\mathrm{prox}(\sigma'),\mathrm{m})}} \right], \end{aligned}$$

where hereafter, $[u]$ stands for the integer part of $u$.

A consequence of Lemma 3 is the following:

**Lemma 11** *For all $N = 2n+4$, for all $\sigma \neq \eta \in \mathcal{A}_N$ such that $\sigma(1,2,3,4) = \eta(1,2,3,4) = (1,2,2,2)$, $\mathrm{prox}(\eta) = \mathrm{prox}(\sigma) = \mathrm{k}$, and $\iota(\sigma) \neq \iota(\eta)$,*

$$\inf_{x \in \Gamma^\sigma, y \in \Gamma^\eta} |x_2 - y_2| \geq \frac{1}{2}(|\iota(\sigma) - \iota(\eta)| - 1) a^{2k+2}. \tag{46}$$

**Proof.** For a given $k$, $0 \leq k \leq n+1$, call $\mathcal{S}_k$ the set of all the maps $\mu \in \mathcal{A}_N$ such that $\mu(1,2,3,4) = (1,2,2,2)$ and $\mathrm{prox}(\mu) = \mathrm{k}$. If $k < n+1$, the lines $\Gamma^\mu$, $\mu \in \mathcal{S}_k$ are either horizontal or vertical. If $\iota(\mu) < \iota(\nu)$, any $x \in \Gamma^\mu$ and $x' \in \Gamma^\nu$ satisfy $x'_2 < x_2$. We introduce the notations

$$x_{2,\min}(i) = \min_{x \in \Gamma^\mu,\, \mu \in \mathcal{S}_k,\, \iota(\mu) = i} x_2, \quad \text{and} \quad x_{2,\max}(i) = \max_{x \in \Gamma^\mu,\, \mu \in \mathcal{S}_k,\, \iota(\mu) = i} x_2.$$

Therefore, assuming $\iota(\eta) > \iota(\sigma)$,

$$\inf_{x \in \Gamma^\sigma, y \in \Gamma^\eta} |x_2 - y_2| \geq \sum_{i=\iota(\sigma)}^{\iota(\eta)-1} \left( x_{2,\min}(i) - x_{2,\max}(i+1) \right). \tag{47}$$

By an elementary geometrical argument, (cf Figure 5), we see that $x_{2,\min}(2j) - x_{2,\max}(2j+1) > 2a^{2k+2}$. This and (47) yield (46).
Similarly, if $k = n+1$, then the lines $\Gamma^\mu$, $\mu \in \mathcal{S}_k$ are vertical and aligned, and if $\mu, \nu \in \mathcal{S}_k$ satisfy $\iota(\mu) < \iota(\nu)$, $\min_{x \in \Gamma^\mu} x_2 - \max_{x \in \Gamma^\nu} x_2 > 0$. Therefore, (47) is still valid. By an elementary geometrical argument, we see that $x_{2,\min}(2j) - x_{2,\max}(2j+1) > 2a^{2(n+1)+2}$. This and (47) yield (46). ■

**Remark 13** *The estimate (46) is very far from optimal, but for what follows, we do not need a better one.*

**Lemma 12** *For all $N = 2n+4$, for all $\sigma, \eta \in \mathcal{A}_N$ such that $\sigma(1,2,3,4) = \eta(1,2,3,4) = (1,2,2,2)$ and $\mathrm{prox}(\eta) < \mathrm{prox}(\sigma)$,*

$$\inf_{x \in \Gamma^\sigma, y \in \Gamma^\eta} |x_2 - y_2| \geq \frac{1}{2} |\iota(\sigma') - \iota(\eta)| a^{2\mathrm{prox}(\eta)+2},$$

*where $\sigma' \in \mathcal{A}_{2m+4}$, $m = \mathrm{prox}(\sigma)$ is defined by $\sigma'(i) = \sigma(i)$, $i = 1, \dots, 2m+4$ and*

$$\mathrm{prox}(\sigma') = \mathrm{prox}(\eta) + 1 \quad \text{and} \quad \iota(\sigma') = \left[ \frac{\iota(\sigma)}{2^{\min(\mathrm{prox}(\sigma),\mathrm{n}) - \mathrm{prox}(\eta)}} \right].$$

**Proof.** The first elementary observation is that if $\iota(\eta) = \iota(\sigma') \pm 1$, then

$$\inf_{x \in \Gamma^\sigma, y \in \Gamma^\eta} |x_2 - y_2| > a^{2\mathrm{prox}(\eta)+2}, \tag{48}$$

see Figure 5.
The second observation is that if for example $\iota(\eta) > \iota(\sigma') + 1$, then calling $\mu$ a map in $\mathcal{A}_N$ such that $\mathrm{prox}(\mu) = \mathrm{prox}(\eta)$ and $\iota(\mu) = \iota(\sigma') + 1$, we have

$$\begin{aligned} \inf_{x \in \Gamma^\sigma, y \in \Gamma^\eta} |x_2 - y_2| &\geq \inf_{x \in \Gamma^\sigma, y \in \Gamma^\mu} |x_2 - y_2| + \inf_{x \in \Gamma^\mu, y \in \Gamma^\eta} |x_2 - y_2| \\ &\geq a^{2\mathrm{prox}(\eta)+2} + \frac{1}{2}\big(\iota(\eta) - \iota(\sigma') - 2\big)\, a^{2\mathrm{prox}(\eta)+2} \\ &= \frac{1}{2}\big(\iota(\eta) - \iota(\sigma')\big)\, a^{2\mathrm{prox}(\eta)+2}, \end{aligned}$$

where we have used (48) and (46).
The argument is the same if $\iota(\eta) < \iota(\sigma') - 1$. ■

**Corollary 4** *There exists a constant $c$ such that, for all $N = 2n+4$, for all $\sigma, \eta \in \mathcal{A}_N$ such that $\sigma(1,2,3,4) = (1,2,2,2)$, $\eta(1,2,3,4) = (2,1,1,1)$ and $\mathrm{prox}(\sigma) > \mathrm{prox}(\eta)$, we have*

$$d(\Gamma^\sigma, \Gamma^\eta) \geq c\sqrt{(a^{2\mathrm{prox}(\sigma)} + a^{2\mathrm{prox}(\eta)})^2 + \left(\iota(\eta) - \left[\frac{\iota(\sigma)}{2^{\min(\mathrm{prox}(\sigma),\mathrm{n})-\mathrm{prox}(\eta)}}\right]\right)^2 a^{4\mathrm{prox}(\eta)}}, \tag{49}$$

**Proof.** Take $x \in \Gamma^\sigma$ and $y \in \Gamma^\eta$. We know from Lemma 4 that $|x_1| \geq \delta' a^{2\mathrm{prox}(\sigma)}$ and $|y_1| \geq \delta' a^{2\mathrm{prox}(\eta)}$. Since $x_1 < 0 < y_1$, we have $y_1 - x_1 \geq \delta'(a^{2\mathrm{prox}(\sigma)} + a^{2\mathrm{prox}(\eta)})$.
The lower estimate for $|y_2 - x_2|$ comes from Lemma 12. ■

## B.2 Proof of Lemma 8

The proof of Lemma 8 will be decomposed into several steps:

**Assertion 1**

$$\text{If } \tau > \tfrac{d_H}{2}, \qquad \text{then} \qquad \lim_{n\to\infty} (2a)^{-(4n+8)} \int_{\Xi^{N,1}} \int_{\Xi^{N,2}} \frac{1}{d(x,y)^{d_H+\tau}} dx\, dy = \infty. \tag{50}$$

**Assertion 2**

$$\text{If } \tau > \tfrac{d_H}{2}, \qquad \text{then} \qquad \int_{F_1(\Gamma^\infty)} \int_{F_2(\Gamma^\infty)} \frac{1}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y) = \infty. \tag{51}$$

**Assertion 3**

$$\text{If } \tau < \tfrac{d_H}{2}, \qquad \text{then} \qquad \lim_{n\to\infty} (2a)^{-(4n+8)} \int_{\Xi^{N,1}} \int_{\Xi^{N,2}} \frac{1}{d(x,y)^{d_H+\tau}} dx\, dy = 0. \tag{52}$$

**Assertion 4**

$$\text{If } \tau < \tfrac{d_H}{2}, \qquad \text{then} \qquad (2a)^{-4n} \sum_{\sigma \in \mathcal{A}_N, \sigma(1)=1} \sum_{\sigma' \in \mathcal{A}_N, \sigma'(1)=2} \int_{\Gamma^\sigma} \int_{\Gamma^{\sigma'}} \frac{dx\, dy}{d(x,y)^{d_H+\tau}} \leq C, \tag{53}$$

where $C > 0$ is independent of $n$.

**Assertion 5**

$$\text{If } \tau < \frac{d_H}{2}, \qquad \text{then} \qquad \int_{F_1(\Gamma^\infty)} \int_{F_2(\Gamma^\infty)} \frac{1}{d(x,y)^{d_H+\tau}} d\mu(x) d\mu(y) < \infty. \tag{54}$$

In order to support assertions 1 and 4, we plot in the top of Figure 6

$$(2a)^{-2N} \sum_{\sigma \in \mathcal{A}_N, \sigma(1)=1} \sum_{\sigma' \in \mathcal{A}_N, \sigma'(1)=2} \int_{\Gamma^\sigma} \int_{\Gamma^{\sigma'}} \frac{dx\, dy}{d(x_\sigma, x_{\sigma'})^{d_H+t}}$$

versus $N$, for $a = 0.593465 \approx a^*$ where $x_\sigma$ is the midpoint of $\Gamma^\sigma$, for $t = 0.4 d_H$, $t = 0.5 d_H$ and $t = 0.6 d_H$. We see that for large values of $N$ the function is convex for $t = 0.6 d_H$ and concave if $t = 0.4 d_H$. On the bottom part of Figure 6, we plot the same quantity for $a = 0.58673 < a^*$ so $d_H = 1.3$ instead of 1.3284371, and $t = d_H$, and we see that the function tends to a constant value as $N$ tends to infinity. This supports the theoretical results obtained in § 4.1 in the sub-critical case $a < a^*$.

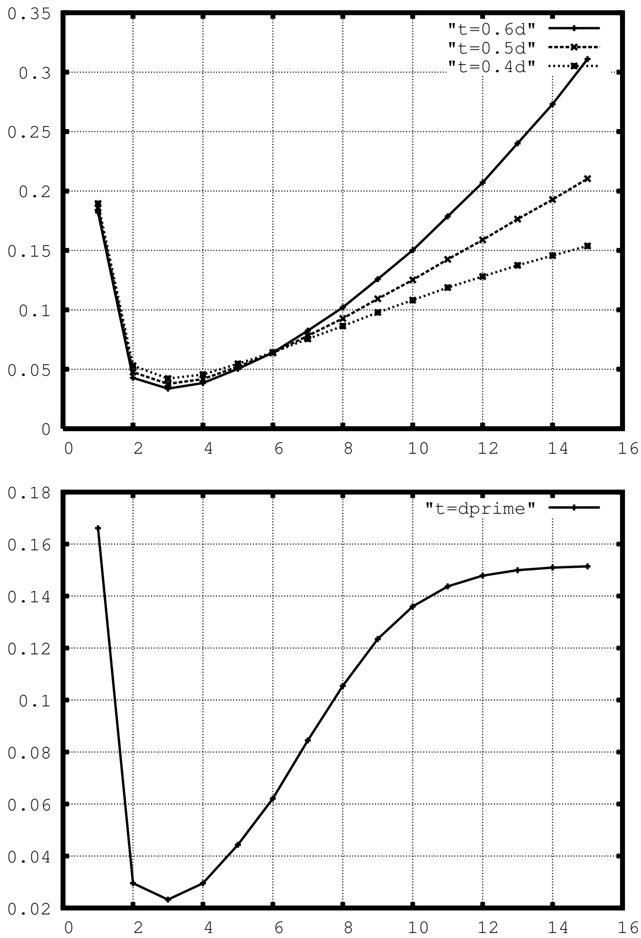


Figure 6: The value of $(2a)^{-2N} \sum_{\sigma \in \mathcal{A}_N, \sigma(1)=1} \sum_{\sigma' \in \mathcal{A}_N, \sigma'(1)=2} \int_{\Gamma^\sigma} \int_{\Gamma^{\sigma'}} \frac{dx\, dy}{d(x_\sigma, x_{\sigma'})^{d_H+t}}$ versus $N$ for $d_H = 1.3284371$, $t = 0.4 d_H$, $t = 0.5 d_H$ and $t = 0.6 d_H$ (top) and $d_H = 1.3$, $t = d_H$ (bottom).

**Proof of Assertions 1 and 3.** As an easy consequence of Lemma 3, there exist two positive numbers $c$ and $C$, such that for all $n > 0$ and for all $\sigma \in \mathcal{A}_n$, $\forall x \in \Xi^{\sigma,1}$,

$$ca^{2n} \leq d(x,y) \leq Ca^{2n}, \quad \forall y \in \Xi^{\sigma,2}, \tag{55}$$

$$ca^{2(n-k)} \leq d(x,y) \leq Ca^{2(n-k)}, \quad \forall y \in \Xi^{\sigma',2},\ \sigma' \in \mathcal{A}_n \text{ s.t. } \begin{cases} \sigma(i) = \sigma'(i), & \forall i < n-k, \\ \sigma(n-k) \neq \sigma'(n-k). \end{cases} \tag{56}$$

Observe that if $\sigma \in \mathcal{A}_n$, for all $k \in \{0, \dots, n-1\}$, there exists $2^k$ maps $\sigma' \in \mathcal{A}_n$ such that $\sigma(i) = \sigma'(i)$ for $i < n-k$, and $\sigma(n-k) \neq \sigma'(n-k)$. From (56), we deduce that

$$\begin{aligned}\frac{a^{-4n-8}}{4} \sum_{\sigma \in \mathcal{A}_n} \sum_{\sigma' \in \mathcal{A}_n} \int_{\Xi^{\sigma,1}} \int_{\Xi^{\sigma',2}} \frac{dx\, dy}{d(x,y)^{d_H+\tau}} &\geq \sum_{\sigma \in \mathcal{A}_n} \sum_{k=0}^{n-1} 2^k \frac{1}{(Ca^{2(n-k)})^{d_H+\tau}} = \sum_{k=0}^{n-1} \frac{2^{n+k}}{(Ca^{2(n-k)})^{d_H+\tau}} \\ &= C^{-d_H-\tau} \sum_{k=0}^{n-1} 2^{n+k} 2^{2(n-k)(\frac{\tau}{d_H}+1)} \\ &= C^{-d_H-\tau} 2^{n(3+\frac{2\tau}{d_H})} \sum_{k=0}^{n-1} 2^{-k(1+\frac{2\tau}{d_H})},\end{aligned}$$

where we have used the fact that $a^{d_H} = 1/2$. From the bound above, we see that if $\tau > \frac{d_H}{2}$, then there exists a positive number $\nu$ such that $3 + \frac{2\tau}{d_H} \geq 4 + \nu$ and

$$a^{-4n-8} \sum_{\sigma \in \mathcal{A}_n} \sum_{\sigma' \in \mathcal{A}_n} \int_{\Xi^{\sigma,1}} \int_{\Xi^{\sigma',2}} \frac{dx\, dy}{d(x,y)^{d_H+\tau}} \gtrsim 2^{(4+\nu)n},$$

which yields (50).
On the other hand, from the lower bounds in (55) and (56), we see in a similar way that

$$\frac{a^{-4n-8}}{4} \sum_{\sigma \in \mathcal{A}_n} \sum_{\sigma' \in \mathcal{A}_n} \int_{\Xi^{\sigma,1}} \int_{\Xi^{\sigma',2}} \frac{dx\, dy}{d(x,y)^{d_H+\tau}} \leq c^{-d_H-\tau} 2^{n(3+\frac{2\tau}{d_H})} + c^{-d_H-\tau} 2^{n(3+\frac{2\tau}{d_H})} \sum_{k=0}^{n-1} 2^{-k(1+\frac{2\tau}{d_H})}.$$

From this, we see that if $\tau < \frac{d_H}{2}$, there exists some positive number $\nu$ such that

$$a^{-4n-8} \sum_{\sigma \in \mathcal{A}_n} \sum_{\sigma' \in \mathcal{A}_n} \int_{\Xi^{\sigma,1}} \int_{\Xi^{\sigma',2}} \frac{dx\, dy}{d(x,y)^{d_H+\tau}} \lesssim 2^{(4-\nu)n},$$

which yields (52). ■

**Proof of Assertion 2.** We need the notations

$$\Gamma^{\infty,\sigma,1} = G_1 \circ F_2 \circ F_2 \circ G_{\sigma(1)} \circ \dots \circ G_{\sigma(n)}(\Gamma^\infty \backslash \mathcal{N}^\infty), \quad \text{for } \sigma \in \mathcal{A}_n, \tag{57}$$

$$\Gamma^{\infty,\sigma,2} = G_1^c \circ F_2^c \circ F_2^c \circ G_{\sigma(1)}^c \circ \dots \circ G_{\sigma(n)}^c(\Gamma^\infty \backslash \mathcal{N}^\infty), \quad \text{for } \sigma \in \mathcal{A}_n, \tag{58}$$

$$\Gamma^{\infty,N,1} = \cup_{\sigma \in \mathcal{A}_n} \Gamma^{\infty,\sigma,1}, \quad \text{and} \quad \Gamma^{\infty,N,2} = \cup_{\sigma \in \mathcal{A}_n} \Gamma^{\infty,\sigma,2}. \tag{59}$$

We define the mapping $\aleph^{N,1} : \Gamma^{\infty,N,1} \to \Xi^{N,1}$ by

$$\aleph^{N,1}(x) = G_1 \circ F_2 \circ F_2 \circ G_{\sigma(1)} \circ \dots \circ G_{\sigma(n)}(\aleph(y)), \quad \text{if} \quad x = G_1 \circ F_2 \circ F_2 \circ G_{\sigma(1)} \circ \dots \circ G_{\sigma(n)}(y),\ y \in \Gamma^\infty,$$

where the function $\aleph$ has been defined in (19). We define in a similar way the mapping $\aleph^{N,2} : \Gamma^{\infty,N,2} \to \Xi^{N,2}$. A consequence of (20) is that

$$\frac{1}{2}(2a)^{-2n-4} \int_{\Xi^{N,i}} f(x)dx = \int_{\Gamma^{\infty,N,i}} f \circ \aleph^{N,i} d\mu, \qquad \forall f \in L^1(\Xi^{N,i}),\ i = 1,2. \tag{60}$$

Simple geometric considerations yield that there exists a constant $C$ such that

$$d(x,\aleph^{N,i}(x)) \le Ca^{2n+4} \qquad \forall x \in \Gamma^{\infty,N,i},\ i=1,2,$$

and that

$$d(x,y) \le d\left(\aleph^{N,1}(x),\aleph^{N,2}(y)\right) + Ca^{2n+4} \qquad \forall x \in \Gamma^{\infty,N,1}, y \in \Gamma^{\infty,N,2}.$$

Therefore

$$\begin{aligned}\int_{\Gamma^{\infty,N,1}}\int_{\Gamma^{\infty,N,2}} \frac{1}{d(x,y)^{d_H+\tau}} d\mu(x)d\mu(y) &\ge \int_{\Gamma^{\infty,N,1}}\int_{\Gamma^{\infty,N,2}} \frac{1}{\left(d\left(\aleph^{N,1}(x),\aleph^{N,2}(y)\right)+Ca^{2n+4}\right)^{d_H+\tau}} d\mu(x)d\mu(y)\\ &= \frac{1}{4}(2a)^{-4n-8}\int_{\Xi^{N,1}}\int_{\Xi^{N,2}} \frac{1}{\left(d\left(x,y\right)+Ca^{2n+4}\right)^{d_H+\tau}} dxdy,\end{aligned}$$

and a slight modification to the proof of Assertion 1 yields that if $\tau > \frac{d_H}{2}$, then

$$\lim_{n\to\infty}\int_{\Gamma^{\infty,N,1}}\int_{\Gamma^{\infty,N,2}} \frac{1}{d(x,y)^{d_H+\tau}} d\mu(x)d\mu(y) = +\infty,$$

which implies (51). ■

**Proof of Assertion 4.** Note that

$$\begin{aligned}&(2a)^{-4n-8}\sum_{\sigma\in\mathcal{A}_N,\sigma(1)=1}\ \sum_{\sigma'\in\mathcal{A}_N,\sigma'(1)=2}\int_{\Gamma^{\sigma}}\int_{\Gamma^{\sigma'}} \frac{dx\ dy}{d(x,y)^{d_H+\tau}}\\ \le& 2^{-4n-8}\sum_{\sigma\in\mathcal{A}_N,\sigma(1)=1}\ \sum_{\sigma'\in\mathcal{A}_N,\sigma'(1)=2}\frac{1}{d(\Gamma^{\sigma},\Gamma^{\sigma'})^{d_H+\tau}}.\end{aligned} \tag{61}$$

First of all observe that if $\text{prox}(\sigma)=0$ or $\text{prox}(\sigma')=0$, $d(\Gamma^{\sigma},\Gamma^{\sigma'}) > c$ for some positive constant $c$ independent of $n$ and the contribution of such $(\sigma,\sigma')$ to the right hand side of (61) is bounded by a constant independent of $n$.

We are left with finding a bound for

$$2^{-4n-8}\sum_{\sigma\in\mathcal{A}_N,\sigma(1)=1,\text{prox}(\sigma)>0}\ \sum_{\sigma'\in\mathcal{A}_N,\sigma'(1)=2,\text{prox}(\sigma')>0}\frac{1}{d(\Gamma^{\sigma},\Gamma^{\sigma'})^{d_H+\tau}} = 2^{-4n-8}(2I+II),$$

where

$$\begin{aligned}I &= \sum_{j=1}^{n+1}\sum_{k=1}^{j-1}\ \sum_{\sigma\in\mathcal{A}_N,\sigma(1)=1,\text{prox}(\sigma)=\text{j}}\ \sum_{\sigma'\in\mathcal{A}_N,\sigma'(1)=1,\text{prox}(\sigma')=\text{k}}\frac{1}{d(\Gamma^{\sigma},\Gamma^{\sigma'})^{d_H+\tau}},\\ II &= \sum_{j=1}^{n+1}\ \sum_{\sigma\in\mathcal{A}_N,\sigma(1)=1,\text{prox}(\sigma)=\text{j}}\ \sum_{\sigma'\in\mathcal{A}_N,\sigma'(1)=1,\text{prox}(\sigma')=\text{j}}\frac{1}{d(\Gamma^{\sigma},\Gamma^{\sigma'})^{d_H+\tau}}.\end{aligned}$$

But, from (49),

$$
\begin{aligned}
I &\lesssim \sum_{j=1}^{n+1}\sum_{k=1}^{j-1}\sum_{i=0}^{2^{\min(n,j)}-1}\sum_{\ell=0}^{2^k-1}\frac{2^{2(n-\min(j,n))}2^{2(n-k)}}{\left((a^{2j}+a^{2k})^2+\left(\ell-\left[\frac{i}{2^{\min(j,n)-k}}\right]\right)^2 a^{4k}\right)^{\frac{d_H+\tau}{2}}}\\
&\lesssim \sum_{j=1}^{n+1}\sum_{k=1}^{j-1}\frac{2^{4n-2k-2\min(j,n)}}{a^{2k(d_H+\tau)}}\sum_{i=0}^{2^{\min(n,j)}-1}\sum_{p=1}^{2^k}p^{-(d_H+\tau)}\\
&\lesssim \sum_{j=1}^{n+1}\sum_{k=1}^{j-1}\frac{2^{4n-2k-\min(j,n)}}{a^{2k(d_H+\tau)}}=\sum_{j=1}^{n}\sum_{k=1}^{j-1}\frac{2^{4n-2k-j}}{a^{2k(d_H+\tau)}}+\sum_{k=1}^{n}\frac{2^{3n-2k}}{a^{2k(d_H+\tau)}}\\
&= 2^{4n}\left(\sum_{j=1}^{n}\sum_{k=1}^{j-1}2^{-j+\frac{2\tau}{d_H}k}+\sum_{k=1}^{n}2^{-n+\frac{2\tau}{d_H}k}\right).
\end{aligned}
$$

From this, it is easy to prove that if $\tau < \frac{d_H}{2}$, then $2^{-4n}I$ is bounded by a constant independent of $n$.
A similar argument can be used for proving that if $\tau < \frac{d_H}{2}$, then $2^{-4n}II$ is bounded by a constant independent of $n$. ■

**Proof of Assertion 5.** We use the mapping $\aleph^{(N)}:\Gamma^\infty\backslash\mathcal{N}^\infty\to\Gamma^N$ (where $\mathcal{N}^\infty$ is defined by (18)) by

$$\aleph^{(N)}(x)=\mathcal{M}_\sigma(\aleph(y)) \qquad \text{if} \quad x=\mathcal{M}_\sigma(y), \quad \text{with} \quad y\in\Gamma^\infty\backslash\mathcal{N}^\infty \quad \text{and} \quad \sigma\in\mathcal{A}_N,$$

with $\aleph$ has been defined in (19). As a consequence of (20),

$$\frac{1}{2}(2a)^{-2n-4}\int_{\Gamma^N}f(x)dx=\int_{\Gamma^\infty\backslash\mathcal{N}^\infty}f\circ\aleph^N d\mu, \qquad \forall f\in L^1(\Gamma^N).$$

It is also easy to see that for all $x\in\Gamma^\infty\backslash\mathcal{N}^\infty$, $\lim_{n\to\infty}d(\aleph^N(x),x)=0$ and that

$$\lim_{n\to\infty}\frac{1}{d(\aleph^N(x),\aleph^N(y))^{d_H+\tau}}=\frac{1}{d(x,y)^{d_H+\tau}} \qquad \forall x\in F_1(\Gamma^\infty)\backslash\mathcal{N}^\infty, y\in F_2(\Gamma^\infty)\backslash\mathcal{N}^\infty. \tag{62}$$

Therefore,

$$
\begin{aligned}
&\frac{1}{4}(2a)^{-4n-8}\sum_{\sigma\in\mathcal{A}_N,\sigma(1)=1}\ \sum_{\sigma'\in\mathcal{A}_N,\sigma'(1)=2}\int_{\Gamma^\sigma}\int_{\Gamma^{\sigma'}}\frac{dx\,dy}{d(x,y)^{d_H+\tau}}\\
=&\frac{1}{4}(2a)^{-4n-8}\int_{F_1(\Gamma^{N-1})}\int_{F_2(\Gamma^{N-1})}\frac{dx\,dy}{d(x,y)^{d_H+\tau}}\\
=&\int_{F_1(\Gamma^\infty)\backslash\mathcal{N}^\infty}\int_{F_2(\Gamma^\infty)\backslash\mathcal{N}^\infty}\frac{1}{d(\aleph^N(x),\aleph^N(y))^{d_H+\tau}}d\mu(x)d\mu(y).
\end{aligned}
$$

From this, (53), (62) and Fatou lemma, we deduce that

$$\int_{F_1(\Gamma^\infty)\backslash\mathcal{N}^\infty}\int_{F_2(\Gamma^\infty)\backslash\mathcal{N}^\infty}\frac{1}{d(x,y)^{d_H+\tau}}d\mu(x)d\mu(y)<\infty,$$

and we obtain (54) because

$$\int_{F_1(\Gamma^\infty)}\int_{F_2(\Gamma^\infty)}\frac{1}{d(x,y)^{d_H+\tau}}d\mu(x)d\mu(y)=\int_{F_1(\Gamma^\infty)\backslash\mathcal{N}^\infty}\int_{F_2(\Gamma^\infty)\backslash\mathcal{N}^\infty}\frac{1}{d(x,y)^{d_H+\tau}}d\mu(x)d\mu(y).$$

■

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